

# A Univariate, Bivariate and Multivariate Extensions for the Inverse Rayleigh Model with Properties and Applications to the Univariate Version

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## Abstract

A new univariate extension of the Inverse Rayleigh distribution is proposed and studied. Some of its fundamental statistical properties such as some stochastic properties, ordinary and incomplete moments, moments generating functions, residual life and reversed residual life functions, order statistics, quantile spread ordering, Rényi, Shannon and q-entropies are derived. A simple type Copula based construction via Morgenstern family and via Clayton copula is employed to derive many bivariate and multivariate extensions of the new model. We assessed the performance of the maximum likelihood estimators using a simulation study. The importance of the new model is shown via two applications to real data sets.

**Keywords:** Inverse Rayleigh Model; Morgenstern Family; Simulation; Clayton Copula; Modeling; Copula; Entropies; Wright Generalized Hypergeometric Function.

## 1. Introduction and motivation

A rv  $W$  is said to have the Inverse Rayleigh (IR) distribution if its probability density function (PDF), cumulative distribution function (CDF) are given by

$$h_{\lambda}(\omega) = 2 \frac{\lambda^2}{\omega^3} \exp(-\lambda^2 \omega^{-2}), \quad (1)$$

and

$$H_{\lambda}(\omega) = \exp(-\lambda^2 \omega^{-2}). \quad (2)$$

The cumulative distribution function (CDF) and probability density function (PDF) of the Generalized Odd Generalized Exponential G (GOGE-G) family are given, respectively, by

$$F_{\alpha,\beta}(\omega) = \left\{ 1 - \exp \left[ \frac{-H_{\psi}(\omega)^{\alpha}}{1 - H_{\psi}(\omega)^{\alpha}} \right] \right\}^{\beta}, \quad (3)$$

$$f_{\alpha,\beta}(\omega) = \frac{\alpha\beta h_{\psi}(\omega) H_{\psi}(\omega)^{\alpha-1}}{[1 - H_{\psi}(\omega)^{\alpha}]^2} \exp \left[ \frac{-H_{\psi}(\omega)^{\alpha}}{1 - H_{\psi}(\omega)^{\alpha}} \right] \left\{ 1 - \exp \left[ \frac{-H_{\psi}(\omega)^{\alpha}}{1 - H_{\psi}(\omega)^{\alpha}} \right] \right\}^{\beta-1}, \quad (4)$$

where  $H_\Psi(\omega)$  is the baseline CDF depending on a parameter vector  $\Psi$  and  $h_\Psi(\omega) = \frac{d}{d\omega}H_\Psi(\omega)$  is its corresponding PDF and  $\alpha, \beta > 0$  are two additional shape parameters. Using (3) and (2) the CDF of the GOGIEIR can be derived as

$$F_{\alpha,\beta,\lambda}(\omega) = \left( 1 - \exp \left\{ \frac{-\exp(-\alpha\lambda^2\omega^{-2})}{1 - \exp(-\alpha\lambda^2\omega^{-2})} \right\} \right)^\beta, \tag{5}$$

the corresponding PDF of (5) can be expressed as

$$f_{\alpha,\beta,\lambda}(\omega) = \frac{2\alpha\beta\lambda^2\omega^{-3}}{[1 - \exp(-\alpha\lambda^2\omega^{-2})]^2} \exp \left[ \frac{-\exp(-\alpha\lambda^2\omega^{-2})}{1 - \exp(-\alpha\lambda^2\omega^{-2})} \right] \times \exp(-\alpha\lambda^2\omega^{-2}) \left\{ 1 - \exp \left[ \frac{-\exp(-\alpha\lambda^2\omega^{-2})}{1 - \exp(-\alpha\lambda^2\omega^{-2})} \right] \right\}^{\beta-1}. \tag{6}$$

Henceforth,  $W \sim \text{GOGIEIR}(\alpha, \beta, \lambda)$  denotes a random variable having density function (6). The hazard rate function (HRF) of  $W$  can be derived using the well-known relationship

$$f_{\alpha,\beta,\lambda}(\omega) / [1 - F_{\alpha,\beta,\lambda}(\omega)].$$

Plots of the GOGIEIR HRF at some parameters value are presented in Figure 2 to show the flexibility of the new model. For simulating data from the GOGIEIR model, if  $u \sim u(0,1)$ , then

$$\omega_u = \lambda \sqrt{\left\{ -\ln \left[ \frac{-\log(1 - \sqrt[\beta]{u})}{1 - \log(1 - \sqrt[\beta]{u})} \right]^{\frac{1}{\alpha}} \right\}^{-1}} \tag{7}$$

has CDF (5). Now, we provide a useful representation for (6). Using the series expansion

$$(1 - c)^\varphi = \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(1 + \varphi)}{m! \Gamma(1 + \varphi - m)} c^m,$$

which holds for  $|c| < 1$  and  $\varphi > 0$  real non-integer and using the power series, the PDF of the GOGIEIR in (6) can be expressed as

$$f(\omega) = \alpha\beta\lambda^2\omega^{-3} \sum_{i,j_1=0}^{\infty} \frac{(-1)^{i+j_1} (i+1)^{j_1}}{j_1!} [1 - \exp(-\alpha\lambda^2\omega^{-2})]^{-(j_1+2)} \times \binom{\beta-1}{i} [\exp(-\lambda^2\omega^{-2})]^{\alpha(j_1+1)}.$$

Using the series expansion again we arrive at

$$f(\omega) = \sum_{j_1,j_2=0}^{\infty} v_{j_1,j_2} h_{\alpha(1+j_1+j_2)}(\omega), \tag{8}$$

where  $h_{\alpha(1+j_1+j_2)}(\omega)$  is the IR density with scale parameter  $\lambda[\alpha(1+j_1+j_2)]^{\frac{1}{2}}$  and

$$v_{j_1,j_2} = \alpha\beta \frac{(-1)^{j_1+j_2}}{j_1! [\alpha(1+j_1+j_2)]} \binom{-(j_1+2)}{j_2} \sum_{i=0}^{\infty} (-1)^i (i+1)^{j_1} \binom{\beta-1}{i}.$$

The CDF of the GOGAIR model can also be expressed as a mixture of the IR density CDFs. By integrating (8), we obtain the same mixture representation

$$F(\omega) = \sum_{j_1, j_2=0}^{\infty} v_{j_1, j_2} H_{\alpha(1+j_1+j_2)}(\omega),$$

where  $H_{\alpha(1+j_1+j_2)}(\omega)$  is the CDF of the IR density with scale parameter  $\lambda[\alpha(1 + j_1 + j_2)]^{\frac{1}{2}}$ .

Some important extensions of the Fr model can be cited Nadarajah and Kotz (2003), Barreto-Souza et al. (2011), Mahmoud and Mandouh (2013), Krishna et.al. (2013), Yousof et el. (2015), Mead et al. (2016), Afify et al. (2016a), Weibull Fr Afify et al. (2016b), Yousof et al. (2016), Korkmaz et al. (2017), Yousof et al. (2018a), Yousof et al. (2018b), Brito et al. (2017), Hamedani et al. (2017), Cordeiro et al. (2018), Chakraborty et al. (2018), Hamedani et al. (2018), Korkmaz et al. (2018), Korkmaz et al. (2019) and Hamedani et al. (2019).

