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The Odd Log-Logistic Poisson Inverse Rayleigh Distribution: Statistical Properties and Applications

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Abstract

In this work, a new extension of the Inverse Rayleigh model is proposed and studied. We derive some of its fundamental properties. We assess the performance of the maximum likelihood estimators via a simulation study. The importance of the new model is shown via two applications to real data sets. The new model is better fit than other important competitive models based on two real data sets.

Key Words: Odd Log-Logistic-G Family; Inverse Rayleigh Distribution; Simulation; Generating Function, Maximum Likelihood.

Mathematical Subject Classification: 62N01; 62N02; 62E10.

1. Introduction and interpretation

A random variable (rv) X is said to have the Inverse Rayleigh (IR) distribution if its probability density function (PDF), cumulative distribution function (CDF) are given by

$$g_{\eta}(x) = 2\eta^2 x^{-3}\, exp[-\eta^2 x^{-2}], \qquad (1)$$

and

$$G_{\eta}(x) = exp[-\eta^2 x^{-2}]. \qquad (2)$$

In this study, we provide and study a new extesion of the Fréchet model called odd log-logistic Poisson Inverse Rayleigh (OLLP-IR) model using the odd log-logistic Poisson G (OLLP-G) family of continuous distributions originally developed by Alizadeh et al. (2018). The cumulative distribution function (CDF) of the OLLP-G family is given by

$$F_{\alpha,\beta,\Phi}(x) = \frac{\left[\frac{exp[\beta G_{\Phi}(x)]-1}{\tau_{(\beta)}}\right]^{\alpha}}{\left[\frac{exp[\beta G_{\Phi}(x)]-1}{\tau_{(\beta)}}\right]^{\alpha} + \left[1-\frac{exp[\beta G_{\Phi}(x)]-1}{\tau_{(\beta)}}\right]^{\alpha}}, \qquad (3)$$

where $\alpha, \beta > 0$ and Φ is the vector of parameter for baseline CDF. The corresponding PDF is given by

$$F_{\alpha,\beta,\Phi}(x) = \alpha\beta\tau_{(\beta)}^{-1} g_{\Phi}(x)\, exp[\beta G_{\Phi}(x)] \times \frac{\{exp[\beta G_{\Phi}(x)]-1\}^{\alpha-1}}{\{exp(\beta)-exp[\beta G_{\Phi}(x)]\}^{-\alpha+1}} \begin{pmatrix} \{exp[\beta G_{\Phi}(x)]-1\}^{\alpha} \\ +\{exp(\beta)-exp[\beta G_{\Phi}(x)]\}^{\alpha} \end{pmatrix}^{-2}. \qquad (4)$$

using (1) and (3) we can derive OLLP-IR CDF as

$$F_{\alpha,\beta,\eta}(x) = \frac{\left(\frac{exp\{\beta\, exp[-\eta^2 x^{-2}]\}-1}{\tau_{(\beta)}}\right)^{\alpha}}{\left(\frac{exp\{\beta\, exp[-\eta^2 x^{-2}]\}-1}{\tau_{(\beta)}}\right)^{\alpha} + \left(1-\frac{exp\{\beta\, exp[-\eta^2 x^{-2}]\}-1}{\tau_{(\beta)}}\right)^{\alpha}}, \qquad (5)$$

The corresponding PDF is given by

$$f_{\alpha,\beta,\eta}(x)=\frac{2\alpha\beta\eta^2\tau_{(\beta)}^{-1}x^{-3}\,exp[-\eta^2x^{-2}]\,[exp\{\beta\,exp[-\eta^2x^{-2}]\}-1]^{\alpha-1}}{exp\{-\beta\,exp[-\eta^2x^{-2}]\}\,[exp(\beta)-exp\{\beta\,exp[-\eta^2x^{-2}]\}]^{-\alpha+1}}$$
$$\times\left\{\left[\frac{1}{\tau_{(\beta)}}(exp\{\beta\,exp[-\eta^2x^{-2}]\}-1)\right]^{\alpha}+\left[1-\frac{1}{\tau_{(\beta)}}(exp\{\beta\,exp[-\eta^2x^{-2}]\}-1)\right]^{\alpha}\right\}^{-2}, \quad (6)$$

where $\tau_{(\beta)}=exp(\beta)-1$. The survival function can be derived as

$$S_{\alpha,\beta,\eta}(x)=1-\frac{\left(\frac{exp\{\beta\,exp[-\eta^2x^{-2}]\}-1}{\tau_{(\beta)}}\right)^{\alpha}}{\left[\frac{1}{\tau_{(\beta)}}(exp\{\beta\,exp[-\eta^2x^{-2}]\}-1)\right]^{\alpha}+\left[1-\frac{1}{\tau_{(\beta)}}(exp\{\beta\,exp[-\eta^2x^{-2}]\}-1)\right]^{\alpha}}$$

The hazard rate functions (HRF)is defined as $f_{\alpha,\beta,\eta}(x)/[1-F_{\alpha,\beta,\eta}(x)]$ can be easily derived. When $\beta\to 0,$ we have the odd log-logistic Inverse Rayleigh (OLL-IR) model and when $\alpha=1,$ we have the Poisson Inverse Rayleigh (P-IR) model.

