The Nadarajah Haghighi Topp Leone-G Family of Distributions with Mathematical Properties and Applications

Hesham Reyad
Department of Information Systems and Production Management
Qassim University, Kingdom of Saudi Arabia
Email: hesham_reyad@yahoo.com

Mahmoud Ali Selim
Community College, King Khalid University, Kingdom of Saudi Arabia
Faculty of Commerce, Al-Azhar University, Egypt
Email: selim.one@gmail.com

Soha Othman
Department of Applied Statistics, Cairo University, Egypt
Email: soha_othman@yahoo.com

Abstract

Based on the Nadarajah Haghighi distribution and the Topp Leone-G family in view of the T-X family, we introduce a new generator of continuous distributions with three extra parameters called the Nadarajah Haghighi Topp Leone-G family. Three sub-models of the new class are discussed. Main mathematical properties of the new family are investigated such as; quantile function, raw and incomplete moments, Bonferroni and Lorenz curves, moment and probability generating functions, stress strength model, Shanon and Rényi entropies, order statistics and probability weighted moments. The model parameters of the new family is estimated by using the method of maximum likelihood and the observed information matrix is also obtained. We introduce two real applications to show the importance of the new family.

Keywords: Entropies, Nadarajah Haghighi distribution, Maximum Likelihood, Probability Weighted Moments, Stress Strength Model, T-X Family.

1. Introduction

In the last decade, the famous distributions such as Lomax, Pareto, Gumbel, Lindley,…etc are extensively used for fitting data in various sciences such as medicine, engineering, agriculture and economics. However, in many applied areas such as environmental sciences, reliability studies and finance, there exist a great need to generalize these distributions to gain more acuracyand flexibility. Based on this reason, the statistican propose new methods for generating new generators of distributions to gain more desirable properties in the resulting distributions. Some of the recent families are : The transmuted Gompertz-G by Reyad et al. (2018a), Marshall-Olkin generalized G Poisson by Korkmaz et al. (2018a), generalized odd Weibull-G by Korkmaz et al. (2018b), generalized transmuted Poisson-G by Yousof et al.(2018), exponentiated odd log-Logistic-G by Alizadeh et al. (2018), Topp Leone odd Lindley-G by Reyad et al. (2018b), extended odd Frechet by Yousof et al.(2019), transmuted generalized odd generalized exponential-G by Reyad et al. (2019a), odd Lomax-G by Cordeiro et al. (2019), Marshall-Olkin alpha power-G by Mazen er al. (2019), exponentiated generalized Topp Leone-G by Reyad et al. (2019b) odd inverse Pareto-G by Aldahlan et al. (2019); among others.

Let $g(x;\phi)$ and $G(x;\phi)$ denote the probability density function (pdf) and cumulative distribution function (cdf) of a baseline model with parameter vector ϕ. Al-Shomrani et al. (2016) proposed the Topp Leone-G (TL-G) family of distributions with cdf by

$$H(x;\phi)=\left\{1-\bar{G}(x;\phi)^2\right\}^{\theta}, \quad \theta>0, \quad x\in R, \tag{1}$$

where, $\bar{G}(x;\phi)=1-G(x;\phi)$. Moreover, the Nadarajah Haghighi distribution has pdf given by

$$f(t)=\alpha\lambda\left(1+\lambda t\right)^{\alpha-1}e^{1-\left(1+\lambda t\right)^{\alpha}}, x>0,\alpha,\lambda>0 \tag{2}$$

Based on the idea of T-X family pioneerd by Alzaatreh et al. (2013), TL-G class and the Nadarajah Haghighi distribution, we introduce a new class of continuous distributions called the Nadarajah Haghighi Topp Leone-G (NHTL-G for short) family with cdf given by

$$\begin{aligned} F(x;\alpha,\lambda,\theta,\phi) &= \int_0^{H(x;\phi)} f(t)dt \\ &= \int_0^{\left\{1-\bar{G}(x;\phi)^2\right\}^{\theta}} \alpha\lambda\left(1+\lambda t\right)^{\alpha-1}e^{1-\left(1+\lambda t\right)^{\alpha}}dt \\ &= 1-e^{1-\left\{1+\lambda\left(1-\bar{G}(x;\phi)^2\right)^{\theta}\right\}^{\alpha}}, \quad x\in R, \end{aligned} \tag{3}$$

The pdf of the NHTL-G family is given by

$$\begin{aligned} f(x;\alpha,\lambda,\theta,\phi) &= 2\alpha\lambda\theta g(x;\phi)\bar{G}(x;\phi)\left(1-\bar{G}(x;\phi)^2\right)^{\theta-1} \\ &\times\left\{1+\lambda\left(1-\bar{G}(x;\phi)^2\right)^{\theta}\right\}^{\alpha-1} e^{1-\left\{1+\lambda\left(1-\bar{G}(x;\phi)^2\right)^{\theta}\right\}^{\alpha}}, \quad x\in R, \end{aligned} \tag{4}$$

We will denote a random variable X with pdf (4) by X~NHTL-G(α,λ,θ,∅). The hazard function $\tau(x)$ for the NHTL-G family is given by

$$\begin{aligned} \tau(x;\alpha,\lambda,\theta,\phi) &= 2\alpha\lambda\theta g(x;\phi)\bar{G}(x;\phi)\left[1-\bar{G}(x;\phi)^2\right]^{\theta-1}\left\{1+\lambda\left[1-\bar{G}(x;\phi)^2\right]^{\theta}\right\}^{\alpha-1} \\ &\times\left\{1+\lambda\left[1-\bar{G}(x;\phi)^2\right]^{\theta}\right\}^{\alpha-1}, \quad x\in R, \end{aligned} \tag{5}$$

The quantile function of the NHTL-G family, say $Q(u)=F^{-1}(u)$, for $u\in(0,1),\alpha\neq 0,\lambda\neq 0$ and $\theta\neq 0$ is the solution of the non-linear equation

$$Q(u)=G^{-1}\left(1-\left\{1-\left[\frac{1}{\lambda}\left\{\left[1-\log\left(1-u\right)\right]^{1/\alpha}-1\right\}\right]^{1/\theta}\right\}^{1/2}\right), \tag{6}$$

This paper is organized as follows: An important expansions of the pdf and cdf corresponding to the new family are derived in Section 2. In Section 3, three special sub-models of the NHTL-G family are discussed. Main mathematical properties of the NHTL-G family are studies in Section 4. The model parameters of the new generator is estimated by using the method of maximum likelihood and the observed information matrix is also obtained in Section 5. In Section 6, two real applications of the NHTL-G are introduced. Concluding remarks are given in Section 7.

2. Linear Representation

In this section, we obtain the pdf and cdf of the new family as mixture repsentations of the exponentiated-G distribution. The pdf (4) can be expressed as

$$f(x)=2\alpha\lambda\theta e\sum_{w=0}^{\infty}\frac{(-1)^w}{w!}g(x;\phi)\bar{G}(x;\phi)\left[1-\bar{G}(x;\phi)^2\right]^{\theta-1}\left\{1+\lambda\left[1-\bar{G}(x;\phi)^2\right]^{\theta}\right\}^{\alpha(w+1)-1}$$

$$=2\alpha\lambda\theta e\sum_{w=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^w\lambda^j}{w!}\binom{\alpha(w+1)-1}{j}g(x;\phi)\bar{G}(x;\phi)\left[1-\bar{G}(x;\phi)^2\right]^{\theta(j+1)-1}$$

$$=2\alpha\lambda\theta e\sum_{w=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^{w+i}\lambda^j}{w!}\binom{\alpha(w+1)-1}{j}\times\binom{\theta(j+1)-1}{i}g(x;\phi)\left[1-G(x;\phi)\right]^{2i+1}$$

$$=2\alpha\theta e\sum_{w=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\ell=0}^{2i+1}\frac{(-1)^{w+i+\ell}\lambda^{j+1}}{w!}\binom{\alpha(w+1)-1}{j}\times\binom{\theta(j+1)-1}{i}\binom{2i+1}{\ell}g(x;\phi)G(x;\phi)^{\ell}$$

Or equivalently

$$f(x)=\sum_{\ell=0}^{2i+1}\delta_{\ell}\,\pi_{\ell+1}(x), \tag{7}$$

where, $\delta_{\ell} = 2\alpha\theta e \sum_{w=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \frac{(-1)^{w+i+\ell}\lambda^{j+1}}{(\ell+1)w\,!}\binom{\alpha(w+1)-1}{j}\binom{\theta(j+1)-1}{i}\binom{2i+1}{\ell}$

and $\pi_{\ell+1}(x) = (\ell+1)\,g(x)G(x)^{\ell}$ is the exponentiated-G distribution with power parameter $\ell+1$.

