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A New Compound Version of the Generalized Lomax Distribution for Modeling Failure and Service Times

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Abstract

The main goal of this article is to introduce a new extension of the continuous Lomax distribution with a strong physical motivation. Some of its statistical properties such as moments, incomplete moments, moment generating function, quantile function, random number generation, quantile spread ordering and moment of the reversed residual life are derived. Two applications are provided to illustrate the importance and flexibility of the new model.

Key Words: Maximum Likelihood Estimation; Quantile Function; Generating Function; Moments; Zero Truncated Poisson.

Mathematical Subject Classification: 62E10; 62N01; 62N02.

1. Introduction

1.Introduction and physical motivation

In this work, we develop and study a new univariate extension of the generalized Lomax (GLx) model by compounding the Rayleigh generalized Lomax (RGLx) model with the zero truncated Poisson (ZTP) distribution. Due to Yousof et al. (2017), a random variable (r.v.) W is said to have the RGLx distribution if its cumulative distribution function (CDF) is given

$$F_{RGLx}^{(c,a,b)}(w) = 1 - exp\left[-\left(\left\{1-\left[1+\left(\frac{w}{b}\right)\right]^{-a}\right\}^{-c} - 1\right)^{-2}\right], \tag{1}$$

where the CDF and probability density function (PDF) of the GLx distribution are

$$\Pi_c(a,b) = G_{GLx}^{(c,a,b)}(w) = \left\{1-\left[1+\left(\frac{w}{b}\right)\right]^{-a}\right\}^{c},$$

and

$$\pi_c(a,b) = g_{GLx}^{(c,a,b)}(w) = cab^{-1}\left[1+\left(\frac{w}{b}\right)\right]^{-a-1}\left\{1-\left[1+\left(\frac{w}{b}\right)\right]^{-a}\right\}^{c-1},$$

respectively.

For $c = 1$, we have the standard two-parameters Lx distribution (see Lomax (1954)). When $c = b = 1$, we have the standard one-parameters Lx distribution. The probability mass function (PMF) of U (where $U \sim \text{ZTP}(\lambda)$) is given by

$$PMF_{ZTP}^{(\lambda)}(U|U=n) = [exp(-\lambda)\lambda^n]/\big(n!\,\Upsilon_{[\lambda]}\big)|_{(n=1,2,\dots)}, \tag{2}$$

where $\Upsilon_{[\lambda]} = [-exp(-\lambda)+1]$. Then the conditional CDF of W given U where

$$W|U = min\{Y_1, Y_2, \cdots, Y_U\},$$

using (1) is

$$F(w \mid U) = 1 - Pr(W > w \mid U) = -\left[1 - F_{RGLx}^{(c,a,b)}(w)\right]^U + 1. \tag{3}$$

The unconditional CDF of the PRGLx PDF can be written as

$$F(w) = F_{PRGLx}^{(\lambda,c,a,b)}(w) = \frac{1}{\Upsilon_{[\lambda]}}\left[1 - exp\left(-\lambda\left\{1 - exp\left[-\left(\left\{1-\left[1+\left(\frac{w}{b}\right)\right]^{-a}\right\}^{-c} - 1\right)^{-2}\right]\right\}\right)\right], \tag{4}$$

with corresponding PDF as

$$f(w) = f_{PRGLx}^{(\lambda,c,a,b)}(w) = \frac{2\lambda ca\left\{1-\left[1+\left(\frac{w}{b}\right)\right]^{-a}\right\}^{2c-1}}{\Upsilon_{[\lambda]}b\left(1-\left\{1-\left[1+\left(\frac{w}{b}\right)\right]^{-a}\right\}^{c}\right)^{3}}$$
$$\times\left[1+\left(\frac{w}{b}\right)\right]^{-a-1} exp\left[-\left(\left\{1-\left[1+\left(\frac{w}{b}\right)\right]^{-a}\right\}^{-c} - 1\right)^{-2}\right]$$
$$\times\, exp\left(-\lambda\left\{1 - exp\left[-\left(\left\{1-\left[1+\left(\frac{w}{b}\right)\right]^{-a}\right\}^{-c} - 1\right)^{-2}\right]\right\}\right). \tag{5}$$

The plots of the PRGLx PDF and HRF are displayed in Fig.s 1 and 2 for selected parameter values. From Fig.s 1 and 2 we conclude that the PRGLx distribution may be suitable in modeling right skewed, left skewed, symmetric and unimodal data sets. The HRF of the PRGLx may have the increasing, **J**-shapes, upside down then bathtub, upside down then increasing, upside down and bathtub (**U**-shape).

