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The Generalized Odd Generalized Exponential Fréchet Model: Univariate, Bivariate and Multivariate Extensions with Properties and Applications to the Univariate Version

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Abstract

A new univariate extension of the Fréchet distribution is proposed and studied. Some of its fundamental statistical properties such as stochastic properties, ordinary and incomplete moments, moments generating functions, residual life and reversed residual life functions, order statistics, quantile spread ordering, Rényi, Shannon and q-entropies are derived. A simple type Copula based construction using Morgenstern family and via Clayton Copula is employed to derive many bivariate and multivariate extensions of the new model. We assessed the performance of the maximum likelihood estimators using a simulation study. The importance of the new model is shown by means of two applications to real data sets.

Key Words: Fréchet Model; Morgenstern Family; Simulation; Clayton Copula; Modeling; Copula; Entropies; Wright Generalized Hypergeometric Function.

Mathematical Subject Classification: 62N01; 62N02; 62E10.

1. Introduction and motivation

The extreme value theory (EVT) is very popular in the statistical literature, it is devoted to study stochastic series of independent and identically distributed (IID) random variables. In EVT, we study the behavior of extreme value even though these values have a very low chance to occur but can turn out to have a very high impact on the observed system. The EVT plays an important role in research fields of finance and insurance. The study of EVT started in the last century as an equivalent to the central limit theory (CLT), which is dedicated to the study of asymptotic distribution for average of a sequence of random variables (RVs). The CLT states that sum and mean of the RVs from an arbitrary distribution are normally distributed under the condition that the sample size (n) is sufficiently large. However, in some other studies we are looking for the limiting distribution of maximum or minimum values rather than the average. Assume that $Z_1, Z_2, \dots, Z_n$ is a sequence of IID RV distributed with cumulative distribution function (CDF) denote by $F(z)$. One of the most interesting statistics in research is the sample maximum $S_n = max\{Z_1, Z_2, \dots, Z_n\}$. The EVT study the behavior of S_n as the sample size n increases to ∞ where $Pr(S_n \leq z) = Pr\{Z_1 \leq z, Z_2 \leq z, \dots, Z_n \leq z,\} = Pr\{Z_1 \leq z\}Pr\{Z_2 \leq z\}\dots Pr\{Z_n \leq z\} = F(z)^n$. Suppose there exist two sequences of constants $(C_n > 0)$ and (D_n) such that $p_r\left\{\frac{S_n - D_n}{C_n} \leq x\right\} \to G(z)$ as $n \to \infty$. If $G(z)$ is a non-degenerate distribution function, it will belong to one of the three following fundamental types of classic extreme value family, the Gumbel distribution (Type **I**); the Fréchet distribution (Type **II**); the Weibull distribution (Type **III**). A RV X is said to have the Fréchet (Fr) distribution if its probability density function (PDF) and CDF are given by

$$g_{a,b}(x) = a^b b x^{-(b+1)}\, exp\left[-\left(\frac{a}{x}\right)^b\right], \tag{1}$$

and

$$G_{a,b}(x) = exp\left[-\left(\frac{a}{x}\right)^b\right]. \tag{2}$$

The cumulative distribution function (CDF) and probability density function (PDF) of the Generalized Odd Generalized Exponential G (GOGE-G) family are given, respectively, by

$$F_{\alpha,\beta}(x) = \left\{1 - exp\left[\frac{-G_\varphi(x)^\alpha}{1-G_\varphi(x)^\alpha}\right]\right\}^\beta, \tag{3}$$

$$f_{\alpha,\beta}(x) = \frac{\alpha\beta g_\varphi(x) G_\varphi(x)^{\alpha-1}}{\left[1-G_\varphi(x)^\alpha\right]^2} exp\left[\frac{-G_\varphi(x)^\alpha}{1-G_\varphi(x)^\alpha}\right]\left\{1 - exp\left[\frac{-G_\varphi(x)^\alpha}{1-G_\varphi(x)^\alpha}\right]\right\}^{\beta-1}, \tag{4}$$

where $G_\varphi(x)$ is the baseline CDF depending on a parameter vector φ, $g_\varphi(x) = \frac{d}{dx}G_\varphi(x)$ is the corresponding PDF and $\alpha, \beta > 0$ are two additional shape parameters. Using (2) and (3) the CDF of the GOGEFr can be derived as

$$F_{\underline{\Psi}}(x) = \left(1 - exp\left(\frac{-\mathcal{E}_{x;\alpha,a,b}}{1-\mathcal{E}_{x;\alpha,a,b}}\right)\right)^\beta, \tag{5}$$

where $\mathcal{E}_{x;\alpha,a,b} = exp\left[-\alpha\left(\frac{a}{x}\right)^b\right]$ and the corresponding PDF of (5) can be expressed as

$$f_{\underline{\Psi}}(x) = \frac{\alpha\beta a^b b x^{-(b+1)}\mathcal{E}_{x;\alpha,a,b}}{\left(1-\mathcal{E}_{x;\alpha,a,b}\right)^2} exp\left(\frac{-\mathcal{E}_{x;\alpha,a,b}}{1-\mathcal{E}_{x;\alpha,a,b}}\right)\left[1 - exp\left(\frac{-\mathcal{E}_{x;\alpha,a,b}}{1-\mathcal{E}_{x;\alpha,a,b}}\right)\right]^{\beta-1}. \tag{6}$$

Henceforth, $X \sim$ GOGEFr ($\underline{\Psi}$)| $\underline{\Psi} = \alpha, \beta, a, b$ denotes a RV with density function in (6). The hazard rate function (HRF) of X can be derived using the well-known relationship $f_{\underline{\Psi}}(x)/\left[1 - F_{\underline{\Psi}}(x)\right]$. Plots of the GOGEFr HRF at some parameters value are presented in Figure 2 to show the flexibility of the new model. The GOGEFr model can be simulated and X values are generated using inversion of (5) as

$$x_u = a\left\{-\ln\left[\frac{-\log\left(1-u^{\frac{1}{\beta}}\right)}{1-\log\left(1-u^{\frac{1}{\beta}}\right)}\right]^{\frac{1}{\alpha}}\right\}^{-\frac{1}{b}}, \tag{7}$$

where $U \sim u(0,1)$. Now, we provide a useful representation for (6). Using the series expansion $(1-z)^\Upsilon = \sum_{w=0}^{\infty} \frac{(-1)^w \Gamma(1+\Upsilon)}{w!\,\Gamma(1+\Upsilon-w)} z^w$, which holds for $|z| < 1$ and $\Upsilon > 0$ real non-integer and using the power series, the PDF of the GOGEFr density in (6) can be expressed as

$$f(x) = \alpha\beta a^b b x^{-(b+1)} \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}(i+1)^j}{j!\left(1-\mathcal{E}_{x;\alpha,a,b}\right)^{j+2}}\binom{\beta-1}{i}\left(\mathcal{E}_{x;\alpha,a,b}\right)^{\alpha(j+1)}.$$

Using the series expansion again we obtain

$$f(x) = \sum_{j,k=0}^{\infty} \xi_{j,k}\, h_{\alpha^*}(x)|_{\alpha^*=\alpha(1+j+k)}, \tag{8}$$

where $h_{\alpha^*}(x)$ is the Fr density with scale parameter $a\sqrt[b]{\alpha_*}$ and shape parameter b and

$$\xi_{j,k} = \alpha\beta\frac{(-1)^{j+k}}{j!\,\alpha^*}\binom{-(j+2)}{k}\sum_{i=0}^{\infty}(-1)^i(i+1)^j\binom{\beta-1}{i}.$$

The CDF of the GOGEFr model can also be expressed as a mixture of the Fr density CDFs. By integrating (8), we obtain the same mixture representation $F(x) = \sum_{j,k=0}^{\infty} \xi_{j,k}\, H_{\alpha^*}(x)$ where $H_{\alpha^*}(x)$ is the CDF of the Fr density with scale parameter $a\sqrt[b]{\alpha_*}$ and shape parameter b. Figure 1 shows that the new density function can take unimodal, symmetric and right skewed shapes. Figure 2 shows that the HRF may be “increasing-constant”, “decreasing”, “increasing”, “upside-down” or “constant” failure rate function. Many useful mathematical tools can be found in Cordeiro and

Lemonte (2014), MirMostafaee et al. (2015, 2016 and 2017). and Hamedani et al. (2019). The new distribution can be used for evaluating entrepreneurial opportunities, see Adel Rastkhiz et al (2019). for more information on this regard.