The rest of the paper is outlined as follows. In Section 2, we derive some mathematical properties for the new model including some stochastic properties, ordinary and incomplete moments, moments generating functions, residual life and reversed residual life functions, order statistics, quantile spread ordering, Rényi, Shannon and q-entropies. A simple type Copula based construction is derived in Section 3. Maximum likelihood estimation of the model parameters is addressed in Section 4. In Section 5, we assess the performance of the maximum likelihood estimators using a Monte Carlo simulation study. In Section 6, we provide two applications to real data to illustrate the importance of the GOGAIR model. Finally, some concluding remarks are presented in Section 7.

**2.Mathematical properties**

**2.1Moments, cumulants and numerical results**

The  $r^{th}$  ordinary moment of  $W$  is given by

$$\mu'_r = E(W^r) = \int_{-\infty}^{\infty} \omega^r f(\omega) d\omega.$$

Then we obtain

$$\mu'_r |_{(2>r)} = \sum_{j_1, j_2=0}^{\infty} v_{j_1, j_2} [\alpha(1 + j_1 + j_2)]^{\frac{r}{2}} K_{\left(1-\frac{r}{2}\right)}^{(\lambda, r)}, \tag{9}$$

where

$$\begin{aligned} \Gamma(1 + \vartheta) |_{(\vartheta \in R^+)} &= \vartheta! \\ &= \prod_{h=0}^{\vartheta-1} (\vartheta - h) \\ &= \int_0^{\infty} t^{\vartheta} \exp(-t) dt, \end{aligned}$$

is the complete gamma function and

$$K_{\Gamma(\cdot)}^{(\lambda, r)} = \lambda^r \Gamma(\cdot)$$

. Setting  $r = 1$  in (9), we have the mean of  $W$

$$\mu'_1 = E(W) = K_{\left(1-\frac{1}{2}\right)}^{(\lambda,1)} \sum_{j_1, j_2=0}^{\infty} v_{j_1, j_2} \sqrt{\alpha(1 + j_1 + j_2)}$$

The last integration can be computed numerically for most parent distributions. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. The  $n^{\text{th}}$  central moment of  $W$ , say  $\mu_n$ , follows as

$$\mu_n = E(W - \mu)^n = \sum_{h=0}^n (-1)^h \binom{n}{h} (\mu'_1)^n \mu'_{n-h}.$$

The cumulants ( $\kappa_n$ ) of  $W$  follow recursively from

$$\kappa_n = \mu'_n - \sum_{r=0}^{n-1} \kappa_r \binom{n-1}{r-1} \mu'_{n-r},$$

where

$$\begin{aligned} \kappa_1 &= \mu'_1, \\ \kappa_2 &= \mu'_2 - \mu_1'^2, \text{ and} \\ \kappa_3 &= \mu'_3 - 3\mu'_2\mu'_1 + \mu_1'^3 \end{aligned}$$

The mean of  $W$ , variance ( $V(W)$ ) skewness ( $S(W)$ ) and kurtosis ( $K(W)$ ) measures also can be calculated from the ordinary moments using well-known relationships. The mean, variance, skewness and kurtosis of the GOGAIR distribution are computed numerically for some selected values using the R software. The numerical values displayed in Table 1 indicate that the skewness of the GOGAIR distribution is always positive and can range in the interval (0.59,1.016). The spread for its kurtosis is ranging from 1.05 to 3.6. With fixing the values of  $\beta$  and  $\lambda$ , we note that the ( $S(W)$ ) and ( $K(W)$ ) has no changes.

$\alpha$	$\beta$	$\lambda$	E(W)	V(W)	S(W)	K(W)
1	3	5	7.331273	3.424783	0.6140166	3.488065
2			10.36799	6.849566	0.6140166	3.488065
5			16.39322	17.12391	0.6140166	3.488065
10			23.18352	34.24783	0.6140166	3.488065
50			51.83993	171.2391	0.6140166	3.488065
100			73.31273	342.4783	0.6140166	3.488065
500			163.9322	1712.391	0.6140166	3.488065
1	10	6	10.89284	3.792974	0.6172596	3.63095
10			34.44618	37.92974	0.6172596	3.63095
50			77.02399	189.6487	0.6172598	3.63095
100			108.9284	379.2974	0.6172596	3.63095
100	1	10	111.757	1598.442	0.7912121	3.622083
	10		181.5473	1053.604	0.6172596	3.63095
	50		221.4869	758.3086	0.7064480	3.876657
	100		236.7447	672.022	0.7437847	3.974214
	150		4.0873 $\times 10^{-6}$	60769.17	1.0162270	1.044547
2	5	1	2.290886	0.2459378	0.599869	3.535318
		10	22.90886	24.59378	0.599869	3.535318
		100	229.0886	2459.378	0.599869	3.535318

		250	572.7216	15371.11	0.599869	3.535318
		500	1145.443	61484.45	0.599869	3.535318

**2.2 Incomplete moment**

The  $r$ th incomplete moment, say  $\phi_r(t)$ , of  $W$  can be expressed, from (8), as

$$\begin{aligned} \phi_r(t) &= \int_{-\infty}^t \omega^r f(\omega) d\omega \\ &= \sum_{j_1, j_2=0}^{\infty} v_{j_1, j_2} [\alpha(1 + j_1 + j_2)]^{\frac{r}{2}} K_{\gamma\left(1-\frac{r}{2}, [\alpha(1+j_1+j_2)]\left(\frac{\lambda}{t}\right)^2\right)}^{(\lambda, r)} \Big|_{(2>r)} \end{aligned}$$

where  $\gamma(\vartheta, u)$  is the incomplete gamma function.

$$\begin{aligned} \gamma(\vartheta, u) |_{(\vartheta \neq 0, -1, -2, \dots)} &= \int_0^u t^{\vartheta-1} \exp(-t) dt \\ &= \frac{u^\vartheta}{\vartheta} \{1F_1[\vartheta; \vartheta + 1; -u]\} \\ &= \sum_{\eta=0}^{\infty} \frac{(-1)^\eta}{\eta! (\vartheta + \eta)} u^{\vartheta+\eta}, \end{aligned}$$

$$K_{\gamma(\cdot, \cdot)}^{(\lambda, r)} = \lambda^r \gamma(\cdot, \cdot)$$

and  $1F_1[\cdot, \cdot, \cdot]$  is a confluent hypergeometric function. The first incomplete moment can be calculated by setting  $r = 1$  in  $\phi_r(t)$  as

$$\phi_1(t) = \sum_{j_1, j_2=0}^{\infty} v_{j_1, j_2} [\alpha(1 + j_1 + j_2)]^{\frac{1}{2}} K_{\gamma\left(1-\frac{1}{2}, [\alpha(1+j_1+j_2)]\left(\frac{\lambda}{t}\right)^2\right)}^{(\lambda, 1)}$$

**2.3 The moment generating function (MGF)**

The MGF  $M_W(t) = E(e^{tW})$  of  $\omega$  can be derived from equation (4) as

$$M_W(t) |_{(2>r)} = \sum_{j_1, j_2, r=0}^{\infty} \frac{t^r}{r!} v_{j_1, j_2} [\alpha(1 + j_1 + j_2)]^{\frac{r}{2}} K_{\left(1-\frac{r}{2}\right)}^{(\lambda, r)}$$

Another alternative method for deriving the MGF can be introduced by the Wright generalized hypergeometric function (WHGF) which is defined by

$$({}_\tau)\Psi_{(u)} \left[ \begin{matrix} a_1, A_1, \dots, a_\tau, A_\tau \\ b_1, B_1, \dots, b_u, B_u \end{matrix}; \omega \right] = \sum_{n=0}^{\infty} \frac{\prod_{j_1=1}^{\tau} \Gamma(a_{j_1} + A_{j_1} n)}{\prod_{j_1=1}^u \Gamma(b_{j_1} + B_{j_1} n)} \frac{\omega^n}{n!}$$

Then, the MGF of (1) can be defined as

$$M_W(t) = ({}_1)\Psi_{(0)} \left[ \left(1, -\frac{1}{2}\right); \lambda t \right]. \tag{10}$$

Combining expressions (8) and (10), we obtain the MGF of the GOGAIR as

$$M_W(t) = \sum_{j_1, j_2=0}^{\infty} v_{j_1, j_2} \left\{ ({}_1)\Psi_{(0)} \left[ \left(1, -\frac{1}{2}\right); t\lambda\sqrt{\alpha(1 + j_1 + j_2)} \right] \right\}. \tag{11}$$

Equations (9) and (11) can be easily evaluated by scripts of the Maple, Matlab and Mathematica platforms.