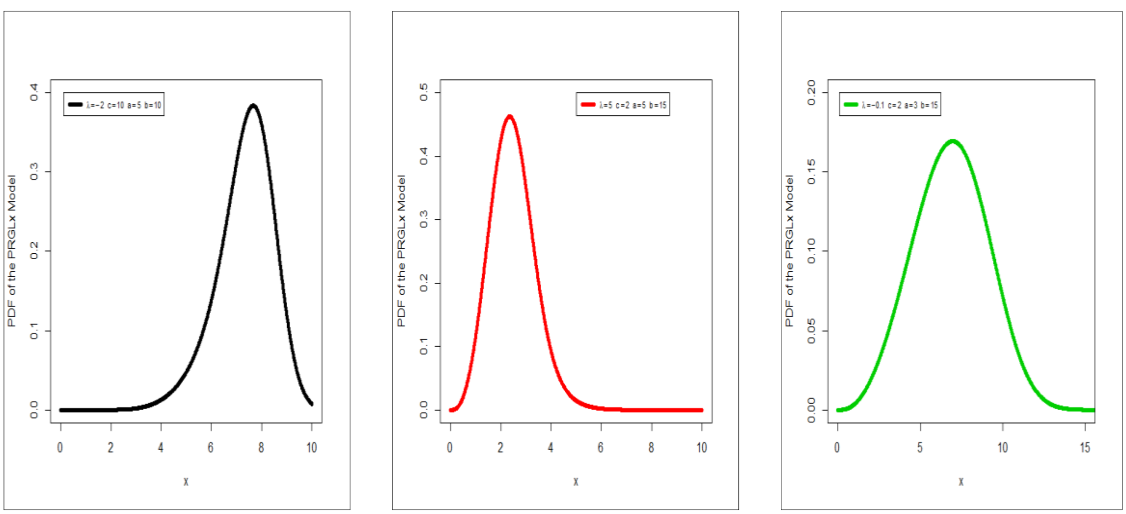

Figure 1: Some PDF plots for the PRGLx model.