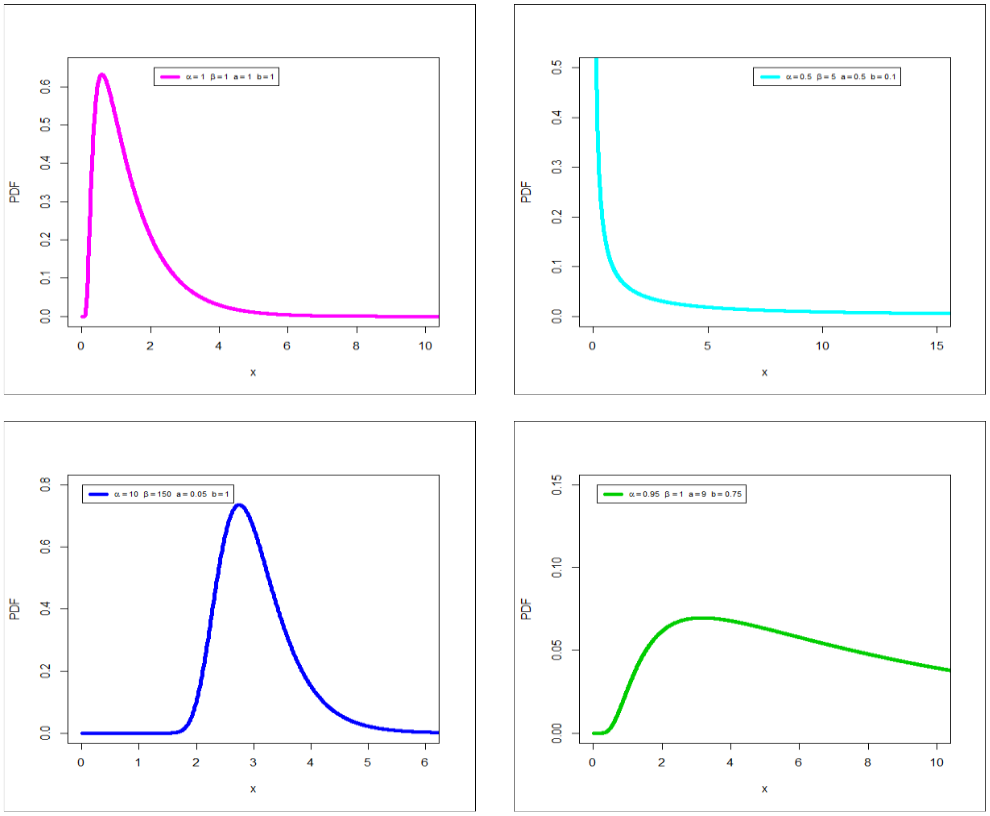

Figure 1: PDF plots of the GOGEFr for some parameter's values.

Some useful extensions of the Fr model can be cited as: transmuted exponentiated generalized Fr (2015) by Yousof et el. (2015), Kumaraswamy transmuted Marshall-Olkin Fr by Yousof et al. (2016), Weibull Fr by Afify et al. (2016b), Kumaraswamy Marshall-Olkin Fr by Afify et al. (2016a), Odd Lindley Fr by Korkmaz et al. (2017), Transmuted Topp Leone Fr by Yousof et al. (2018b), odd log-logistic Fr by Yousof et al. (2018a). Other extension can be found in Cordeiro and Lemonte (2014), MirMostafaee et al. (2015, 2016 and 2017), Brito et al. (2017), Hamedani et al. (2017), Cordeiro et al. (2018), Chakraborty et al. (2018), Hamedani et al. (2018), Korkmaz et al. (2018), Ibrahim (2019) and Korkmaz et al. (2019).

2. Mathematical properties

2.1 Moments and cumulants

The rth ordinary moment of X is given by $\mu'_r = E(X^r) = \int_{-\infty}^{\infty} f(x)\, dx$. Then, we obtain

$$\mu'_r|_{(b>r)} = a^r \Gamma\left(1-\frac{r}{b}\right)\sum_{j,k=0}^{\infty} \xi_{j,k}\,\alpha_*^{\frac{r}{b}}, \quad (9)$$

where $\Gamma(1+\varphi) = \varphi! = \prod_{h=0}^{\varphi-1}(\varphi-h) = \int_0^{\infty} t^{\varphi}\, exp(-t)\, dt$ is the complete gamma function. Setting $r=1$ in (9), we have the mean of X. The last integration can be computed numerically for most parent distributions. The skewness and kurtosis measures can be calculated from the ordinary moments using the well-known relationships. The skewness and kurtosis measures can also be calculated from the ordinary moments using the well-known relationships. The mean, variance, skewness and kurtosis of the GOGEFr distribution are computed numerically for some selected parameter values using the R software. The numerical values displayed in Table 1 indicate that the skewness of the GOGEFr distribution is always positive and can range in the interval $(0.23,149.38)$. The spread of its kurtosis is much larger ranging from 1.085 to 22316.32.

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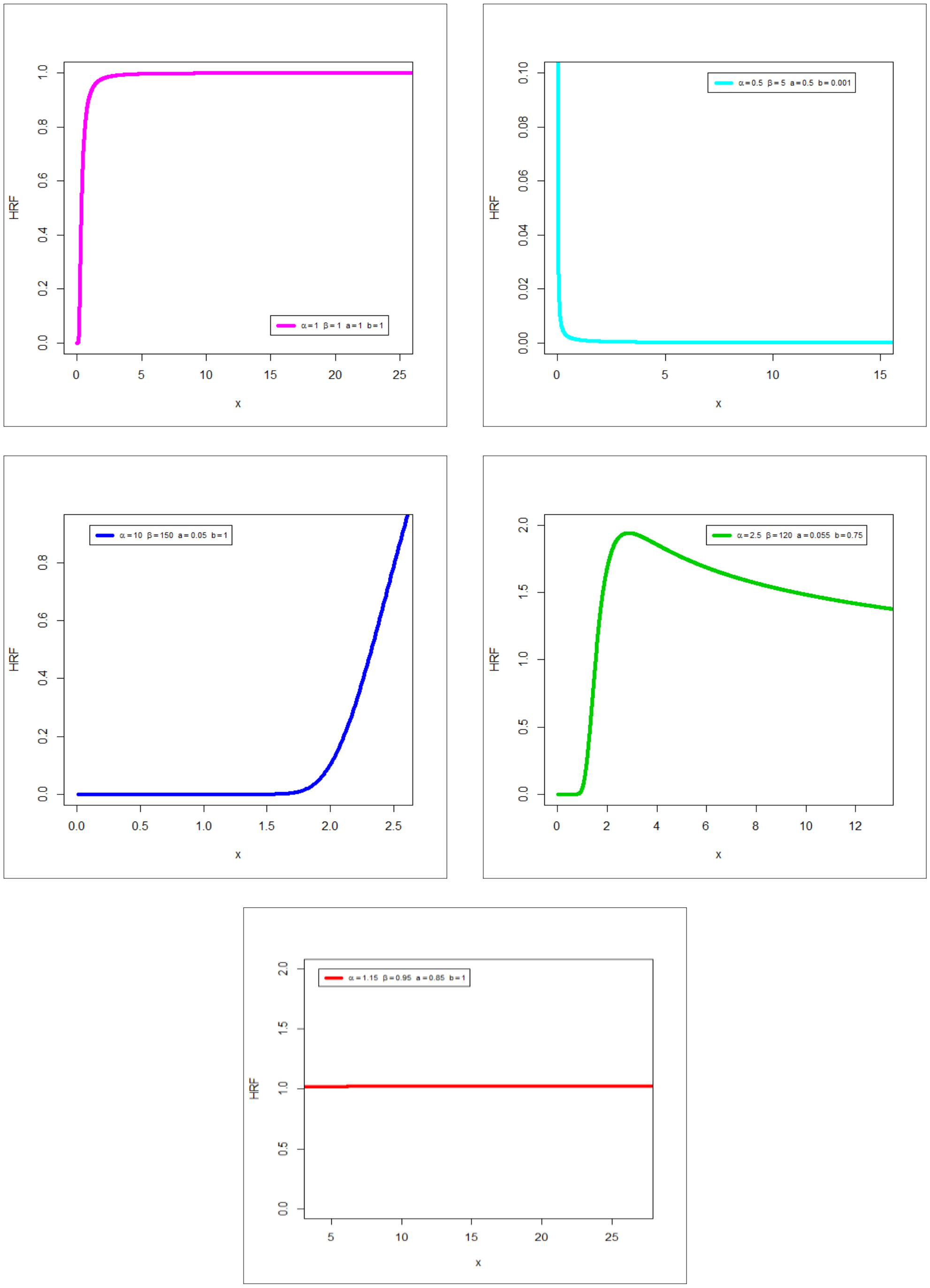

Figure 2: HRF plots of the GOGEFr for some parameter's values.