Let T be a rv describing a stochastic system by the CDF, $G_{\beta,\eta}(x)=\frac{exp\{\beta\,exp[-\eta^2x^{-2}]\}-1}{\tau_{(\beta)}}$, which is the CDF of the compound Poisson Inverse Rayleigh model. If the rv X represents the odds ratio, the risk that the system following the lifetime T will be not working at time x is given by

$$\vartheta_{\beta,\eta}(x)\,\frac{\frac{1}{\tau_{(\beta)}}(exp\{\beta\,exp[-\eta^2x^{-2}]\}-1)}{1-\frac{1}{\tau_{(\beta)}}(exp\{\beta\,exp[-\eta^2x^{-2}]\}-1)}.$$

If we are interested in modeling the randomness of the odds ratio, $\vartheta_{\beta,\eta}(x)$, by the exponentiated half-logistic CDF

$$R(t)=\frac{t^{\alpha}}{1+t^{\alpha}}|_{(t>0)},$$

the CDF of X is given by

$$Pr(X\le x)=R\left(\frac{\frac{1}{\tau_{(\beta)}}(exp\{\beta\,exp[-\eta^2x^{-2}]\}-1)}{1-\frac{1}{\tau_{(\beta)}}(exp\{\beta\,exp[-\eta^2x^{-2}]\}-1)}\right)=\frac{(exp\{\beta\,exp[-\eta^2x^{-2}]\}-1-1)^{\alpha}}{\left[\begin{array}{c}(exp\{\beta\,exp[-\eta^2x^{-2}]\}-1-1)^{\alpha}\\+(exp(\beta)-exp\{\beta\,exp[-\eta^2x^{-2}]\})^{\alpha}\end{array}\right]}.$$

Furthermore, the basic motivations for using the OLLP-IR model in practice are the following: to make the kurtosis more flexible compared to the baseline Fr model; to produce a skewness for symmetrical distributions; to construct heavy-tailed distributions that are not longer-tailed for modeling real data; to generate distributions with symmetric, left-skewed, right-skewed and reversed-J shaped; to define special models with flexible types of the HRF; to provide consistently better fits than other generated models under the same baseline distribution. Although, we have stated that $\beta\in(0,\infty)$, equation (5) is still a CDF if $\beta<0$. Hence, we can consider the OLLP-IR model defined for any $\beta\in R-\{0\}$. Many extensions of the IR model can be cited as: beta IR (BIR) by Barreto-Souza et al. (2011), Marshall-Olkin IR (MOIR) by Krishna et.al. (2013), Kumaraswamy transmuted Marshall-Olkin IR (KTMOIR) by Yousof et al. (2016), Odd Lindley IR (OLIR) by Korkmaz et al. (2017), odd log-logistic IR (OLLIR) by Yousof et al. (2018a), Transmuted Topp Leone IR (TTLIR) by Yousof et al. (2018b), among others. Many other extensions can be found in Brito et al. (2017), Chakraborty et al. (2018), Cordeiro et al. (2018), Korkmaz et al. (2018) and Korkmaz et al. (2019).

2. Linear representations

In this section, mixture representations for Equations (5) and (6) are obtained, firstly we have

$$\left[\frac{1}{\tau_{(\beta)}}(exp\{\beta\,exp[-\eta^2x^{-2}]\}-1)\right]^{\alpha}=\sum_{j=0}^{\infty}a_j\left[\frac{1}{\tau_{(\beta)}}(exp\{\beta\,exp[-\eta^2x^{-2}]\}-1)\right]^{j},$$

where $a_j=\sum_{i=j}^{\infty}(-1)^{i+j}\binom{\alpha}{i}\binom{i}{j}$ and

$$\left\{\begin{array}{l} \left[\frac{1}{\tau_{(\beta)}}(exp\{\beta\, exp[-\eta^2 x^{-2}]\}-1)\right]^{\alpha} \\ +\left[1-\frac{1}{\tau_{(\beta)}}(exp\{\beta\, exp[-\eta^2 x^{-2}]\}-1)\right]^{\alpha} \end{array}\right\} = \sum_{j=0}^{\infty} b_j \left(\frac{exp\{\beta\, exp[-\eta^2 x^{-2}]\}-1}{\tau_{(\beta)}}\right)^j,$$

where $b_j = a_j + (-1)^j \binom{\alpha}{j}$. Then, the OLLP-IR CDF in (5) can be written as

$$F(x) = \frac{\sum_{j=0}^{\infty} a_j \left(\frac{exp\{\beta\, exp[-\eta^2 x^{-2}]\}-1}{\tau_{(\beta)}}\right)^j}{\sum_{j=0}^{\infty} b_j \left(\frac{exp\{\beta\, exp[-\eta^2 x^{-2}]\}-1}{\tau_{(\beta)}}\right)^j} = \sum_{j=0}^{\infty} c_j \left(\frac{exp\{\beta\, exp[-\eta^2 x^{-2}]\}-1}{\tau_{(\beta)}}\right)^j,$$

where $c_0 = \frac{a_0}{b_0}$ and for $j \geq 1$, we have $c_j = \frac{1}{b_0}\left(a_j - \frac{1}{b_0}\sum_{r=1}^{j} b_r c_{j-r}\right)$ then

$$F(x) = \sum_{k=0}^{\infty} \xi_k\, H_{k+1}(x), \tag{7}$$

where $\xi_k = \sum_{j=0}^{\infty}\sum_{l=0}^{j} \frac{(1+j)c_{j+1}(-1)^{j+l}(1+l)^k}{\tau_{(\beta)}^{1+j}(k+1)!}\binom{j}{l}$ and $H_\delta(x) = exp[-(k+1)\eta^2 x^{-2}]$is CDF of the IR distribution with scale parameter $\eta(k+1)^{\frac{1}{2}}$. The corresponding OLLP-IR density function is obtained by differentiating (7) as

$$f(x) = \xi_k h_{k+1}(x), \tag{8}$$

where $h_{k+1}(x) = 2(k+1)\eta^2 x^{-3}\, exp[-(k+1)\eta^2 x^{-2}]$is PDF of the IR distribution with scale parameter $\eta\ (k+1)^{\frac{1}{2}}$. Equation (7) and (8) reveal that PDF of OLLP-IR is a linear combination of IR densities. Thereby, some properties of the proposed family such as moments and generating function can be determined by means of IR distribution.