### 2.4 Entropies

The Rényi entropy of a random variable  $W$  represents a measure of variation of the uncertainty. The Rényi entropy is defined by

$$R_{\vartheta}(W)|_{(\vartheta>0 \text{ and } \vartheta \neq 1)} = \frac{\log \int_{-\infty}^{\infty} f(\omega)^{\vartheta} d\omega}{1 - \vartheta}.$$

Using the PDF (6), we can write

$$f(\omega)^{\vartheta} = \sum_{j_1, j_2=0}^{\infty} \eta_{j_1, j_2}^{(\vartheta)} \omega^{-3\vartheta} \exp \left[ -[\alpha(j_1 + j_2 + \vartheta)] \left(\frac{\lambda}{\omega}\right)^2 \right],$$

where

$$\eta_{j_1, j_2}^{(\vartheta)} = (2\alpha\beta\lambda^2)^{\vartheta} \frac{(-1)^{j_1+j_2}}{j_1!} \binom{-(j_1+2)}{j_2} \sum_{i=0}^{\infty} (-1)^i (i + \vartheta)^{j_1} \binom{\vartheta(\beta-1)}{i}.$$

Then, the Rényi entropy of the GOGIEIR model is given by

$$R_{\vartheta}(W)|_{(\vartheta>0 \text{ and } \vartheta \neq 1)} = \frac{\log \left\{ \sum_{j_1, j_2=0}^{\infty} \eta_{j_1, j_2}^{(\vartheta)} [(\vartheta)I_0^{\infty}] \right\}}{1 - \vartheta},$$

where

$$(\vartheta)I_0^{\infty} = \int_0^{\infty} \omega^{-3\vartheta} \exp \{ -[\alpha(j_1 + j_2 + \vartheta)] \lambda^2 \omega^{-2} \} d\omega$$

The q-entropy, say  $Q_{\delta}(W)$ , can be obtained as

$$Q_{\delta}(W)|_{(\delta>0 \text{ and } \delta \neq 1)} = \frac{\log \left( 1 - \left\{ \sum_{j_1, j_2=0}^{\infty} \eta_{j_1, j_2}^{(\delta)} [(\delta)I_0^{\infty}] \right\} \right)}{\delta - 1},$$

where

$$\eta_{j_1, j_2}^{(\delta)} = (2\alpha\beta\lambda^2)^{\delta} \frac{(-1)^{j_1+j_2}}{j_1!} \binom{-(j_1+2)}{j_2} \sum_{i=0}^{\infty} (-1)^i (i + \delta)^{j_1} \binom{\delta(\beta-1)}{i},$$

and

$$(\delta)I_0^{\infty} = \int_0^{\infty} \omega^{-3\delta} \exp \{ -[\alpha(j_1 + j_2 + \delta)] \lambda^2 \omega^{-2} \} d\omega$$

The Shannon entropy of a random variable  $W$ , say  $SE$ , is defined by

$$SE = E \{ -[\log f(W)] \}.$$

It is the special case of the Rényi entropy,  $R_{\vartheta}(W)|_{(\vartheta>0 \text{ and } \vartheta \neq 1)}$ , when  $\vartheta \uparrow 1$ .

### 2.5 Order statistics

Let  $W_1, W_2, \dots, W_n$  be a random sample (RS) from the GOGIEIR distribution and let  $W_1 : n, W_2 : n, \dots, W_n : n$  be the corresponding order statistics. The PDF of the  $i^{\text{th}}$  order statistic, say  $W_i : n$ , can be written as

$$f_{i : n}(\omega) = B^{-1}(i, 1 + n - i) \sum_{j_1=0}^{n-i} (-1)^{j_1} \binom{n-i}{j_1} f(\omega) F^{j_1+i-1}(\omega), \tag{12}$$

where  $B(\cdot, \cdot)$  is the beta function. Substituting (1) and (2) in equation (12) and using a power series expansion, we have

$$f(\omega) F(\omega)^{j_1+i-1} = \sum_{w, j_2=0}^{\infty} \tau_{w, j_2} h_{\alpha(w+j_2+1)}(\omega),$$

where

$$\tau_{w, j_2} = \frac{\alpha\beta(-1)^{w+j_2}}{w! [\alpha(w+m+1)]} \binom{-(w+2)}{j_2} \sum_{l=0}^{\infty} (-1)^l (l+1)^w \binom{\beta(i+j_1)-1}{l}.$$

Then, the PDF of  $\omega_{i : n}$  can be expressed as

$$f_{i : n}(\omega) = \sum_{j_1=0}^{n-i} \sum_{w, j_2=0}^{\infty} (-1)^{j_1} B^{-1}(i, 1 + n - i) \binom{n-i}{j_1} \tau_{w, j_2} h_{\alpha(w+j_2+1)}(\omega).$$

The density function of the GOGIEIR order statistics is a mixture of IR densities. Based on the last equation, the moments of  $W_{i : n}$  can be expressed as

$$E(W_{i : n}^{\delta} |_{(2>\delta)}) = \sum_{w, j_2=0}^{\infty} \sum_{j_1=0}^{n-i} \frac{(-1)^{j_1} B^{-1}(i, 1 + n - i) \binom{n-i}{j_1} \tau_{w, j_2} K_{(1-\frac{\delta}{2})}^{(\lambda, \delta)}}{[\alpha(w+j_2+1)]^{\frac{-\delta}{2}}}.$$

### 2.6 Quantile spread order

The quantile spread ( $Y(\vartheta)$ ) of the rv  $W \sim \text{GOGIEIR}(\alpha, \beta, \lambda)$  having the CDF (5) is given by

$$Y_{\omega}(\vartheta) |_{(\vartheta \in (\frac{1}{2}, 1))} = [F^{-1}(\vartheta)] - [F^{-1}(1 - \vartheta)],$$

which implies

$$Y_{\omega}(\vartheta) = [S^{-1}(1 - \vartheta)] - [S^{-1}(\vartheta)],$$

where  $F^{-1}(\vartheta) = S^{-1}(1 - \vartheta)$  and  $S(\cdot) = 1 - F(\cdot)$  is the survival function. The  $Y(\vartheta)$  of a distribution describes how the probability mass is placed symmetrically about its median and hence can be used to formalize concepts such as peakedness and tail weight traditionally associated with kurtosis. So, it allows us to separate concepts of kurtosis and peakedness for asymmetric models.