Table 1: E(X), Var(X), Ske(X) and Ku(X) of the GOGEFr distribution.

α	β	a	b	mean	variance	skewness	kurtosis
0.005	0.75	0.5	0.5	9.59×10^{-8}	2.05×10^{-10}	149.3827	22316.3
0.01				3.31×10^{-5}	6.99×10^{-8}	7.972852	69.7448
0.05				0.003072	3.63×10^{-5}	6.191395	77.2233
0.1				0.012292	0.000581	6.182813	77.1168
0.5				0.076836	0.022688	6.182308	77.1127
0.75				0.691525	1.837693	6.182282	77.1123
0.95				0.888226	3.031821	6.182283	77.1123
1	1	2	2	2.235140	0.639377	0.791212	3.62208
	2			2.678900	0.590066	0.651933	3.48356
	5			3.239802	0.491876	0.599869	3.53532
	10			3.630946	0.421442	0.617260	3.63095
	50			4.429738	0.303323	0.706449	3.87665
	100			4.734894	0.268809	0.743785	3.97437
	200			5.110491	0.233254	0.786577	4.09045
	500			5.377014	0.211949	0.814033	4.16822
	1000			5.630711	0.194205	0.837785	4.23801
	5000			6.179000	0.162655	0.881684	4.37394
	10000			6.400429	0.152049	0.896859	4.42321
3	10	1	2.5	2.494324	0.126232	0.506266	3.41228
		5		12.47162	3.155788	0.506266	3.41228
		20		49.88648	50.49261	0.506266	3.41228
		60		149.6594	454.4335	0.506266	3.41228
		100		249.4324	1262.315	0.506267	3.41228
		350		873.0130	15463.36	0.506266	3.41228
		400		997.7300	20197.05	0.506266	3.41228
3	4	10	1	82.41150	1347.000	1.308852	5.91950
				28.05750	36.89070	0.599871	3.53532
			3	19.79160	8.081260	0.385470	3.17539
			4	16.65430	3.209200	0.280390	3.05748
			5	15.02520	1.669670	0.217800	3.00533

2.2 Incomplete moment

The r th incomplete moment, say $\phi_r(t)$, of X can be expressed, from (8), as

$$\phi_r(t) = \int_{-\infty}^{t} x^r f(x)dx = a^r \sum_{j,k=0}^{\infty} \xi_{j,k} \alpha_*^{\frac{r}{b}} \gamma\left(1 - \frac{r}{b}, \alpha^* \left(\frac{a}{t}\right)^b\right)|_{(b>r)},$$

where $\gamma(\varphi, u)$ is the incomplete gamma function.

$$\gamma(\varphi, u)|_{(\varphi\neq 0,-1,-2,\dots)} = \int_0^u t^{\varphi-1} exp(-t)\, dt = \frac{u^\varphi}{\varphi}\{1F_1[\varphi; \varphi+1; -u]\} = \sum_{\zeta=0}^{\infty} \frac{(-1)^\zeta}{\zeta!\,(\varphi+\zeta)} u^{\varphi+\zeta},$$

and $1F_1[\cdot,\cdot,\cdot]$ is a confluent hypergeometric function. The first incomplete moment can be calculated by setting $r = 1$ in $\phi_r(t)$ as

$$\phi_1(t) = a \sum_{j,k=0}^{\infty} \xi_{j,k} \sqrt[b]{\alpha_*} \Gamma\left(1 - \frac{1}{b}, \alpha^* \left(\frac{a}{t}\right)^b\right)|_{(b>1)}.$$

2.3 The moment generating function (MGF)

The MGF $M_X(t) = E(e^{tX})$ of X can be derived from equation (4) as $M_X(t)|_{(b>r)} = \sum_{j,k,r=0}^{\infty} \frac{t^r}{r!} \xi_{j,k} a^r \alpha_*^{\frac{r}{b}} \Gamma\left(1 - \frac{r}{b}\right)$.

An alternative method for deriving the MGF can be introduced by the Wright generalized hypergeometric function (WHGF) which is defined by

$$(c)\Psi_{(v)}\begin{bmatrix} a_1, A_1, \dots, a_c, A_c \\ b_1, B_1, \dots, b_v, B_v \end{bmatrix}; x\bigg] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{c} \Gamma(a_j + A_j n)}{\prod_{j=1}^{v} \Gamma(b_j + B_j n)} \frac{x^n}{n!}.$$

Then, the MGF of (1) can be defined as

$$M_X(t) = {}_{(1)}\Psi_{(0)}\left[\left(1, -\frac{1}{b}\right); at\right]. \quad (10)$$

By combining expressions (8) and (10), we obtain the MGF of GOGEFr as

$$M_X(t) = \sum_{j,k=0}^{\infty} \xi_{j,k} \left\{(1)\Psi_{(0)}\left[\left(1, -\frac{1}{b}\right); a\sqrt[b]{\alpha_*}t\right]\right\}. \quad (11)$$

Equations (9) and (11) can be easily evaluated by scripts of the Maple, Matlab and Mathematica platforms.

2.4 Residual life and reversed residual life functions

The n th moment of the residual life, say $z_n(t) = E[(X-t)^n] \mid_{(X>t \text{ and } n=1,2,\dots)}$, uniquely determines the CDF $F(x)$. The n th moment of the residual life of X is given by $z_n(t) = \frac{\int_t^{\infty}(x-t)^n dF(x)}{1-F(t)}$. Therefore,

$$z_n(t)|_{(b>n)} = \frac{a^n}{1-F(t)} \sum_{j,k=0}^{\infty} A_{j,k}\alpha_*^{\frac{n}{b}} \Gamma\left(1-\frac{n}{b}, \alpha^*\left(\frac{a}{t}\right)^b\right),$$

where $A_{j,k} = \xi_{j,k}\sum_{r=0}^{n} (-t)^{n-r}\binom{n}{r}$, $\Gamma(\varphi,u)|_{(x>0)} = \int_u^{\infty} t^{\varphi-1} exp(-t)\, dt$ and $\Gamma(\varphi,u) = \Gamma(\omega) - \gamma(\varphi,u)$. The n th moment of the reversed residual life, say $Z_n(t) = E[(t-X)^n] \mid_{(X\le t, t>0 \text{ and } n=1,2,\dots)}$ uniquely determines the CDF $F(x)$. We obtain $Z_n(t) = \frac{\int_0^t (t-x)^n dF(x)}{F(t)}$. Then, the n th moment of the reversed residual life of X becomes

$$Z_n(t)|_{(b>n)} = \frac{a^n}{F(t)} \sum_{j,k=0}^{\infty} B_{j,k}\alpha_*^{\frac{n}{b}} \gamma\left(1-\frac{n}{b}, \alpha^*\left(\frac{a}{t}\right)^b\right),$$

where $B_{j,k} = \xi_{j,k}\sum_{r=0}^{n} (-1)^r t^{n-r}\binom{n}{r}$.