Similarly, the cdf (3) can be written as

$$F(x) = \sum_{\ell=0}^{2i+1} \delta_{\ell}\, \Pi_{\ell+1}(x), \tag{8}$$

where, $\Pi_{\ell+1}(x = G(x)^{\ell+1}$.

3. The Sub-Models of the New Family

In this section, we introduce three special sub-models of the NHTL-G family.

3.1 The NHTL-Lomax (NHTLLx) Model

Suppose the cdf and pdf of the Lomax distribution are the following $G(x) = 1-(1+bx)^{-a}$, $x \geq 0$, and $g(x) = ab((1+bx)^{-a}, x>0,\ a,b>0,$ respectively. Then, the pdf and cdf of NHTLLx distribution are given, respectively, by

$$f(x) = 2\alpha\lambda\theta ab(1+bx)^{-2a-1}\left(1-(1+bx)^{-2a}\right)^{\theta-1}$$

$$\times\left\{1+\lambda\left(1-(1+bx)^{-2a}\right)^{\theta}\right\}^{\alpha-1} e^{1-\left\{1+\lambda\left(1-(1+bx)^{-2a}\right)^{\theta}\right\}^{\alpha}},\ \ x>0.$$

and

$$F(x) = 1-e^{1-\left\{1+\lambda\left(1-(1+bx)^{-2a}\right)^{\theta}\right\}^{\alpha}},\ \ x \geq 0.$$

The plots of the density and hazard functions are given in Figure 1.

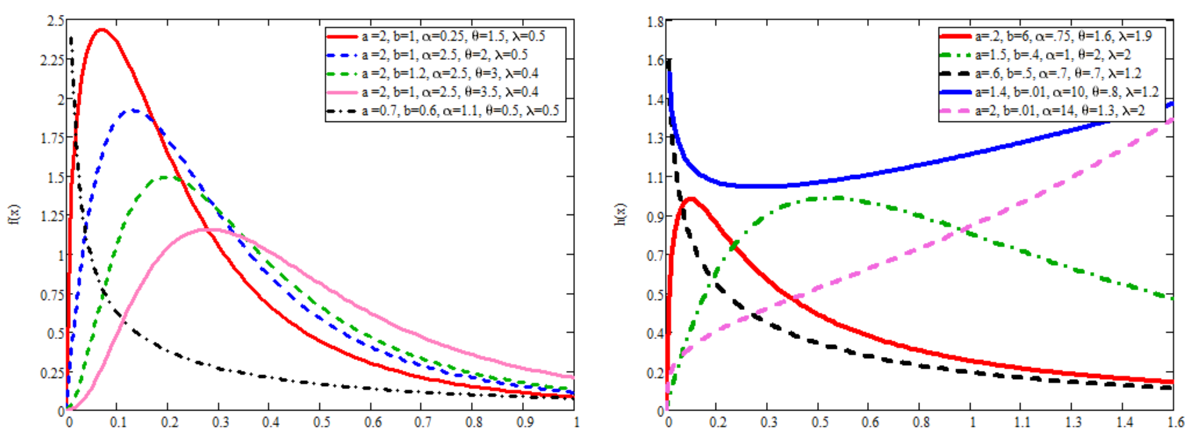

Figure 1: Plots of the NHTLLx pdf and hrf for selected values of parameters.

3.2 The NHTL-Kumaraswamy (NHTLKw) Model

Consider the cdf and pdf of the Kumaraswamy distribution $G(x)=1-(1-x^{a})^{b}, 0\le x\le 1,$ and $g(x)=abx^{a-1}(1-x^{a})^{b-1}, 0<x<1,\ a,b>0,$ respectively. Then, the pdf and cdf of NHTLKw distribution are given, respectively, by

$$f(x)=2\alpha\lambda\theta abx^{a-1}(1-x^{a})^{2b-1}\left(1-(1-x^{a})^{2b}\right)^{\theta-1}$$

$$\times\left\{1+\lambda\left(1-(1-x^{a})^{2b}\right)^{\theta}\right\}^{\alpha-1}e^{1-\left\{1+\lambda\left(1-(1-x^{a})^{2b}\right)^{\theta}\right\}^{\alpha}},\ 0<x<1.$$

and

$$F(x)=1-e^{1-\left\{1+\lambda\left(1-(1-x^{a})^{2b}\right)^{\theta}\right\}^{\alpha}}, 0\le x\le 1.$$

The plots of the density and hazard functions are given in Figure 2.

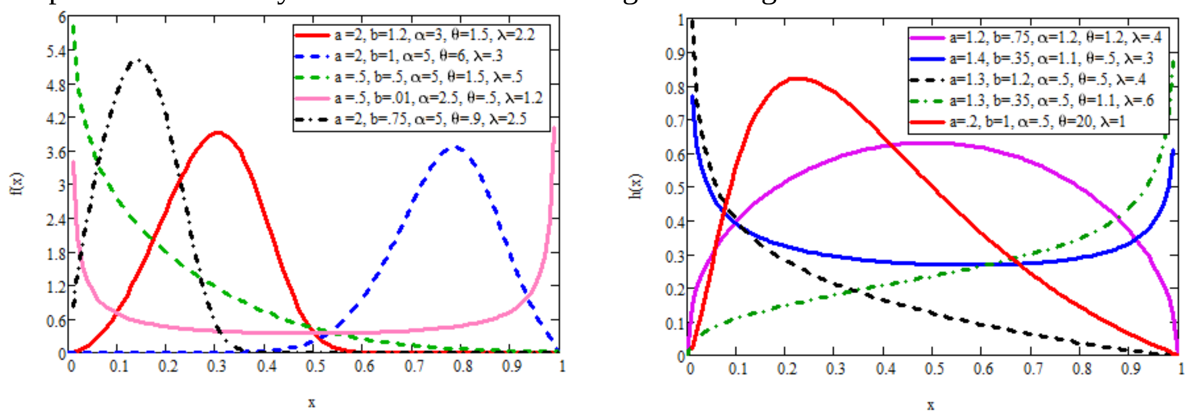

Figure 2: Plots of the NHTLKw pdf and hrf for selected values of parameters.

3.3 The NHTL-Exponential (NHTLEx) Model

Consider the cdf and pdf of the exponential distribution $G(x)=1-e^{-ax}, x\ge 0,$ and $g(x)=ae^{-ax}, x>0,\ a>0,$ respectively. Then, the pdf and cdf of NHTLEx distribution are given, respectively, by

$$f(x)=2\alpha\lambda\theta ae^{-2ax}\left(1-e^{-2ax}\right)^{\theta-1}\left\{1+\lambda\left(1-e^{-2ax}\right)^{\theta}\right\}^{\alpha-1}e^{1-\left\{1+\lambda\left(1-e^{-2ax}\right)^{\theta}\right\}^{\alpha}},\ x>0.$$

and

$$F(x)=1-e^{1-\left\{1+\lambda\left(1-e^{-2ax}\right)^{\theta}\right\}^{\alpha}}, x\ge 0.$$

The plots of the density and hazard functions are given in Figure 2.

