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The Weighted Power Lindley Distribution with Applications on the Life Time Data

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Abstract

In this paper, we propose a new version of the power Lindley distribution known as weighted power Lindley distribution. The different structural properties of the new model are studied such as moments, generating functions, likelihood ratio test, entropy measures and order statistics. The maximum likelihood estimators of the parameters and the Fisher's information matrix are discussed. It also provides more flexibility to analyse complex real data sets. Applications of the model to real data sets are provided using the new distribution, which shows that the weighted power Lindley distribution can be used quite effectively in analysing real life time data.

Key Words: Weighted distribution, Power Lindley distribution, Reliability, Entropy, Order Statistics, Maximum likelihood estimator

1. Introduction

The weighted distributions are applied in various research areas related to biomedicine, reliability, ecology and branching processes. The concept of weighted distributions is traceable to the work of Fisher (1934) in respect of his studies on how methods of ascertainment can affect the form of distribution of recorded observations. Later, it was introduced and formulated in a more general way by Rao (1965) with respect to modelling statistical data where the routine practice of using standard distributions for the purpose was dismissed as inappropriate. The weighted distribution reduces to length biased distribution when the weight function considers only the length of the units. The concept of length biased sampling was first introduced by Cox (1969) and Zelen (1974). More generally, when the sampling mechanism selects units with probability proportional to some measure of the unit size, the resulting distribution is called size-biased. There are various good sources which provide the detailed description of weighted distributions. Many newly introduced distributions along with their weighted versions exist in literature whose statistical behaviour is extensively studied during decades. Das and Kundu (2016) discussed on various statistical properties of the weighted exponential distribution and its length biased version. Dar *et al.* (2018) obtained the weighted transmuted power distribution and discussed its properties and applications. Rather and Subramanian (2019) derived the weighted sushila distribution with various statistical properties and its applications.

A two-parameter power Lindley (PL) distribution was suggested by Ghitany *et al.* (2013). They introduced a new generalization of the Lindley distribution by considering the power transformation of the random variable $X=T^{1/\beta}$. Nadarajah *et al.* (2011) discussed another generalization of the Lindley distribution named as the generalized Lindley distribution. Ashour and Eltehiwy (2014) derived the exponentiated PL distribution with properties and its applications. Alizadeh *et al.* (2017) obtained a new extension of the PL distribution, namely odd log-logistic PL distribution, for analysing bimodal data and discussed its properties.

In this paper, we introduce a new distribution with three parameters, namely as weighted power Lindley (WPL) distribution, with the hope that it will attract many applications in different disciplines such as reliability, survival analysis, biology and others. On applying the weighted version, the third parameter indexed to this distribution makes it more flexible to describe different types of real data than its sub-models. The WPL distribution, due to its flexibility in accommodating different forms of the hazard function, seems to be more suitable distribution that can be used in a variety of problems in fitting survival data.

The paper is organized as follows: In Section 2, we define the proposed WPL distribution. Some structural properties are discussed in Section 3. The likelihood ratio test is given in Section 4. Then, Renyi and Tsallis entropy measures of the WPL distribution are obtained in Section 5. Order statistics are obtained in Section 6. Finally, the real life time data has been fitted and the fit has been found to be good.

2. Weighted Power Lindley (WPL) Distribution

2.1. Density and Cumulative Density Functions

The probability density function (pdf) of the PL distribution with parameters β and θ and is defined by

$$f(x) = \frac{\beta\theta^2(1+x^\beta)x^{\beta-1}e^{-\theta x^\beta}}{(\theta+1)}, x > 0, \beta, \theta > 0 \qquad (1)$$

Suppose X is a non-negative random variable with pdf$f(x)$. Let $w(x)$ be the non-negative weight function, then the pdf of the weighted random variable $X_w$is given by

$$f_l(x) = \frac{w(x)f(x)}{E(w(x))}, x > 0$$

where$w(x)$ is a non-negative weight function and$E(w(x)) = \int w(x)f(x)dx$.

In this paper, we will consider the weight function as $w(x) = x^c$, and using the definition of weighted distribution, the pdf of the WPL distribution is given as

$$f_w(x) = \frac{x^c f(x)}{E(x^c)}, \qquad (2)$$

where c > 0 is the weight parameter and the expected value is defined as

$$E(x^c) = \int_0^\infty x^c f(x)\, dx$$

$$= \frac{1}{(\theta+1)}\left(\frac{1}{\theta}\right)^{\frac{\beta+c}{\beta}-2}\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right) \qquad (3)$$