Let  $Z_1$  and  $\omega_2$  be two random variables following the GOGIEIR model with quantile spreads  $Y_{Z_1}$  and  $Y_{Z_2}$ , respectively. Then  $Z_1$  is called smaller than  $Z_2$  in quantile spread order, denoted as

$$Z_1 \leq_{[Y]} Z_2,$$

when

$$Y_{Z_1}(\vartheta) \leq Y_{Z_2}(\vartheta) |_{(\vartheta \in (\frac{1}{2}, 1))}.$$

The following properties of the quantile spread order can be obtained:

- The order  $\leq_{[Y]}$  is location-Free

$$Z_1 \leq_{[\gamma]} Z_2 \text{ if } (Z_1 + \eta) \leq_{[\gamma]} Z_2 \forall \eta \in R.$$

- The order  $\leq_{[\gamma]}$  is dilative  
 $Z_1 \leq_{[\gamma]} \eta Z_1$  whenever  $\eta \geq 1$  and  $Z_2 \leq_{[\gamma]} \eta Z_2 \mid_{\eta \geq 1}$ .

- Let  $F_{Z_1}$  and  $F_{Z_2}$  be symmetric, then

$$Z_1 \leq_{[\gamma]} Z_2 \text{ if, and only if } F_{Z_1}^{-1}(\vartheta) \leq F_{Z_2}^{-1}(\vartheta), \forall \vartheta \in \left(\frac{1}{2}, 1\right).$$

- The order  $\leq_{[\gamma]}$  implies ordering of the mean absolute deviation around the median, say  $\omega(\cdot)$ ,

$$\omega(Z_1) = E[|Z_1 - \text{Median}(Z_1)|]$$

and

$$\omega(Z_2) = E[|Z_2 - \text{Median}(Z_2)|],$$

where

$$Z_1 \leq_{[\gamma]} Z_2 \Rightarrow \omega(Z_1) \leq_{[\gamma]} \omega(Z_2).$$

Finally

$$Z_1 \leq_{[\gamma]} Z_2 \text{ if, and only if } -Z_1 \leq_{[\gamma]} -Z_2.$$

## 2.7 Residual life and reversed residual life functions

The  $n^{\text{th}}$  moment of the residual life, say

$$a_n(t) = E[(W - t)^n] \mid_{(W > t \text{ and } n=1,2,\dots)},$$

uniquely determines the CDF  $F(\omega)$ . The  $n^{\text{th}}$  moment of the residual life of  $W$  is given by

$$a_n(t) = \frac{\int_t^\infty (\omega - t)^n dF(\omega)}{1 - F(t)}.$$

Therefore

$$a_n(t) \mid_{(2 > n)} = \frac{1}{1 - F(t)} \sum_{j_1, j_2=0}^\infty \varphi_{j_1, j_2} [\alpha(1 + j_1 + j_2)]^n K_{\Gamma\left(1 - \frac{n}{2}, [\alpha(1 + j_1 + j_2)]\left(\frac{\lambda}{t}\right)^2\right)}^{(\lambda, n)},$$

where

$$\varphi_{j_1, j_2} = v_{j_1, j_2} \sum_{r=0}^n (-t)^{n-r} \binom{n}{r},$$

$$\Gamma(\vartheta, u) \mid_{(\omega > 0)} = \int_u^\infty t^{\vartheta-1} \exp(-t) dt,$$

$$\Gamma(\vartheta, u) = \Gamma(\omega) - \gamma(\vartheta, u)$$

and

$$K_{\Gamma(\cdot, \cdot)}^{(\lambda, n)} = \lambda^n \Gamma(\cdot, \cdot)$$

Another interesting function is the mean residual life (MRL) function or the life expectation at age  $t$  defined by



$$z_1(t) = E[(W - t)^n \mid (W > t \text{ and } n=1),$$

which represents the expected additional life length for a unit which is alive at age  $t$ . The MRL of  $W$  can be obtained by setting  $n = 1$  in the last equation. The  $n^{\text{th}}$  moment of the reversed residual life, say

$$Z_n(t) = E[(t - W)^n \mid (W \leq t, t > 0 \text{ and } n=1,2,\dots)$$

uniquely determines the CDF  $F(\omega)$ . We obtain

$$A_n(t) = \frac{\int_0^t (t - \omega)^n dF(\omega)}{F(t)}.$$

Then, the  $n^{\text{th}}$  moment of the reversed residual life of  $W$  becomes

$$A_n(t) |_{(2>n)} = \frac{1}{F(t)} \sum_{j_1, j_2=0}^{\infty} \phi_{j_1, j_2} [\alpha(1 + j_1 + j_2)]^{\frac{n}{2}} K_{\left(1-\frac{n}{2}, [\alpha(1+j_1+j_2)]\left(\frac{\lambda}{t}\right)^2\right)}^{(\lambda, n)},$$

where

$$\phi_{j_1, j_2} = v_{j_1, j_2} \sum_{r=0}^n (-1)^r t^{n-r} \binom{n}{r}.$$

The mean inactivity time (MIT) or mean waiting time (MWT) also called the mean reversed residual life function is given by

$$A_1(t) = E[(t - W)^n \mid (W \leq t, t > 0 \text{ and } n=1)$$

and it represents the waiting time elapsed since the failure of an item on condition that occurred in  $(0, t)$ . The MIT of the GOGAIR distribution can be obtained easily by setting  $n = 1$  in the above equation.

### 3.Simple type Copula based construction

In this Section, we consider several approaches to construct the bivariate and the multivariate GOGAIR type distributions via copula (or with straightforward bivariate CDFs form, in which we just need to consider two different GOGAIR CDFs).

#### 3.1 Via Morgenstern family

**First, we start with CDF for Morgenstern family of two random variables  $(W_1, W_2)$  which has the following form**

$$F_{\lambda}(W_1, \omega_2) |_{(|\lambda| \leq 1)} = F_1(\omega_1)F_2(\omega_2)\{1 + \lambda[1 - F_1(\omega_1)][1 - F_2(\omega_2)]\},$$

setting

$$F_{\alpha_1, \beta_1, \lambda}(\omega_1) = \left\{ 1 - \exp \left[ \frac{-\exp(-\alpha_1 \lambda^2 \omega_1^{-2})}{1 - \exp(-\alpha_1 \lambda^2 \omega_1^{-2})} \right] \right\}^{\beta_1},$$

and

$$F_{\alpha_2, \beta_2, \lambda}(\omega_2) = \left\{ 1 - \exp \left[ \frac{-\exp(-\alpha_2 \lambda^2 \omega_2^{-2})}{1 - \exp(-\alpha_2 \lambda^2 \omega_2^{-2})} \right] \right\}^{\beta_2},$$

then we have a seven-dimension parameter model. The estimation will be a big issue here. Estimation via Bayesian paradigm may be done, but again, the choice of appropriate priors will be challenging.

### 3.2 Via Clayton copula

#### The bivariate extension

The bivariate extension via Clayton copula can be considered as a weighted version of the Clayton copula, which is of the form

$$C(u, v) = [u^{-(\delta_1+\delta_2)} + v^{-(\delta_1+\delta_2)} - 1]^{-\frac{1}{\delta_1+\delta_2}}.$$

This is indeed a valid copula. Next, let us assume that  $W \sim \text{GOGIEIR}(\alpha_1, \beta_1, \lambda)$  and  $Y \sim \text{GOGIEIR}(\alpha_2, \beta_2, \lambda)$ . Then, setting

$$u = \left\{ 1 - \exp \left[ \frac{-\exp(-\alpha_1 \lambda^2 \omega_1^{-2})}{1 - \exp(-\alpha_1 \lambda^2 \omega_1^{-2})} \right] \right\}^{\beta_1}$$

and

$$v = F(y) = \left\{ 1 - \exp \left[ \frac{-\exp(-\alpha_2 \lambda^2 y_2^{-2})}{1 - \exp(-\alpha_2 \lambda^2 y_2^{-2})} \right] \right\}^{\beta_2},$$

the associated CDF bivariate GOGIEIR type distribution will be

$$H(\omega, y) = \left( \begin{array}{c} \left\{ 1 - \exp \left[ \frac{-\exp(-\alpha_1 \lambda^2 \omega_1^{-2})}{1 - \exp(-\alpha_1 \lambda^2 \omega_1^{-2})} \right] \right\}^{-\beta_1(\delta_1+\delta_2)} \\ + \left\{ 1 - \exp \left[ \frac{-\exp(-\alpha_2 \lambda^2 y_1^{-2})}{1 - \exp(-\alpha_2 \lambda^2 y_2^{-2})} \right] \right\}^{-\beta_2(\delta_1+\delta_2)} \\ -1 \\ -d \end{array} \right)^{\frac{1}{\delta_1+\delta_2}}.$$

Note: Depending on the specific baseline CDF, one may construct various bivariate GOGIEIR type model in which  $(\delta_1 + \delta_2) \geq 0$ .

