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The Generalized Odd Log-Logistic Fréchet Distribution for Modeling Extreme Values

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Abstract

We introduce a new extension of the Fréchet distribution for modeling the extreme values. The new model generalizes eleven distributions at least, five of them are quite new. Some important mathematical properties of the new model are derived. We assess the performance of the maximum likelihood estimators (MLEs) via a simulation study. The new model is better than some other important competitive models in modeling the breaking stress data, the glass fibers data and the relief time data.

Key Words: Fréchet distribution; Extreme Values; Moments; Estimation; Odd Log-Logistic Family.

Mathematical Subject Classification: 60E05, 62E15.

1. Introduction

A statistical field known as extreme value theory or extreme value analysis (EVA) studies extreme departures from the median of probability distributions. It aims to determine the likelihood of events that are more extreme than any previously recorded events from a given ordered sample of a given random variable. Extreme value analysis is frequently employed in a variety of fields, including geological engineering, finance, earth sciences, and structural and structural engineering. The EVA, for instance, might be used in the hydrology profession to calculate the likelihood of an exceptionally significant flooding occurrence, like the 100-year flood. Similar to this, a coastal engineer would aim to determine the 50-year wave and construct the structure appropriately while designing a breakwater (for more details, see Kotz and Nadarajah (2000), Afify et al. (2016a,b) and Salah et al. (2020)). The Fisher-Tippet-Gnedenko theorem only states that if the distribution of a normalised maximum converges, then the limit has to be one of a specific class of distributions. The role of the extremal types theorem for maxima is similar to that of the central limit theorem for averages, with the exception that the central limit theorem applies to the average of a sample from any distribution with finite variance. It doesn't say that the normalised maximum distribution converges.

The Fisher-Tippett-Gnedenko theorem in statistics is a broad conclusion of the extreme value theory concerning the asymptotic distribution of extreme order statistics. It is also known as the Fisher-Tippett theorem or the extreme value theorem.Only one of three alternative distributions, the Gumbel distribution, the Fréchet distribution, or the Weibull distribution, can be reached by the maximum of a sample of iid random variables after sufficient renormalization. The Fréchet (Fr) model is one of the most important distributions in modeling extreme values. The Fr model was originally proposed by Fréchet (1927). It has many applications in ranging, accelerated life testing, earthquakes, the floods, the wind speeds, the horse racing, the rainfall, queues in supermarkets and sea waves. .Some new Fréchet versions can be cited by Jahanshahi et al. (2019) (for the Burr X Fréchet extension for modeling the extreme values with some mathematical properties, classical and Bayesian analysis), Al-Babtain et al. (2020a,b), Elsayed and Yousof

(2020) (for the generalized odd generalized exponential Fréchet model with univariate, bivariate and multivariate extensions, properties and real data applications to the univariate version), Ibrahim et al. (2021b) (for new three-parameter xgamma Fréchet distribution with different methods of estimation and some real-life applications), Ahmed and Yousof (2022) (for some new group acceptance sampling plans based on Percentiles for the Weibull Fréchet Model), however, for more other Fréchet versions see Yousof et al. (2016, 2018a, 2018b, 2019, 2020), Haq et al. (2017), Korkmaz et al. (2017), Salah et al. (2020).

A RV X is said to have the Fr distribution if its probability density function (PDF) and cumulative distribution function (CDF) are given by

$$g_{a,b}(x) = ba^b x^{-(b+1)} \exp\left[-\left(\frac{a}{x}\right)^b\right]|_{x\geq 0} \tag{1}$$

and

$$G_{a,b}(x) = \exp\left[-\left(\frac{a}{x}\right)^b\right]|_{x\geq 0}, \tag{2}$$

where $a > 0$ is a scale parameter and $b > 0$ is a shape parameters. For $b = 2.$we get the Inverse Rayleigh (IR) model. For $a = 1$ we get the Inverse Exponential (IEx) model.

Recently, Cordeiro et al. (2016) proposed a new class of distributions called the generalized odd log-logistic-G (GOLL-G) family with two extra shape parameters. For an arbitrary baseline CDF $G_\xi(x)$, the CDF of the the GOLL-G family is given by

$$F_{\alpha,\theta,\xi}(x) = \frac{G_\xi(x)^{\alpha\theta}}{G_\xi(x)^{\alpha\theta} + \left[1 - G_\xi(x)^\theta\right]^\alpha}. \tag{3}$$

The PDF corresponding to (3) is given by

$$f_{\alpha,\theta,\xi}(x) = \frac{\alpha\theta g_\xi(x)\, G_\xi(x)^{\alpha\theta-1}\left[1 - G_\xi(x)^\theta\right]^{\alpha-1}}{\left\{G_\xi(x)^{\alpha\theta} + \left[1 - G_\xi(x)^\theta\right]^\alpha\right\}^2}. \tag{4}$$

For $\theta = 1$ we get the OLL-G family (Gleaton and Lynch (2006)). For $\alpha = 1$ we get the Proportional reversed hazard rate G family (Gupta and Gupta (2007)). Here, we define a new Fr model based on Cordeiro et al. (2016) called generalized odd log-logistic Fr (GOLLFr) family and provide some plots of its PDF and hazard rate function (HRF) $[h_{\alpha,\theta,a,b}(x)]$. The GOLLFr CDF is given by

$$F_{\alpha,\theta,a,b}(x) = \frac{\exp\left[-\alpha\theta\left(\frac{a}{x}\right)^b\right]}{\exp\left[-\alpha\theta\left(\frac{a}{x}\right)^b\right] + \left\{1 - \exp\left[-\theta\left(\frac{a}{x}\right)^b\right]\right\}^\alpha}. \tag{5}$$

The new CDF in (5) can be used for presenting a new discrete model for modeling the count data (see Aboraya et al. (2020), Chesneau et al. (2022), Ibrahim et al. (2021a) and Yousof et al. (2021) for more details). The PDF corresponding to (5) is given by

$$f_{\alpha,\theta,a,b}(x) = \frac{\alpha\theta ba^b x^{-(b+1)} \exp\left[-\alpha\theta\left(\frac{a}{x}\right)^b\right]\left\{1 - \exp\left[-\theta\left(\frac{a}{x}\right)^b\right]\right\}^{\alpha-1}}{\left(\exp\left[-\alpha\theta\left(\frac{a}{x}\right)^b\right] + \left\{1 - \exp\left[-\theta\left(\frac{a}{x}\right)^b\right]\right\}^\alpha\right)^2}. \tag{6}$$

The new model in (6) can be used in regression modeling and in assessing various estimations methods as recently presented by Altun et al. (2022), Yousof et al. (2022), Korkmaz et al. (2022) and Aboraya et al. (2022). The HRF for the new model can be get from $f_{\alpha,\theta,a,b}(x) / [1 - F_{\alpha,\theta,a,b}(x)]$. Let $\tau = \inf\left\{x|_{G(x;\xi)>0}\right\}$, the asymptotics of the CDF, PDF and HRF as $x \to \tau$ are given by

$$F_{\alpha,\theta,a,b}(x) \sim \exp\left[-\alpha\theta\left(\frac{a}{x}\right)^b\right]|_{x\to\tau},$$

$$f_{\alpha,\theta,a,b}\left(x\right) \sim \alpha\theta c b a^{b} x^{-(b+1)} \text{exp } \left[-\alpha\theta\left(\frac{a}{x}\right)^{b}\right]|_{x\rightarrow\tau}$$

and

$$h_{\alpha,\theta,a,b}\left(x\right) \sim \alpha\theta b a^{b} x^{-(b+1)} \text{exp } \left[-\alpha\theta\left(\frac{a}{x}\right)^{b}\right]|_{x\rightarrow\tau}.$$