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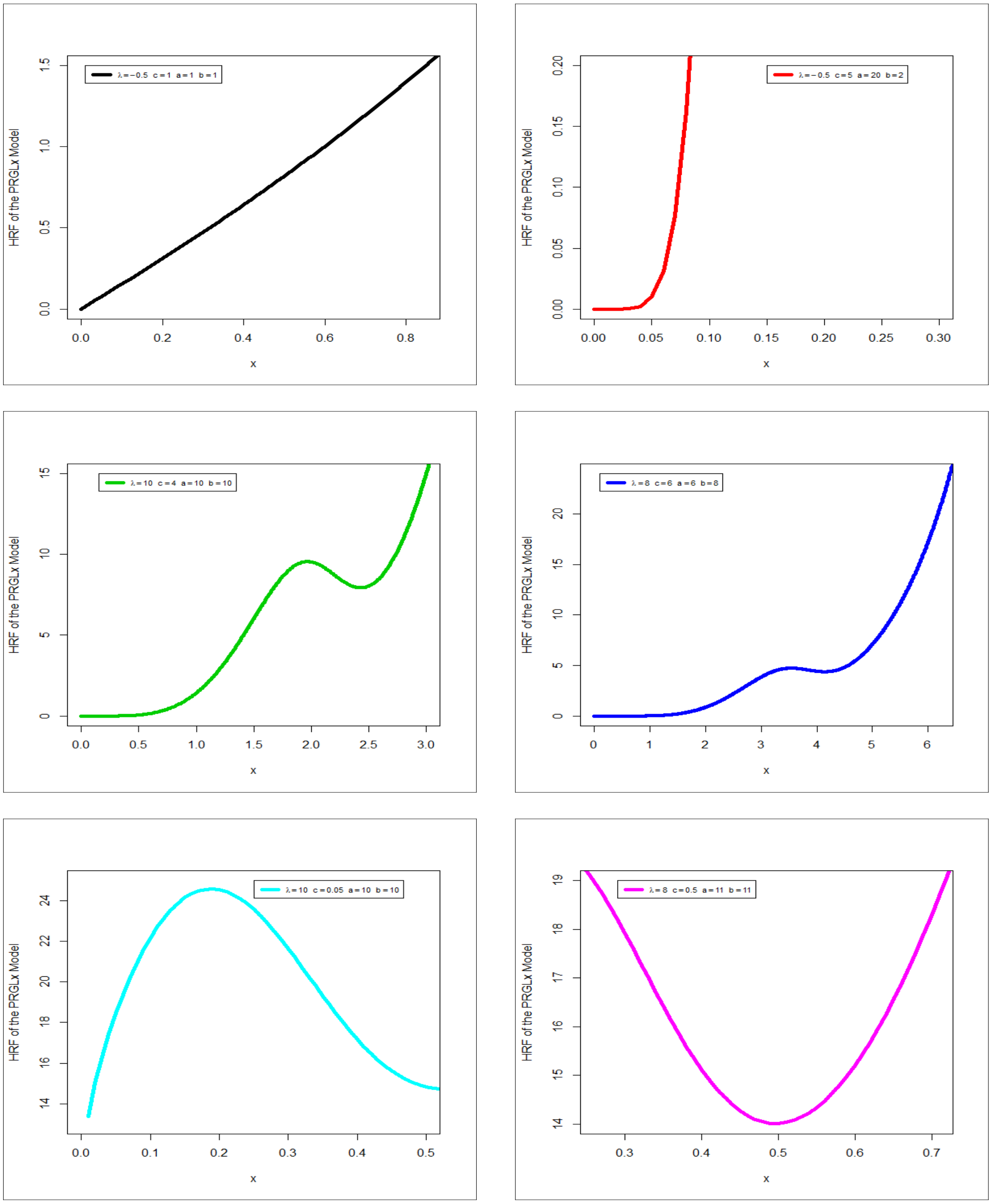

Figure 2: Some HRF plots for the PRGLx model.

Some useful versions of the Lx model have been recently developed by Yousof et al. (2015), Brito et al. (2017), Cordeiro et al. (2018), Altun et al. (2018a), Altun et al. (2018b), Yousof et al. (2018a-e, 2019a-b), Ibrahim et al. (2019), Gad et al. (2019) and Goual and Yousof (2019), Hamedani et al. (2017, 2018, 2019), Ibrahim, M. (2019, 2020a-b) Korkmaz et al. (2017, 2018, 2019), and Ibrahim et al. (2019, 2020).

2. Mathematical properties

2.1 Useful expansions

Using the power series

$$exp(\varphi_1) = \sum_{p=0}^{\infty} \frac{1}{p!} {\varphi_1}^p, \tag{6}$$

the PDF in (6) can be written as

$$\begin{aligned} f(w) &= 2ca \sum_{\zeta_3=0}^{\infty} \lambda^{1+\zeta_3} \frac{(-1)^{\zeta_3}}{\zeta_3!\, b\Upsilon_{[\lambda]}} \\ &\times \frac{\left[1+\left(\frac{w}{b}\right)\right]^{-a-1} \left\{1-\left[1+\left(\frac{w}{b}\right)\right]^{-a}\right\}^{2c-1}}{\left(1-\left\{1-\left[1+\left(\frac{w}{b}\right)\right]^{-a}\right\}^{c}\right)^{3}} \\ &\times exp\left[-\left(\left\{1-\left[1+\left(\frac{w}{b}\right)\right]^{-a}\right\}^{-c}-1\right)^{-2}\right] \\ &\times \left\{1-exp\left[-\left(\left\{1-\left[1+\left(\frac{w}{b}\right)\right]^{-a}\right\}^{-c}-1\right)^{-2}\right]\right\}^{\zeta_3}. \end{aligned} \tag{7}$$