3. Mathematical properties

Asymptotics

Let $\varepsilon = inf\{x|G(x) > 0\}$, the asymptotics of equations (3), (4) and (5) as $x \to d$ are given by

$$F(x) \sim \left[\frac{\beta\, exp[-\eta^2 x^{-2}]}{\tau_{(\beta)}}\right]^{\alpha} |_{(x\to\varepsilon)}, f(x) \sim \frac{2\alpha\beta^{\alpha}}{\tau_{(\beta)}^{\alpha}}\eta^2 x^{-(2+1)}\, exp[-2\eta^2 x^{-2}]\ |_{(x\to\varepsilon)},$$

and

$$\tau(x) \sim \frac{2\alpha\beta^{\alpha}}{\tau_{(\beta)}^{\alpha}}\eta^2 x^{-(2+1)}\, exp\left[-\alpha\left(\frac{a}{x}\right)^2\right] |_{(x\to\varepsilon)},$$

The asymptotics of equations (3), (4) and (5) as $x \to \infty$ are given by

$$1 - F(x) \sim \left(\frac{\beta\{1-exp[-\eta^2 x^{-2}]\}}{\tau_{(\beta)}}\right)^{\alpha} |_{(x\to\infty)}, f(x) \sim \frac{2\eta^2\alpha\beta^{\alpha}\{1-exp[-\eta^2 x^{-2}]\}}{x^3\, exp[\eta^2 x^{-2}]\,\tau_{(\beta)}^{\alpha}} |_{(x\to\infty)},$$

and

$$\tau(x) \sim \frac{2\alpha\eta^2 x^{-(2+1)}\, exp[-\eta^2 x^{-2}]}{1-exp[-\eta^2 x^{-2}]} |_{(x\to\infty)},$$

These equations show the effect of parameters on tails of OLLP-IR distribution.

Moments, incomplete moments and generating function

The r th ordinary moment of X is given by $\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx$. Then we obtain

$$\mu'_r = \sum_{k=0}^{\infty} \xi_k\, \eta^r (k+1)^{\frac{r}{2}} \Gamma\left(1-\frac{r}{2}\right) |(r<2), \tag{9}$$

where $\Gamma(1+\delta)|_{(\delta\in R^+)} = \delta! = \prod_{h=0}^{\delta-1}(\delta-h) = \int_0^{\infty} t^{\delta}\, exp(-t)\, dt$ is the complete gamma function. Setting $r = 1$ in (9), we have the mean of X. The last integration can be computed numerically for most parent distributions. The r th incomplete moment, say $I_r(t)$, of X can be expressed from (7) as

$$I_r(t) = \int_{-\infty}^{t} x^r f(x)dx = \sum_{k=0}^{\infty} \xi_k\, \eta^r (k+1)^{\frac{r}{2}} \gamma\left(1-\frac{r}{2}, (k+1)\eta^2 x^{-2}\right) |(r<2), \tag{10}$$

where $\gamma(\delta,\vartheta)$ is the incomplete gamma function.

$$\gamma(\delta,\vartheta)|_{(\delta\neq 0,-1,-2,\ldots)} = \int_0^{\vartheta} t^{\delta-1} exp(-t)\, dt = \frac{\vartheta^{\delta}}{\delta}\{1F_1[\delta;\delta+1;-\vartheta]\} = \sum_{\zeta=0}^{\infty} \frac{(-1)^{\zeta}}{\zeta!\,(\delta+\zeta)} \vartheta^{\delta+\zeta},$$

and $1F_1[\cdot,\cdot,\cdot]$ is a confluent hypergeometric function. The first incomplete moment $I_1(t)$ given by (9) with $r = 1$. The moment generating function (mgf) $M_X(t) = E(e^{tX})$ of X can be derived from equation (8) as

$$M_X(t) = \sum_{k,r=0}^{\infty} \frac{t^r}{r!} \xi_k \eta^r (k+1)^{\frac{r}{2}} \Gamma\left(1-\frac{r}{2}\right) |(r<2),$$

Skewness and kurtosis for the OLLP-IR model can be calculated from the ordinary moments by using well-known relationships. The mean, variance, skewness and kurtosis of the OLLP-IR distribution are computed numerically using the R software and reported in Table 1. Table 1 indicate that the skewness of the OLLP-IR distribution is always positive. The kurtosis is always more than 3.

Table 1: Mean, variance, skewness and kurtosis of the OLLP-IR distribution.

α	β	η	mean	variance	skewness	kurtosis
2	0.5	1.5	0.3099428	9.819942	426.3933	6811390
1.5			0.5919179	14.35429	346.1332	457474.1
1			1.2393480	18.58084	299.4811	348174.4
1.5	0.1	1.5	0.0009350	0.021149	8695.573	29912471
	0.5		0.5919180	14.35429	346.1332	457474.1
	0.75		4.1792060	91.85036	159.0526	83479.36
	1		19.855450	150.9925	373.4145	153301.3
1.5	0.5	0.5	0.1973479	1.825138	849.4490	3145095
		0.75	0.2960061	3.915383	607.9861	1537535
		1	0.3946538	6.719541	480.6036	927976.9
		2.5	0.9863203	37.19714	230.2226	189177.9
		5	1.9715930	134.2666	133.9198	58031.79
		10	3.9389940	478.9951	79.10100	18209.54
		25	9.8160510	2514.283	40.55250	4110.307
		50	19.527330	8608.742	25.08110	1390.652
		200	75.597090	91679.62	10.38719	185.8402
		500	176.51360	386813.8	6.240796	58.02135
		2000	477.21200	21397430	3.557830	16.25156

Quantile function

The OLLP-G family can easily be simulated by inverting (5) as follows: if $U \sim U(0,1)$, then the random variable X_U can be obtained from the baseline qf, say $Q_G(u) = G^{-1}(u)$. In fact, the random variable

$$X_U = \sqrt{-\eta^2\left(ln\left\{\beta^{-1}\, ln\left[1+\frac{\tau_{(\beta)}\sqrt[\alpha]{u}}{\sqrt[\alpha]{u}+\sqrt[\alpha]{1-u}}\right]\right\}\right)^{-1}}, \qquad (11)$$

has CDF (5). The effects of the shape parameters on the skewness and kurtosis can be based on quantile measures.