The asymptotics of CDF, PDF and HRF as $x \rightarrow \infty$ are given by

$$1 - F_{\alpha,\theta,a,b}(x) \sim \left(\theta\left\{1 - \text{exp}\left[-(\frac{a}{x})^{b}\right]\right\}\right)^{\alpha}|_{x\rightarrow\infty},$$

$$f_{\alpha,\theta,a,b}\left(x\right) \sim \alpha\theta^{\alpha} b a^{b} x^{-(b+1)} \text{exp } \left[-\left(\frac{a}{x}\right)^{b}\right]\left\{1 - \text{exp } \left[-\left(\frac{a}{x}\right)^{b}\right]\right\}^{\alpha-1}|_{x\rightarrow\infty}$$

and

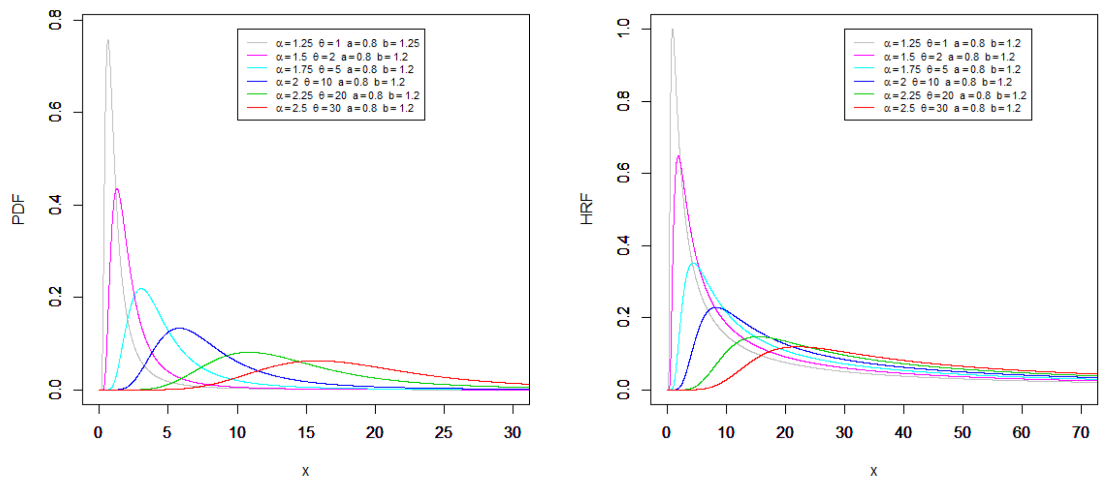
$$h_{\alpha,\theta,a,b}(x) \sim \frac{\alpha b a^{b} x^{-(b+1)} \text{exp}\left[-(\frac{a}{x})^{b}\right]}{1 - \text{exp}\left[-(\frac{a}{x})^{b}\right]}|_{x\rightarrow\infty}.$$

Table 1 provides the some sub-models of the GOLLFr model. As illustrated in Table 1, the new model generalizes eleven sub-model, five of them are quite new.

Table 1: Sub-models of the GOLLFr model.

N	α	θ	a	b	Reduced model	Reduced CDF	Author
1				2	GOLLIR	$\frac{\text{exp}\left[-\alpha\theta\left(\frac{a}{x}\right)^{2}\right]}{\text{exp}\left[-\alpha\theta\left(\frac{a}{x}\right)^{2}\right]+\left\{1-\text{exp}\left[-\theta\left(\frac{a}{x}\right)^{2}\right]\right\}^{\alpha}}$	New
2				1	GOLLIEx	$\frac{\text{exp}\left[-\alpha\theta\left(\frac{a}{x}\right)\right]}{\text{exp}\left[-\alpha\theta\left(\frac{a}{x}\right)\right]+\left\{1-\text{exp}\left[-\theta\left(\frac{a}{x}\right)\right]\right\}^{\alpha}}$	New
3	1				PRHFr	$\text{exp }\left[-\theta\left(\frac{a}{x}\right)^{b}\right]$	Gusmao et al. (2011)
4	1			2	PRHIR	$\text{exp }\left[-\theta\left(\frac{a}{x}\right)^{2}\right]$	Gusmao et al. (2011)
5	1			1	PRHIEx	$\text{exp }\left[-\theta\left(\frac{a}{x}\right)\right]$	Gusmao et al. (2011)
6		1			OLLFr	$\frac{\text{exp}\left[-\alpha\left(\frac{a}{x}\right)^{b}\right]}{\text{exp}\left[-\alpha\left(\frac{a}{x}\right)^{b}\right]+\left\{1-\text{exp}\left[-\left(\frac{a}{x}\right)^{b}\right]\right\}^{\alpha}}$	Yousof et al. (2018)
7		1		2	OLLIR	$\frac{\text{exp}\left[-\alpha\left(\frac{a}{x}\right)^{2}\right]}{\text{exp}\left[-\alpha\left(\frac{a}{x}\right)^{2}\right]+\left\{1-\text{exp}\left[-\left(\frac{a}{x}\right)^{2}\right]\right\}^{\alpha}}$	Yousof et al. (2018a)
8		1		1	OLLIEx	$\frac{\text{exp}\left[-\alpha\left(\frac{a}{x}\right)\right]}{\text{exp}\left[-\alpha\left(\frac{a}{x}\right)\right]+\left\{1-\text{exp}\left[-\left(\frac{a}{x}\right)\right]\right\}^{\alpha}}$	Yousof et al. (2018a)
9	1	1			Fr	$\text{exp }\left[-\left(\frac{a}{x}\right)^{b}\right]$	Fréchet (1927)
10	1	1		2	IR	$exp\left[-\left(\frac{a}{x}\right)^{2}\right]$	Trayer (1964)
11					IEx	$exp\left[-\left(\frac{a}{x}\right)\right]$	Keller and Kamath (1982)

Some plots of the GOLLFr PDF and HRF are given in Figure 1 to illustrate some of its characteristics.



(a) (b)

Figure 1: Plots of the GOLLFr PDF and HRF.

For simulation of this new model, we obtain the quantile function (QF) of X (by inverting (5)), say $x_u = Q(u) = F^{-1}(u)$, as

$$x_u = a\left(-\ln\left\{\left[\frac{\left(\frac{u}{1-u}\right)^{\frac{1}{\alpha}}}{1+\left(\frac{u}{1-u}\right)^{\frac{1}{\alpha}}}\right]^{\frac{1}{\theta}}\right\}\right)^{-\frac{1}{b}}. \tag{7}$$

Equation (7) is used for simulating the new model (see Section 5).

2. Mathematical properties

2.1. Useful representations

Based on generalized binomial expansions and after some algebra, the PDF in (6) can be expressed as

$$f(x) = \sum_{k=0}^{\infty} b_k\, \pi_{(1+k)}(x; a, b), \tag{8}$$

where

$$b_k = \frac{\alpha\theta}{1+k} \sum_{i,j=0}^{\infty} \sum_{k=0}^{l} (-1)^{j+k+l} \binom{-2}{i} \binom{-\alpha\,(i+1)}{j} \binom{\alpha\theta\,(i+1)+\theta j-1}{l} \binom{l}{k},$$

and $\pi_{(1+k)}(x; a, b)$ is the PDF of the Fr model with scale parameter $a\,[(1+k)]^{\frac{1}{b}}$ and shape parameter b. So, the new density (6) can be expressed as a double linear mixture of the Fr density.Then, several structural properties of the new model can be obtained from Equation (8) and those properties of the Fr model. By integrating Equation (8), the CDF of X becomes

$$F(x) = \sum_{k=0}^{\infty} b_k\, \Pi_{(1+k)}(x; a, b), \tag{9}$$

where $\Pi_{(1+k)}(x; a, b)$ is the CDF of the Fr distribution with scale parameter $a\,(ck)^{\frac{1}{b}}$ and shape parameter b.
The r^{th} ordinary moment of X is given by

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r\, f\,(x)\, dx,$$

then we obtain

$$\mu_r' = \sum_{k=0}^{\infty} b_k a^r\ (1+k)^{\frac{r}{b}}\, \Gamma\left(1 - \frac{r}{b}\right), \forall\, b > r, \tag{10}$$

where

$$\Gamma\,(1+\omega)\,|_{(\omega \in \mathbb{R}^+)} = \omega! = \prod_{h=0}^{\omega-1} (\omega - h)\,.$$

Setting $r = 1, 2, 3$ and 4 in (10), we have

$$\begin{aligned} E(X) &= \mu_1' = \sum_{k=0}^{\infty} b_k a\ (1+k)^{\frac{1}{b}}\, \Gamma\left(1 - \frac{1}{b}\right), \forall\, b > 1, \\ E(X^2) &= \mu_2' = \sum_{k=0}^{\infty} b_k a^2\ (1+k)^{\frac{2}{b}}\, \Gamma\left(1 - \frac{2}{b}\right), \forall\, b > 2, \\ E(X^3) &= \mu_3' = \sum_{k=0}^{\infty} b_k a^3\ (1+k)^{\frac{3}{b}}\, \Gamma\left(1 - \frac{3}{b}\right), \forall\, b > 3, \end{aligned}$$

and

$$E(X^4) = \mu_4' = \sum_{k=0}^{\infty} b_k a^4\ (1+k)^{\frac{4}{b}}\, \Gamma\left(1 - \frac{4}{b}\right), \forall\, b > 4,$$

where $E(X) = \mu_1'$ is the mean of X. The skewness (Skew(X)) and kurtosis (Kur(X)) measures can be calculated via (10) using well-known relationships. $E(X)$, variance (Var(X)), Skew(X) and Kur(X) of the GOLLFr distribution are computed numerically for some selected values of parameter α, θ, a and b using the R software. We conclude that, the Skew (X) can range in the interval $(1, 19706)$ and always positive, whereas the Kur(X) varies in the interval $(1, 388338)$. The parameters θ and a don't control neither the Skew(X)) nor the Kur(X) as illustrated below in Table 2, on the other hand parameters α and b control Skew(X)) and Kur(X) . $E(X)$ decreases as α and b increases. $E(X)$ increases as a and θ increases.