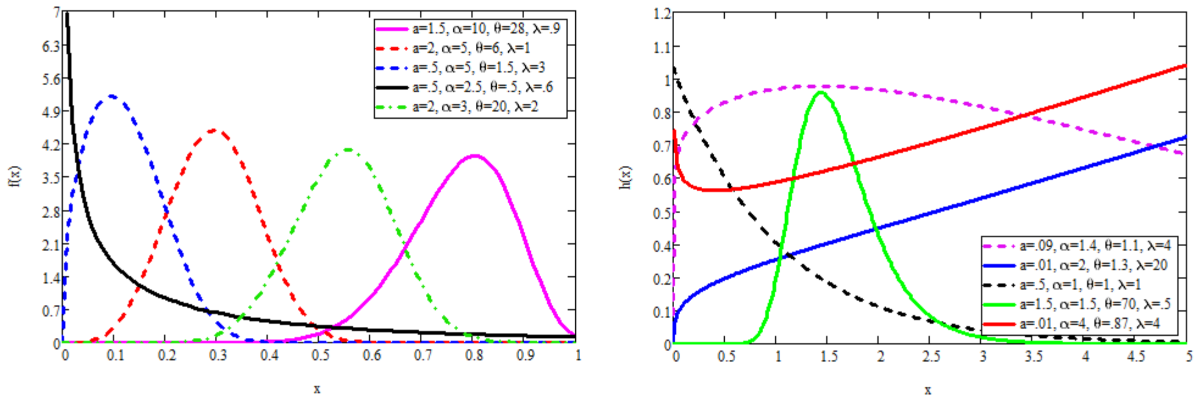

Figure 3: Plots of the NHTLEx pdf and hrf for selected values of parameters.

4. Mathematical Properties

This section studies some main properties of the NHTL-G family.

4.1. Order Statistics

Let $X_1, X_2, ..., X_n$ be a simple random sample from the NHTL-G family with cdf (3), pdf (4) and $X_{1:n}, X_{2:n}, ..., X_{n:n}$ be the corresponding order statistics. The pdf of $X_{k:n}$, the kth order statistic, is given by

$$f_{X_{q:n}}(x) = \frac{1}{\beta(k, n-k+1)} \sum_{w=0}^{n-k} (-1)^w \binom{n-k}{w} f(x) F(x)^{k+w-1}, \tag{9}$$

where, $\beta(.,.)$ is the beta function. We can obtain from (3) and (4)

$$f(x)F(x)^{k+w-1} = 2\alpha\lambda\theta \sum_{j=0}^{k+w-1} (-1)^j \binom{k+w-1}{j} g(x;\phi)\bar{G}(x;\phi)\left[1-\bar{G}(x;\phi)^2\right]^{\theta-1}$$
$$\times\left\{1+\lambda\left[1-\bar{G}(x;\phi)^2\right]^{\theta}\right\}^{\alpha-1} e^{-(j+1)\left(1-\left\{1+\lambda\left[1-\bar{G}(x;\phi)^2\right]^{\theta}\right\}^{\alpha}\right)}$$
$$= 2\alpha\lambda\theta \sum_{j=0}^{k+w-1} \sum_{i=0}^{\infty} \frac{(-1)^{j+i}(j+1)^i e^{j+1}}{i!} \binom{k+w-1}{j}$$
$$\times g(x;\phi)\bar{G}(x;\phi)\left[1-\bar{G}(x;\phi)^2\right]^{\theta-1}\left\{1+\lambda\left[1-\bar{G}(x;\phi)^2\right]^{\theta}\right\}^{\alpha(i+1)-1}$$
$$= 2\alpha\theta \sum_{j=0}^{k+w-1} \sum_{i,\ell=0}^{\infty} \frac{(-1)^{j+i}(j+1)^i e^{j+1}\lambda^{\ell+1}}{i!} \binom{k+w-1}{j}$$
$$\times \binom{\alpha(i+1)-1}{\ell} g(x;\phi)\bar{G}(x;\phi)\left[1-\bar{G}(x;\phi)^2\right]^{\theta(\ell+1)-1}$$
$$= 2\alpha\theta \sum_{j=0}^{k+w-1} \sum_{i,\ell,d=0}^{\infty} \sum_{m=0}^{2d+1} \frac{(-1)^{j+i+d+m}(j+1)^i e^{j+1}\lambda^{\ell+1}}{i!} \binom{k+w-1}{j}$$
$$\times \binom{\alpha(i+1)-1}{\ell} \binom{\theta(\ell+1)-1}{d} \binom{2d+1}{m} g(x;\phi)\bar{G}(x;\phi)^m$$

Or equivalently,

$$f(x)F(x)^{k+w-1}=\sum_{m=0}^{2d+1}\mu_m\,\pi_{m+1}(x), \tag{10}$$

where, $\mu_m=2\alpha\theta\sum_{j=0}^{k+w-1}\sum_{i,\ell,d=0}^{\infty}\frac{(-1)^{j+i+d+m}(j+1)^i e^{j+1}\lambda^{\ell+1}}{i!\,(m+1)}\binom{k+w-1}{j}\binom{\alpha(i+1)-1}{\ell}\binom{\theta(\ell+1)-1}{d}\binom{2d+1}{m}.$

Substituting from (10) into (9), we arrive at

$$f_{X_{k:n}}(x)=\sum_{m=0}^{2d+1}\mu_m^*\,\pi_{m+1}(x), \tag{11}$$

where, $\mu_m^*=\sum_{w=0}^{n-k}\frac{(-1)^w}{\beta(q,n-k+1)}\binom{n-k}{w}\mu_m.$

In addition, the rth moment of the kth order statistic for NHTL-G family is given by

$$E\left(x_{k:n}^r\right)=\sum_{m=0}^{2d+1}\mu_m^{**}\,\pi_{m+1}(x), \tag{12}$$

where, $\mu_m^{**}=(m+1)\mu_m^*.$

4.2. Probability Weighted Moments

The $(r+s)$th PWM of X with NHTL-G distribution denoted as $\upsilon_{r,s}$, is given by

$$\upsilon_{r,s}=E\left(X^rF(x)^s\right)=\int_{-\infty}^{\infty}X^rF(x)^s f(x)dx. \tag{13}$$

By using (3) and (4) and after some simplifications, we arrive at

$$f(x)F(x)^s=\sum_{m=0}^{2\ell+1}a_m\,\pi_{m+1}(x), \tag{14}$$

where, $a_m=2\alpha\theta\sum_{h=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{(-1)^{h+j+\ell+m}(h+1)^j e^{h+1}\lambda^{i+1}}{i!\,(m+1)}\binom{s}{h}\binom{\alpha(j+1)-1}{i}\binom{\theta(i+1)-1}{\ell}\binom{2\ell+1}{m},$

Substituting (14) into (13), we have

$$\upsilon_{r,s}=\sum_{m=0}^{2\ell+1}a_m^*\,\Psi_{r,m}, \tag{15}$$

where, $a_m^*=(m+1)a_m$ and $\Psi_{r,m}=\int_{-\infty}^{\infty}x^r g(x)G(x)^m dx$ is the probability weighted moment of the parent distribution.

4.3. Moments

The raw moment denoted as, μ_r', of a random variable where, X with NHTL-G distribution is given by

$$\mu_r'=E(X^r)=\int_{-\infty}^{\infty}x^r f(x)\,dx$$

$$=\sum_{\ell=0}^{2i+1}\delta_\ell^*\,\Psi_{r,\ell}, \tag{16}$$

where, $\delta_\ell^*=(\ell+1)\delta_\ell.$

The nth central moment of the NHTL-G distribution, say μ_n, can be calculated from

$$\mu_n = \sum_{r=0}^{n} \binom{n}{r} \left(-\mu_1'\right)^{n-r} E(x^r)$$

$$= \sum_{\ell=0}^{2i+1} \delta_\ell^{**} \Psi_{r,\ell}, \tag{17}$$

where, $\delta_\ell^{**} = \sum_{r=0}^{n} \binom{n}{r} \left(-\mu_1'\right)^{n-r} \delta_\ell^{*}.$

The rth incomplete moment of X, denoted by $\varphi_s(t)$, is

$$\varphi_s(t) = \int_{-\infty}^{t} x^s f(x)\, dx$$

$$= \sum_{\ell=0}^{2i+1} \delta_\ell^{*} \Delta_{r,\ell}, \tag{18}$$

where, $\Delta_{r,\ell} = \int_{-\infty}^{t} x^s g(x)\, G(x)^\ell\, dx.$

The Lorenz and Bonferroni curves denoted as $L_F(x)$ and $B(F(x))$ can be calculated from (18) by using $L_F(x) = \varphi_1(t)/\mu_1'$ and $B(F(x)) = L_F(x)/F(x).$