2.5 Entropies

The Rényi entropy of a RV X represents a measure of uncertainty and defined by

$$R_\theta(X)|_{(\theta>0 \text{ and } \theta\neq 1)} = \frac{1}{1-\theta} log \int_{-\infty}^{\infty} f(x)^\theta\, dx.$$

Using the PDF in (6), we obtain

$$f(x)^\theta = \sum_{j,k=0}^{\infty} \zeta_{j,k}^{(\theta)}\, x^{-\theta(b+1)}\, exp\left[-[\alpha(j+k+\theta)]\left(\frac{a}{x}\right)^b\right],$$

where

$$\zeta_{j,k}^{(\theta)} = (\alpha\beta a^b b)^\theta \frac{(-1)^{j+k}}{j!}\binom{-(j+2)}{k}\sum_{i=0}^{\infty} (-1)^i (i+\theta)^j \binom{\theta(\beta-1)}{i}.$$

Then, the Rényi entropy of the GOGEFr model is given by

$$R_\theta(X)|_{(\theta>0 \text{ and } \theta\neq 1)} = \frac{1}{1-\theta} log\left\{\sum_{j,k=0}^{\infty} \zeta_{j,k}^{(\theta)}\, [(\theta)I_0^\infty]\right\},$$

where $(\theta)I_0^\infty = \int_0^\infty x^{-\theta(b+1)}\, exp\left\{-[\alpha(j+k+\theta)]\left(\frac{a}{x}\right)^b\right\} dx$. The q-entropy, say $Q_q(X)$, can be defined as

$$Q_q(X)|_{(q>0 \text{ and } q\neq 1)} = \frac{1}{q-1} log\left(1-\left\{\sum_{j,k=0}^{\infty} \zeta_{j,k}^{(q)}\, [(q)I_0^\infty]\right\}\right),$$

where

$$\zeta_{j,k}^{(q)} = (\alpha\beta a^b b)^q \frac{(-1)^{j+k}}{j!}\binom{-(j+2)}{k}\sum_{i=0}^{\infty} (-1)^i (i+q)^j \binom{q(\beta-1)}{i},$$

and $(q)I_0^{\infty} = \int_0^{\infty} x^{-q(b+1)}\, exp\left\{-[\alpha(j+k+q)]\left(\frac{a}{x}\right)^b\right\} dx$. The Shannon entropy of a RV X, say SE, is defined by $SE = E\{-[log\, f\,(X)]\}$. Where SE is a special case of the Rényi entropy, $R_\theta(X)|_{(\theta>0 \text{ and } \theta\neq 1)}$, when $\theta \uparrow 1$.

2.6 Order statistics

Let $X_1, X_2, \dots, X_n$ be a random sample (RS) from the GOGEFr distribution and let $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ be the corresponding order statistics. The PDF of the ith order statistic, say $X_{i:n}$, can be written as

$$f_{i:n}(x) = B^{-1}(i, 1+n-i)\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j} f(x)F^{j+i-1}(x), \tag{12}$$

where B $(\cdot,\cdot)$ is the beta function. Substituting (1) and (2) in (12) and using a power series expansion, we have

$$f(x)F(x)^{j+i-1} = \sum_{w,k=0}^{\infty} \tau_{w,k}\, h_{\alpha(w+k+1)}(x),$$

where

$$\tau_{w,k} = \frac{\alpha\beta(-1)^{w+k}}{w!\,[\alpha(w+m+1)]}\binom{-(w+2)}{k}\sum_{l=0}^{\infty}(-1)^l(l+1)^w\binom{\beta(i+j)-1}{l}.$$

Then, the PDF of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = \sum_{j=0}^{n-i}\sum_{w,k=0}^{\infty}(-1)^j B^{-1}(i, 1+n-i)\binom{n-i}{j}\tau_{w,k}h_{\alpha(w+k+1)}(x).$$

The density function of the GOGEFr order statistics is a mixture of Fr densities. Based on the previous equation, the moments of $X_{i\,:\,n}$ can be expressed as

$$E\left(X_{i:n}^q\right)|_{(b>q)} = a^q\Gamma\left(1-\frac{q}{b}\right)\sum_{w,k=0}^{\infty}\sum_{j=0}^{n-i}\frac{(-1)^j B^{-1}(i, 1+n-i)}{[\alpha(w+k+1)]^{\frac{-q}{b}}}\binom{n-i}{j}\tau_{w,k}.$$

2.7 Quantile spread order

The quantile spread ($\Lambda_{.}(q)$) of the RV $X \sim$ GOGEFr $(\underline{\Psi})$ having the in CDF (5) and given by $\Lambda_X(q)|_{\left(q\in\left(\frac{1}{2},1\right)\right)} = [F^{-1}(q)] - [F^{-1}(1-q)]$ which implies $\Lambda_X(q) = [S^{-1}(1-q)] - [S^{-1}(q)]$ where $F^{-1}(q) = S^{-1}(1-q)$ and $S(\cdot) = 1 - F(\cdot)$ is the survival function. The $\Lambda_{.}(q)$ of a distribution describes how the probability mass is placed symmetrically about its median and hence can be used to formalize concepts such as peakedness and tail weight traditionally associated with kurtosis. Hence, it allows to separate concepts of kurtosis and peakedness for asymmetric models. Let X_1 and X_2 be two RVs following the GOGEFr model with quantile spreads Λ_{X_1} and Λ_{X_2}, respectively. Then, X_1 is smaller than X_2 in quantile spread order and denoted as $X_1 \leq_{[\Lambda]} X_2$, if $\Lambda_{X_1}(q) \leq \Lambda_{X_2}(q)|_{\left(q\in(\frac{1}{2},1)\right)}$. The following properties of the QS order can be obtained:

- The order $\leq_{[\Lambda]}$ is "location-free": $X_1 \leq_{[\Lambda]} X_2$ if $(X_1 + \zeta) \leq_{[\Lambda]} X_2$ for any real ζ.
- Let F_{X_1} and F_{X_2} be symmetric, then $X_1 \leq_{[\Lambda]} X_2$ if, and only if $F_{X_1}^{-1}(q) \leq F_{X_2}^{-1}(q), \forall\, q \in \left(\frac{1}{2}, 1\right)$.
- The order $\leq_{[\Lambda]}$ implies ordering of the mean absolute deviation around the median, say $\Upsilon(X_i)|_{i=1,2}$, $\Upsilon(X_1) = E[|X_1 - Median(X_1)|]$ and $\Upsilon(X_2) = E[|X_2 - Median(X_2)|]$, where $X_1 \leq_{[\Lambda]} X_2$ implies $\Upsilon(X_1) \leq_{[\Lambda]} \Upsilon(X_2)$.

Finally, $X_1 \leq_{[\Lambda]} X_2$ if, and only if $-X_1 \leq_{[\Lambda]} - X_2$.

3. Simple type Copula based construction

3.1 Bivariate GOGEFr (BivGOGEFr) type via Renyi's entropy

Following Pougaza and Djafari (2011), The joint CDF OF THE Renyi's entropy Copula can be expressed as $C(u,v)|_{u,v\in(0,1)} = x_2u + x_1v - x_1x_2$. Then, the associated bivariate GOGEFr can be derived with setting $u = F_{\underline{\Psi}_1}(x_1)$ and $v = F_{\underline{\Psi}_2}(x_2)$.

3.2 BivGOGEFr type via Farlie Gumbel Morgenstern (FGM) Copula

Consider the joint CDF of the FGM family, then $C_\Delta(u,v) = uv(1 + \Delta\overline{u}\overline{v})$, where the marginal function $u = F_1(y_1)$, $v = F_2(y_2)$, $\Delta \in [-1,1]$ is a dependence parameter and for every $\overline{u}, \overline{v} \in (0,1)$, $\overline{u} = 1-u, \overline{v} = 1-v$, $C_\Delta(u,0) = C_\Delta(0,v) = 0$ which is "grounded minimum" and $C_\Delta(u,1) = u$ and $C_\Delta(1,v) = v$ which is "grounded maximum".

Then, setting $\overline{u} = 1 - F_{\underline{\Psi}_1}(x_1)$ and $\overline{v} = 1 - F_{\underline{\Psi}_2}(x_2)$. Then, we have $F(x_1, x_2) = C\left(F_{\underline{\Psi}_1}(x_1), F_{\underline{\Psi}_2}(x_2)\right)$. The joint PDF can be derived from $c_\Delta(u,v) = 1 + \Delta u^* v^*|_{(u^*=1-2u \text{ and } v^*=1-2v)}$ or from $f(x_1,x_2) = f_{\underline{\Psi}_1}(x_1), f_{\underline{\Psi}_2}(x_2) c(F_{\underline{\Psi}_1}(x_1), F_{\underline{\Psi}_2}(x_2))$. For more details see Morgenstern (1956), Farlie (1960), Gumbel (1960), Gumbel (1961), Johnson and Kotz (1975) and Johnson and Kotz (1977).