Table 2: Mean, variance, skewness and kurtosis of the GOLLFr distribution.

α	θ	a	b	$E(X)$	Var(X)	Skew(X)	Kur(X)
10	10	1	1	14.59026	3.815803	0.8830787	6.295525
			2	3.811433	0.06324338	0.5135545	5.059341
			5	1.706911	0.002005427	0.3176427	4.419477
			10	1.306376	0.0002927583	0.2534783	4.321513
			15	1.195007	0.0001087731	0.2321448	4.309573
			20	1.142944	5.594415×10^{-05}	0.2217203	4.28053
			25	1.112803	3.393171×10^{-05}	0.215345	4.281838
			30	2.256006×10^{-06}	2.577334×10^{-06}	711.7362	50665
5	2.5	1	1.5	2.405262	0.1952397	1.419549	9.958146
		2		4.810524	0.7809586	1.419549	9.958146
		5		12.02631	4.880991	1.419549	9.958146
		10		24.05262	19.52397	1.419549	9.958146
		20		48.10524	78.09586	1.419549	9.958146
		50		120.2631	488.0991	1.419549	9.958146
		100		240.5262	1952.397	1.419549	9.958146
		200		481.0524	7809.586	1.419549	9.958147
		500		1202.631	48809.91	1.419549	9.958146
4	3	2	2.5	3.648525	0.2463512	1.194554	7.830151
	5			4.475657	0.3707095	1.194554	7.830151
	10			5.905664	0.6454427	1.194554	7.83015
	20			7.792571	1.123781	1.194554	7.830151
	50			11.24234	2.339019	1.194554	7.830151
	100			14.83436	4.072468	1.194554	7.83015
	200			19.57405	7.090579	1.194554	7.830151
	500			28.23948	14.75821	1.194554	7.830151
5	2	3	1.5	6.218375	1.30496	1.419549	9.958146
10				6.11425	0.2917586	0.6331622	5.224647
20				6.088745	0.07096327	0.3083424	4.437874
50				6.081641	0.01126739	0.1228647	4.213466
100				1.590397×10^{-05}	36.97685	1.000106	1.000294
200				1.476099×10^{-08}	8.461368×10^{-08}	19706.00	388338

The r^{th} incomplete moment, say $\varphi_r(t)$, of X can be expressed, from (9), as

$$\begin{aligned}\varphi_r(t) &= \int_{-\infty}^{t} x^r f(x)\,dx = \sum_{k=0}^{\infty} b_k \int_{-\infty}^{t} x^r\, \pi_{(1+k)}(x; a, b)\,dx \\ &= \sum_{k=0}^{\infty} b_k a^r \left[(1+k)\right]^{\frac{r}{b}} \gamma(1-\frac{r}{b}, [(1+k)]\,(\frac{a}{t})^b), \forall b > r, \end{aligned} \quad (11)$$

where $\gamma(\omega, q)$ is the incomplete gamma function.

$$\begin{aligned}\gamma(\omega, q)\,|_{(\omega\neq 0,-1,-2,\dots)} &= \int_0^q t^{\omega-1} exp(-t)\,dt \\ &= \frac{q^\omega}{\omega}\left\{{}_1\mathbf{F}_1[\omega; \omega+1; -q]\right\} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(\omega+k)} q^{\omega+k},\end{aligned}$$

and ${}_1\mathbf{F}_1\left[\cdot,\cdot,\cdot\right]$ is a confluent hypergeometric function. The first incomplete moment given by (11) with $r=1$ as

$$\varphi_1\left(t\right)=\sum_{k=0}^{\infty}w_k a\ \left[\left(1+k\right)\right]^{\frac{1}{b}}\,\Gamma\left(1-\frac{1}{b},\left[\left(1+k\right)\right]\left(\frac{a}{t}\right)^b\right),\forall\, b>1.$$

2.2. Moment generating function (MGF)

The MGF $M_X\left(t\right)=E\left(e^{t\,X}\right)$ of X can be derived from equation (8) as

$$M_X\left(t\right)=\sum_{k=0}^{\infty}b_k\,M_{(1+k)}\left(t;a,b\right),$$

where $M_{(1+k)}\left(t;a,b\right)$ is the MGF of the Fr model with scale parameter $a\left[\left(1+k\right)\right]^{\frac{1}{b}}$ and shape parameter b, then

$$M_X\left(t\right)=\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\left(t^r b_k/r!\right)\,a^r\ \left[\left(1+k\right)\right]^{\frac{r}{b}}\,\Gamma\left(1-\frac{r}{b}\right),\forall\, b>r.$$

We aslo can determine the generating function of $g_{a,b}(x)$ by setting $y=x^{-1}$, the MGF can be written as

$$M(t;a,b)=ba^b\int_0^{\infty}\exp\ \left(t/y\right)\ y^{(b-1)}\,exp\left\{-\left(ay\right)^b\right\}.$$

By expanding the first exponential and calculating the integral, we have

$$\begin{aligned}M(t;a,b)\ &=\ ba^b\int_0^{\infty}\sum_{m=0}^{\infty}\left(t^m/m!\right)\exp\ \left(t/y\right)\,y^{b-m-1}\exp\ \left\{-\left(ay\right)^b\right\}\\ &=\ \sum_{m=0}^{\infty}\left(a^m\,t^m/m!\right)\,\Gamma\left(1-\frac{m}{b}\right).\end{aligned}$$

Consider the Wright generalized hypergeometric function (Wright (1935))defined by

$${}_p\Psi_q\left[\begin{array}{c}a_1,A_1,\ldots,a_p,A_p\\ b_1,B_1,\ldots,b_q,B_q\end{array};x\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma\left(a_j+A_j\,n\right)}{\prod_{j=1}^{q}\Gamma\left(b_j+B_j\,n\right)}\frac{x^n}{n!}.$$

Then, $M(t;a,\beta)$ can be written as

$$M(t;a,b)={}_1\Psi_0\left[\begin{array}{c}\left(1,-\frac{1}{b}\right)\\ -\end{array};a\,t\right].\qquad(12)$$

Combining expressions (10) and (12), we obtain the MGF of X, say $M(t)$,as

$$M(t)=\sum_{k=0}^{\infty}b_k\left\{{}_1\Psi_0\left[\begin{array}{c}\left(1,-\frac{1}{b}\right)\\ -\end{array};a\left[c(k+1)\right]^{\frac{1}{b}}\,t\right]\right\}.$$

2.3. Residual life and reversed residual life functions

The n^{th} moment of the residual life

$$m_n(t)=E\left[\left(X-t\right)^n\mid_{X>t,\,n=1,2,\ldots}\right]$$

the n^{th} moment of the residual life of X is given by

$$m_n(t)=\frac{\int_t^{\infty}(x-t)^n dF(x)}{1-F(t)}.$$

Therefore,

$$m_n(t) = \frac{a^n}{1-F(t)} \sum_{k=0}^{\infty} \zeta_k \, [(1+k)]^{\frac{n}{b}} \, \Gamma\left(1-\frac{n}{b}, [(1+k)] \left(\frac{a}{t}\right)^b\right), \forall \, b > n,$$

where

$$\zeta_k = b_k \sum_{r=0}^{n} \binom{n}{r} (-t),$$

$$\Gamma(\omega, q) \,|_{x>0} = \int_q^{\infty} t^{\omega-1} exp(-t) \, dt,$$

and

$$\Gamma(\omega, q) + \gamma(\omega, q) = \Gamma(\omega).$$

The n^{th} moment of the reversed residual life, say

$$M_n(t) = E\left[(t-X)^n \,|_{X \le t,\, t>0 \text{ and } n=1,2,\ldots}\right]$$