Substituting Eqs. (1) and (3) in Eq. (2), we obtain the density function of WPL distribution as follows:

$$f_w(x) = \frac{\beta\theta^{\frac{\beta+c}{\beta}}x^{\beta+c-1}(1+x^\beta)e^{-\theta x^\beta}}{\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)},\ x > 0,\ \beta, \theta, c > 0 \qquad (4)$$

and the cumulative density function (cdf) of the WPL distribution is obtained by

$$F_w(x) = \int_0^x f_w(x)dx$$

$$= \int_0^\infty \frac{\beta\theta^{\frac{\beta+c}{\beta}}x^{\beta+c-1}(1+x^\beta)e^{-\theta x^\beta}}{\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)}dx.$$

After simplification, the cdf of the WPL distribution is given by

$$F_w(x) = \frac{\left(\gamma\left(\left(\frac{c}{\beta}+1\right),x\right)+\frac{1}{\theta}\gamma\left(\left(\frac{\beta+c}{\beta}+1\right),x\right)\right)}{\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\left(\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)\right)} \qquad (5)$$

Figures 1 and 2 represent graphs for the pdf and cdf of the WPL distribution for several values of parameters.

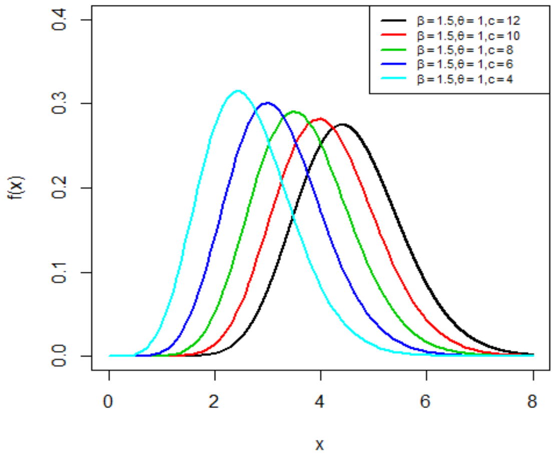

Fig.1:Pdf plot of Weighted Power Lindley distribution

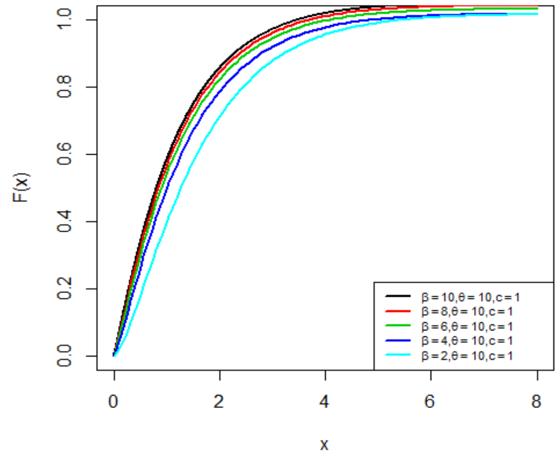

Fig.2:Cdf plot of Weighted Power Lindley distribution

2.2. Survival, Hazard and Reversed Hazard Functions

In this section, we discuss about the survival function, hazard and reverse hazard functions of the WPL distribution. The survival function or the reliability function of the WPL distribution is given by

$$S(x) = 1 - F_w(x)$$
$$S(x) = 1 - \frac{1}{\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\left(\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)\right)}\left(\gamma\left(\left(\frac{c}{\beta}+1\right),x\right)+\frac{1}{\theta}\gamma\left(\left(\frac{\beta+c}{\beta}+1\right),x\right)\right)$$

The hazard function is also known as the hazard rate function, instantaneous failure rate or force of mortality and is given for the WPL distribution as

$$h(x) = \frac{f_W(x)}{S(x)}$$
$$h(x) = \frac{\beta\theta^{\frac{\beta+c}{\beta}}x^{\beta+c-1}(1+x^{\beta})e^{-\theta x^{\beta}}}{\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\left(\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)\right)-\left(\gamma\left(\left(\frac{c}{\beta}+1\right),x\right)+\frac{1}{\theta}\gamma\left(\left(\frac{\beta+c}{\beta}+1\right),x\right)\right)}$$

The reverse hazard function of the WPL distribution is given by

$$h_r(x) = \frac{f_W(x)}{F_W(x)}$$
$$h_r(x) = \frac{\beta\theta^{\frac{\beta+c}{\beta}}x^{\beta+c-1}(1+x^{\beta})e^{-\theta x^{\beta}}}{\left(\gamma\left(\left(\frac{c}{\beta}+1\right),x\right)+\frac{1}{\theta}\gamma\left(\left(\frac{\beta+c}{\beta}+1\right),x\right)\right)}$$

Figures 3 and 4 represent graphs for the Survival function and Hazard rate function of the WPL distribution for several values of parameters.