Moments of residual and reversed residual life

The n th moment of the residual life say $z_n(t) = E[(X-t)^n \mid X > t]$, $n = 1,2,\ldots$ uniquely determines $F(x)$. The n th moment of the residual life of X is given by $z_n(t) = \frac{\int_t^{\infty} (x-t)^n dF(x)}{1-F(t)}$. Therefore

$$z_n(t) = \frac{1}{1-F(t)} \sum_{k=0}^{\infty} \sum_{r=0}^{n} \xi_k (1-t)^n \int_t^{\infty} x^r h_{k+1}(x) dx$$

$$= \frac{1}{1-F(t)} \sum_{k=0}^{\infty} \sum_{r=0}^{n} \xi_k (1-t)^n \eta^n (k+1)^{\frac{n}{2}} \Gamma\left(1-\frac{n}{2}, (k+1)\eta^2 x^{-2}\right) |(n<2),$$

where $\Gamma(\delta,\vartheta)|_{(x>0)} = \int_{\vartheta}^{\infty} t^{\delta-1}\, exp(-t)\, dt$ and $\Gamma(\delta,\vartheta) = \Gamma(\omega) - \gamma(\delta,\vartheta)$. The n$th$ moment of the reversed residual life say, $Z_n(t) = E[(t-X)^n] \mid (X \le t, t>0$ and $n = 1,2,\ldots)$ uniquely determines $F(x)$. We obtain $Z_n(t) = \frac{\int_0^t (t-x)^n dF(x)}{F(t)}$. Then, the n th moment of the reversed residual life (RRL) of X becomes

$$Z_n(t) = \frac{1}{F(t)} \sum_{k=0}^{\infty} \sum_{r=0}^{n} \xi_k (-1)^r \binom{n}{r} t^{n-r} \int_0^t x^r h_{k+1}(x) dx$$
$$= \frac{1}{F(t)} \sum_{k=0}^{\infty} \sum_{r=0}^{n} \xi_k (-1)^r \binom{n}{r} t^{n-r} \eta^n (k+1)^{\frac{n}{2}} \gamma\left(1-\frac{n}{2}, (k+1)\eta^2 x^{-2}\right) |(n<2).$$

The mean residual life (MRL) function or the life expectation at age t defined by $z_1(t) = E[(X-t) \mid X > t]$, which represents the expected additional life length for a unit which is alive at age t . The MRL of X can be obtained when $n = 1$ in $z_n(t)$ equation. The mean inactivity time (MIT) or mean waiting time also called the mean reversed residual life function (MRRL) is given by $Z_1(t) = E[(t-X) \mid X \le t]$and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0,t)$. The MIT of the OLLP-IR distribution can be obtained easily when $n = 1$ in $Z_n(t)$ equation.

Order statistics

Suppose $X_1, \ldots, X_n$ is a random sample from any OLLP-G model, let $X_{i:n}$ denote the ith order statistic. The PDF of $X_{i:n}$ can be expressed as $f_{i:n}(x) = \frac{f(x)}{2(i,n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F(x)^{j+i-1}$. Then, we can write the density function of $X_{i:n}$ as

$$f_{i:n}(x) = \sum_{l,k=0}^{\infty} d_{l,k}\, h_{l+k+1}(x), \qquad (12)$$

where $d_{l,k} = \frac{n!(l+1)(i-1)!d_{l+1}}{(l+k+1)} \sum_{j=0}^{n-i} \frac{(-1)^j \zeta_{j+i-1,k}}{(n-i-j)!j!}$ and the quantities $\zeta_{j+i-1,k}$ can be determined with $\zeta_{j+i-1,0} = d_0^{j+i-1}$ and recursively for $k \ge 1$, $\zeta_{j+i-1,k} = (kd_0)^{-1} \sum_{m=1}^{k} [m(j+i) - k]\, d_m \zeta_{j+i-1,k-m}$. Equation (12) is the main result of this section. It reveals that the PDF of the OLLP-IR order statistics is a linear combination of IR density functions. So, several mathematical quantities of the OLLP-IR order statistics such as ordinary, incomplete and factorial moments, mean deviations and several others can be determined from those quantities of the IR distribution. Then

$$E\left(X_{i:n}^q\right) = \sum_{l,k=0}^{\infty} d_{l,k} \eta^q (k+1)^{\frac{q}{2}} \Gamma\left(1-\frac{q}{2}\right) |(q<2),$$

4. Maximum likelihood estimation

Here, we consider estimation of the unknown parameters of the OLLP-IR distribution by the maximum likelihood method. Let $x_1, \ldots, x_n$ be a random sample from the OLLP-G distribution with a 4×1 parameter vector. The log-likelihood function for Ψ is given by

$$\ell(\Psi) = n\, log\, \alpha + n\, log\, \beta + n2\, log\, \eta + n\, log\, 2 - n\, log\left(\tau_{(\beta)}\right) (2+1) \sum_{i=1}^{n} log(x_i) + \sum_{i=1}^{n} log\left[\zeta_i^{(\eta)}\right]$$
$$+(\alpha-1) \sum_{i=1}^{n} log\left\{exp(\beta) - exp\left[\beta \zeta_i^{(\eta)}\right]\right\} + \beta \sum_{i=1}^{n} \zeta_i^{(\eta)} + (\alpha-1) \sum_{i=1}^{n} log\left\{exp\left[\beta \zeta_i^{(\eta)}\right] - 1\right\}$$
$$-2 \sum_{i=1}^{n} log\left(\left\{\frac{exp\left[\beta \zeta_i^{(\eta)}\right] - 1}{\tau_{(\beta)}}\right\}^{\alpha} + \left\{1 - \frac{exp\left[\beta \zeta_i^{(\eta)}\right] - 1}{\tau_{(\beta)}}\right\}^{\alpha}\right)$$

where

$$\zeta_i^{(\eta)} = exp[-\eta^2 x_i^{-2}].$$

The components of the score vector

$$U(\Psi) = \frac{\partial \ell(\Psi)}{\partial \Psi} = \left(U_\alpha = \frac{\partial \ell(\Psi)}{\partial \alpha}, U_\beta = \frac{\partial \ell(\Psi)}{\partial \beta}, U_\eta = \frac{\partial \ell(\Psi)}{\partial \eta}\right)^{?},$$

can be easily derived. Procedures of the are available in literature so we can ignore this for avoiding the redundant details.