uniquely determines $F(x)$. We obtain

$$M_n(t) = \frac{\int_0^t (t-x)^n dF(x)}{F(t)}.$$

Then, the n^{th} moment of the reversed residual life of X becomes

$$M_n(t) = \frac{a^n}{F(t)} \sum_{k=0}^{\infty} \eta_k \, [(1+k)]^{\frac{n}{b}} \, \gamma\left(1-\frac{n}{b}, [(1+k)] \left(\frac{a}{t}\right)^b\right), \forall \, b > n,$$

where

$$\eta_k = b_k \sum_{r=0}^{n} (-1)^r \binom{n}{r} t^{n-r}.$$

3. Maximum likelihood estimation (MLE)

Let $x_1, \ldots, x_n$ be a random sample from the GOLLFr distribution with parameters α, θ, a and b. Let $\Theta = (\alpha, \theta, a, b)^\intercal$ be the 4×1 parameter vector. For determining the MLE of Θ, we have the log-likelihood function

$$\begin{aligned} \ell \;&= \ell(\Theta) = n \log\left(\alpha\theta cba^b\right) - (b+1) \sum_{i=1}^{n} \log(x_i) - \alpha\theta \sum_{i=1}^{n} \left(\tfrac{a}{x_i}\right)^b \\ &+2 \sum_{i=1}^{n} \log\left(\exp\left[-\alpha\theta\left(\tfrac{a}{x_i}\right)^b\right] + \left\{1-\exp\left[-\theta\left(\tfrac{a}{x_i}\right)^b\right]\right\}^{\alpha}\right) \\ &+ (\alpha-1) \sum_{i=1}^{n} \log\left\{1-\exp\left[-\theta\left(\tfrac{a}{x_i}\right)^b\right]\right\}. \end{aligned}$$

The components of the score vector, $\mathbf{L}_{(\Theta)} = \frac{\partial \ell}{\partial \Theta} = \left(\frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial a}, \frac{\partial \ell}{\partial b}\right)^\intercal$, are available if needed. Setting $\mathbf{L}_\alpha = \mathbf{L}_\theta = \mathbf{L}_a =$ and $\mathbf{L}_b = \mathbf{0}$ and solving them simultaneously yields the MLE $\widehat{\Theta} = (\widehat{\alpha}, \widehat{\theta}, \widehat{a}, \widehat{b})^\intercal$. To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize ℓ. For interval estimation of the parameters, we obtain the 5×5 observed information matrix

$$J(\Theta) = \{\partial^2 \ell / \partial r \, \partial s\} \quad (\forall r, s = \alpha, \theta, a, b),$$

whose elements can be computed numerically.

4. Simulation studies

We simulate the GOLLFr model by taking n=50, 100, 200, 500 and 1000 using (7). For each sample size, we evaluate the sample means and standard deviations (SDs) using the optim function of the R software. Then, we repeat this process 1000 times. Values in Table 3 indicate that the empirical means approach to the true parameter values when the sample size n increases. The SDs decrease when the sample size n increases as expected. These results are in agreement with first-order asymptotic theory.

Table 3: Sample means and SDs for the GOLLFr distribution

Parameters	$n = 50$			
α, θ, a, b	$\widehat{\alpha}$	$\widehat{\theta}$	$\widehat{a}$	$\widehat{b}$
5.5, 1.0, 0.1, 1.1	5.77822	1.08921	0.07795	1.05178
	(0.36779)	(0.39108)	(0.39607)	(0.2883)
1.0, 1.0, 0.1, 1.0	1.17971	1.15373	0.04028	1.10949
	(0.66989)	(0.49877)	(0.48493)	(0.49198)
2.0, 5.0, 1.0, 1.0	1.75894	4.76046	0.94989	0.97457
	(0.39091)	(0.29943)	(0.19713)	(0.44093)
5.0, 3.0, 1.0, 1.5	4.87732	3.08116	1.02135	1.387801
	(0.32755)	(0.56159)	(0.29489)	(0.37205)
1.5, 2.5, 1.5, 1.5	1.34786	2.39884	1.76762	1.55591
	(0.50898)	(0.72777)	(0.59654)	(0.56195)
5.0, 3.0, 5.0, 4.0	4.82595	2.95460	5.21645	3.78133
	(0.79902)	(0.85585)	(1.67871)	(1.14468)
3.0, 2.0, 2.0, 4.0	2.61830	2.18634	1.84569	4.147651
	(0.79719)	(0.83339)	(0.95476)	(0.88011)
	$n = 100$			
5.5, 1.0, 0.1, 1.1	5.69167	1.22982	0.04799	1.08389
	(0.33689)	(0.29022)	(0.29298)	(0.26676)
1.0, 1.0, 0.1, 1.0	1.05001	1.01439	0.03571	0.50965
	(0.50141)	(0.35014)	(0.31279)	(0.31869)
2.0, 5.0, 1.0, 1.0	1.89902	5.09125	1.00715	1.01379
	(0.26497)	(0.32821)	(0.16892)	(0.13199)
5.0, 3.0, 1.0, 1.5	4.66394	3.02911	0.95089	1.53187
	(0.16131)	(0.51262)	(0.1991)	(0.11531)
1.5, 2.5, 1.5, 1.5	1.53171	2.52997	1.52505	1.49444
	(0.51059)	(0.60391)	(0.50031)	(0.3391)
5.0, 3.0, 5.0, 4.0	4.78010	2.90403	5.06941	3.91180
	(0.31342)	(0.50193)	(0.46591)	(0.65915)
3.0, 2.0, 2.0, 4.0	2.95968	2.10771	1.96959	4.11890
	(0.60475)	(0.61737)	(0.58046)	(0.61502)

Parameters		$n = 200$		
5.5, 1.0, 0.1, 1.1	5.55859	0.98477	0.09387	0.98803
	(0.01201)	(0.01705)	(0.01982)	(0.00698)
1.0, 1.0, 0.1, 1.0	1.0123	1.0608	0.0106	1.0502
	(0.0548)	(0.02718)	(0.00997)	(0.10058)
2.0, 5.0, 1.0, 1.0	2.0006	5.0005	0.97923	0.9922
	(0.023955)	(0.14745)	(0.10281)	(0.0780)
5.0, 2.0, 1.0, 1.5	4.9942	2.0129	0.9926	1.5061
	(0.19074)	(0.0269)	(0.0194)	(0.09921)
1.5, 2.5, 1.5, 1.5	1.49944	2.49509	1.49584	1.49602
	(0.3276)	(0.4027)	(0.3247)	(0.2570)
5.0, 3.0, 5.0, 4.0	4.99121	2.97115	5.00987	3.99791
	(0.03298)	(0.02601)	(0.06072)	(0.08832)
3.0, 2.0, 2.0, 4.0	2.87466	2.09689	1.95976	4.05208
	(0.03990)	(0.05271)	(0.08772)	(0.03003)
		$n = 500$		
5.5, 1.0, 0.1, 1.1	5.50283	0.99941	0.10119	0.98818
	(0.00339)	(0.00855)	(0.00543)	(0.00611)
1.0, 1.0, 0.1, 1.0	1.00201	1.00701	0.1087	1.0242
	(0.00231)	(0.00411)	(0.00832)	(0.00311)
2.0, 5.0, 1.0, 1.0	2.00399	5.00178	1.00678	0.99998
	(0.00960)	(0.00711)	(0.00841)	(0.00222)
5.0, 3.0, 1.0, 1.5	4.99787	3.00469	2.00848	1.50292
	(0.00091)	(0.00402)	(0.00101)	(0.00901)
1.5, 2.5, 1.5, 1.5	1.49901	2.55841	1.49986	1.49645
	(0.00851)	(0.00137)	(0.00401)	(0.001904)
5.0, 3.0, 5.0, 4.0	4.9929	2.99647	5.00974	3.98843
	(0.2015)	(0.2962)	(0.4785)	(0.458)
3.0, 2.0, 2.0, 4.0	2.90367	2.00539	2.00438	4.00050
	(0.3165)	(0.3274)	(0.3150)	(0.3233)
		$n = 1000$		
5.5, 1.0, 0.1, 1.1	5.50001	0.99998	0.10009	1.10012
	(0.1515)	(0.0980)	(0.1504)	(0.0884)
1.0, 1.0, 0.1, 1.0	1.00011	1.00021	0.10061	1.00081
	(0.2071)	(0.0979)	(0.1124)	(0.0900)
2.0, 5.0, 1.0, 1.0	1.99996	5.00007	0.99899	1.00101
	(0.0667)	(0.0373)	(0.0315)	(0.0288)
5.0, 3.0, 1.0, 1.5	5.00011	3.00153	0.99999	1.50004
	(0.0139)	(0.1052)	(0.0409)	(0.0312)
1.5, 2.5, 1.5, 1.5	1.49989	2.49888	1.50013	1.49988
	(0.00088)	(0.00909)	(0.0014)	(0.1275)
5.0, 3.0, 5.0, 4.0	5.00088	3.00084	4.99979	4.00111
	(0.00001)	(0.00090)	(0.00047)	(0.00077)
3.0, 2.0, 2.0, 4.0	3.00041	2.00033	2.00050	4.00117
	(0.00021)	(0.00113)	(0.0033)	(0.0001)