The moment and probability generating functions denoted as $M_x(t)$ and $M_{[x]}(t)$ of the NHTL-G distribution are given, respectively, by

$$M_x(t) = E(e^{tx}) = \sum_{r=0}^{\infty} \sum_{\ell=0}^{2i+1} \frac{t^r}{r!} \delta_\ell^{*} \Psi_{r,\ell}, \tag{19}$$

and

$$M_{[x]}(t) = E(t^x) = \sum_{r=0}^{\infty} \sum_{\ell=0}^{2i+1} \frac{(\ln t)^r}{r!} \delta_\ell^{*} \Psi_{r,\ell}. \tag{20}$$

4.4. Entropies

The Rényi entropy can be obtained from

$$I_R(X) = (1-\beta)^{-1} \log \int_{-\infty}^{\infty} f(x)^\beta\, dx,\ \beta > 0,\ \beta \neq 0. \tag{21}$$

Based on (4), we have

$$f(x)^\beta = \sum_{\ell=0}^{\infty} k_\ell\, g(x)^\beta\, G(x)^\ell, \tag{22}$$

where, $k_\ell = (2\alpha\theta e)^\beta \sum_{w=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^{w+i+\ell} \beta^w \lambda^{\beta+j}}{w!} \binom{\alpha(\beta+w)-\beta}{j} \binom{\theta(\beta+j)-\beta}{i} \binom{2i+\beta}{\ell}.$

Inserting (22) into (21), we have

$$I_R(X) = (1-\beta)^{-1} \log\left(\sum_{\ell=0}^{\infty} k_\ell \int_{-\infty}^{\infty} g(x)^\beta\, G(x)^\ell\, dx \right). \tag{23}$$

The Shannon entropy is defined as follows:

$$\Upsilon_x = -E\left\{\log f(x)\right\}. \tag{24}$$

From (4), we have

$$\log f(x) = 1 + \log\left(2\alpha\lambda\theta\right) + \log\left(g(x;\phi)\right) - \log\left(\bar{G}(x;\phi)\right)$$
$$+(\theta-1)\log\left(1-\bar{G}(x;\phi)^2\right) + (\alpha-1)\log\left\{1+\lambda\left(1-\bar{G}(x;\phi)^2\right)^{\theta}\right\}$$
$$-\left\{1+\lambda\left(1-\bar{G}(x;\phi)^2\right)^{\theta}\right\}^{\alpha} \tag{25}$$

Using $\log(1-z) = -\sum_{w=1}^{\infty}\frac{z^w}{w}$ and $\log(1+z) = -\sum_{w=1}^{\infty}\frac{(-1)^{w-1}z^w}{w}, |z|<1$ in (25), we can obtain the following quantities:

$$\log\left(\bar{G}(x;\phi)\right) = -\sum_{w=1}^{\infty}\frac{G(x;\phi)^w}{w}$$
$$\log\left(1-\bar{G}(x;\phi)^2\right) = -\sum_{w=1}^{\infty}\sum_{j=0}^{2w}\frac{(-1)^j}{w}\binom{2w}{j}G(x;\phi)^j$$
$$\log\left\{1+\lambda\left(1-\bar{G}(x;\phi)^2\right)^{\theta}\right\} = -\sum_{w=1}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{2i}\frac{(-1)^{w+i+j-1}\lambda^w}{w}\binom{\theta w}{i}\binom{2i}{j}G(x;\phi)^j$$
$$\left\{1+\lambda\left(1-\bar{G}(x;\phi)^2\right)^{\theta}\right\}^{\alpha} = \sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{2i}(-1)^{i+j}\lambda^m\binom{\alpha}{m}\binom{\theta m}{i}\binom{2i}{j}G(x;\phi)^j$$

Inserting the previous quantities into (25) and then into (24), we arrive at

$$\Upsilon_x = -1 - \log\left(2\alpha\lambda\theta\right) - E\left(\log g(x;\phi)\right) + \sum_{w=1}^{\infty}\frac{E\left(G(x;\phi)^w\right)}{w}$$
$$+(\theta-1)\sum_{w=1}^{\infty}\sum_{j=0}^{2w}\frac{(-1)^j}{w}\binom{2w}{j}E\left(G(x;\phi)^j\right)$$
$$-(\alpha-1)\sum_{w=1}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{2i}\frac{(-1)^{w+i+j-1}\lambda^w}{w}\binom{\theta w}{i}\binom{2i}{j}E\left(G(x;\phi)^j\right)$$
$$+\sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{2i}(-1)^{i+j}\lambda^m\binom{\alpha}{m}\binom{\theta m}{i}\binom{2i}{j}E\left(G(x;\phi)^j\right). \tag{26}$$

4.5. Stress Strength Model

Let X_1 and X_2 be two independent random variables with $\text{NHTL-G}(\alpha_1,\lambda_1,\theta_1,\phi)$. and $\text{NHTL-G}(\alpha_2,\lambda_2,\theta_2,\phi)$. distributions. Then, the stress strength model is given by

$$R = \Pr(X_2 < X_1) = \int_0^{\infty} f_1(\alpha_1,\lambda_1,\theta_1,\phi)F_2(\alpha_2,\lambda_2,\theta_2,\phi)\,dx. \tag{27}$$

Equations (1) and (2) yield

$$f_1(\alpha_1,\lambda_1,\theta_1,\phi)F_2(\alpha_2,\lambda_2,\theta_2,\phi) = 2e\alpha_1\lambda_1\theta_1 g(x;\phi)\bar{G}(x;\phi)\left(1-\bar{G}(x;\phi)^2\right)^{\theta_1-1}$$

$$\times\left\{1+\lambda_1\left(1-\bar{G}(x;\phi)^2\right)^{\theta_1}\right\}^{\alpha_1-1} e^{-\left\{1+\lambda_1\left(1-\bar{G}(x;\phi)^2\right)^{\theta_1}\right\}^{\alpha_1}}$$

$$-2e^2\alpha_1\lambda_1\theta_1 g(x;\phi)\bar{G}(x;\phi)\left(1-\bar{G}(x;\phi)^2\right)^{\theta_1-1}$$

$$\times\left\{1+\lambda_1\left(1-\bar{G}(x;\phi)^2\right)^{\theta_1}\right\}^{\alpha_1-1}$$

$$\times e^{-\left\{1+\lambda_1\left(1-\bar{G}(x;\phi)^2\right)^{\theta_1}\right\}^{\alpha_1}-\left\{1+\lambda_2\left(1-\bar{G}(x;\phi)^2\right)^{\theta_2}\right\}^{\alpha_2}}$$

$$=2e\alpha_1\lambda_1\theta_1\sum_{w=0}^{\infty}\frac{(-1)^w}{w!} g(x;\phi)\bar{G}(x;\phi)$$

$$\times\left(1-\bar{G}(x;\phi)^2\right)^{\theta_1-1}\left\{1+\lambda_1\left(1-\bar{G}(x;\phi)^2\right)^{\theta_1}\right\}^{\alpha_1(w+1)-1}$$

$$-2e^2\alpha_1\lambda_1\theta_1\sum_{w=0}^{\infty}\sum_{h=0}^{\infty}\frac{(-1)^{w+h}}{w!h!}g(x;\phi)\bar{G}(x;\phi)$$

$$\times\left(1-\bar{G}(x;\phi)^2\right)^{\theta_1-1}\left\{1+\lambda_1\left(1-\bar{G}(x;\phi)^2\right)^{\theta_1}\right\}^{\alpha_1(w+1)-1}$$

$$\times\left\{1+\lambda_2\left(1-\bar{G}(x;\phi)^2\right)^{\theta_2}\right\}^{\alpha_2 h}$$

$$=2e\alpha_1\theta_1\sum_{w=0}^{\infty}\sum_{c=0}^{\infty}\frac{(-1)^w\lambda_1^{c+1}}{w!}\binom{\alpha_1(w+1)-1}{c}g(x;\phi)$$

$$\times\bar{G}(x;\phi)\left(1-\bar{G}(x;\phi)^2\right)^{\theta_1(c+1)-1}$$

$$-2e^2\alpha_1\theta_1\sum_{w=0}^{\infty}\sum_{h=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\frac{(-1)^{w+h}\lambda_1^{c+1}\lambda_2^{d}}{w!h!}$$

$$\times\binom{\alpha_1(w+1)-1}{c}\binom{\alpha_2 h}{d}g(x;\phi)$$

$$\times\bar{G}(x;\phi)\left(1-\bar{G}(x;\phi)^2\right)^{\theta_1(c+1)+\theta_2 d-1}$$