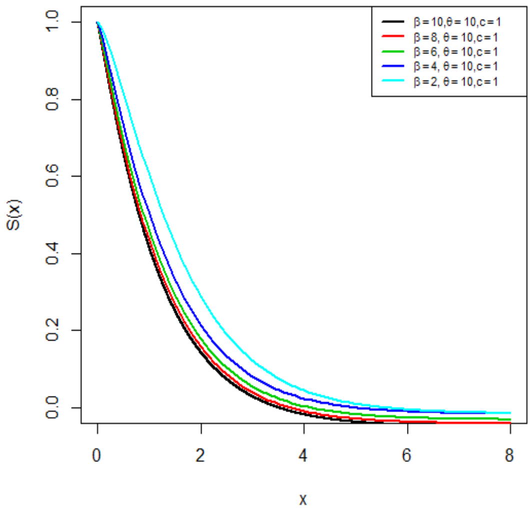

Fig.3: Survival function curves of WPL distribution

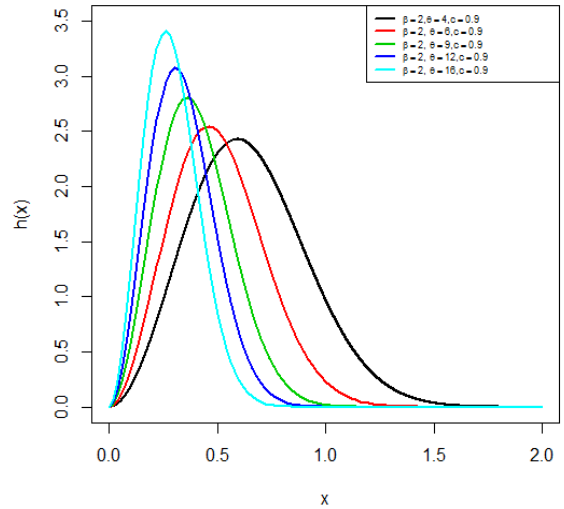

Fig.4: Hazard rate curves of WPL distribution

3. Structural Properties

In this section, we investigate various structural properties of the WPL distribution.

Let X denotes the random variable of WPL distribution with parameters β, θ and c, then its r^{th} order moment $E(X^r)$ about origin is given by

$$E(X^r) = \mu_r{}' = \int_0^\infty x^r f_w(x)dx$$

$$= \int_0^\infty \frac{\beta\theta^{\frac{\beta+c}{\beta}} x^{\beta+c+r-1}(1+x^\beta)e^{-\theta x^\beta}}{\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)}dx.$$

After simplifying the expression, we get

$$E(X^r) = \frac{\Gamma\left(\frac{c+r}{\beta}-1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c+r}{\beta}+1\right)}{\theta^{\frac{r}{\beta}}\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)}.$$

Putting r =1, we get the expected value of WPL distribution as follows:

$$E(X) = \frac{\Gamma\left(\frac{c+1}{\beta}-1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c+1}{\beta}+1\right)}{\theta^{\frac{1}{\beta}}\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)} \quad (6)$$

and putting r = 2, we obtain the second moment as

$$E(X^2) = \frac{\Gamma\left(\frac{c+2}{\beta}-1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c+2}{\beta}+1\right)}{\theta^{\frac{2}{\beta}}\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)}.$$

Therefore, the variance of the WPL distribution is given by

$$V(X) = \frac{\Gamma\left(\frac{c+2}{\beta}-1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c+2}{\beta}+1\right)}{\theta^{\frac{2}{\beta}}\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)} - \left(\frac{\Gamma\left(\frac{c+1}{\beta}-1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c+1}{\beta}+1\right)}{\theta^{\frac{1}{\beta}}\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)}\right)^2.$$

3.1 Harmonic mean

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The harmonic mean of the WPL distributed random variable X can be written as

$$H = E\left(\frac{1}{x}\right) = \int_0^\infty \frac{1}{x} f_w(x)dx = \int_0^\infty \frac{\beta\theta^{\frac{\beta+c}{\beta}} x^{\beta+c-2}(1+x^\beta)e^{-\theta x^\beta}}{\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)} dx.$$

After simplifying the expression, we get

$$H = \frac{\theta^{\frac{1}{\beta}}\Gamma\left(\frac{c+1}{\beta}-1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c+1}{\beta}+1\right)}{\theta^{\frac{1}{\beta}}\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)}.$$

3.2 Moment generating function and Characteristic function

Let X have a WPL distribution, then the MGF of X is obtained as

$$\boldsymbol{M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} f_w(x)dx}$$

Using Taylor's series, we obtain

$$\boldsymbol{M_X(t) = E(e^{tx}) = \int_0^\infty \left(1 + tx + \frac{(tx)^2}{2!} + \cdots\right) f_w(x)dx}$$

$$\boldsymbol{= \int_0^\infty \sum_{j=0}^\infty \frac{t^j}{j!} x^j f_w(x)dx}$$

$$\boldsymbol{= \sum_{j=0}^\infty \frac{t^j}{j!} E(X^j)}$$

$$= \sum_{j=0}^\infty \frac{t^j}{j!}\left(\frac{\Gamma\left(\frac{c+j}{\beta}-1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c+j}{\beta}+1\right)}{\theta^{\frac{j}{\beta}}\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)}\right)$$