#### The Multivariate extension

A straightforward  $d$ -dimensional extension from the above will be

$$H(\omega_1, \omega_2, \dots, \omega_d) = \left( \begin{array}{c} \sum_{i=1}^d \left\{ 1 - \exp \left[ \frac{-\exp(-\alpha_i \lambda^2 \omega_i^{-2})}{1 - \exp(-\alpha_i \lambda^2 \omega_i^{-2})} \right] \right\}^{-\beta_i(\delta_1+\delta_2)} \\ +1 \\ -d \end{array} \right)^{-1/(\delta_1+\delta_2)}.$$

Further future works could be allocated for studying the bivariate and the multivariate extensions of the GOGIEIR model.

### 3.3 Some stochastic properties

Suppose  $W_1 \sim \text{GOGIEIR}(\alpha_1, \beta_1, \lambda)$  and  $W_2 \sim \text{GOGIEIR}(\alpha_2, \beta_2, \lambda)$ . Then  $W_1$  is stochastically smaller than  $W_2$  if  $\alpha_1 > \alpha_2$  and  $\beta_1 > \beta_2$ . Note that for any  $\alpha_1 > \alpha_2$ ,

$$\exp(-\alpha_1 \lambda^2 \omega_1^{-2}) > \exp(-\alpha_2 \lambda^2 \omega_2^{-2}).$$

This is true for both integer and fractional values of  $\alpha_1$  and  $\alpha_2$ . After some algebra, we get the following: Since for  $\alpha_1 > \alpha_2$  we have

$$\exp(-\alpha_1 \lambda^2 \omega_1^{-2}) > \exp(-\alpha_2 \lambda^2 \omega_2^{-2}),$$

then

$$\{1 - \exp(-\alpha_1 \lambda^2 \omega_1^{-2})\} < \{1 - \exp(-\alpha_2 \lambda^2 \omega_2^{-2})\},$$

$$\frac{\exp(-\alpha_1 \lambda^2 \omega_1^{-2})}{1 - \exp(-\alpha_1 \lambda^2 \omega_1^{-2})} > \frac{\exp(-\alpha_2 \lambda^2 \omega_2^{-2})}{1 - \exp(-\alpha_2 \lambda^2 \omega_2^{-2})},$$

and

$$\frac{-\exp(-\alpha_1 \lambda^2 \omega_1^{-2})}{1 - \exp(-\alpha_1 \lambda^2 \omega_1^{-2})} < \frac{-\exp(-\alpha_2 \lambda^2 \omega_2^{-2})}{1 - \exp(-\alpha_2 \lambda^2 \omega_2^{-2})}.$$

Rest of the proof follows immediately from here

$$\begin{aligned} & -\frac{\exp(-\alpha_1 \lambda^2 \omega_1^{-2})}{1 - \exp(-\alpha_1 \lambda^2 \omega_1^{-2})} < -\frac{\exp(-\alpha_2 \lambda^2 \omega_2^{-2})}{1 - \exp(-\alpha_2 \lambda^2 \omega_2^{-2})} \\ & \rightarrow \left(1 - \exp\left\{\frac{-\exp(-\alpha_1 \lambda^2 \omega_1^{-2})}{1 - \exp(-\alpha_1 \lambda^2 \omega_1^{-2})}\right\}\right)^{\beta_1} > \left(1 - \exp\left\{\frac{-\exp(-\alpha_2 \lambda^2 \omega_2^{-2})}{1 - \exp(-\alpha_2 \lambda^2 \omega_2^{-2})}\right\}\right)^{\beta_2} \\ & \rightarrow 1 - \left(1 - \exp\left\{\frac{-\exp(-\alpha_1 \lambda^2 \omega_1^{-2})}{1 - \exp(-\alpha_1 \lambda^2 \omega_1^{-2})}\right\}\right)^{\beta_1} \\ & < 1 - \left(1 - \exp\left\{\frac{-\exp(-\alpha_2 \lambda^2 \omega_2^{-2})}{1 - \exp(-\alpha_2 \lambda^2 \omega_2^{-2})}\right\}\right)^{\beta_2}. \end{aligned}$$

This completes the proof.

#### 4. Estimation

Let  $\omega_1, \dots, \omega_n$  be a RS IRom the GOGAIR distribution with parameters  $\alpha, \beta, \lambda$  and 2. Let  $\Phi = (\alpha, \beta, \lambda)$  be the  $4 \times 1$  parameter vector. For determining the MLE of  $\theta$ , we have the log-likelihood function

$$\begin{aligned} \ell(\Phi) = & n \log \alpha + n \log \beta + n \log 2 + 2n \log \lambda + \\ & -3 \sum_{i=1}^n \log \omega_i - 2 \sum_{i=1}^n \log [1 - \eta_i^{(\alpha, \lambda)}] \\ & + \sum_{i=1}^n \left[ \frac{-\eta_i^{(\alpha, \lambda)}}{1 - \eta_i^{(\alpha, \lambda)}} \right] + \sum_{i=1}^n \log [\eta_i^{(\alpha, \lambda)}] \\ & + (\beta - 1) \sum_{i=1}^n \log \left( 1 - \exp \left[ \frac{-\eta_i^{(\alpha, \lambda)}}{1 - \eta_i^{(\alpha, \lambda)}} \right] \right) \end{aligned}$$

where  $\eta_i^{(\alpha, \lambda)} = \exp(-\alpha \lambda^2 \omega_i^{-2})$ . The components of the score vector

$$U(\Phi) = \frac{\partial \ell}{\partial \Phi} = \left( U_\alpha = \frac{\partial \ell(\Phi)}{\partial \alpha}, U_\beta = \frac{\partial \ell(\Phi)}{\partial \beta}, U_\lambda = \frac{\partial \ell(\Phi)}{\partial \lambda} \right),$$

are easy to derive, setting the nonlinear system of equations  $U_\alpha = U_\beta = U_\lambda = 0$  and  $U_2 =$

0 and solving them simultaneously yields the MLE  $\hat{\alpha}, \hat{\beta}, \hat{\lambda}$ . To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize  $\ell(\Phi)$ . Further works could be devoted using other different methods to estimate the GOGAIR parameters such as least squares, moments, weighted least squares, Anderson-Darling, Jackknife, bootstrap, Cramér-von-Mises, Bayesian analysis, among others, and compare the estimators based on these methods.