5 Simulation studies

Upon (14), we simulate the OLLP-IR model by taking n =20, 50, 200, 500 and 1000. For each sample size, we evaluate the ML estimations (MLEs) of the parameters using the optim function of the R software (see the R code in the Appendix). Then, we repeat this process 1000 times and compute the averages of the estimates (AEs), biases (Bias) and mean squared errors (MSEs). Table 1 gives all simulation results. The values in Table 2 indicate that the MSEs decay toward zero when n increases for all settings of α, β and η, as expected under first-under asymptotic theory. The AEs of the parameters tend to be closer to the true parameter values (**I:** $\alpha = 1.5,\ \beta = 1.5$ and $\eta = 2.5$ and **II:** $\alpha = 2.5,\ \beta = 0.5$ and $\eta = 2.5$) when n increases. This fact supports that the asymptotic normal distribution provides an adequate approximation to the finite sample distribution of the MLEs. Table 2 gives the AEs and MSEs based on 1000 simulations of the OLLP-IR distribution for some values of a and b when by taking $n = 20{,}50{,}150{,}500$ and 1000 .

Table 2: The AEs, biases and MSEs based on 1000 simulations.

n			AE	MSE			AE	MSE
20	**(I)**	α	1.793350	0.861535	**(II)**	α	2.767703	0.990319
		β	1.845321	0.314960		β	0.680413	0.390565
		η	2.679219	0.523719		η	2.683424	0.518902
50		α	1.594410	0.695429		α	2.573102	0.637859
		β	1.697538	0.305324		β	0.606314	0.193455
		η	2.55980	0.359734		η	2.60910	0.396730
200		α	1.525549	0.494399		α	2.547899	0.415617
		β	1.530891	0.225603		β	0.556720	0.099810
		η	2.515467	0.125894		η	2.534517	0.275609
500		α	1.505480	0.191170		α	2.503258	0.225699
		β	1.508474	0.008005		β	0.513240	0.067683
		η	2.501775	0.016781		η	2.505690	0.038110
1000		α	1.500711	0.002130		α	2.500219	0.001103
		β	1.500843	0.000215		β	0.500205	0.006102
		η	2.500111	0.000405		η	2.500136	0.000224

6. Real data modeling

This section presents two applications of the new distribution using real data sets. We shall compare the fit of the new distribution with the Weibull Inverse Weibull (W-IW), exponentiated IW (E-IW), Kumaraswamy IW(Kum-IW), beta IW (B-IW) transmuted IW (T-IW), gamma extended IW (GE-IW), Marshall-Olkin IW (MO-IW), MOKum-IW, generalized MO-IW(GMO-IW), KumMO-IW and IW distributions. The PDFs of the competitive model are available in statistical literature. The unknown parameters of the above PDFs are all positive real numbers except for the T-IW distribution for which $|a| \leq 1$. The $1\ st$ data set consists of 100 observations of breaking stress of carbon fibers given by Nichols and Padgett (2006). The $2\ nd$ data set consists of 63 observations of the strengths of 1.5 cm glass fibers (see Smith and Naylor (1987)), originally obtained by workers at the UK National Physical Laboratory. Unfortunately, the units of measurement are not given in the paper, we consider the AIC (Akaike Information Criterion), CAIC (Consistent Akaike Information Criterion), BIC (Bayesian information criterion) and HQIC (Hannan-Quinn information Criterion). The model with minimum values for these statistics could be chosen as the best model to fit the data. All results are obtained using the R PROGRAM. Figure 1 and 2, respectively, display the box plots and the quantile-quantile (Q-Q) plots. Tables 3 and 5 compare the OLLP-IR model with other important competitive distributions. The OLLP-IR model gives the lowest values for the AIC, BIC, HQIC and CAIC statistics (in bold values) among all competitive fitted models to these data. So, it may be considered as the best model among them. Figure 3-8, respectively, display the TTT plot, the P-P plot, estimated PDF, estimated HRF estimated CDF and Kaplan-Meier survival plot for the 1st and 2nd data sets. These plots reveal that the proposed distribution yields a better fit than other nested and non-nested models for both data sets. For more useful real life data sets see Elbiely and Yousof (2018 and 2019), Gad et al. (2019), Goual and Yousof (2019), Goual et al. (2019), Ibrahim et al. (2019 and 2020), Nascimento et al. (2019), Yadav et al. (2020), Ibrahim, M. (2020a and b), Ansari and Nofal (2020), Ansari et al. (2020) and Mansour et al. (2020a,b).

Table 3: The statistics AIC, BIC, HQIC and CAICvalues for breaking stress data.

Model	Measures			
	AIC	BIC	HQIC	CAIC
OLLP-IR	144.76	152.58	147.92	146.01
OLLP-IE	381.85	390.19	382.06	385.24
W-IW	294.5	304.9	298.7	294.9
E-IW	295.7	303.5	298.9	296.0
Kum-IW	297.1	307.5	301.3	297.5
B-IW	311.1	321.6	315.4	311.6
GE-IW	312.0	332.4	316.2	312.4
IW	348.3	353.5	350.4	348.4
T-IW	350.5	358.3	353.6	350.7
MO-IW	351.3	359.1	354.5	351.6

Table 4: MLEs and their standard errors (in parentheses) for breaking stress of carbon fiber data.

Model	Estimates			
OLLP-IR(α,β,η)	0.87510	1.9025	1.3623	
	(0.000)	0.447244	(0.000)	
OLLP-IR(α,β,η)	0.0670	18.5425	16.4366	
	(0.08291)	(4.3222)	(20.377)	
W-IW(α,β,a,b)	2.2231	0.355	6.9721	4.9179
	(11.409)	(0.411)	(113.811)	(3.756)
Kum-IW(α,β,a,b)	2.0556	0.4654	6.2815	224.18
	(0.071)	(0.00701)	(0.063)	(0.164)
B-IW(α,β,a,b)	1.6097	0.4046	22.0143	29.7617
	(2.498)	(0.108)	(21.432)	(17.479)
GE-IW(α,β,a,b)	1.3692	0.4776	27.6452	17.4581
	(2.017)	(0.133)	(14.136)	(14.818)
E-IW(α,β,a)	69.1489	0.5019	145.3275	
	(57.349)	(0.08)	(122.924)	
T-IW(α,β,a)	1.9315	1.7435	0.0819	
	(0.097)	(0.076)	(0.198)	
MO-IW(α,β,a)	2.3066	1.5796	0.5988	
	(0.498)	(0.16)	(0.3091)	
IW(α,β)	1.8705	1.7766		
	(0.112)	(0.113)		

Table 5: The statistics AIC, BIC, HQIC and CAIC values for glass fiber data.