5. Real data modeling

We consider the Cramér-Von Mises and the Anderson-Darling $\left[\mathbf{W}^{\star}, \mathbf{A}^{\star}\right]$ and theKolmogorov-Smirnov (KS) statistic. The $W^{*}$and $\mathbf{A}^{\star}$ statistics are given by

$$\mathbf{W}^{\star} = (1 + 1/2n)\left[[1/(12n)] + \sum\nolimits_{j=1}^{n} W_j\right],$$

and

$$\mathbf{A}^{\star} = nA_{(n)} + A_{(n)}n^{-1}\sum\nolimits_{j=1}^{n} a_j,$$

where

$$W_j = \left[z_i - (2j-1)\,/\,(2n)\right]^2,$$

$$A_{(n)} = 1 + \frac{9}{4}n^{-2} + \frac{3}{4}n^{-1},$$

and

$$A_j = (2j-1)\log\left[z_i\left(1 - z_{n-j+1}\right)\right],$$

where $z_i = F(y_j)$ and the y_j's values are the ordered observations. We compare the fits of the GOLLFr distribution with other models such as Fr échet (Fr), Kumaraswamy Fréchet (KFr), exponentiated Fréchet (EFr), beta Fréchet (BFr), transmuted Fréchet (TFr), Marshal-Olkin Fr échet (MOFr) and McDonald Fréchet (McFr) distributions given by:

EFr :

$$f_{EFr}(x) = \alpha b a^b x^{-(b+1)} exp\left[-(a/x)^b\right]\left\{1 - exp\left[-(a/x)^b\right]\right\}^{\alpha-1};$$

BFr :

$$f_{BFr}(x) = b a^b B^{-1}(\alpha, c) x^{-(b+1)} exp\left[-\alpha(a/x)^b\right]\left\{1 - exp\left[-(a/x)^b\right]\right\}^{c-1};$$

KFr :

$$f_{KFr}(x) = \alpha c b a^b x^{-(b+1)} exp\left[-\alpha(a/x)^b\right]\left\{1 - exp\left[-\alpha(a/x)^b\right]\right\}^{c-1};$$

TFr :

$$f_{TFr}(x) = b a^b x^{-(b+1)} exp\left[-(a/x)^b\right]\left\{1 + \alpha - 2\alpha exp\left[-(a/x)^b\right]\right\};$$

MOFr :

$$f_{MOFr}(x) = \alpha b a^b x^{-(b+1)} exp\left[-(a/x)^b\right]\left\{\alpha + (1-\alpha) exp\left[-(a/x)^b\right]\right\}^{-2};$$

McFr :

$$\begin{aligned} f_{McFr}(x) &= \lambda b a^b x^{-(b+1)} B^{-1}(\alpha, c) exp\left[-(a/x)^b\right]\left(exp\left[-(a/x)^b\right]\right)^{\alpha\lambda-1} \\ &\times\left(1 - \left(exp\left[-(a/x)^b\right]\right)^{\lambda}\right)^{c-1}; \end{aligned}$$

OLLEFr:

$$\begin{aligned} f_{OLLEFr}(x) &= \theta\beta b a^b x^{-(b+1)} \exp\left[-\beta\left(\tfrac{a}{x}\right)^b\right] \\ &\times\left(\exp\left[-\beta\left(\tfrac{a}{x}\right)^b\right]\left\{1 - \exp\left[-\beta\left(\tfrac{a}{x}\right)^b\right]\right\}\right)^{-1+\theta} \\ &\times\left(\exp\left[-\beta\theta\left(\tfrac{a}{x}\right)^b\right] + \left\{1 - \exp\left[-\beta\left(\tfrac{a}{x}\right)^b\right]\right\}^{\theta}\right)^{-2}, \end{aligned}$$

The parameters of the above densities are all positive real numbers except for the TFr distribution for which $|\alpha| \leq 1$. For more different symmetric and asymmetric real-life data sets see Brito et al. (2017), Merovci et al. (2017, 2020), Nascimento et al. (2019), Hamedani et al. (2017, 2018, 2019, 2021), Shehata and Yousof (2021a,b), Shehata et al. (2021), Chesneau and Yousof (2021), Chesneau et al. (2022), Elgohari and Yousof (2020a,b, 2021), Elgohari et al. (2021), Karamikabir et al. (2020), Korkmaz et al. (2018a,b, 2020) and Hamedani et al. (2022).

5.1. Breaking stress data

The 1st data set is an uncensored data set consisting of 100 observations on breaking stress of carbon fibers (in Gba) given by Nichols and Padgett (2006) and these data are used by Mahmoud and Mandouh (2013) to fit the transmuted Fr distribution. The data are: 0.920, 0.9280, 0.997, 0.99710, 1.061, 1.1170, 1.162, 1.1830, 1.187, 1.1920, 1.196, 1.2130, 1.215, 1.21990, 1.22, 1.2240, 1.225, 1.2280, 1.237, 1.240, 1.244, 1.2590, 1.261, 1.2630, 1.276, 1.310, 1.321, 1.3290, 1.331, 1.3370, 1.351, 1.3590, 1.388, 1.4080, 1.449, 1.44970, 1.45, 1.4590, 1.471, 1.4750, 1.477, 1.480, 1.489, 1.5010, 1.507, 1.5150, 1.53, 1.53040, 1.533, 1.5440, 1.5443, 1.5520, 1.556, 1.562, 1.5660, 1.585, 1.5860, 1.5990, 1.602, 1.6140, 1.6160, 1.6170, 1.6280, 1.6840, 1.7110, 1.7180, 1.733, 1.7380, 1.7430, 1.759, 1.7770, 1.794, 1.7990, 1.806, 1.8140, 1.8160, 1.828, 1.830, 1.884, 1.8920, 1.944, 1.9720, 1.9840, 1.9870, 2.02, 2.03040, 2.0290, 2.03500, 2.037, 2.0430, 2.0460, 2.059, 2.1110, 2.165, 2.6860, 2.778, 2.9720, 3.504, 3.8630 and 5.3060. Figure 2 gives the total time test (TTT) plot for data set **I**. It indicates that the empirical HRFs of data sets **I** is increasing.