### 5. Simulations

Using (7), we simulate the GOGAIR model via taking  $n = 20, 50, 150, 500$  and  $1000$ . We evaluate the MLEs of the parameters for each sample size. Then, repeating this process  $N = 1000$  times and calculate the averages of the estimates (AEs), mean squared errors (MSEs). Table 2 gives all simulation results. The numerical results in Table 2 indicate that MSEs of  $\hat{\alpha}$ ,  $\hat{\beta}$  and  $\hat{\lambda}$  decay toward 0 when  $n$  increases for all initial values of  $\alpha, \beta$  and  $\lambda$ . The AEs of the parameters tend to be closer to the true (initial) parameter values (**I**:  $\alpha = 1.75, \beta = 1.5$  and  $\lambda = 2.25$  and **II**:  $\alpha = 2.5, \beta = 2.25$  and  $\lambda = 1.25$ ) when  $n$  increases. These results supports that the asymptotic normal model provides an adequate approximation to the finite sample distribution of the MLEs. Table 2 below gives the AEs and MSEs based on  $N = 1000$  simulations of the GOGAIR model for some values of  $\alpha, \beta$  and  $\lambda$ .

Table 2: The AEs and MSEs based on N=1000 simulations.

n		$\Phi$	AEs		MSEs		$\Phi$	AEs		MSEs
	I					II				
20		$\alpha$	1.96717		0.91203		$\alpha$	2.45471		1.05401
		$\beta$	1.88852		0.02123		$\beta$	2.45784		0.19173
		$\lambda$	2.47091		0.04563		$\lambda$	1.22858		0.13019
50		$\alpha$	1.73284		0.29813		$\alpha$	2.55139		0.44856
		$\beta$	1.81745		0.00789		$\beta$	2.35185		0.11015
		$\lambda$	2.35153		0.01607		$\lambda$	1.24736		0.10311
150		$\alpha$	1.74199		0.18674		$\alpha$	2.52722		0.11051
		$\beta$	1.55308		0.00283		$\beta$	2.26009		0.01125
		$\lambda$	2.29044		0.00526		$\lambda$	1.248323		0.03483
500		$\alpha$	1.75981		0.02537		$\alpha$	2.49993		0.04112
		$\beta$	1.50158		0.00081		$\beta$	2.25033		0.00312
		$\lambda$	2.25266		0.00162		$\lambda$	1.24845		0.01164
1000		$\alpha$	1.75051		0.00229		$\alpha$	2.500942		0.00493
		$\beta$	1.50012		0.00045		$\beta$	2.250011		0.00003
		$\lambda$	2.25039		0.00083		$\lambda$	1.250023		0.00081

### 6. Real data modeling

This section presents two applications of the new distribution using real data sets. We shall compare the fit of the new distribution with the Weibull Inverse Weibull (W-IW), exponentiated IW (E-IW), Kumaraswamy IW(Kum-IW), beta IW (B-IW) transmuted IW (T-IW), gamma extended IW (GE-IW), Marshall-Olkin IW (MO-IW), MOKum-IW, generalized MO-IW(GMO-IW), KumMO-IW and IW distributions. The PDFs of the competitive model are available in statistical literature. The unknown parameters of the above PDFs are all positive real numbers except for the T-IW distribution for which  $|\lambda| \leq 1$ .

The 1<sup>st</sup> data set consists of 100 observations of breaking stress of carbon fibers given by Nichols and Padgett (2006) {0.920, 0.9280, 0.997, 0.9971, 1.0610, 1.117, 1.1620, 1.183, 1.187, 1.1920, 1.196, 1.2130, 1.215, 1.2199, 1.220, 1.2240, 1.225, 1.2280, 1.237, 1.240,

1.244, 1.259, 1.2610, 1.263, 1.276, 1.310, 1.3210, 1.3290, 1.3310, 1.337, 1.351, 1.359, 1.388, 1.4080, 1.449, 1.4497, 1.450, 1.459, 1.471, 1.475, 1.477, 1.480, 1.489, 1.501, 1.507, 1.515, 1.530, 1.5304, 1.533, 1.544, 1.5443, 1.552, 1.556, 1.5620, 1.566, 1.585, 1.586, 1.599, 1.602, 1.6140, 1.6160, 1.617, 1.6280, 1.6840, 1.7110, 1.7180, 1.733, 1.7380, 1.7430, 1.7590, 1.777, 1.7940, 1.799, 1.806, 1.814, 1.8160, 1.8280, 1.830, 1.884, 1.892, 1.944, 1.972, 1.9840, 1.987, 2.02, 2.0304, 2.0290, 2.0350, 2.0370, 2.0430, 2.0460, 2.0590, 2.111, 2.165, 2.686, 2.778, 2.972, 3.504, 3.863, 5.3060}.

The 2<sup>nd</sup> data set consists of 63 observations of the strengths of 1.5 cm glass fibers (see Smith and Naylor (1987)) {1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.880, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585}.

In order to compare the distributions, we consider the following criteria: the  $\hat{\ell}_n$  (Maximized Log-Likelihood), AIC (Akaike Information Criterion), CAIC (Consistent Akaike Information Criterion), Bc (Bayesian information criterion) and HQIC (Hannan-Quinn information Criterion). The model with minimum values for these statistics could be chosen as the best model to fit the data. Total time test (TTT) plot (see Aarset (1987) and Figure 1) is an important graphical approach to verify whether our data can be applied to a specific model or not. The TTT plots the two real data sets is presented in Figure 1. This plot indicates that the empirical HRFs of the the two data sets are increasing.

Table 3: The statistics AIC, BIC, HQIC and CAIC values for breaking stress data.

Model	Measures			
	AIC	BIC	HQIC	CAIC
GOGAIR	143.76	151.58	146.92	144.01
W-IW	294.5	304.9	298.7	294.9
E-IW	295.7	303.5	298.9	296.0
Kum-IW	297.1	307.5	301.3	297.5
B-IW	311.1	321.6	315.4	311.6
GE-IW	312.0	332.4	316.2	312.4
IW	348.3	353.5	350.4	348.4
T-IW	350.5	358.3	353.6	350.7
MO-IW	351.3	359.1	354.5	351.6

Table 4: MLEs and their standard errors (in parentheses) for breaking stress of carbon fiber data.

Model	Estimates			
	$\alpha$	$\beta$	$\lambda$	$a$
GOGAIR( $\alpha, \beta, \lambda$ )	0.88 (0.000)	1.903 (0.44)	1.36 (0.000)	
W-IW( $\alpha, \beta, a, b$ )	2.2231 (11.409)	0.355 (0.411)	6.9721 (113.811)	4.9179 (3.756)
Kum-IW( $\alpha, \beta, a, b$ )	2.0556 (0.071)	0.4654 (0.00701)	6.2815 (0.063)	224.18 (0.164)
B-IW( $\alpha, \beta, a, b$ )	1.6097	0.4046	22.0143	29.7617

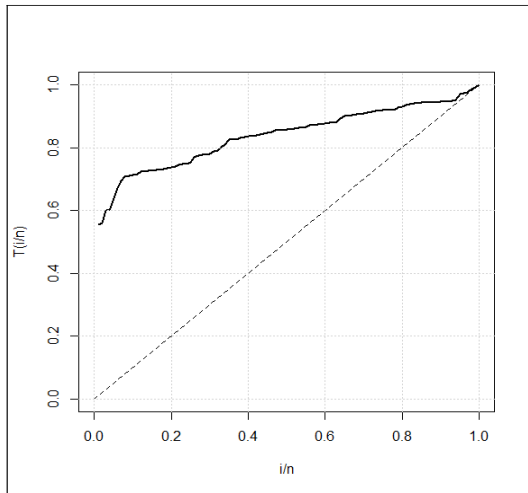
	(2.498)	(0.108)	(21.432)	(17.479)
GE-IW( $\alpha, \beta, a, b$ )	1.3692	0.4776	27.6452	17.4581
	(2.017)	(0.133)	(14.136)	(14.818)
E-IW( $\alpha, \beta, a$ )	69.1489	0.5019	145.3275	
	(57.349)	(0.08)	(122.924)	
T-IW( $\alpha, \beta, \lambda$ )	1.9315	1.7435	0.0819	
	(0.097)	(0.076)	(0.198)	
MO-IW( $\alpha, \beta, a$ )	2.3066	1.5796	0.5988	
	(0.498)	(0.16)	(0.3091)	
IW( $\alpha, \beta$ )	1.8705	1.7766		
	(0.112)	(0.113)		

Table 5: The statistics  $AI_c$ ,  $BI_c$ ,  $HQI_c$  and  $CAI_c$  values for glass fiber data.