Model	Measures			
	AIC	BIC	HQIC	CAIC
OLLP-IR	66.72	73.15	69.25	67.12
OLLP-IE	101.3	109.1	104.4	101.5
B-IW	68.60	77.20	72.00	69.30
GE-IW	69.60	78.10	72.90	70.30
IW	97.70	1020	99.40	97.90
T-IW	100.1	106.5	102.6	100.5
MO-IW	101.7	108.2	104.2	102.1

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Table 5: The statistics AIC, BIC, HQIC and CAIC values for glass fiber data.

Model	Estimates			
OLLP-IR(η,θ,α)	1.0125	3.9473	1.0418	
	(4.772)	(1.2276)	(2.455)	
OLLP-IE(η,θ,α)	0.8419	4.2072	1.207	
	(0.000)	(0.8754)	(0.000)	
B-IW(α,β,a,b)	2.0518	0.6466	15.0756	36.9397
	(0.986)	(0.163)	(12.057)	(22.649)
GE-IW(α,β,a,b)	1.6625	0.7421	32.112	13.2688
	(0.952)	(0.197)	(17.397)	(9.967)
T-IW(α,β,a)	1.3068	2.7898	0.1298	
	(0.034)	(0.165)	(0.208)	
MO-IW(α,β,a)	1.5441	2.3876	0.4816	
	(0.226)	(0.253)	(0.252)	
IW(α,β)	1.264	2.888		
	(0.059)	(0.234)		

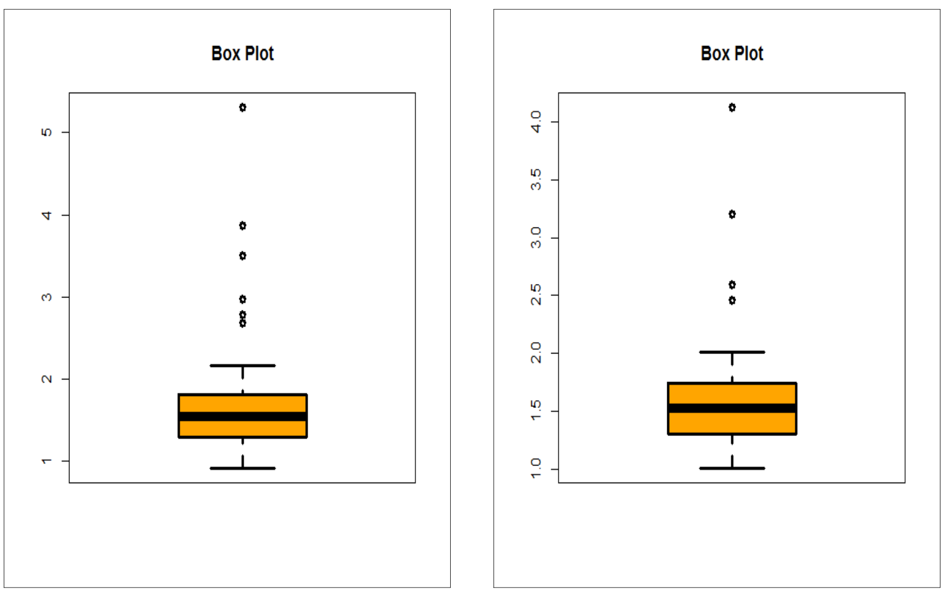

Figure 1: Box plots.

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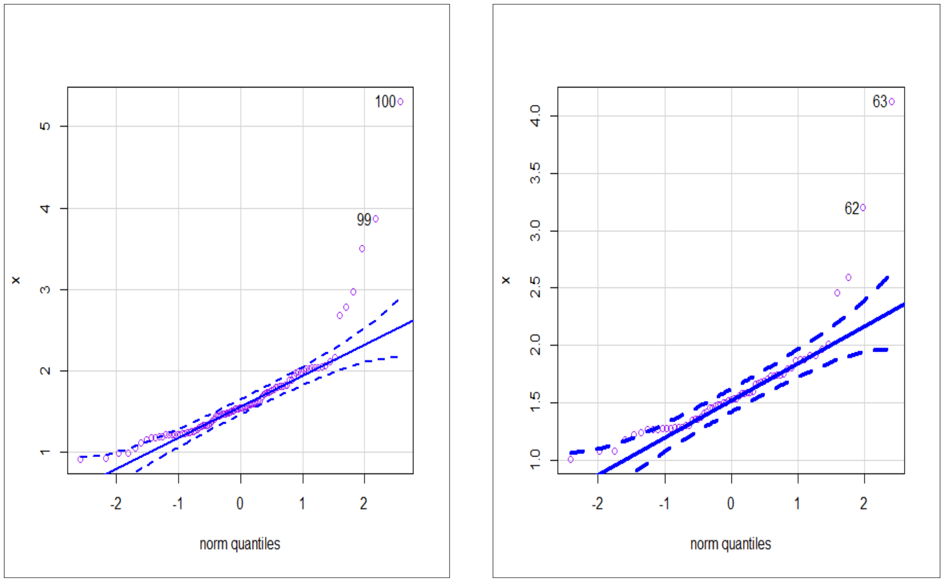

Figure 2: Q-Q plots.