Similarly, the characteristic function of the WPL distribution can be obtained as

$$\phi_x(t) = M_x(it) = \sum_{j=0}^\infty \frac{(it)^j}{j!}\left(\frac{\Gamma\left(\frac{c+j}{\beta}-1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c+j}{\beta}+1\right)}{\theta^{\frac{j}{\beta}}\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)}\right).$$

4. Likelihood Ratio Test

Let $X_1, X_2, \ldots, X_n$ be a random sample from the WPL distribution. We use the hypothesis

$$H_o: f(x) = f(x;\beta,\theta) \text{against} H_1: f(x) = f_w(x;\beta,\theta,c)$$

In order to test whether the random sample of size n comes from the PL distribution or the WPL distribution. Then, the following test statistic is used

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^n \frac{f_w(x;\beta,\theta,c)}{f(x;\beta,\theta)} = \prod_{i=1}^n \frac{\theta^{\frac{\beta+c}{\beta}-2} {x_i}^c(\theta+1)}{\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)}$$

$$\Delta = \left(\frac{\theta^{\frac{\beta+c}{\beta}-2}(\theta+1)}{\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)}\right)^n \prod_{i=1}^n {x_i}^c$$

We reject the null hypothesis, if

$$\Delta = \left(\frac{\theta^{\frac{\beta+c}{\beta}-2}(\theta+1)}{\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)}\right)^{n} \prod_{i=1}^{n} x_i{}^{c} > k$$

$$\Delta = \prod_{i=1}^{n} x_i{}^{c} > k\left(\frac{\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)}{\theta^{\frac{\beta+c}{\beta}-2}(\theta+1)}\right)^{n}$$

$$\text{or}\Delta^{*} = \prod_{i=1}^{n} x_i{}^{c} > k^{*}\text{where } k^{*} = k\left(\frac{\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)}{\theta^{\frac{\beta+c}{\beta}-2}(\theta+1)}\right)$$

For large sample size *n*, *2 log*Δ is distributed as chi-square distribution with one degree of freedom and also *p*-value is obtained from the chi-square distribution. Thus, we reject the null hypothesis, when the probability value is given by

$$p(\Delta^{*} > \alpha^{*})$$

where $\alpha^{*} = \prod_{i=1}^{n} x_i{}^{c}$is less than a specified level of significance and $\prod_{i=1}^{n} x_i{}^{c}$is the observed value of th statistic Δ^{*}.

5. Entropy Measures

The concept of entropy is important in different areas such as probability and statistics, physics, communication theory and economics. Entropy measures quantify the diversity, uncertainty, or randomness of a system. Entropy of a random variable *X* is a measure of variation of the uncertainty.

5.1 Renyi Entropy

The Rényi entropy is important in ecology and statistics as index of diversity. It was proposed by Rényi (1957). The Rényi entropy of order α for a random variable *X* is given by

$$e(\alpha) = \frac{1}{1-\alpha} log\left(\int f^{\alpha}(x)dx\right)$$

where $\alpha > 0$ and $\alpha \neq 1$. Then, we have

$$e(\alpha) = \frac{1}{1-\alpha} log \int_0^{\infty} \left(\frac{\beta\theta^{\frac{\beta+c}{\beta}} x^{\beta+c-1}(1+x^{\beta})e^{-\theta x^{\beta}}}{\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)}\right)^{\alpha} dx$$

$$= \frac{1}{1-\alpha} log\left(\left(\frac{\beta\theta^{\frac{\beta+c}{\beta}}}{\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)}\right)^{\alpha} \int_0^{\infty} \left(x^{\beta+c-1}(1+x^{\beta})e^{-\theta x^{\beta}}\right)^{\alpha} dx\right).$$

After simplifying the expression, we get

$$e(\alpha) = \frac{1}{1-\alpha} log\left(\frac{1}{\alpha\beta}\left(\frac{\beta}{\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)}\right)^{\alpha} \sum_{i=0}^{\infty} \binom{\alpha}{i} \left(\frac{1}{\theta}\right)^{\left(\frac{i\beta-\alpha+1}{\beta}\right)} \Gamma\left(\frac{\alpha(\beta+c-1)+i\beta+1}{\alpha\beta}\right)\right)$$