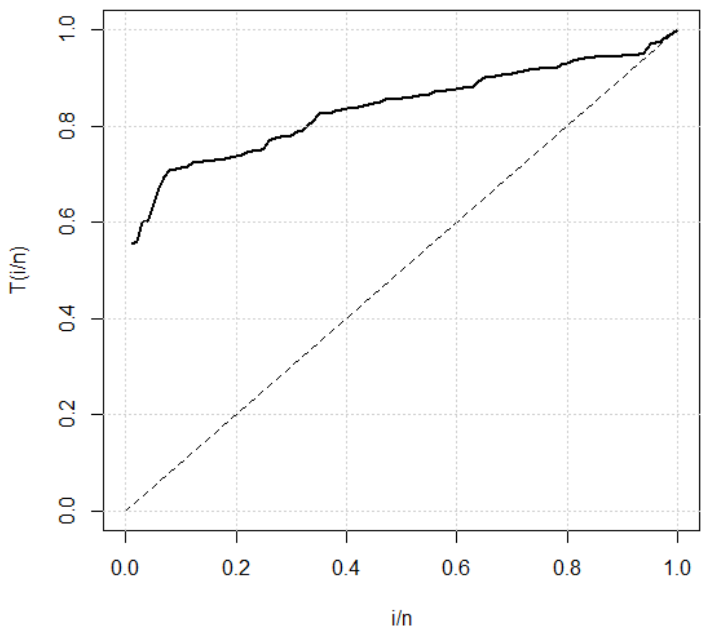


Figure 2: TTT plot for data set **I**

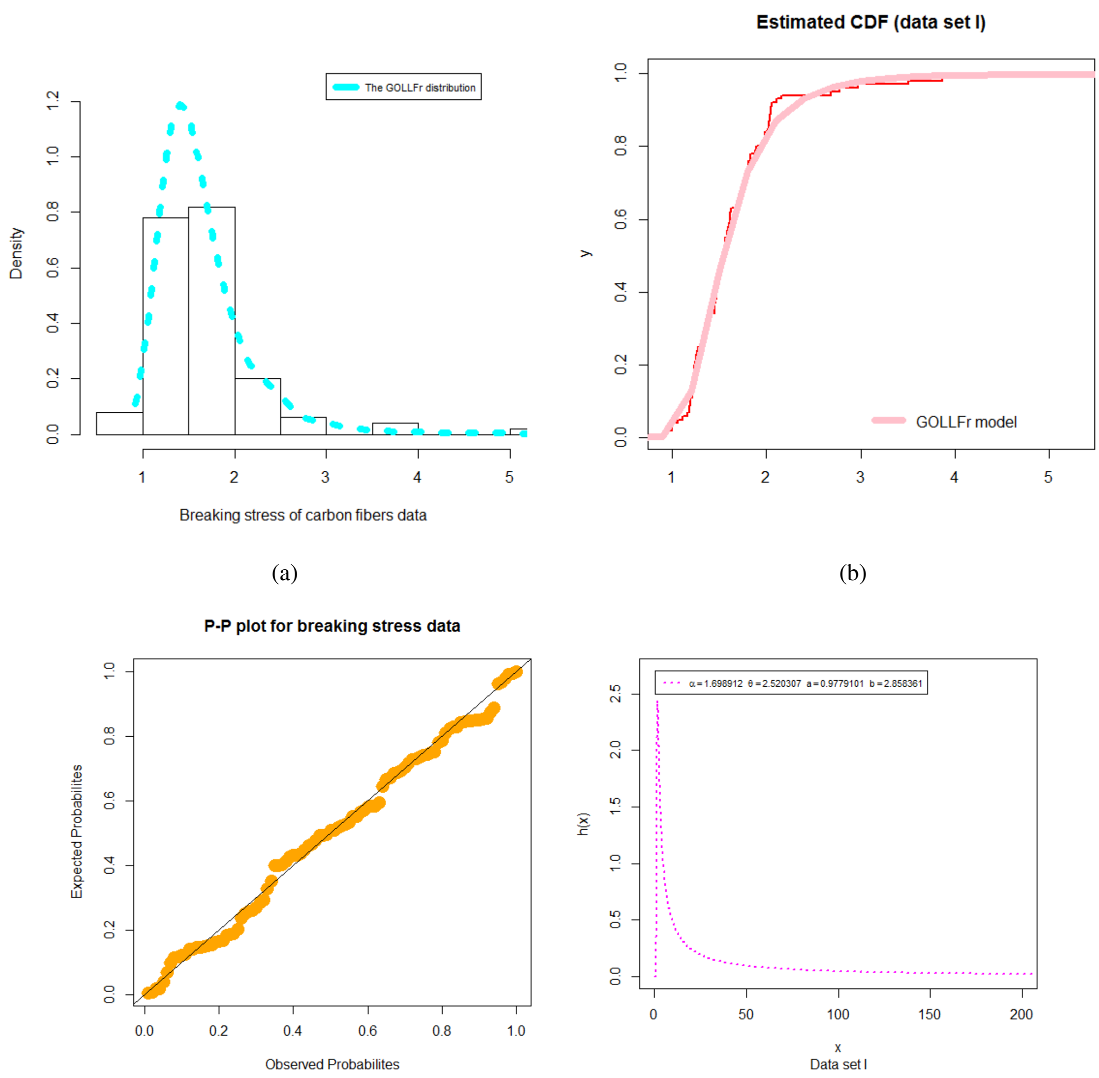
The statistics ($W^{\star}$, $A^{\star}$, K-S and p-value) of all fitted models are presented in Table 4. The MLEs and corresponding standard errors are given in Table 5. The GOLLFr distribution in Table 4 gives the lowest values the$W^{\star}$, $A^{\star}$, K-S and the biggest value of the p-value statistics as compared to other extensions of the Fr models, and therefore the new one can be chosen as the best model. Figure 3 gives the estimated density, estimated CDF, P-P plot and estimated HRF for data set **I**.

Table 4: $W^{\star}$, $A^{\star}$, K-S and p-value for data set **I**.

Model	Goodness of fit criteria			
	$W^{\star}$	$A^{\star}$	K-S	p-value
GOLLFr	**0.0624**	**0.4837**	**0.0638**	**0.8101**
OLLEFr	0.1203	0.9639	0.5561	$2.2\times e^{-16}$
OLLEIR	0.1553	1.21197	0.65497	$2.2\times e^{-16}$
OLLIR	0.15532	1.21201	0.6550	$2.2\times e^{-16}$
Fr	0.1090	0.7657	0.0874	0.4282
KFr	0.0812	0.6217	0.0759	0.6118
EFr	0.1091	0.7658	0.0874	0.4287
BFr	0.0809	0.6207	0.0757	0.6147
TFr	0.0871	0.6209	0.0782	0.5734
MOFr	0.0886	0.6142	0.0763	0.5168
McFr	0.1333	1.0608	0.0807	0.5332

Table 5: MLEs and their standard errors (in parentheses) for data set **I**.

Model	Estimates				
	$\widehat{\alpha}$	$\widehat{\theta}$	$\widehat{c}$	$\widehat{a}$	$\widehat{b}$
GOLLFr	1.6989	2.5203		0.9779	2.8584
	(0.655)	(27.6)		(3.75)	(1.0037)
OLLEFr	0.1351		3.7216	0.9296	21.319
	(0.011)		(0.0034)	(0.0033)	(0.0034)
OLLEIR	0.4946		0.067	1.74262	2
	(0.04135)		(0.7195)	(9.3007)	
OLLIR	0.49459			0.45242	2
	0.04135			0.03869	
Fr				1.3968	4.3724
				(0.0336)	(0.3278)
KFr		0.8489	1.6239	1.6341	3.4208
		(16.083)	(0.6979)	(9.049)	(0.7635)
EFr		0.9395		1.4169	0.9395
		(3.543)		(2.568)	(0.3278)
BFr		0.7346	1.5830	1.6684	3.5112
		(1.5290)	(0.7132)	(0.7662)	(0.9683)
TFr	-0.7166			1.2656	4.7121
	(0.2616)			(0.0579)	(0.3657)
MOFr		0.0033		6.2296	1.2419
		(0.0009)		(1.0134)	(0.1181)
McFr	0.8503	44.423	19.859	0.0203	46.974
	(0.1353)	(25.100)	(6.706)	(0.0060)	(21.871)



(c) (d)

Figure 3: Estimated density, estimated CDF, P-P plot and estimated HRF for data set **I**

5.2. Glass fibers data

The 2nd data set is generated data to simulate the strengths of glass fibers which was given by Smith and Naylor (1987). The data set is: 1.0140, 1.0810, 1.082, 1.1850, 1.2230, 1.2480, 1.2670, 1.2710, 1.2720, 1.2750, 1.2760, 1.278, 1.2860, 1.288, 1.2920, 1.304, 1.3060, 1.355, 1.361, 1.3640, 1.379, 1.4090, 1.426, 1.4590, 1.460, 1.4760, 1.481, 1.4840, 1.501, 1.5060, 1.5240, 1.5260, 1.5350, 1.541, 1.5680, 1.579, 1.5810, 1.591, 1.5930, 1.602, 1.6660, 1.67, 1.684, 1.6910, 1.704, 1.7310, 1.735, 1.7470, 1.748, 1.7570, 1.8000, 1.806, 1.8670, 1.876, 1.878, 1.910, 1.9160, 1.9720, 2.0120, 2.456, 2.5920, 3.1970 and 4.1210. Figure 4 gives the TTT plot for data set **II**. It indicates that the empirical HRFs of data sets **II** is increasing.