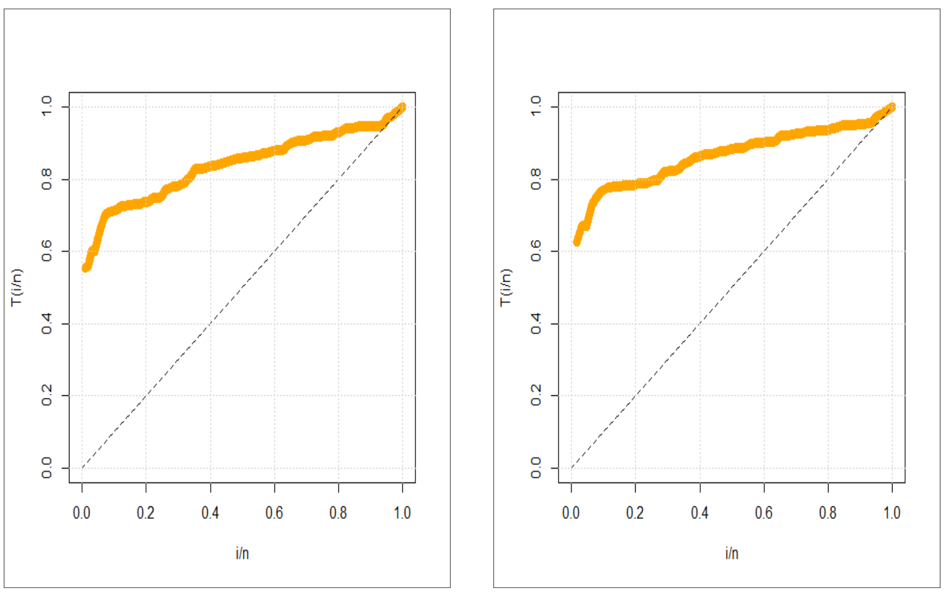

Figure 3: TTT plots.

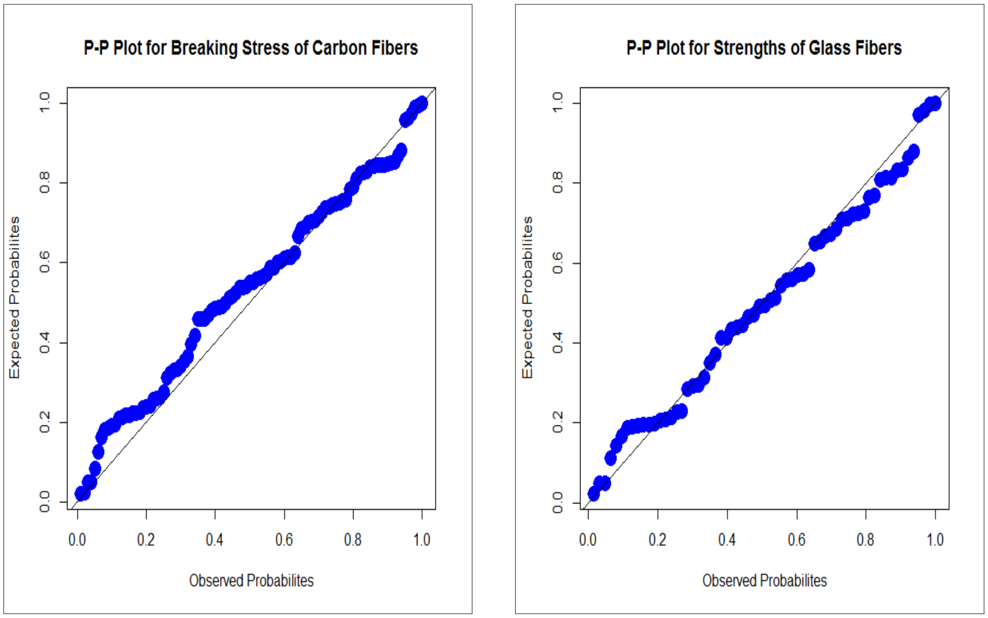

Figure 4: P-P plots.

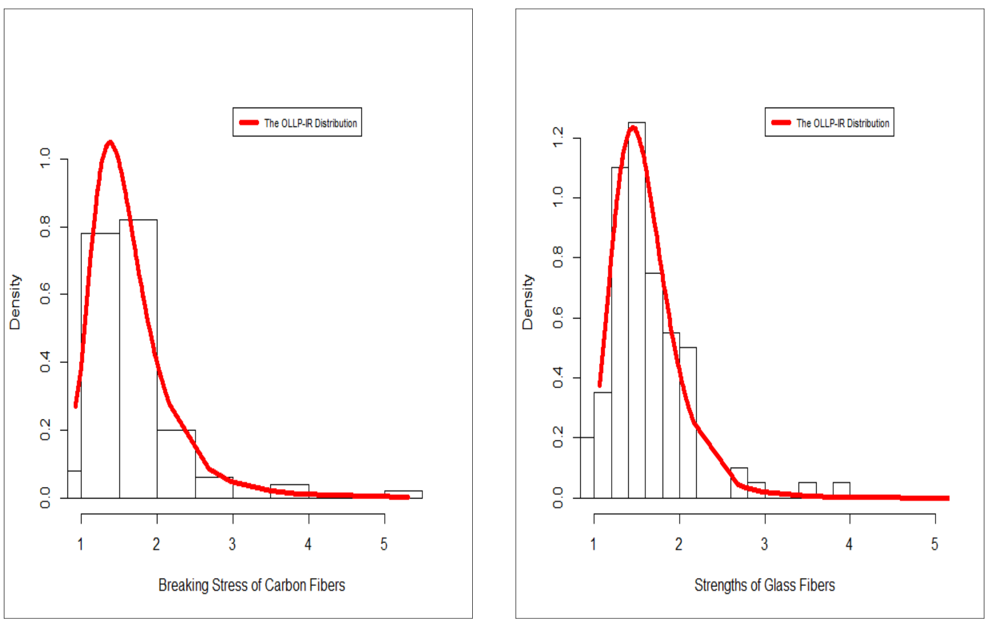

Figure 5: Estimated PDFs.