5.2: Tsallis Entropy

A generalization of Boltzmann-Gibbs (B-G) statistical mechanics initiated by Tsallis has focussed a great deal to attention. This generalization of B-G statistics was proposed firstly by introducing the mathematical expression of Tsallis entropy (Tsallis, 1988) for a continuous random variable. Tsallis entropy of order λ of the WPL distribution is given by

$$S_\lambda = \frac{1}{\lambda-1}\left(1-\int_0^\infty f^\lambda(x)dx\right)$$

$$= \frac{1}{\lambda-1}\left(1-\int_0^\infty \left(\frac{\beta\theta^{\frac{\beta+c}{\beta}}x^{\beta+c-1}(1+x^\beta)e^{-\theta x^\beta}}{\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)}\right)^\lambda dx\right).$$

After simplifying the expression, we get

$$S_\lambda = \frac{1}{\lambda-1}\left(1-\frac{1}{\lambda\beta}\left(\frac{\beta}{\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)}\right)^\lambda \sum_{i=0}^\infty \binom{\lambda}{i}\left(\frac{1}{\theta}\right)^{\left(\frac{i\beta-\lambda+1}{\beta}\right)}\Gamma\left(\frac{\lambda(\beta+c-1)+i\beta+1}{\lambda\beta}\right)\right)$$

6. Order Statistics

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order statistics of a random sample $X_1, X_2, \ldots, X_n$ drawn from the continuous population with pdf $f_x(x)$ and cdf $F_x(x)$, then the pdf of r^{th} order statistic $X_{(r)}$ is given by

$$f_{X(r)}(x) = \frac{n!}{(r-1)!(n-r)!}f_X(x)[F_X(x)]^{r-1}[1-F_X(x)]^{n-r} \tag{7}$$

Using Eqs (4) and (5) in Eq. (7), the pdf of r^{th} order statistic $X_{(r)}$ of the WPL distribution is given by

$$f_{X(r)}(x) = \frac{n!}{(r-1)!\,(n-1)!}\left(\frac{\beta\theta^{\frac{\beta+c}{\beta}}x^{\beta+c-1}(1+x^\beta)e^{-\theta x^\beta}}{\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)}\right)$$

$$\times\left(\frac{1}{\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\left(\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)\right)}\left(\gamma\left(\left(\frac{c}{\beta}+1\right),x\right)+\frac{1}{\theta}\gamma\left(\left(\frac{\beta+c}{\beta}+1\right),x\right)\right)\right)^{r-1}$$

$$\times\left(1-\frac{1}{\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\left(\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)\right)}\left(\gamma\left(\left(\frac{c}{\beta}+1\right),x\right)+\frac{1}{\theta}\gamma\left(\left(\frac{\beta+c}{\beta}+1\right),x\right)\right)\right)^{n-r}$$

Therefore, the pdf of the higher order statistic $X_{(n)}$ can be obtained as

$$f_{X(n)}(x) = n\left(\frac{\beta\theta^{\frac{\beta+c}{\beta}}x^{\beta+c-1}(1+x^\beta)e^{-\theta x^\beta}}{\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)}\right)$$

$$\times\left(\frac{1}{\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\left(\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)\right)}\left(\gamma\left(\left(\frac{c}{\beta}+1\right),x\right)+\frac{1}{\theta}\gamma\left(\left(\frac{\beta+c}{\beta}+1\right),x\right)\right)\right)^{n-1}$$

and the pdf of the first order statistic $X_{(1)}$ can be obtained as

$$f_{X(1)}(x)=n\left(\frac{\beta\theta^{\frac{\beta+c}{\beta}}x^{\beta+c-1}(1+x^{\beta})e^{-\theta x^{\beta}}}{\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)}\right)$$

$$\times\left(1-\frac{1}{\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\left(\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)\right)}\left(\gamma\left(\left(\frac{c}{\beta}+1\right),x\right)+\frac{1}{\theta}\gamma\left(\left(\frac{\beta+c}{\beta}+1\right),x\right)\right)\right)^{n-1}$$

7. Income Distribution Curve

The Bonferroni and the Lorenz curves are not only used in economics in order to study the income and poverty, but it is also being used in other fields like reliability, medicine, insurance and demography. The Bonferroni and Lorenz curves are given by

$$B(p)=\frac{1}{p\mu_1{}'}\int_0^q xf(x)dx$$

and

$$L(p)=PB(p)=\frac{1}{\mu_1{}'}\int_0^q xf(x)dx$$

Here, we define the first raw moment as

$$\mu_1{}'=\frac{\Gamma\left(\frac{c+1}{\beta}-1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c+1}{\beta}+1\right)}{\theta^{\frac{1}{\beta}}\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)}$$

and $q=F^{-1}(p)$.Then, we have

$$B(p)=\frac{1}{p\mu_1{}'}\int_0^q\frac{\beta\theta^{\frac{\beta+c}{\beta}}x^{\beta+c}(1+x^{\beta})e^{-\theta x^{\beta}}}{\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)}dx.$$