3.3 BivGOGEFr type via modified FGM Copula

Due to Rodriguez-Lallena and Ubeda-Flores (2004), the modified joint CDF of the bivariate FGM Copula can be expressed as $C_\Delta(u,w) = uv + \Delta\widetilde{\mathbf{O}(u)}\widetilde{\boldsymbol{\varphi}(w)}$, where $\widetilde{\mathbf{O}(u)} = u\overline{\mathbf{O}(u)}$, and $\widetilde{\boldsymbol{\varphi}(w)} = w\overline{\boldsymbol{\varphi}(w)}$. The functions $\mathbf{O}(u)$ and $\boldsymbol{\varphi}(w)$ are two absolutely continuous functions on the interval $(0,1)$ with the following conditions:

(**I**)-The boundary condition: $\mathbf{O}(0) = \mathbf{O}(1) = \boldsymbol{\varphi}(0) = \boldsymbol{\varphi}(1) = 0$.

(**II**)-Let

$$p = inf\left\{\frac{\partial}{\partial u}\widetilde{\mathbf{O}(u)} : \boldsymbol{\mathcal{L}}_1(u)\right\} < 0, q = sup\left\{\frac{\partial}{\partial u}\widetilde{\mathbf{O}(u)} : \boldsymbol{\mathcal{L}}_1(u)\right\} < 0,$$

$$w = inf\left\{\frac{\partial}{\partial v}\widetilde{\boldsymbol{\varphi}(w)} : \boldsymbol{\mathcal{L}}_2(w)\right\} > 0, m = sup\left\{\frac{\partial}{\partial v}\widetilde{\boldsymbol{\varphi}(w)} : \boldsymbol{\mathcal{L}}_2(w)\right\} > 0.$$

Then, $min(pq, wm) \geq 1$. Where $\frac{\partial}{\partial u}\widetilde{\mathbf{O}(u)} = \mathbf{O}(u) + u\frac{\partial}{\partial u}\mathbf{O}(u)$,

$\boldsymbol{\mathcal{L}}_1(u) = \left\{u \in (0,1) \;:\; \frac{\partial}{\partial u}\widetilde{\mathbf{O}(u)} \text{ exists}\right\}$ and $\boldsymbol{\mathcal{L}}_2(w) = \left\{v \in (0,1) \;:\; \frac{\partial}{\partial v}\widetilde{\boldsymbol{\varphi}(w)} \text{ exists}\right\}$.

3.3.1 BivGOGEFr-FGM (Type I) model

The BivGOGEFR-FGM (Type **I**) can be derived using $C_\Delta(u,w) = uv + \Delta\widetilde{\mathbf{O}(u)}\widetilde{\boldsymbol{\varphi}(w)}$, where $\widetilde{\mathbf{O}(u)} = u\overline{\mathbf{O}(u)} = u\left[1 - F_{\underline{\Psi}_1}(u)\right]$, and $\widetilde{\boldsymbol{\varphi}(w)} = w\overline{\boldsymbol{\varphi}(w)} = w\left[1 - F_{\underline{\Psi}_2}(w)\right]$.

3.3.2 BivGOGEFr-FGM (Type II) model:

Consider the following functional form for both $\mathbf{O}(u)$ and $\boldsymbol{\varphi}(w)$ which satisfy all the conditions stated earlier where

$$\mathbf{O}(u)|_{(\Delta_1>0)} = u^{\Delta_1}(1-u)^{1-\Delta_1} \text{ and } \boldsymbol{\varphi}(w)|_{(\Delta_2>0)} = v^{\Delta_2}(1-w)^{1-\Delta_2}.$$

The corresponding BivGOGEFr-FGM (Type **II)** Copula can be derived from

$$C_{\Delta,\Delta_1,\Delta_2}(u,w) = uw[1 + \Delta u^{\Delta_1} w^{\Delta_2}(1-u)^{1-\Delta_1}(1-w)^{1-\Delta_2}].$$

3.3.3 BivGOGEFr-FGM (Type III) model:

Consider the following functional form for both $\mathbf{O}(u)$ and $\boldsymbol{\varphi}(w)$ which satisfy all the conditions stated earlier where

$$\mathbf{O}(u) = u[log(1+\overline{u})] \text{ and } \boldsymbol{\varphi}(w) = w[log(1+\overline{w})].$$

In this case, one can also derive a closed form expression for the associated CDF of the BivGOGEFr-FGM (Type **III**).

3.3.4 BivGOGEFr-FGM (Type IV) model:

due to Ghosh and Ray (2016) the CDF of the BivGOGEFR-FGM (Type **IV**) model can be derived from

$$C(u,w) = uF^{-1}(w) + wF^{-1}(u) - F^{-1}(u)F^{-1}(w).$$

3.4 Bivariate GOGEFr type via Clayton Copula

The Clayton Copula can be considered as $C(u_1,u_2) = (u_1^{-\nabla} + u_2^{-\nabla} - 1)^{-\frac{1}{\nabla}}|_{\nabla\in[0,\infty]}$. Let us assume that $Z \sim$ GOGEFR ($\underline{\Psi}_1$) and $W \sim$ GOGEFr ($\underline{\Psi}_2$). Then, setting $u_1 = F_{\underline{\Psi}_1}(z)$ and $u_2 = F_{\underline{\Psi}_2}(w)$. Then, the bivariate GOGEFR type distribution can be derived easily.

3.5 Multivariate GOGEFr extension via Clayton Copula

A straightforward d-dimensional extension using the Clayton Copula can expressed using

$$H(v_i) = \left[\sum_{i=1}^{d} u_i^{-\nabla} + 1 - d\right]^{-\frac{1}{\nabla}}.$$

4. Some stochastic properties

Suppose X_1 ~GOGEFr(α_1,β_1,a,b) and X_2 ~GOGEFr (α_2,β_2,a,b). Then X_1 is stochastically smaller than X_2 if $\alpha_1>\alpha_2$ and $\beta_1>\beta_2$. For $\alpha_1>\alpha_2$, $\mathcal{E}_{x_1;\alpha_1,a,b}>\mathcal{E}_{x_2;\alpha_2,a,b}$. This is true for both integer and fractional values of α_1 and α_2 and then we obtain $\mathcal{E}_{x_1;\alpha_1,a,b}>\mathcal{E}_{x_2;\alpha_2,a,b}$. Then, we have $\{1-\mathcal{E}_{x_1;\alpha_1,a,b}\}<\{1-\mathcal{E}_{x_2;\alpha_2,a,b}\}$,

$$\Rightarrow \frac{\mathcal{E}_{x_1;\alpha_1,a,b}}{1-\mathcal{E}_{x_1;\alpha_1,a,b}}>\frac{\mathcal{E}_{x_2;\alpha_2,a,b}}{1-\mathcal{E}_{x_2;\alpha_2,a,b}},\ \Rightarrow \frac{-\mathcal{E}_{x_1;\alpha_1,a,b}}{1-\mathcal{E}_{x_1;\alpha_1,a,b}}<\frac{-\mathcal{E}_{x_2;\alpha_2,a,b}}{1-\mathcal{E}_{x_2;\alpha_2,a,b}},$$

$$\Rightarrow \left[1-exp\left(\frac{-\mathcal{E}_{x_1;\alpha_1,a,b}}{1-\mathcal{E}_{x_1;\alpha_1,a,b}}\right)\right]^{\beta_1}>\left[1-exp\left(\frac{-\mathcal{E}_{x_2;\alpha_2,a,b}}{1-\mathcal{E}_{x_2;\alpha_2,a,b}}\right)\right]^{\beta_2},$$

then

$$1-\left[1-exp\left(\frac{-\mathcal{E}_{x_1;\alpha_1,a,b}}{1-\mathcal{E}_{x_1;\alpha_1,a,b}}\right)\right]^{\beta_1}<1-\left[1-exp\left(\frac{-\mathcal{E}_{x_2;\alpha_2,a,b}}{1-\mathcal{E}_{x_2;\alpha_2,a,b}}\right)\right]^{\beta_2}.$$

5. Estimation

Let $x_1,\ldots,x_n$ be a RS from the GOGEFr distribution with parameters α,β,a and b. Let $\underline{\Psi}$ be the 4×1 parameter vector. For determining the MLE of $\underline{\Psi}$, the log-likelihood function is

$$\ell=\ell(\underline{\Psi})=n\,log\,\alpha+n\,log\,\beta+n\,log\,b+nb\,log\,a-(b+1)\sum_{i=1}^{n}log\,x_i-2\sum_{i=1}^{n}log\left(1-\mathcal{E}_{x_i;\alpha,a,b}\right)$$
$$+\sum_{i=1}^{n}d_i+\sum_{i=1}^{n}log\,\mathcal{E}_{x_i;\alpha,a,b}+(\beta-1)\sum_{i=1}^{n}log[1-exp(d_i)],$$

where $d_i=\frac{-\mathcal{E}_{x_i;\alpha,a,b}}{1-\mathcal{E}_{x_i;\alpha,a,b}}$. The score vector components are easy to be derived, setting the nonlinear system of equations $\mathbf{U}_\alpha=\mathbf{U}_\beta=\mathbf{U}_a=0$ and $\mathbf{U}_b=0$ and solving them simultaneously yields the MLE.