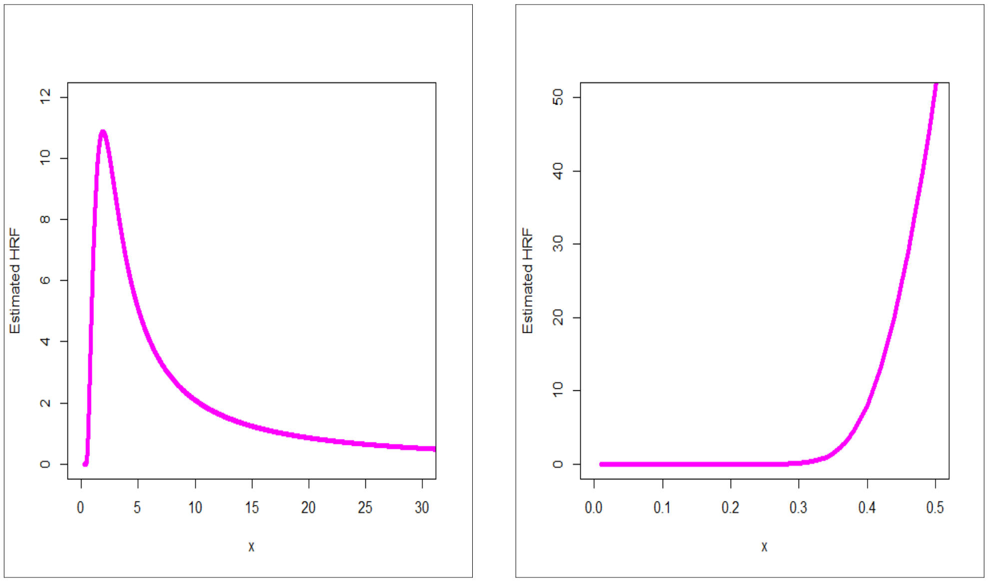

Figure 6: Estimated HRFs.

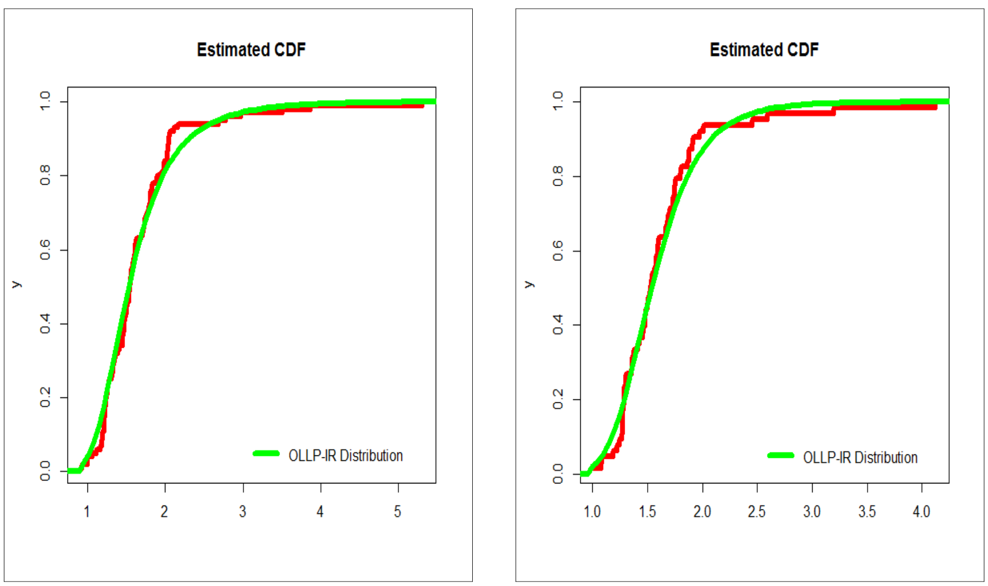

Figure 7: Estimated CDFs.

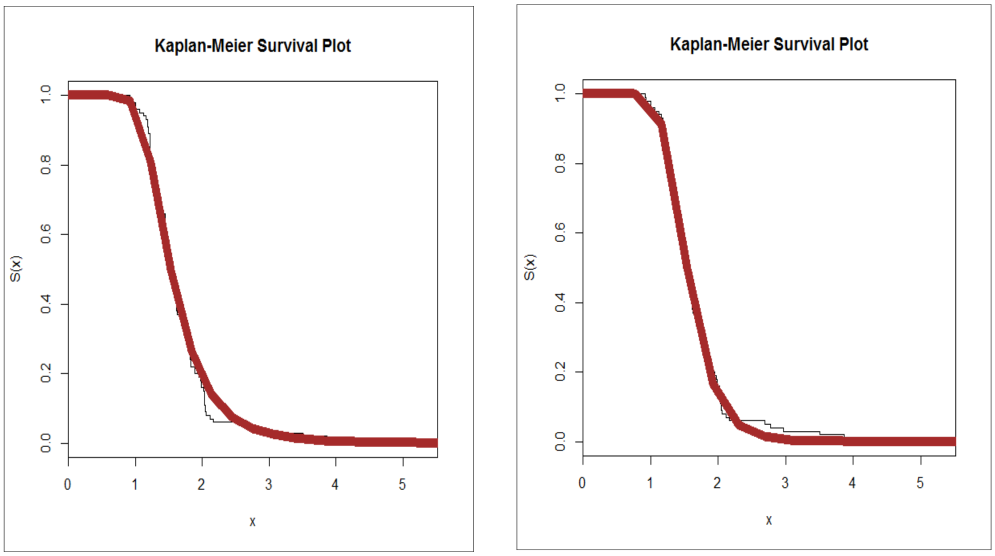

Figure 8: Kaplan-Meier Survival plots.

7. Conclusions

A new extension of the Inverse Rayleigh model is proposed and studied. Some of its fundamental properties are derived. We assessed the performance of the maximum likelihood estimators via a simulation study. The mean, variance, skewness and kurtosis of the new distribution are computed numerically using the R software. The skewness of the new distribution is always positive, the kurtosis is always more than 3. The importance of the new model is shown via two applications to real data sets. The new model is better fit than other important competitive models based on two real data sets.

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