After simplification, we get

$$B(p)=\frac{1}{p\left(\Gamma\left(\frac{c+1}{\beta}-1\right)+\frac{1}{\theta}\left(\Gamma\left(\frac{\beta+c+1}{\beta}+1\right)\right)\right)}\left(\gamma\left(\left(\frac{\beta+c+1}{\beta}\right),\theta q^{\beta}\right)+\frac{1}{\theta}\gamma\left(\left(\frac{2\beta+c+1}{\beta}\right),\theta q^{\beta}\right)\right)$$

Similarly, Lorenz curve is obtained as

$$L(p)=\frac{1}{\left(\Gamma\left(\frac{c+1}{\beta}-1\right)+\frac{1}{\theta}\left(\Gamma\left(\frac{\beta+c+1}{\beta}+1\right)\right)\right)}\left(\gamma\left(\left(\frac{\beta+c+1}{\beta}\right),\theta q^{\beta}\right)+\frac{1}{\theta}\gamma\left(\left(\frac{2\beta+c+1}{\beta}\right),\theta q^{\beta}\right)\right)$$

8. Estimation

In this section, we will discuss the maximum likelihood estimators (MLEs) of the parameters of the WPL distribution. Consider $X_1, X_2, ..., X_n$ be the random sample of size n from the WPL distribution, then the likelihood function is given by

$$L(x;\beta,\theta,c)=\frac{\beta^n\theta^{n\left(\frac{\beta+c}{\beta}\right)}}{\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)^n}\prod_{i=1}^n\left(x_i{}^{\beta+c-1}(1+x_i{}^{\beta})e^{-\theta x_i{}^{\beta}}\right)$$

The log likelihood function is obtained as

$$log\,L=n\,log\,\beta+n\left(\frac{\beta+c}{\beta}\right)log\,\theta-n\,log\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)+\sum_{i=1}^n log(1+x_i{}^{\beta})$$
$$+(c+\beta-1)\sum_{i=1}^n log\,x_i-\theta\sum_{i=1}^n x_i{}^{\beta}\ (8)$$

The MLEs of β, θ, c can be obtained by differentiating Eq. (8) with respect to β, θ, c and must satisfy the normal equation

$$\frac{\partial\,log\,L}{\partial\beta}=\frac{n}{\beta}-\frac{nc}{\beta^2}log\,\theta-n\psi\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)$$
$$+\sum_{i=1}^n\frac{x_i{}^{\beta}\,log\,x_i}{(1+x_i{}^{\beta})}+\sum_{i=1}^n log\,x_i-\theta\sum_{i=1}^n x_i{}^{\beta}\,log\,x_i=0$$

$$\frac{\partial\,log\,L}{\partial c}=\frac{n}{\beta}log\,\theta-n\psi\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)+\sum_{i=1}^n log\,x_i=0$$

$$\frac{\partial\,log\,L}{\partial\theta}=\frac{n}{\theta}\left(\frac{\beta+c}{\beta}\right)-\frac{n\Gamma\left(\frac{\beta+c}{\beta}+1\right)}{\theta^2\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)}+\sum_{i=1}^n x_i{}^{\beta}=0$$

where ψ(.) is the digamma function. Because of the complicated form of the above likelihood equations, algebraically it is very difficult to solve the system of nonlinear equations. Therefore, we use R and Wolfram Mathematica for estimating the required parameters.

To obtain confidence interval we use the asymptotic normality results. We have that, if $\hat{\lambda}=(\hat{\beta},\hat{c},\hat{\theta})$ denotes the MLE of $\lambda=(\beta,c,\theta)$, we can state the results as follows

$$\sqrt{n}(\hat{\lambda}-\lambda)\rightarrow N_3(0, I^{-1}(\lambda))$$

where I(λ) is Fisher's information matrix given by

$$I(\lambda) = -\frac{1}{n}\begin{pmatrix} E\left(\frac{\partial^2 \log L}{\partial\beta^2}\right) E\left(\frac{\partial^2 \log L}{\partial\beta\partial\theta}\right) E\left(\frac{\partial^2 \log L}{\partial\beta\partial c}\right) \\ E\left(\frac{\partial^2 \log L}{\partial\theta\partial\beta}\right) E\left(\frac{\partial^2 \log L}{\partial\theta^2}\right) E\left(\frac{\partial^2 \log L}{\partial\theta\partial c}\right) \\ E\left(\frac{\partial^2 \log L}{\partial c\partial\beta}\right) E\left(\frac{\partial^2 \log L}{\partial c\partial\theta}\right) E\left(\frac{\partial^2 \log L}{\partial c^2}\right) \end{pmatrix}$$