6. Graphical assessment

Graphically, we can perform the simulation experiments to assess of the finite sample behavior of the MLEs. The assessment was based on the following algorithm:

- Use (7) we generate $N=10000$ samples of size n from the GOGEFR distribution;
- Compute the MLEs for the 1000 samples;
- Compute the SEs of the MLEs for the 1000 samples;
- Compute the biases and mean squared errors given for $\Psi=\alpha,\beta,a_1,a_2$. We repeated these steps for $n=50{,}100,\ldots,150$ with $\alpha=\beta=a=b=1$, so computing biases $(B_\Psi(n))$, mean squared errors ($MSEs$) $(MSE_h(n))$ for α,β,a,b and $n=50{,}100,\ldots,150$.

Figure 2 (left panel) shows how the four biases vary with respect to n . Figure 2 (right panel) shows how the four MSEs vary with respect to m . The broken lines in Figure 2 corresponds to the biases being 0 . From Figure 2, the biases for each parameter are generally negative and decrease to zero as $n\rightarrow\infty$, the MSEs for each parameter decrease to zero as $n\rightarrow\infty$.

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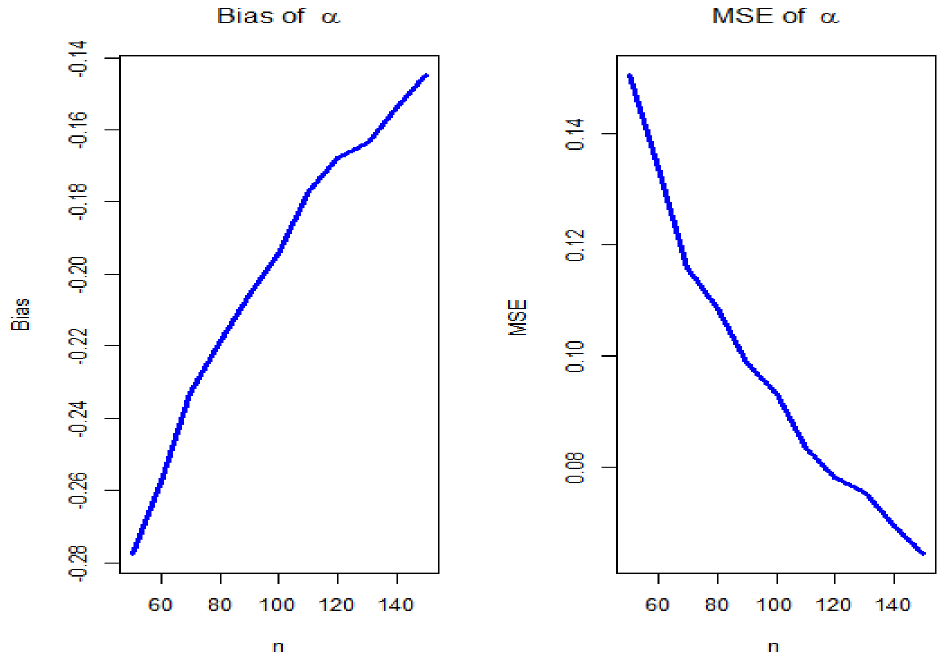


Figure 3: biases and mean squared errors for the parameter α.

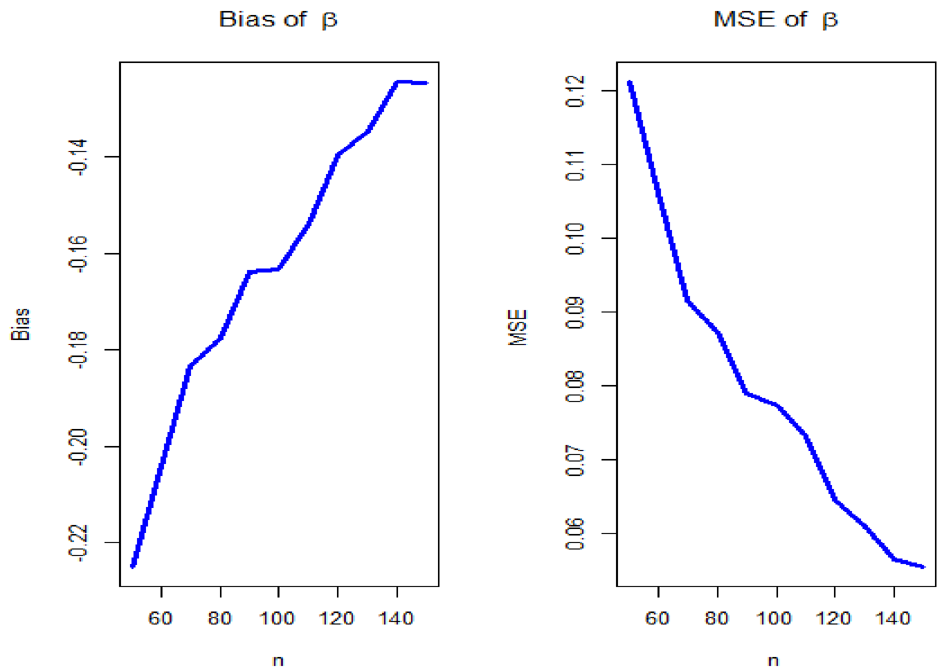


Figure 4: biases and mean squared errors for the parameter β.

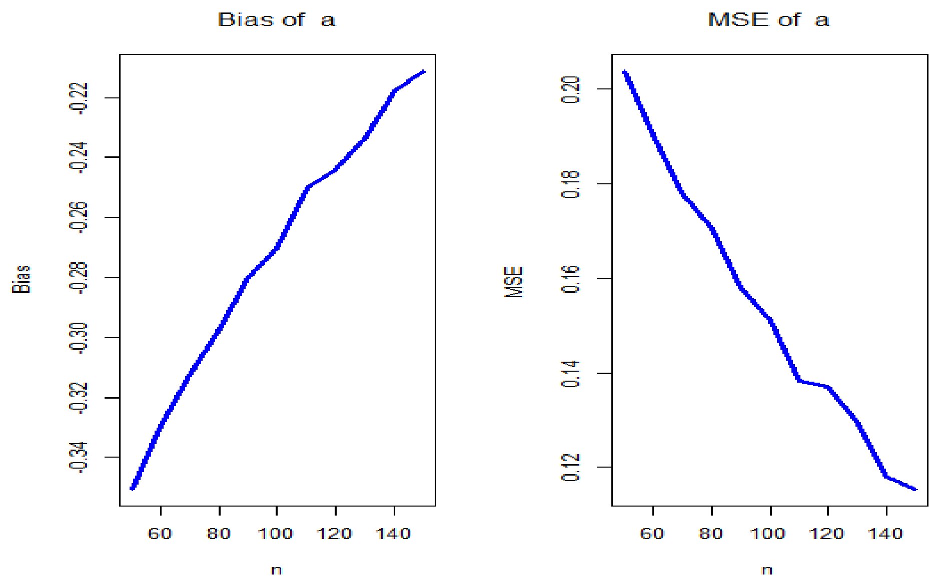


Figure 5: biases and mean squared errors for the parameter a.