$$=2e\alpha_1\theta_1\sum_{w=0}^{\infty}\sum_{c=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{(-1)^{w+\ell}\lambda_1^{c+1}}{w!}$$

$$\times\binom{\alpha_1(w+1)-1}{c}\binom{\theta_1(c+1)-1}{\ell}g(x;\phi)\bar{G}(x;\phi)^{2\ell+1}$$

$$-2e^2\alpha_1\theta_1\sum_{w=0}^{\infty}\sum_{h=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{(-1)^{w+h+\ell}\lambda_1^{c+1}\lambda_2^{d}}{w!h!}$$

$$\times\binom{\alpha_1(w+1)-1}{c}\binom{\alpha_2 h}{d}\binom{\theta_1(c+1)+\theta_2 d-1}{\ell}$$

$$\times g(x;\phi)\bar{G}(x;\phi)^{2\ell+1}$$

$$= 2e\alpha_1\theta_1 \sum_{w=0}^{\infty}\sum_{c=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=0}^{2\ell+1} \frac{(-1)^{w+\ell+m}\lambda_1^{c+1}}{w!} \binom{\alpha_1(w+1)-1}{c} \times \binom{\theta_1(c+1)-1}{\ell}\binom{2\ell+1}{m} g(x;\phi)G(x;\phi)^m$$
$$-2e^2\alpha_1\theta_1 \sum_{w=0}^{\infty}\sum_{h=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{w+h+\ell+m}\lambda_1^{c+1}\lambda_2^{d}}{w!h!} \times \binom{\alpha_1(w+1)-1}{c}\binom{\alpha_2 h}{d}\binom{\theta_1(c+1)+\theta_2 d-1}{\ell} \times \binom{2\ell+1}{m} g(x;\phi)G(x;\phi)^m$$

Or

$$f_1(\alpha_1,\lambda_1,\theta_1,\phi)F_2(\alpha_2,\lambda_2,\theta_2,\phi) = \sum_{m=0}^{2\ell+1}\left(\varepsilon_m + \Omega_m\right)\pi_{m+1}(x), \tag{28}$$

where, $\varepsilon_m = 2e\alpha_1\theta_1 \sum_{w=0}^{\infty}\sum_{c=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{(-1)^{w+\ell+m}\lambda_1^{c+1}}{(m+1)\,w!}\binom{\alpha_1(w+1)-1}{c}\binom{\theta_1(c+1)-1}{\ell}\binom{2\ell+1}{m}$,

and $\Omega_m = 2e^2\alpha_1\theta_1 \sum_{w=0}^{\infty}\sum_{h=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{\ell=0}^{\infty}\binom{\alpha_1(w+1)-1}{c}\binom{\alpha_2 h}{d}\binom{\theta_1(c+1)+\theta_2 d-1}{\ell}\binom{2\ell+1}{m}$.

Inserting (28) into (27), we arrive at

$$R = \sum_{m=0}^{2\ell+1}\left(\varepsilon_m + \Omega_m\right). \tag{29}$$

5. Estimation of Parameters

This section deals with the maximum likelihood estimates (MLEs) of the model parameters of the NHTL-G distribution from complete samples. Let $x_1, x_2, ..., x_n$ be an observed values of a random sample from the NHTL-G family with parameter vector $\Phi = (\alpha,\lambda,\theta,\phi)^T$, then the log-likelihood function is given by

$$\ell = n\left\{1+\log(2)+\log(\alpha)+\log(\lambda)+\log(\theta)\right\} + \sum_{i=1}^{n}\log\left(g(x_i,\phi)\right) + \sum_{i=1}^{n}\log\left(\bar{G}(x_i,\phi)\right) + (\theta-1)\sum_{i=1}^{n}\log(a_i) - \sum_{i=1}^{n}\left\{1+\lambda a_i^{\theta}\right\}^{\alpha} + (\alpha-1)\sum_{i=1}^{n}\log\left\{1+\lambda a_i^{\theta}\right\}, \tag{30}$$

where, $a_i = 1-\bar{G}(x_i,\phi)^2$.

By differentiating (30) with respect to α, λ, θ and ϕ, we obtain the following nonlinear system of equations

$$\frac{\partial \ell}{\partial \alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\log\left\{1+\lambda a_i^{\theta}\right\}-\sum_{i=1}^{n}\left\{\left(1+\lambda a_i^{\theta}\right)\log\left(1+\lambda a_i^{\theta}\right)\right\} \tag{31}$$

$$\frac{\partial \ell}{\partial \lambda}=\frac{n}{\lambda}+\left(\alpha-1\right)\sum_{i=1}^{n}\left\{\frac{a_i^{\theta}}{1+\lambda a_i^{\theta}}\right\}-\alpha\sum_{i=1}^{n}\left\{a_i^{\theta}\left(1+\lambda a_i^{\theta}\right)^{\alpha-1}\right\}, \tag{32}$$

$$\frac{\partial \ell}{\partial \theta}=\frac{n}{\theta}+\sum_{i=1}^{n}\log\left(a_i\right)++\lambda\left(\alpha-1\right)\sum_{i=1}^{n}\left\{\frac{a_i^{\theta}\log\left(a_i\right)}{1+\lambda a_i^{\theta}}\right\}$$
$$-\alpha\lambda\sum_{i=1}^{n}\left\{a_i^{\theta}\left(1+\lambda a_i^{\theta}\right)^{\alpha-1}\log\left(a_i\right)\right\} \tag{33}$$

and

$$\frac{\partial \ell}{\partial \phi}=\sum_{i=1}^{n}\left(\frac{g'(x_i,\phi)}{g(x_i,\phi)}\right)-\sum_{i=1}^{n}\left(\frac{G'(x_i,\phi)}{\bar{G}(x_i,\phi)}\right)+2\left(\theta-1\right)\sum_{i=1}^{n}\left\{\frac{\bar{G}(x_i,\phi)G'(x_i,\phi)}{a_i}\right\}$$
$$+2\lambda\theta\left(\alpha-1\right)\sum_{i=1}^{n}\left\{\frac{\bar{G}(x_i,\phi)G'(x_i,\phi)a_i^{\theta-1}}{1+\lambda a_i^{\theta}}\right\}$$
$$-2\lambda\theta\alpha\sum_{i=1}^{n}\left\{\bar{G}(x_i,\phi)G'(x_i,\phi)a_i^{\theta-1}\left(1+\lambda a_i^{\theta}\right)^{\alpha-1}\right\}. \tag{34}$$

where, $g'(x_i,\phi)=\partial g(x_i,\phi)/\partial\phi$ and $G'(x_i,\phi)=\partial G(x_i,\phi)/\partial\phi$.

Equating (31), (32), (33) and (34) to zero and solving for α,λ,θ and ϕ simultaneously yields the MLEs, say $\hat{\Phi}=(\hat{\alpha},\hat{\lambda},\hat{\theta},\hat{\phi})$ of $\Phi=(\alpha,\lambda,\theta,\phi)^T$. The components of the observed information matrix are derived in Appendix A.

6. Applications

In this section, we provide two applications to real data by using the NHTLLx distribution, which serve to illustrate the importance of the NHTL-G family.

The first real data set consists of 63 observations of the strengths of 1.5 cm glass fibres which obtained by workers at the UK National Physical Laboratory. The data are: 0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49,1.49, 1.50, 1.50, 1.51, 1.52, 1.53,1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61,1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68,1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76,1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24.

The second data set represents 13 observations of cotton average yield in different years in Lodhran district in Pakistan, see Beig (2017). The data are: 26.90, 24.16, 21.52, 16.75, 19.37, 17.46, 23.97, 21.92, 21.38, 23.73, 12.50, 20.69, 20.00. These two datasets are recently studied by (Reyad et al. 2018b).

Initially, we fit the NHTLLx model to these data sets and compare it with some competitive models such as; beta Lomax (BLx) (Lemonte and Cordeiro 2013), Kumaraswamy Lomax (KwLx) (Shams 2013), exponentiated Lomax (ELx) (Abdul-Moniem and Abdel-Hameed 2012), Odd-Lindley-Lomax (OLLx) (Gomes-Silva et al. 2017), Topp-Leone Lomax (TLLx) (Al-Shomrani et al. 2016), Lomax (Lx) and Nadarajah Haghighi (NH) (Nadarajah and Haghighi 2011) distributions.