Here, we define

$$\frac{\partial^2 \log L}{\partial\beta^2} = -\frac{n}{\beta^2} - \frac{2nc}{\beta^3}\log\theta - n\psi'\left(\Gamma\left(\frac{c}{\beta}+1\right) + \frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right) + \sum_{i=1}^{n}\frac{(1+x_i{}^{\beta})x_i{}^{\beta}(\log x_i)^2 - (x_i{}^{\beta}\log x_i)^2}{(1+x_i{}^{\beta})^2} - \theta\sum_{i=1}^{n}\left(x_i{}^{\beta}\log x_i\right)^2$$

$$\frac{\partial^2 \log L}{\partial\theta^2} = -\frac{n}{\theta^2}\left(\frac{\beta+c}{\beta}\right) - \frac{2n\Gamma\left(\frac{\beta+c}{\beta}+1\right)}{\theta^3\left(\Gamma\left(\frac{c}{\beta}+1\right)+\frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)} - n$$

$$\frac{\partial^2 \log L}{\partial c^2} = -\psi'\left(\Gamma\left(\frac{c}{\beta}+1\right) + \frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)$$

Also, $$\frac{\partial \log L}{\partial\beta\partial c} = -\frac{n}{\beta^2}\log\theta - n\psi'\left(\Gamma\left(\frac{c}{\beta}+1\right) + \frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)$$

$$\frac{\partial \log L}{\partial\beta\partial\theta} = -\frac{nc}{\theta\beta^2} - n\psi'\left(\Gamma\left(\frac{c}{\beta}+1\right) + \frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right) - \sum_{i=1}^{n} x_i{}^{\beta}\log x_i$$

$$\frac{\partial \log L}{\partial c\partial\theta} = -\frac{n}{\theta\beta} - n\psi'\left(\Gamma\left(\frac{c}{\beta}+1\right) + \frac{1}{\theta}\Gamma\left(\frac{\beta+c}{\beta}+1\right)\right)$$

where $\psi(.)'$ is the first order derivative of digamma function. Since λ being unknown, we estimate $I^{-1}(\lambda)$ by $I^{-1}(\hat{\lambda})$ and this can be used to obtain asymptotic confidence intervals for β, θ and c.

9. Application

In this section, we have used two real lifetime data sets for fitting WPL distribution and the model has been compared with the PL, Exponential and Lindley distributions.

The first data set represents the survival times of 121 patients with breast cancer obtained from a large hospital which is widely reported in literatures like Ramos *et al*. (2013). The data set is given as follows:

The second data set corresponding to remission times (in months) of a random sample of 124 bladder cancer patients given in Lee and Wang (2003). The data set is given as follows:

0.08	2.22	7.09	14.24	0.81	5.32	10.66	43.01	4.33	11.64	4.40	12.02
2.09	3.52	9.22	25.82	2.62	7.39	15.96	1.19	5.49	17.36	5.85	2.02
2.73	4.98	13.80	0.51	3.82	10.34	36.66	2.75	7.66	1.40	8.26	3.31
3.48	6.99	25.74	2.54	5.32	14.83	1.05	4.26	11.25	3.02	11.98	4.51
4.87	9.02	0.50	3.70	7.32	34.26	2.69	5.41	17.14	4.34	19.13	6.54
6.94	13.29	2.46	5.17	10.06	0.90	4.23	7.63	79.05	5.71	1.76	8.53

8.66	0.40	3.64	7.28	14.77	2.69	5.41	17.12	1.35	7.93	3.25	12.03
13.11	2.26	5.09	9.74	32.15	4.18	7.62	46.12	2.87	11.79	4.50	20.28
23.63	3.57	7.26	14.76	2.64	5.34	10.75	1.26	5.62	18.10	6.25	2.02
0.2	5.06	9.47	26.31	3.88	7.59	15.62	2.83	7.87	1.46	8.37	3.36
6.93	8.65	12.63	22.69								

In order to compare the WPL distribution with the PL, Exponential and Lindley distributions, we consider the criteria like Bayesian information criterion *(BIC)*, Akaike Information Criterion *(AIC)*, Akaike Information Criterion Corrected *(AICC)* and *-2 logL*. The better distribution is which corresponds to lower values of *AIC,BIC, AICC* and *–2 log L*. For calculating *AIC, BIC, AICC* and *-2 logL* can be evaluated by using the formulas as follows:

$$AIC = 2K - 2logL, \qquad BIC = klogn - 2logL, \qquad AICC = AIC + \frac{2k(k+1)}{(n-k-1)}$$

where *k* is the number of parameters, *n* is the sample size and *-2 logL* is the maximized value of log-likelihood function and are showed in table 1 and table 2.

Table1: Parameter estimations and goodness of fit test statistics.