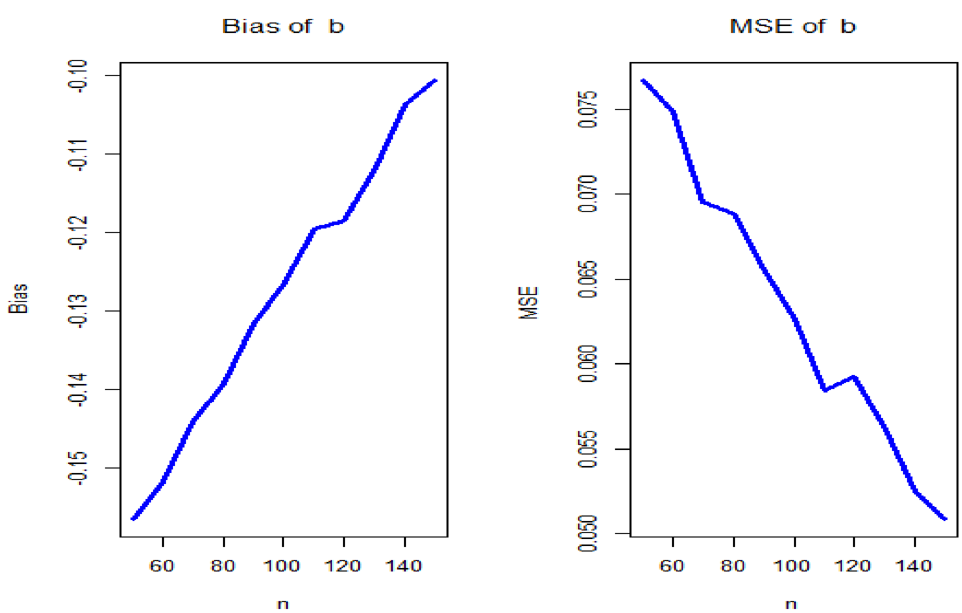


Figure 6: biases and mean squared errors for the parameter b.

7. Real data modeling

This section presents two applications of the new distribution using real data sets. We compare the fit of the new distribution with the Weibull Inverse Weibull (W-Fr), exponentiated Fr (E-Fr), Kumaraswamy Fr (Kum-Fr), beta Fr (B-Fr) transmuted Fr (T-Fr), gamma extended Fr (GE-Fr), Marshall-Olkin Fr (MO-Fr), MOKum-Fr, generalized MO-Fr(GMO-Fr), KumMO-Fr and Fr distributions. The PDFs of the competitive models are available in statistical literature. All the unknown parameters of the above PDFs are positive real numbers except for the T-Fr distribution for which $|\lambda| \leq 1$. The 1st data set consists of 100 observations of breaking stress of carbon fibers given by Nichols and Padgett (2006). The 2nd data set consists of 63 observations of the strengths of 1.5 cm glass fibers (see Smith and Naylor (1987)). In order to compare the distributions, we consider the AIC (Akaike Information Criterion), CAIC (Consistent Akaike Information Criterion), BIC (Bayesian Information Criterion) and HQIC (Hannan-Quinn Information Criterion) for comparing models. Total time test (TTT) plot (see Figure 7) is an important graphical approach to verify whether our data can be applied to a specific model. The TTT plots of the two real data sets are presented in Figure 7 (first row). These plots indicate that the empirical HRFs of the two data sets are "increasing HRF". The box plots of the two real data sets are presented in Figure 7 (second row). The normal Q-Q plots of the two real data sets are presented in Figure 7 (third row).

Table 3: Statistics of the AIC, BIC, HQIC and CAIC values for breaking stress data.

Model	Measures			
	AIC	BIC	HQIC	CAIC
GOGEFr	**118.5**	**128.9**	**122.8**	**118.9**
PBX-Fr	122.6	133.1	126.8	123.1
W-Fr	294.5	304.9	298.7	294.9
E-Fr	295.7	303.5	298.9	296.0
Kum-Fr	297.1	307.5	301.3	297.5
B-Fr	311.1	321.6	315.4	311.6
GE-Fr	312.0	332.4	316.2	312.4
Fr	348.3	353.5	350.4	348.4
T-Fr	350.5	358.3	353.6	350.7
MO-Fr	351.3	359.1	354.5	351.6

Table 4: MLEs and their standard errors for breaking stress of carbon fiber data.

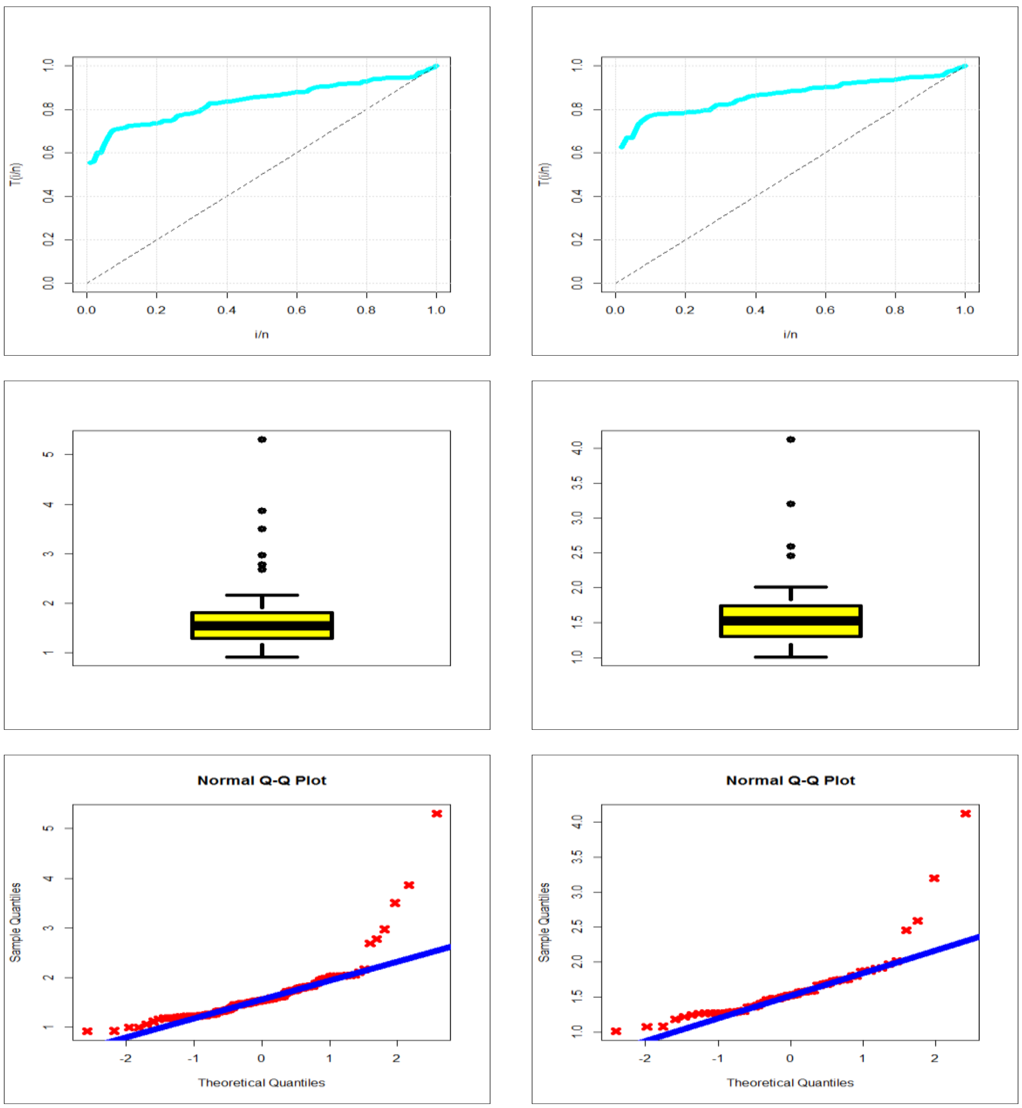
Model	Estimates			
GOGEFr(α,β,a,b)	2.4085	120.55	0.055	0.783
	(1.003)	(83.85)	(0.029)	(0.102)
PBX-Fr($\lambda,\theta,\alpha,\beta$)	4.900	3.4523	1.0310	0.742
	(1.247)	(1.024)	(0.193)	(0.117)
W-Fr(α,β,a,b)	2.2231	0.355	6.9721	4.9179
	(11.41)	(0.411)	(113.8)	(3.756)
Kum-Fr(α,β,a,b)	2.0556	0.4654	6.2815	224.18
	(0.071)	(0.007)	(0.063)	(0.164)
B-Fr(α,β,a,b)	1.6097	0.4046	22.014	29.762
	(2.498)	(0.108)	(21.43)	(17.48)
GE-Fr(α,β,a,b)	1.3692	0.4776	27.645	17.458
	(2.017)	(0.133)	(14.14)	(14.82)
E-Fr(α,β,a)	69.149	0.5019	145.33	
	(57.35)	(0.08)	(122.9)	
T-Fr(α,β,λ)	1.9315	1.7435	0.0819	
	(0.097)	(0.076)	(0.198)	
MO-Fr(α,β,a)	2.3066	1.5796	0.5988	
	(0.498)	(0.16)	(0.309)	
Fr(α,β)	1.8705	1.7766		
	(0.112)	(0.113)		

Table 5: Statistics of the AIC, BIC, HQIC and CAIC values for glass fibre data.