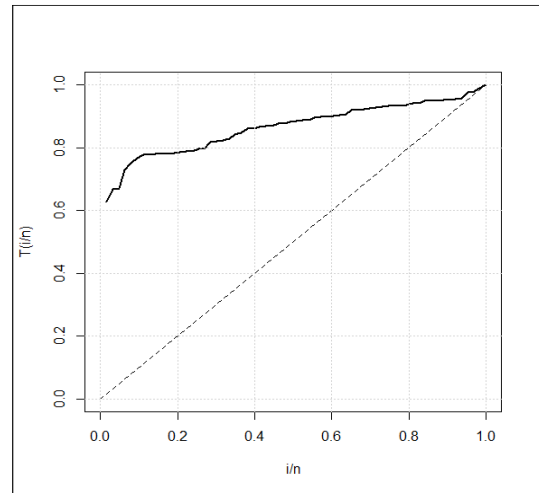
Model	Measures			
	AIC	BIC	HQIC	CAIC
GOGAIR	65.72	72.15	68.25	66.13
B-IW	68.6	77.2	72.0	69.3
GE-IW	69.6	78.1	72.9	70.3
IW	97.7	102	99.4	97.9
T-IW	100.1	106.5	102.6	100.5
MO-IW	101.7	108.2	104.2	102.1

Table 6: MLEs and their standard errors for glass fiber data.

Model	Estimates			
	GOGAIR( $\alpha, \beta, \lambda$ )	1.013	3.95	1.042
	(4.77)	(1.23)	(2.46)	
B-IW( $\alpha, \beta, a, b$ )	2.0518	0.647	15.076	36.94
	(0.986)	(0.163)	(12.06)	(22.65)
GE-IW( $\alpha, \beta, a, b$ )	1.663	0.7421	32.112	13.269
	(0.95)	(0.197)	(17.397)	(9.967)
T-IW( $\alpha, \beta, a$ )	1.3068	2.79	0.13	
	(0.034)	(0.17)	(0.21)	
MO-IW( $\alpha, \beta, a$ )	1.544	2.388	0.482	
	(0.226)	(0.253)	(0.25)	
IW( $\alpha, \beta$ )	1.264	2.888		
	(0.059)	(0.234)		

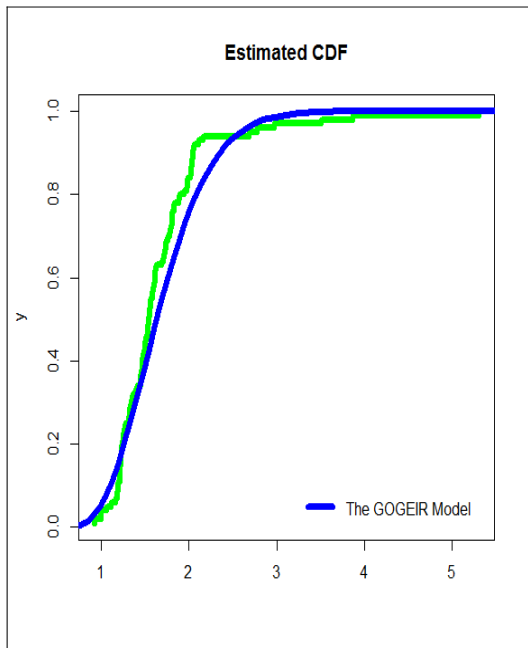


The 1<sup>st</sup> data.

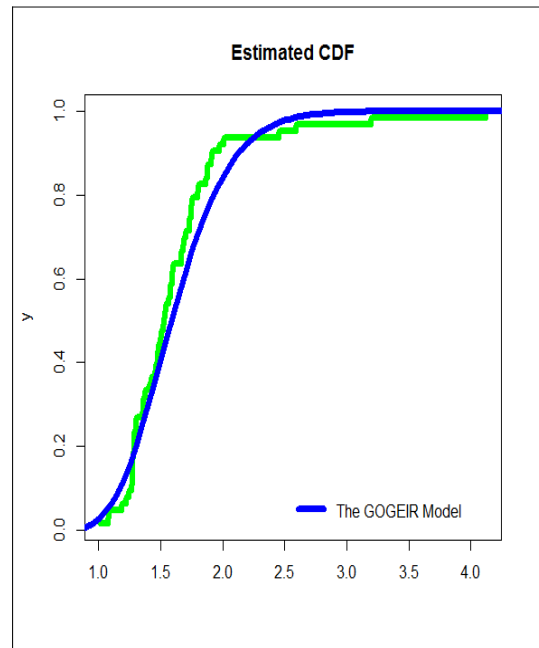


The 2<sup>nd</sup> data.

Figure 1: TTT plots.

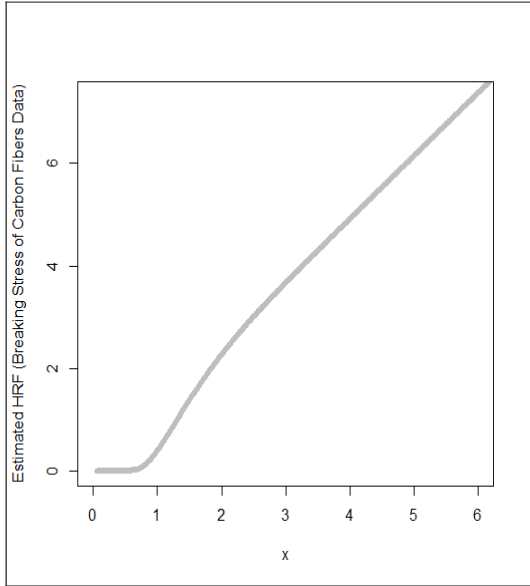


The 1<sup>st</sup> data.

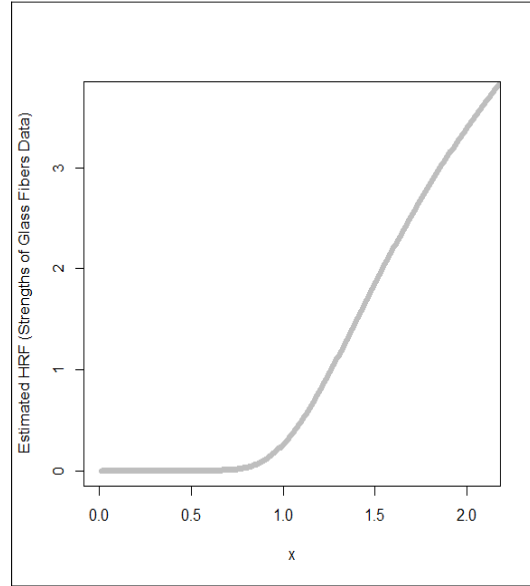


The 2<sup>nd</sup> data.

Figure 2: Estimated CDFs.