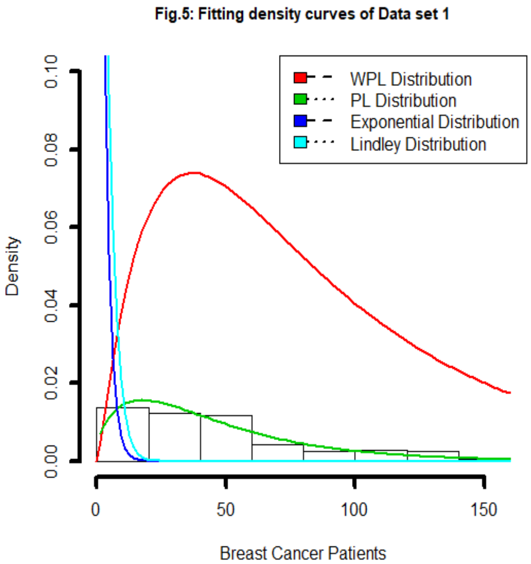
Data Set	Distribution	MLE	S.E	-2 logL	AIC	BIC	AICC
1	**WPL**	$\hat{c}$=1.59953956 $\hat{\beta}$ =0.58700034 $\hat{\theta}$=0.35120295	$\hat{c}$=0.19458134 $\hat{\beta}$ =0.01995351 $\hat{\theta}$=0.02661878	726.8712	732.8712	741.2586	733.0764
	PL	$\hat{\beta}$ =0.90725769 $\hat{\theta}$=0.6127261	$\hat{\beta}$ =0.05679699 $\hat{\theta}$=0.01432534	1158.367	1162.367	1167.958	1162.572
	Exponential	$\hat{\theta}$=0. 021597653	0.001959203	1170.256	1172.256	1175.051	1172.2896
	Lindley	$\hat{\theta}$=0. 042301604	0.002718848	1160.863	1162.863	1165.659	1162.8966

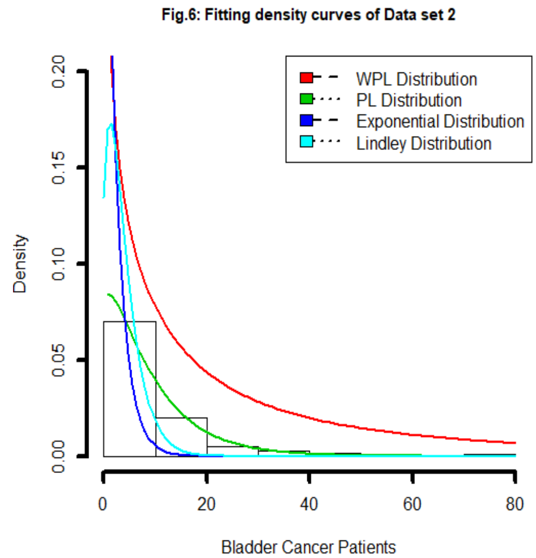
Table2: Parameter estimations and goodness of fit test statistics.

Data Set	Distribution	MLE	S.E	-2 logL	AIC	BIC	AICC
2	**WPL**	$\hat{c}$=0.24419247 $\hat{\beta}$ =0.45437968 $\hat{\theta}$=0.57932961	$\hat{c}$=0.07474640 $\hat{\beta}$ =0.0399462 7 $\hat{\theta}$=0.05048331	574.622	580.622	589.0828	580.822
	PL	$\hat{\beta}$ =0.82536680 $\hat{\theta}$=0.29920792	$\hat{\beta}$ =0.0476219 4 $\hat{\theta}$=0.3795285	799.5421	803.5421	809.1827	803.7421
	Exponential	$\hat{\theta}$=0. 107404293	0.009644354	801.3337	803.3337	806.154	803.36544
	Lindley	$\hat{\theta}$=0. 19711910	0.01260246	812.3593	814.3593	817.1796	814.39104

From table 1 and table 2, we can see that the WPL distribution have the lower *AIC, BIC, AICC* and *-2 logL* values as compared to PL, Exponential and Lindley distributions. Hence, we can conclude that the WPL distribution leads to better fit than the PL, Exponential and Lindley distributions.

Figures 5 and 6 represent graphs for the density curves of data set 1 and 2.





10. Conclusion

In the present study, a new version of the power Lindley (PL) distribution is introduced named as weighted power Lindley (WPL) distribution with three parameters and its different statistical properties are investigated and studied. The subject distribution is generated by using the weighted technique and the parameters have been obtained by using maximum likelihood estimator. The main motivation behind the completion of this manuscript is to make one realize how important are the new extensions in expressing some random processes even though when we have already a number of existing distributions. It is observed that for the considered data sets mostly the new cases of models proved to be best fit rather than the baseline distribution i.e. PL distribution. Finally the distribution has been fitted to a real life data and the fit has been found to be good.

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