Model	Measures			
	AIC	BIC	HQIC	CAIC
GOGEFr	**55.96**	**64.52**	**59.33**	**56.65**
B-Fr	68.62	77.24	72.02	69.33
GE-Fr	69.64	78.12	72.92	70.34
Fr	97.73	102.0	99.43	97.93
T-Fr	100.1	106.5	102.6	100.5
MO-Fr	101.7	108.2	104.2	102.1

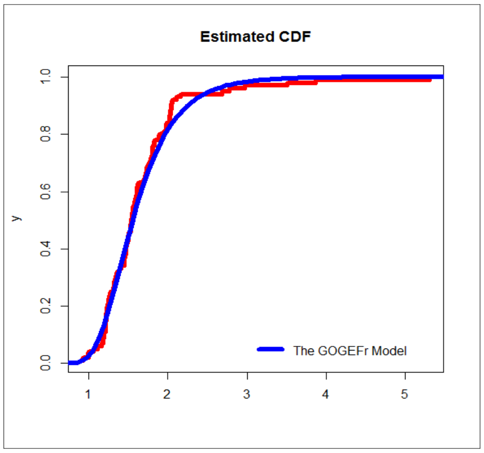
Table 6: MLEs and their standard errors for glass fibre data.

Model	Estimates			
GOGEFr(α,β,a,b)	0.901	32.693	0.4579	1.112
	(0.000)	(20.60)	(0.000)	(0.157)
B-Fr(α,β,a,b)	2.0518	0.6466	15.076	36.940
	(0.986)	(0.163)	(12.06)	(22.65)
GE-Fr(α,β,a,b)	1.6625	0.7421	32.112	13.269
	(0.952)	(0.197)	(17.397)	(9.967)
T-Fr(α,β,a)	1.3068	2.7898	0.1298	
	(0.034)	(0.165)	(0.208)	
MO-Fr(α,β,a)	1.5441	2.3876	0.4816	
	(0.226)	(0.253)	(0.252)	
Fr(α,β)	1.264	2.888		
	(0.059)	(0.234)		

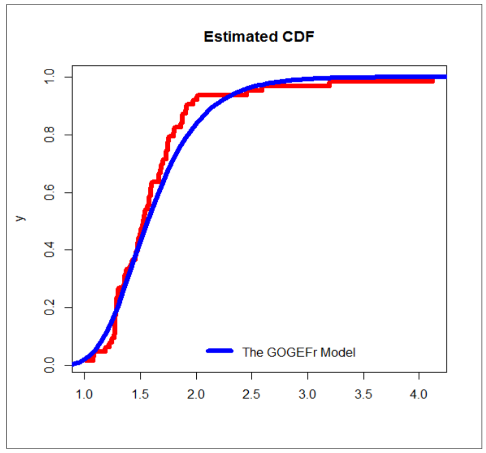


The 1st data set. The 2nd data set.

Figure 7: TTT plots.

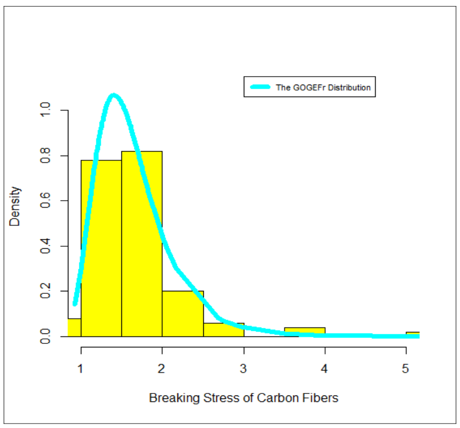


The 1st data set.

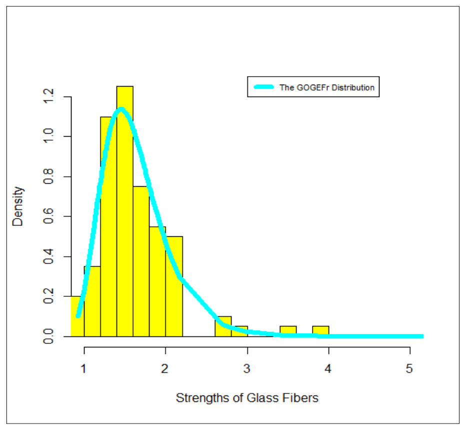


The 2nd data set.

Figure 8: Estimated CDFs.

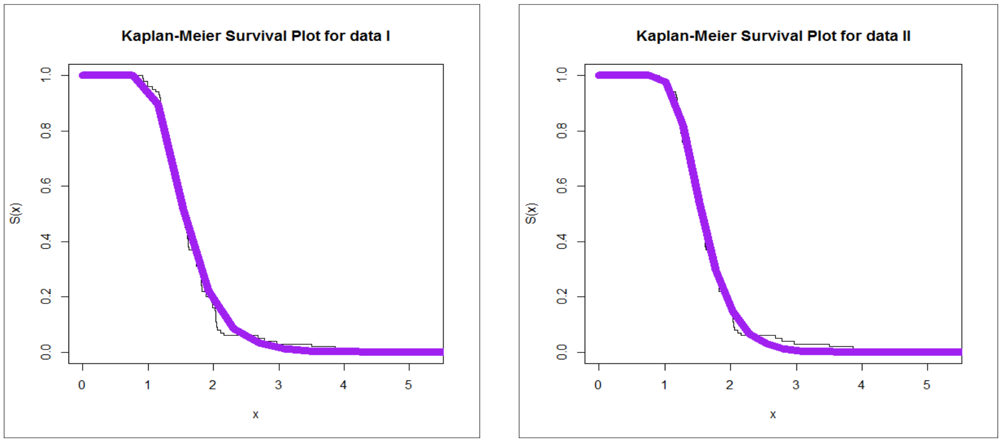


The 1st data set.



The 2nd data set.

Figure 9: Estimated PDFs.



The 1st data set. The 2nd data set.

Figure 10: Kaplan-Meier Survival Plots.

Tables 3 and 5 compare the GOGEFr model with other important competitive Fr distributions. The GOGEFr model gives the lowest values for the AIC, BIC, HQIC and CAIC statistics (in bold) among all fitted Fr versions to these data. Hence, it may be considered as the best model among them. Figures 8-10, respectively, display the plots of estimated CDFs, estimated PDFs and Kaplan-Meier survival plots for the two data sets. These plots reveal that the proposed distribution gives adequate fit for both data sets.

8. Conclusions

A new extension of the Fréchet model is proposed and studied. Some of its fundamental statistical properties such as, some stochastic properties, ordinary and incomplete moments, moments generating functions, residual life and reversed residual life functions, order statistics, quantile spread ordering, Rényi, Shannon and q-entropies are derived. A simple type Copula based construction via Renyi's entropy Copula Farlie Gumbel Morgenstern Copula, modified Farlie Gumbel Morgenstern Copula, Clayton Copula is employed to derive many bivariate and multivariate extensions of the new model. We assessed the performance of the maximum likelihood estimators using a simulation study. The importance of the new model is shown via two applications of real data sets.

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