In order to compare these models and to verify the quality of the fits, we consider some of the well-known goodness-of-fit statistics like, Cramér-von Mises (W^*), Anderson Darling (A^*), Kolmogorov-Smirnov (KS), -Loglikelihood ($-L$), Akaike Information

Criterion (AIC), Consistent Akaike Information Criterion ($CAIC$), Bayesian information criterion (BIC) and Hannan-Quinn criterion (HQC). The model with a minimum value of Goodness-of-Fit statistics is the best model to fit the data. The required numerical evaluations are implemented using the BFGS quasi-Newton method in R software.

The results of the MLEs and their standard errors of the fitted models for the two data sets are displayed in Tables 1 and 3 respectively. In addition, the values of W^*, A^*, $KS, -L, AIC, CAIC, BIC$ and $HOIQ$ statistics of all fitted models for the two data sets are displayed in Tables 2 and 4 respectively.

Table 1: The MLEs and their standard errors (in parentheses) for the first data set

Models	Estimates				
	a	b	α	λ	θ
NHTLLx	0.9152 (1.1542)	0.2090 (0.2762)	25.0302 (46.8917)	5.9320 (36.5898)	6.1208 (8.1393)
BLx	17.8478 (3.4499)	40.1913 (60.2638)	77.5816 (222.1545)	19.3027 (41.0907)	---
KwLx	27.1531 (67.0525)	46.1506 (125.0250)	9.0710 (2.3638)	91.7567 (89.5981)	---
ELx	43.2618 (53.4628)	116.1829 (139.3477)	32.7701 (10.3498)	---	---
OLLx	1.8079 (0.1482)	79.9901 (22.3521)	29.5177 (8.8291)	---	---
TLLx	44.4017 (54.9352)	59.6955 (72.1453)	32.8566 (10.1778)	---	---
Lx	211.2298 (322.7742)	140.5679 (214.5268)	---	---	---
NH	139.801 (66.6059)	0.0037 (0.0017)	---	---	---

Table 2: Goodness of fit statistics for all fitted models for the first data set.

Models	A^*	W^*	*-L*	AIC	CAIC	BIC	HQC	K–S	P-value
NHTLLx	1.2323	0.2207	15.5482	41.097	42.149	51.8122	45.311	0.1548	0.0974
BLx	3.1526	0.57484	24.1456	56.291	56.981	64.864	59.663	0.2165	0.0054
KwLx	1.7966	0.3281	17.2688	42.538	43.227	51.11	45.909	0.1731	0.0457
ELx	4.3439	0.7969	31.7373	69.475	69.881	75.904	72.003	0.2285	0.0027
OLLx	2.4574	0.4481	69.9755	145.951	146.358	152.38	148.48	0.3724	0.0000
TLLx	4.3452	0.7972	31.7284	69.457	69.864	75.886	71.986	0.2300	0.0025
Lx	3.1354	0.5717	89.0439	182.088	182.288	186.374	183.774	0.4179	0.000
NH	2.3152	0.4223	68.7617	141.523	141.723	145.810	143.209	0.4519	0.000

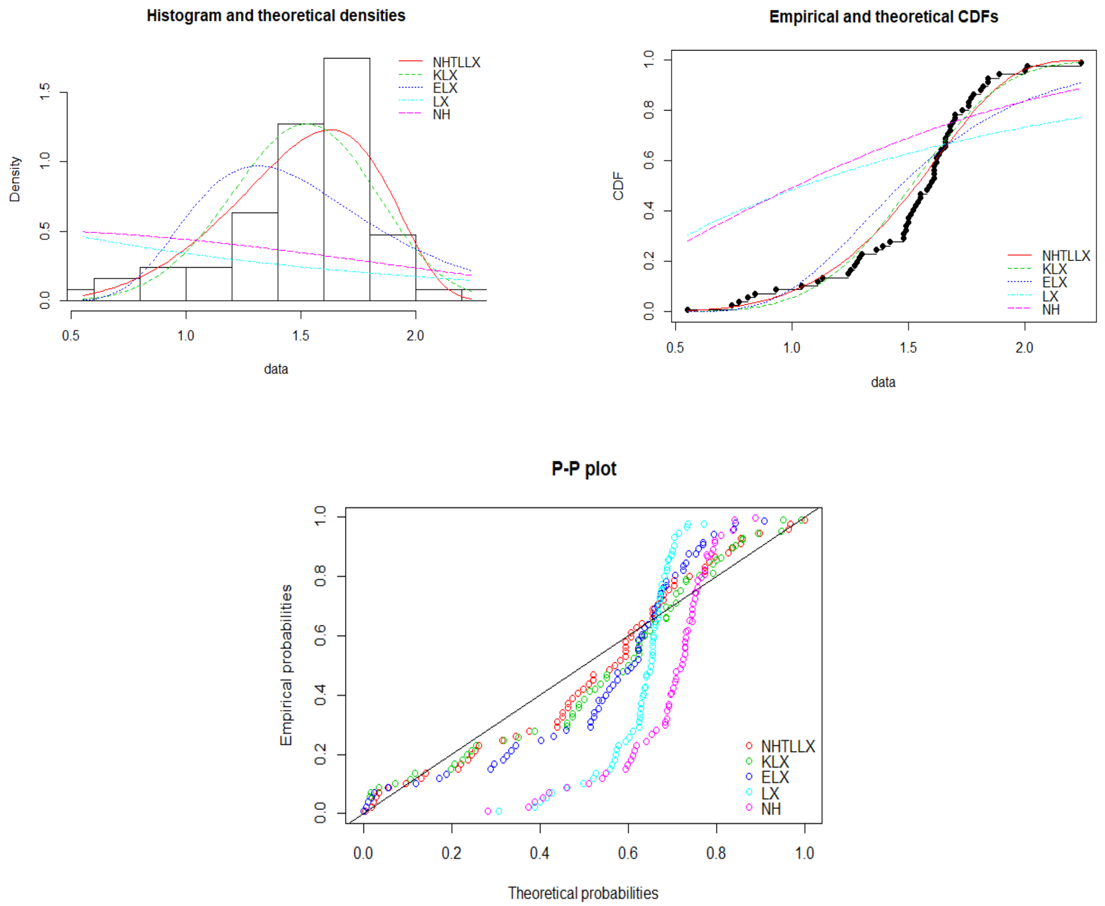

Figure. 4: Histogram and estimated densities (left upper panel); Theoretical and estimated CDFs (right upper panel); P-P plots (lower panel) for the first data set.

Table 3: The MLEs and their standard errors (in parentheses) for the second data set

Models	Estimates				
	a	b	α	λ	θ
NHTLLx	3.6485 (4.9110)	0.0146 (0.0021)	3.6599 (6.4367)	3.7324 (15.4892)	20.9750 (32.7064)
BLx	36.2399 (28.3572)	30.4202 (93.6734)	58.1401 (147.5679)	2.5363 (7.2874)	---
KwLx	2.8402 (1.3255)	20.2466 (13.6450)	18.5246 (9.4360)	9.4511 (6.1925)	---
ELx	232.6174 (511.1264)	59.5700 (121.6153)	115.4825 (109.1986)	---	---
OLLx	12.6601 (4.1220)	11.2461 (9.2156)	0.0785 (2.3131)	---	---
TLLx	188.5994 (379.5414)	24.5373 (45.1973)	119.1881 (114.7701)	---	---
Lx	235.7117 (229.6951)	11.3362 (10.8514)	---	---	---
NH	9.7418 (6.7881)	0.0040 (0.0028)	---	---	---

Table 4: Goodness of fit statistics for all fitted models for the second data set.