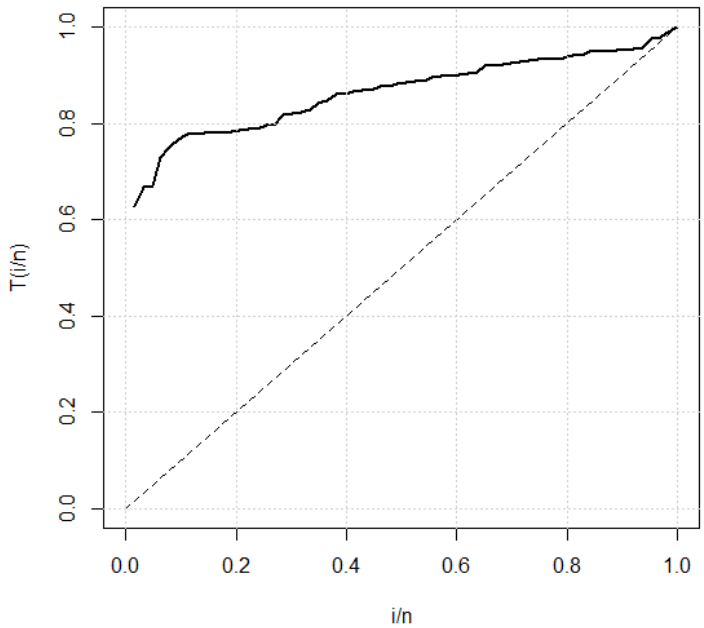


Figure 4: TTT plot for data set **II**

The statistics ($W^{\star}$, $A^{\star}$, K-S and p-value) of the fitted models are provided in Table 6. The MLEs and corresponding standard errors are given in Table 7. From Table 6, the GOLLFr distribution gives the lowest values the $W^{\star}$, $A^{\star}$, K-S and the biggest value of the p-value statistics as compared to further Fr models, and therefore the new one can be chosen as the best model. Figure 5 gives the estimated density, estimated CDF, P-P plot and estimated HRF for data set **II**.

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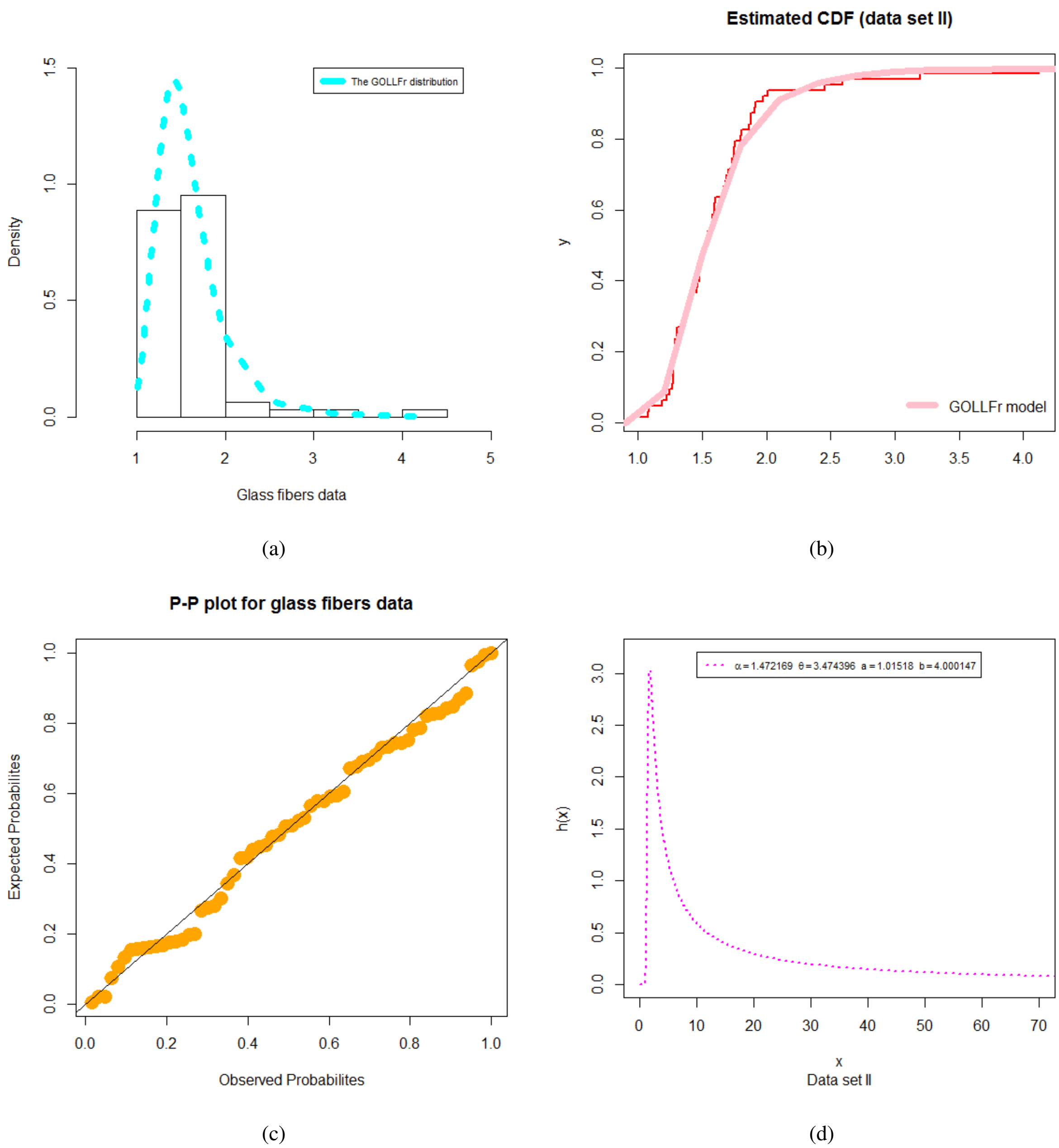
Table 6: $W^\star$, $A^\star$, K-S and p-value for data set **II**.

Model	Goodness of fit criteria			
	$W^\star$	$A^\star$	K-S	p-value
GOLLFr	**0.05084**	**0.4117**	**0.06901**	**0.9040**
GOLLIR	0.05250	0.4530	0.070334	0.8926
OLLEFr	0.10487	0.8325	0.55196	$6.661\times e^{-16}$
OLLEIR	0.1502	1.14697	0.67949	$6.661\times e^{-16}$
OLLIR	0.15021	1.14697	0.67951	$6.661\times e^{-16}$
Fr	0.0707	0.5332	0.0772	0.8185
KFr	0.0634	0.4981	0.0715	0.8810
EFr	0.0707	0.5332	0.0772	0.8187
BFr	0.0640	0.5008	0.0716	0.8804
TFr	0.0655	0.4939	0.0735	0.8470
MOFr	0.0629	0.4902	0.0813	0.7685
McFr	0.1161	0.9193	0.0831	0.7455

Table 7: MLEs and their standard errors (in parentheses) for data set **II**.

Model	**Estimates**				
	$\widehat{\alpha}$	$\widehat{\theta}$	$\widehat{c}$	$\widehat{a}$	$\widehat{b}$
GOLLFr	1.472169	3.474396		1.015180	4.000147
	(0.63)	(63.63)		(4.65)	(1.52)
OLLEFr	0.1449		0.00879	1.2997	24.878
	(0.0129)		(0.000)	(0.000)	(0.000)
GOLLIR	3.032	2.1568		0.86599	2
	(0.325)	(9.145)		(1.836)	
OLLEIR	0.5025		0.0716	1.7048	2
	(0.0529)		(1.13062)	(13.47)	
OLLIR	0.50251			0.45599	2
	0.052946			0.048652	
Fr				1.4108	5.4377
				(0.0344)	(0.5192)
KFr		0.2855	1.2824	1.9142	4.7731
		(9.1338)	(0.6388)	(12.836)	(1.3134)
EFr		0.9059		1.4367	5.4379
		(2.764)		(4.324)	(0.5193)
BFr		1.2996	1.2649	1.3945	4.7927
		(4.4378)	(0.6640)	(0.9304)	(1.4641)
TFr	0.7778			1.5491	4.3139
	(0.2477)			(0.0655)	(0.5849)
MOFr		0.0023		5.2383	1.4537
		(0.0004)		(0.8209)	(0.1650)
McFr		56.227	14.953	0.0073	29.104
		(30.539)	(4.733)	(0.0013)	(11.304)



(a) (b) (c) (d)

Figure 5: Estimated density, estimated CDF, P-P plot and estimated HRF for data set **II.**

5.3. Relief time data

The 3rd data set (wingo data) represents a complete sample from a clinical trial describe a relief time (in hours) for 50 arthritic patients. The data set is: 0.700, 0.84, 0.58, 0.500, 0.55, 0.82, 0.590, 0.71, 0.720, 0.610, 0.62, 0.49, 0.54, 0.36, 0.360, 0.71, 0.35, 0.64, 0.84, 0.550, 0.59, 0.29, 0.75, 0.460, 0.46, 0.60, 0.60, 0.360, 0.52, 0.68, 0.800, 0.55, 0.84, 0.340, 0.34, 0.700, 0.490, 0.56, 0.710, 0.61, 0.570, 0.73, 0.75, 0.440, 0.44, 0.81, 0.80, 0.870, 0.29 and 0.500. Figure 6 gives the TTT plot for data set **III**. It indicates that the empirical HRFs of data sets **III** is increasing.