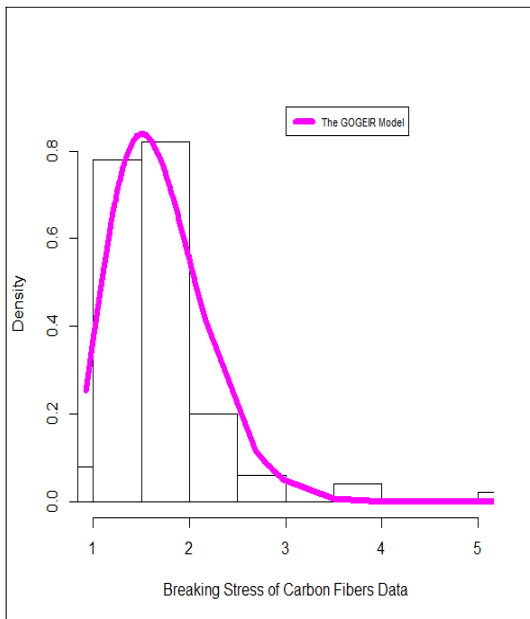


The 1<sup>st</sup> data.

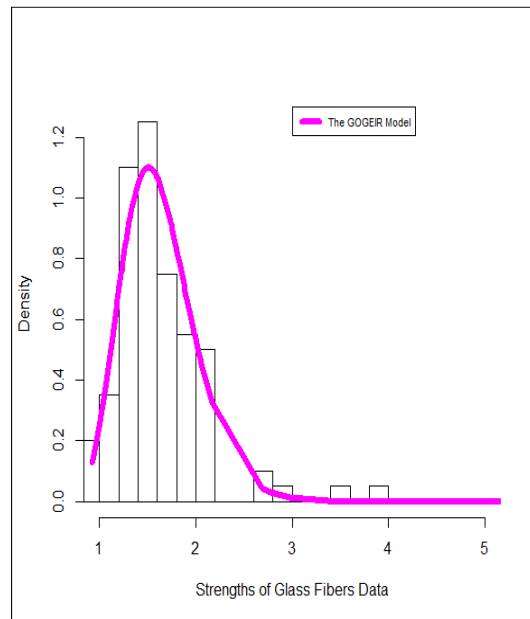


The 2<sup>nd</sup> data.

Figure 3: Estimated HRFs.



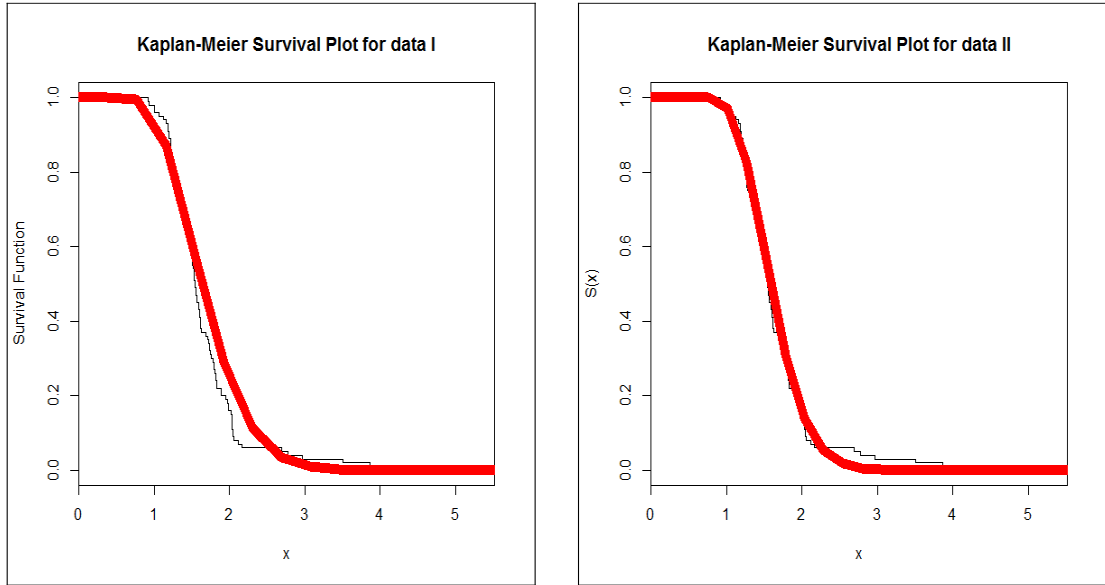
The 1<sup>st</sup> data.



The 2<sup>nd</sup> data.

Figure 4: Estimated PDFs.

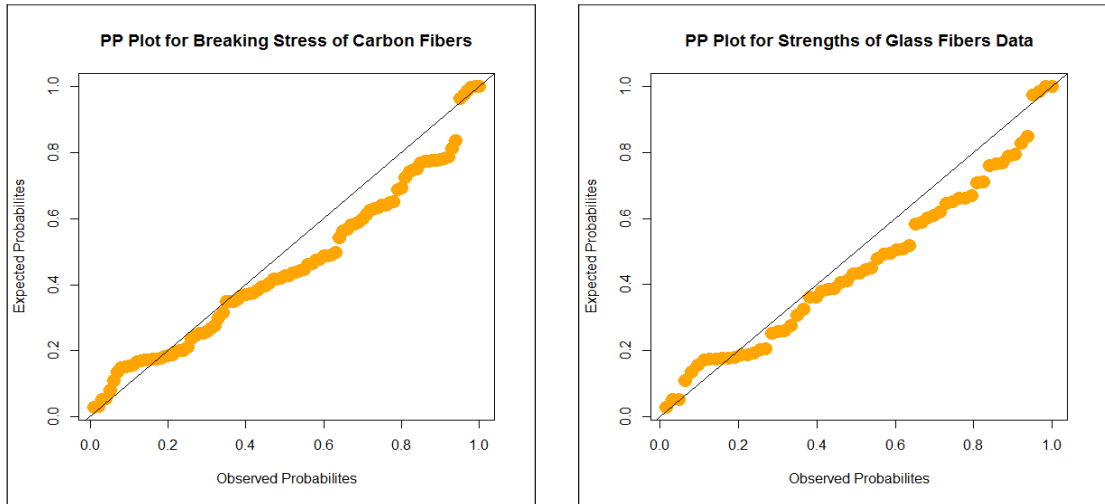




The 1<sup>st</sup> data.

The 2<sup>nd</sup> data.

Figure 5: Kaplan-Meier Survival Plots.



The 1<sup>st</sup> data.

The 2<sup>nd</sup> data.

Figure 6: P-P plots.

Tables 3 and 5 compare the GOGIEIR model with other important competitive distributions. The GOGIEIR model gives the lowest values for the  $AI_c$ ,  $BI_c$ ,  $HQI_c$  and  $CAI_c$  statistics (in bold values) among all fitted models to these data. So, it may be considered as the best model among them. Figures 2-6, respectively, display the plots of estimated CDFs, estimated PDFs, estimated HRFs, P-P plots and Kaplan-Meier survival plots for the 1<sup>st</sup> and 2<sup>nd</sup> data. These plots reveal that the proposed distribution yields a sufficient fit for both data sets.

## 7. Conclusions

A new extension of the Inverse Rayleigh model is proposed and studied. Some of its fundamental statistical properties such as some stochastic properties, ordinary and incomplete moments, moments generating functions, residual life and reversed residual life functions, order statistics, quantile spread ordering, Rényi, Shannon and q-entropies are derived. A simple type Copula based construction via Morgenstern family and via Clayton copula is employed to derive many bivariate and multivariate extensions of the new model. We assessed the performance of the maximum likelihood estimators using a simulation study. The importance of the new model is shown via two applications to real data sets.

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