Model	A^*	W^*	-*L*	AIC	CAIC	BIC	HQC	K–S	P-value
NHTLLx	0.2175	0.0311	34.8134	79.627	88.198	82.452	79.046	0.1012	0.9973
BLx	0.3986	0.0648	40.0340	88.068	93.068	90.328	87.604	0.1745	0.7248
KwLx	0.3613	0.0591	40.9954	89.991	94.991	92.251	89.526	0.1979	0.5758
ELx	0.6169	0.0985	41.2033	88.407	91.073	90.101	88.058	0.1993	0.5672
OLLx	0.4397	0.0710	62.9155	131.831	134.498	133.526	131.483	0.5111	0.0006
TLLx	0.6215	0.0993	41.2299	88.46	91.126	90.155	88.111	0.1996	0.5652
Lx	0.3687	0.0602	57.5090	119.018	120.218	120.148	118.786	0.4693	0.0023
NH	0.3090	0.0448	48.6385	101.277	102.477	102.407	101.045	0.5094	0.0012

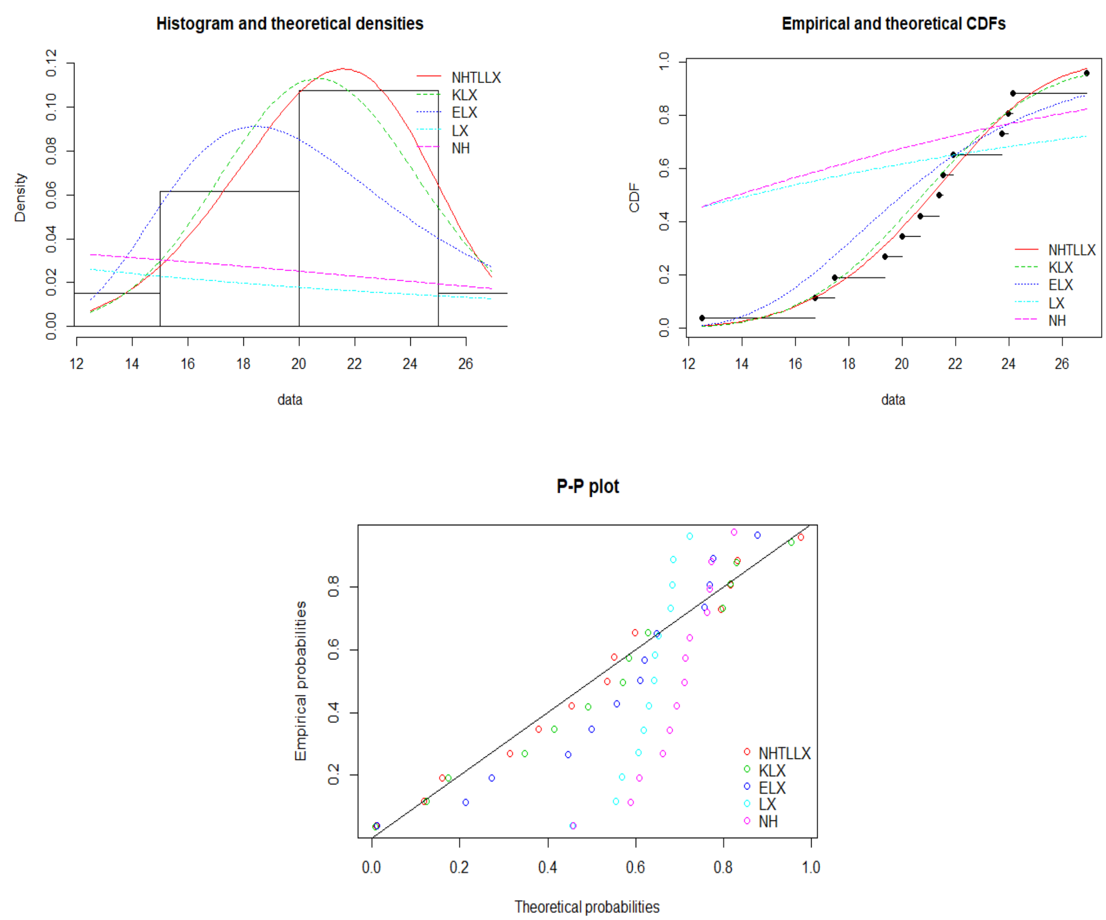

Figure. 5: Histogram and estimated densities (left upper panel); Theoretical and estimated CDFs (right upper panel); P-P plots (lower panel) for the second data set.

The values in Tables 2, 4 show that the NHTLLx distribution gives the smallest values for the Goodness-of-Fit statistics and the greatest of p-value of KS-test. This means that NHTLLx distribution provides better fit than the rest of the distributions. Additionally, Figures 4, 5 display the plots of fitted and empirical densities of competitive distributions for datasets I and II, respectively. These figures confirm that the NHTLLx distribution is more appropriate to represent the two datasets than the rest distributions.

7. Conclusions

We propose and study a new generator of continuous distributions, also obtained. We introduce two real applications to show the importance of the new family called the Nadarajah Haghighi Topp Leone-G family based on the Nadarajah Haghighi distribution, the T-X and TL-G classes. We provide three sub-models corresponding to the new family and derive an useful expansion of its density and distribution functions in terms of the exponential-G class. Moreover, we introduc an extensive treatment of the mathematical properties of the NHTL-G family such as quantile function, raw and incomplete moments, order statistics, stress strenght model, generating functions, entropies and probability weighted moments. The model parameters of the new family are estimated via maximum likelihood estimation and the observed information matrix is obtained. Two real applications are provided to show the effectivness of the new family. Numerical calculations showed that some sub-models of the NHTL-G family can give better fit than similar models generated by other families.

References

1. Aldahlan, M. A., Afify, A, Z. and Ahmed, A. N. (2019). The odd inverse Pareto-G class: properties and applications. Journal of Nonlinear Sciences and Applications; 12: 278-290.
2. Alizadeh, M., Tahmasebi, S. and Haghbin, H. (2018). The exponentiated odd log-Logistic family of distributions: properties and applications. Journal of Statistical Modeling:Theory and Applications; 1(2): 1-24.
3. Alzaatreh, A, Lee, C. and Famoye, F. (2013). A new method for generating families of continuous distributions. Metron; 71(1):63–79.
4. Al-Shomrani, A., Arif, O., Shawky, K., Hanif, S. and Shahbaz, M. Q. (2016). Topp–Leone family of distributions: some properties and application, Pak.j.stat.oper.res.; 12:443-451.
5. Abdul-Moniem, I., and H. Abdel-Hameed. (2012). ON exponentiated Lomax distribution. International Journal of Mathematical Archive EISSN 2229-5046 3 (5).
6. Beig, M. A. (2017). Crop Reporting Service, Multan Road, Lahore, Pakistan. Director crop Reporting Service www.crs.agripunjab.gov.pk.
7. Cordeiro, G. M., Afify, A. Z., Ortega, E. M. M., Suzuki, A. K. and Mead, M. E. (2019). The odd Lomax generator of distributions: properties, estimation and applications. Journal of Computational and Applied Mathematics; 347:222-237.
8. Lemonte, A., Cordeiro, G. (2013). An extended lomax distribution, Statistics, 47: 800-816.
9. Gomes-Silva, F. S., A. Percontini, E. de Brito, M. W. Ramos, R. Venâncio, and Cordeiro, G. M. (2017). The odd Lindley-G family of distributions. Austrian Journal of Statistics 46 (1):65-87.
10. Korkmaz, M. C., Alizadeh, M., Yousof, H.M. and Butt, N. S. (2018b). The generalized odd Weibull generated family of distributions: statistical properties and applications. Pak.j.stat.oper.res. vol.XIV. No. 3:541-556.