If $|\varphi_1| < 1$ and $\varphi_2 > 0$ is a real non-integer, then

$$(1-\varphi_1)^{\varphi_2 - 1} = \sum_{m=0}^{\infty} \frac{1}{m!\, \Gamma(\varphi_2 - m)} (-\varphi_1)^m \Gamma(\varphi_2). \tag{8}$$

Applying (8) to (7) we have

$$\begin{aligned} f(w) &= \frac{2ca\lambda^{1+\zeta_3}}{b\Upsilon_{[\lambda]}} \left(1-\left\{1-\left[1+\left(\frac{w}{b}\right)\right]^{-a}\right\}^{c}\right)^{3} \\ &\times \left[1+\left(\frac{w}{b}\right)\right]^{-a-1} \left\{1-\left[1+\left(\frac{w}{b}\right)\right]^{-a}\right\}^{2c-1} \\ &\times \sum_{\zeta_3,\zeta_4=0}^{\infty} \frac{(-1)^{\zeta_3+\zeta_4} \Gamma\left(c(\zeta_3+1)\right)}{\zeta_4!\, \Gamma(c(\zeta_3+1)-\zeta_4)} \\ &\times exp\left[-(\zeta_4+1)\left(\left\{1-\left[1+\left(\frac{w}{b}\right)\right]^{-a}\right\}^{-c}-1\right)^{-2}\right]. \end{aligned} \tag{9}$$

Applying the power series to

$$exp\left[-(\zeta_4+1)\left(\left\{1-\left[1+\left(\frac{w}{b}\right)\right]^{-a}\right\}^{-c}-1\right)^{-2}\right],$$

Equation (9) can be written as

$$\begin{aligned} f(w) &= \sum_{\zeta_3,\zeta_4,\zeta_1=0}^{\infty} 2\lambda^{1+\zeta_3} \frac{(-1)^{\zeta_3+\zeta_4+\zeta_1} (\zeta_4+1)^{\zeta_1} \Gamma(\zeta_3+1)}{\zeta_4!\, \zeta_1!\, \Upsilon_{[\lambda]} \Gamma(\zeta_3+1-\zeta_4)} \\ &\times ab^{-1}c \left[1+\left(\frac{w}{b}\right)\right]^{-a-1} \left\{1-\left[1+\left(\frac{w}{b}\right)\right]^{-a}\right\}^{c-1} \\ &\times \frac{\left(\left\{1-\left[1+\left(\frac{w}{b}\right)\right]^{-a}\right\}^{c}\right)^{2\zeta_1+1}}{\left(1-\left\{1-\left[1+\left(\frac{w}{b}\right)\right]^{-a}\right\}^{c}\right)^{2\zeta_1+3}}. \end{aligned} \tag{10}$$

Consider the series expansion

$$(1-\varphi_1)^{-\varphi_2}\,|_{(|\varphi_1|<1,\ \varphi_2>0)}=\sum_{w=0}^{\infty}\frac{\Gamma(\varphi_2+w)}{w!\,\Gamma(\varphi_2)}{\varphi_1}^{w}. \tag{11}$$

Applying (11) to

$$\left(1-\left\{1-\left[1+\left(\frac{w}{b}\right)\right]^{-a}\right\}^{c}\right)^{-2\zeta_1-3},$$

Equation (10) becomes

$$\begin{aligned}f(w)=&\sum_{\zeta_1,\zeta_2,\zeta_3,\zeta_4=0}^{\infty}2\lambda^{1+\zeta_3}\frac{(-1)^{\zeta_3+\zeta_4+\zeta_1}(\zeta_4+1)^{\zeta_1}}{\zeta_4!\,\zeta_1!\,\zeta_2!\,\Upsilon_{[\lambda]}}\\&\times\frac{\Gamma(\zeta_3+1)\Gamma(3+2\zeta_1+\zeta_2)}{\Gamma(\zeta_3+1-\zeta_4)\Gamma(2\zeta_1+3)[2(\zeta_1+1)+\zeta_2]}\\&\times ab^{-1}c[2(\zeta_1+1)+\zeta_2]\left[1+\left(\frac{w}{b}\right)\right]^{-a-1}\\&\times\left\{1-\left[1+\left(\frac{w}{b}\right)\right]^{-a}\right\}^{c[2(\zeta_1+1)+\zeta_2]-1}.\end{aligned}$$