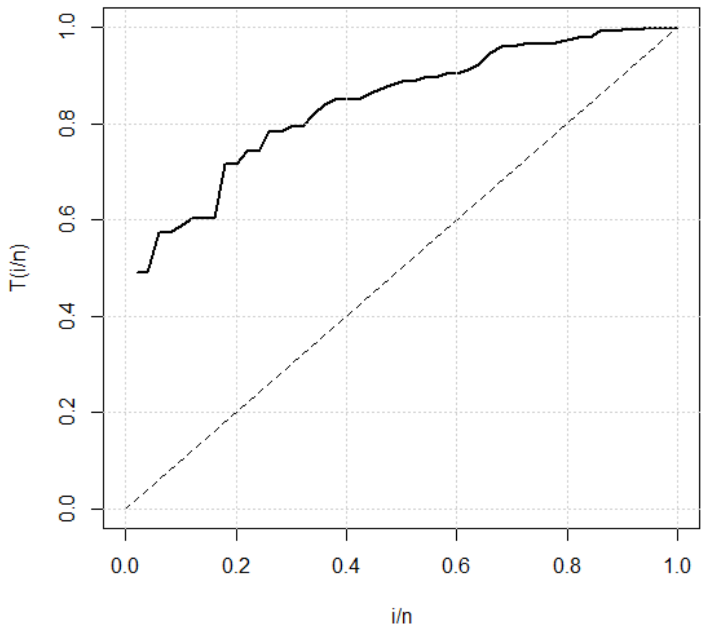


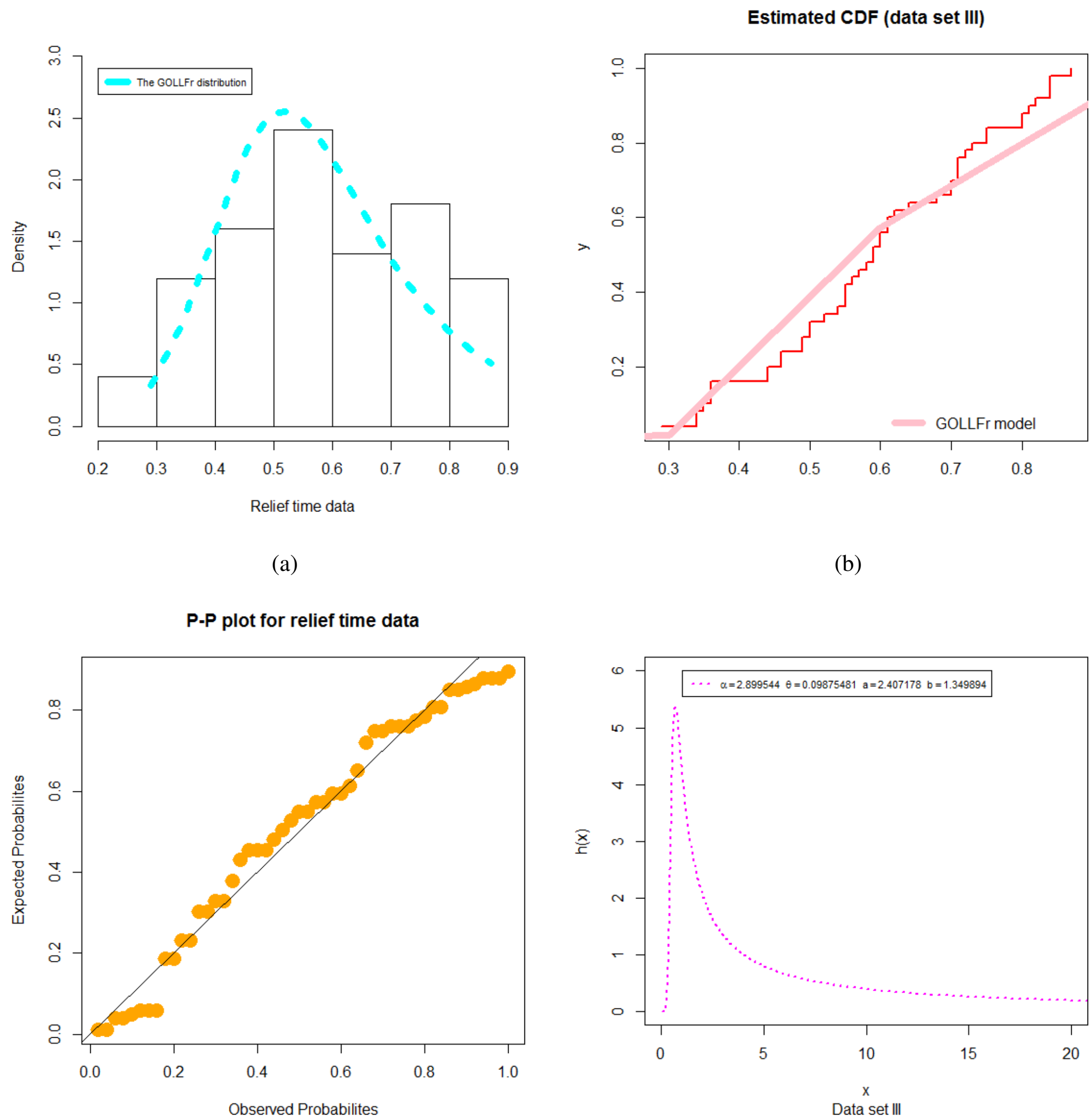
Figure 6: TTT plot for data set **III**

Table 8: $W^{\star}$, $A^{\star}$, K-S and p-value for data set **III**.

Model	Goodness of fit criteria			
	$W^{\star}$	$A^{\star}$	K-S	p-value
GOLLFr	**0.15615**	**1.1276**	**0.10476**	**0.6427**
GOLLIR	0.19551	1.3498	0.11008	0.5797
OLLEFr	0.1577	1.09876	0.53498	$7.436\times e^{-13}$
Fr	0.3233	2.0301	0.1506	0.2066
EFr	0.3233	2.0301	0.1506	0.2064
TFr	0.2823	1.8152	0.1370	0.3045

Table 9: MLEs and their standard errors (in parentheses) for data set **III**.

Model	Estimates				
	$\widehat{\alpha}$	$\widehat{\theta}$	$\widehat{c}$	$\widehat{a}$	$\widehat{b}$
GOLLFr	2.899	0.09875		2.4072	1.34989
	(0.8263)	(0.05666)		(0.8460)	(0.34177)
GOLLIR	1.961	0.111		1.4123	2
	(0.234)	(0.000)		(0.000)	
OLLEFr	0.0669		0.00459	0.3558	32.561
	(0.0076)		(0.0028)	(0.0047)	(0.006)
Fr				0.4859	3.2078
				(0.0227)	(0.3263)
EFr			0.9047	0.5013	3.2077
			(18.784)	(3.2444)	(0.3263)
TFr	−0.5816			0.4400	3.4974
	(0.2787)			(0.0290)	(0.3527)



(a) (b) (c) (d)

Figure 7: Estimated density, estimated CDF, P-P plot and estimated HRF for data set **III**

The statistics ($W^\star$, $A^\star$, K-S and p-value) of all fitted models are presented in Table 8. The MLEs and corresponding standard errors are given in Table 9. From Table 8, the GOLLFr distribution gives the lowest values the$W^\star$, $A^\star$, K-S and the biggest value of the p-value statistics as compared to further Fr models, and therefore the new one can be chosen as the best model. Figure 5 gives the estimated density, estimated CDF, P-P plot and estimated HRF for data set **III**.

6. Concluding remarks

We introduce a new distribution called GOLLFr distribution for modeling the extreme values. The proposed model provides generalization for eleven distributions at least, five of them are quite new. The sample mean and the standard deviations are evaluated using a maximum likelihood method via a simulation study The MLEs for GOLLFr distribution provides satisfying results. Some important mathematical properties of the new model are derived. The skewness of the new model always positive, whereas the kurtosis can be more than (or less than) 3. The new model is better than some other important competitive versions of the Fréchet model in modeling the breaking stress data, the glass fibers data and the relief time data. In our upcoming work, we can apply many new useful goodness-of-fit tests for right censored validation such as the Nikulin-Rao-Robson goodness-of-fit test and Bagdonavičius-Nikulin goodness-of-fit test as performed by Ibrahim et al. (2019), Goual et al. (2019, 2020), Mansour et al. (2020a-f), Yadav et al. (2020), Goual and Yousof (2020), Aidi et al. (2021) Yadav et al. (2020, 2022) and Ibrahim et al. (2021a, 2022), among others.

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