11. Korkmaz, M. C., Yousof, H. M., Hamedani, G. G. and Ali, M. M. (2018a). The Marshall-Olkin generalized G Poison family of distributions. Pakistan Journal of Statistics; 34(3):251-267.
12. Mazed, N., Kumar, D., Dey, S. and Cordeiro, G. M. (2019). The Marshall-Plkin alpha power family of distributions with applications. Journal of Computational and Applied Mathematics; 351: 41-53.
13. Nadarajah, S., and F. Haghighi. (2011). An extension of the exponential distribution. Statistics 45 (6):543-558.
14. Reyad, H., Jamal, F., Othman, S. and Hamedani, G. G. (2018a). The transmuted Gompertz-G family of distributions: properties and applications. Tbilist Mathematical Journal; 11(3): 47-67.
15. Reyad, H., Alizadeh, M., Jamal, F. and Othman, S. (2018b). The Topp Leone odd Lindley-G family of distributions: properties and applications. Journal of Statistics and Management Systems; 21(7): 1273-1297.
16. Reyad, H., Alizadeh, M., Jamal, F., Othman, S. and Hamedani, G. G (2019b). The exponentiated generalized Topp Leone-G family of distributions: theory and applications. Pak.j.stat.res. XV (1); 1-24.
17. Reyad, H., Othman, S. and Ul Haq, M. A. (2019a). The transmuted generalized odd generalized exponential-G family of distributions: theory and applications. Journal of Data science; 17(2):279-300.
18. Shams, T. (2013). The Kumaraswamy-generalized Lomax distribution. Middle-East Journal of Scientific Research; 17 (5):641-646.
19. Yousof, H. M., Afify, A. Z., Alizadeh, M., Hamedani, G. G., Jahanshahi, S. M. A. and Ghosh, I. (2018). The generalized transmuted Poisson-G family of distributions. Pak.j.stat.res. XIV (4); 759-779.
20. Yousof, H. M., Rasekhi, M., Altun, E., & Alizadeh, M. (2019). The extended odd Fréchet family of distributions: properties, applications and regression modeling. Int J Math Comput, 30, 1-16.

Appendix A

The observed information matrix has the following components:

$$\frac{\partial^2 \ell}{\partial \alpha^2} = \frac{-n}{\alpha^2} - \sum_{i=1}^{n} \left\{ \left(1+\lambda a_i^{\theta}\right)^{\alpha} \left[\log\left(1+\lambda a_i^{\theta}\right) \right]^2 \right\},$$

$$\frac{\partial^2 \ell}{\partial \lambda \, \partial \alpha} = \sum_{i=1}^{n} \left\{ \frac{a_i^{\theta}}{1+\lambda a_i^{\theta}} \right\} - \sum_{i=1}^{n} \left\{ a_i^{\theta} \left(1+\lambda a_i^{\theta}\right)^{\alpha-1} \left[1+\log\left(1+\lambda a_i^{\theta}\right)^{\alpha} \right] \right\},$$

$$\frac{\partial^2 \ell}{\partial \theta \, \partial \alpha} = \lambda \sum_{i=1}^{n} \left\{ \frac{a_i^{\theta} \log\left(a_i\right)}{1+\lambda a_i^{\theta}} \right\} - \sum_{i=1}^{n} \left\{ a_i^{\theta} \left(1+\lambda a_i^{\theta}\right)^{\alpha-1} \log\left(a_i\right) \left[1+\log\left(1+\lambda a_i^{\theta}\right)^{\alpha} \right] \right\},$$

$$\frac{\partial^2 \ell}{\partial \phi \, \partial \alpha} = 2\lambda\theta \sum_{i=1}^{n} \left\{ \frac{\bar{G}(x_i,\phi) G'(x_i,\phi) a_i^{\theta-1}}{1+\lambda a_i^{\theta}} \right\}$$

$$-2\lambda\theta\sum_{i=1}^{n}\left\{\bar{G}(x_i,\phi)G'(x_i,\phi)a_i^{\theta-1}\left(1+\lambda a_i^{\theta}\right)^{\alpha-1}\left[1+\log\left(1+\lambda a_i^{\theta}\right)^{\alpha}\right]\right\},$$

$$\frac{\partial^2\ell}{\partial\lambda^2}=\frac{-n}{\lambda^2}-(\alpha-1)\sum_{i=1}^{n}\left\{\frac{a_i^{2\theta}}{\left(1+\lambda a_i^{\theta}\right)^2}\right\}-\alpha(\alpha-1)\sum_{i=1}^{n}\left\{a_i^{2\theta}\left(1+\lambda a_i^{\theta}\right)^{\alpha-2}\right\},$$

$$\frac{\partial^2\ell}{\partial\theta\,\partial\lambda}=(\alpha-1)\sum_{i=1}^{n}\left\{\frac{a_i^{\theta}\log(a_i)}{\left(1+\lambda a_i^{\theta}\right)^2}\right\}+(\alpha-1)\sum_{i=1}^{n}\left\{a_i^{\theta}\left\{1+(\lambda+\alpha-1)a_i^{\theta}\right\}\log(a_i)\left(1+\lambda a_i^{\theta}\right)^{\alpha-2}\right\},$$

$$\frac{\partial^2\ell}{\partial\phi\,\partial\lambda}=2\theta(\alpha-1)\sum_{i=1}^{n}\left\{\frac{\bar{G}(x_i,\phi)G'(x_i,\phi)a_i^{\theta-1}}{\left(1+\lambda a_i^{\theta}\right)^2}\right\}$$

$$-2\alpha\theta\sum_{i=1}^{n}\left\{\bar{G}(x_i,\phi)G'(x_i,\phi)a_i^{\theta-1}\left(1+\lambda a_i^{\theta}\right)^{\alpha-2}\left(1+\alpha\lambda a_i^{\theta}\right)\right\},$$

$$\frac{\partial^2\ell}{\partial\theta^2}=\frac{-n}{\theta^2}+\lambda(\alpha-1)\sum_{i=1}^{n}\left\{\frac{a_i^{\theta}\left(\log(a_i)\right)^2}{\left(1+\lambda a_i^{\theta}\right)^2}\right\}-\lambda\alpha\sum_{i=1}^{n}\left\{a_i^{\theta}\left(\log(a_i)\right)^2\left(1+\alpha\lambda a_i^{\theta}\right)\left(1+\lambda a_i^{\theta}\right)^{\alpha-2}\right\},$$

$$\frac{\partial^2\ell}{\partial\phi\,\partial\theta}=2\sum_{i=1}^{n}\left\{\frac{\bar{G}(x_i,\phi)G'(x_i,\phi)}{a_i}\right\}+2\lambda(\alpha-1)\sum_{i=1}^{n}\left\{\frac{\bar{G}(x_i,\phi)G'(x_i,\phi)a_i^{\theta-1}\left[1+\lambda a_i^{\theta}+\theta\log(a_i)\right]}{\left(1+\lambda a_i^{\theta}\right)^2}\right\}$$

$$-2\alpha\lambda\sum_{i=1}^{n}\left\{\bar{G}(x_i,\phi)G'(x_i,\phi)a_i^{\theta-1}\left(1+\lambda a_i^{\theta}\right)^{\alpha-2}\left[1+\theta+\lambda(1+\theta\alpha)a_i^{\theta}\right]\right\},$$

$$\frac{\partial^2\ell}{\partial\phi^2}=\sum_{i=1}^{n}\left(\frac{g(x_i,\phi)g''(x_i,\phi)-g'(x_i,\phi)^2}{g(x_i,\phi)^2}\right)-\sum_{i=1}^{n}\left(\frac{\bar{G}(x_i,\phi)G''(x_i,\phi)+G'(x_i,\phi)^2}{\bar{G}(x_i,\phi)^2}\right)$$

$$-2(\theta-1)\sum_{i=1}^{n}\left\{\frac{a_i\,b_i-2c_i}{a_i^2}\right\}+2\lambda\theta(\alpha-1)\sum_{i=1}^{n}\left\{\frac{a_i^{\theta-2}\,z_i}{\left(1+\lambda a_i^{\theta}\right)^2}\right\}$$

$$+2\lambda\theta\alpha\sum_{i=1}^{n}\left\{a_i^{\theta-2}\left(\rho_i-v_i\right)\left(1+\lambda a_i^{\theta}\right)^{\alpha-2}\right\}$$

where,

$b_i=\bar{G}(x_i,\phi)G''(x_i,\phi)-G'(x_i,\phi)^2,$

$c_i=\bar{G}(x_i,\phi)^2G'(x_i,\phi)^2,$

$z_i=\left(1+\lambda a_i^{\theta}\right)\left[2(\theta-1)c_i+a_i\,b_i\right]-2\lambda\theta c_i\,a_i^{\theta},$

$\rho_i=2c_i\left\{1-\theta+\lambda(1-\theta\alpha)a_i^{\theta}\right\},$

$v_i=a_i\,b_i\left(1+\lambda a_i^{\theta}\right),$

$g''(x_i,\phi)=\partial^2 g(x_i,\phi)/\partial\phi^2$

and

$G''(x_i,\phi)=\partial^2 G(x_i,\phi)/\partial\phi^2$.