This can be written as

$$f(w)=\sum_{\zeta_1,\zeta_2=0}^{\infty}\upsilon_{[\zeta_1,\zeta_2]}\pi_{\mathbf{c}^*}(w;a,b), \tag{12}$$

where $\mathbf{c}^*=c[2(1+\zeta_1)+\zeta_2]$ and

$$\begin{aligned}\upsilon_{[\zeta_1,\zeta_2]}=&\frac{2\lambda^{1+\zeta_3}(-1)^{\zeta_1}\Gamma(3+2\zeta_1+\zeta_2)}{\zeta_1!\,\zeta_2!\,\Upsilon_{[\lambda]}\Gamma(2\zeta_1+3)[2(1+\zeta_1)+\zeta_2]}\\&\times\sum_{\zeta_3,\zeta_4=0}^{\infty}\frac{(-1)^{\zeta_3+\zeta_4}\Gamma(\zeta_3+1)(\zeta_4+1)^{\zeta_1}}{\zeta_4!\,\Gamma(\zeta_3+1-\zeta_4)},\end{aligned}$$

and

$$\pi_{\mathbf{c}^*}(w;\phi)=\frac{a\mathbf{c}^*}{b}\left[1+\left(\frac{w}{b}\right)\right]^{-a-1}\left\{1-\left[1+\left(\frac{w}{b}\right)\right]^{-a}\right\}^{\mathbf{c}^*-1},$$

which is the PDF of the GLx model with power parameter $\mathbf{c}^*$. Similarly

$$F(w)=\sum_{\zeta_1,\zeta_2=0}^{\infty}\upsilon_{[\zeta_1,\zeta_2]}\ \Pi_{\mathbf{c}^*}(w;a,b), \tag{13}$$

where

$$\Pi_{\mathbf{c}^*}(w;a,b)=\left\{1-\left[1+\left(\frac{w}{b}\right)\right]^{-a}\right\}^{\mathbf{c}^*}$$

is the CDF of the GLx model with power parameter $\mathbf{c}^*$.

2.2 Moments and incomplete moments

Let Y be a continuous r.v. having the GLx model with power parameter φ, then the q^{th} ordinary and incomplete moments ($\mu'_q(y)$ and $m_q(y)$) of the GLx r.v. (defined in this subsection) given by

$$\mu'_q(y)=\sum_{h=0}^{q}\ \varphi b^q(-1)^h\binom{q}{w}B\left(\varphi,\frac{h-q}{a}+1\right)|_{(a>q)},$$

and

$$m_q(y)=\sum_{h=0}^{q}\ \varphi b^q(-1)^h\binom{q}{w}B_t\left(\varphi,\frac{h-q}{a}+1\right)|_{(a>q)},$$

where

$$B(a_1,a_2)|_{[a_1,a_2\notin(0,-1,-2,\ldots)]}=\int_0^1 u^{a_1-1}(1-u)^{a_2-1}du,$$

is the complete beta function and

$$B_z(a_1,a_2)|_{[a_1,a_2\notin(0,-1,-2,\ldots)]}=\int_0^z u^{a_1-1}(1-u)^{a_2-1}du,$$

is the incomplete beta function. The r^{th} ordinary moment of W, say μ_r', follows from (12) as

$$\mu_r'=E(W^r)=\sum_{\zeta_1,\zeta_2=0}^{\infty}\sum_{h=0}^{r}\xi_{\zeta_1,\zeta_2,h}^{\{\mathbf{c}^*,r\}}B\left(\mathbf{c}^*,\frac{h-r}{a}+1\right)|_{(a>r)}, \quad (14)$$

where

$$\xi_{\zeta_1,\zeta_2,w}^{\{\mathbf{c}^*,r\}}=\xi_{\zeta_1,\zeta_2}\,\mathbf{c}^*b^r(-1)^h\binom{r}{h}.$$

Setting $r=1$ in (14) gives the mean of W as

$$\mu_1'=E(W)=\sum_{\zeta_1,\zeta_2=0}^{\infty}\sum_{h=0}^{1}\xi_{\zeta_1,\zeta_2,h}^{\{\mathbf{c}^*,1\}}B\left(\mathbf{c}^*,\frac{h-1}{a}+1\right)|_{(a>1)},$$

Setting $r=2,3$ and 4 in (14), we have the $2\,nd$, $3\,rd$ and the $4\,th$ moments about the origin

$$\mu_2'=E(W^2)=\sum_{\zeta_1,\zeta_2=0}^{\infty}\sum_{h=0}^{2}\xi_{\zeta_1,\zeta_2,h}^{\{\mathbf{c}^*,2\}}B\left(\mathbf{c}^*,\frac{h-2}{a}+1\right)|_{(a>2)},$$

$$\mu_3'=E(W^3)=\sum_{\zeta_1,\zeta_2=0}^{\infty}\sum_{h=0}^{3}\xi_{\zeta_1,\zeta_2,h}^{\{\mathbf{c}^*,3\}}B\left(\mathbf{c}^*,\frac{h-3}{a}+1\right)|_{(a>3)}$$

and

$$\mu_4'=E(W^4)=\sum_{\zeta_1,\zeta_2=0}^{\infty}\sum_{h=0}^{4}\xi_{\zeta_1,\zeta_2,h}^{\{\mathbf{c}^*,4\}}B\left(\mathbf{c}^*,\frac{h-4}{a}+1\right)|_{(a>4)}.$$

The r^{th} incomplete moment of W is defined by

$$m_r(q)=\int_{-\infty}^{q}u^rf(u)du.$$

We can write from (12)

$$m_r(w)=\sum_{\zeta_1,\zeta_2=0}^{\infty}\sum_{h=0}^{r}\xi_{\zeta_1,\zeta_2,h}^{\{\mathbf{c}^*,r\}}B_t\left(\mathbf{c}^*,\frac{h-r}{a}+1\right)|_{(a>r)},$$

The first incomplete moment $m_1(y)$ can derived as

$$m_1(y)=\sum_{\zeta_1,\zeta_2=0}^{\infty}\sum_{h=0}^{1}\xi_{\zeta_1,\zeta_2,h}^{\{\mathbf{c}^*,1\}}B_t\left(\mathbf{c}^*,\frac{h-1}{a}+1\right)|_{(a>1)}.$$

2.3 Generating function

The moment generating function (mgf) of W, say $M_W(t)=E(exp(tW))$, is obtained from (12) as

$$M_W(t)=\sum_{\zeta_1,\zeta_2,r=0}^{\infty}\sum_{h=0}^{r}\frac{t^r}{r!}\xi_{\zeta_1,\zeta_2,h}^{\{\mathbf{c}^*,r\}}B\left(\mathbf{c}^*,\frac{h-r}{a}+1\right)|_{(a>r)}.$$

2.4 Quantile function (qf) and random number generation

The qf of W, where $W\sim$ PRGLx (λ,c,a,b), is obtained by inverting (4) as

$$Q(u)=b\left\{\left[1-\left(1+\left\{-ln\left[1+\frac{ln\left(1-u\Upsilon_{[\lambda]}\right)}{\lambda}\right]\right\}^{\frac{1}{2}}\right)^{\frac{1}{c}}\right]^{-\frac{1}{a}}-1\right\}$$

2.5Moment of the reversed residual life

The n^{th} moment of the reversed residual, say

$$\Upsilon_n(t)=E[(t-W)^n]\,|_{(W\leq t\,,\,t>0)}^{(n=1,2,\dots)}.$$

Then, we have

$$\Upsilon_n(t)=F^{-1}(t)\int_0^t (t-w)^n dF(w).$$

Then

$$\Upsilon_n(t)=F^{-1}(t)\sum_{\zeta_1,\zeta_2=0}^{\infty}\sum_{h=0}^{n}\sum_{r=0}^{n}\xi_{\zeta_1,\zeta_2,wh,r}^{\{\mathbf{c}^*,n\}}B_t\left(\mathbf{c}^*,\frac{h-n}{a}+1\right)|_{(a>n)},$$

where

$$\xi_{\zeta_1,\zeta_2,w,r}^{\{\mathbf{c}^*,n\}}=\xi_{\zeta_1,\zeta_2}(-1)^r t^{n-r}\mathbf{c}^* b^n(-1)^h\binom{n}{h}\binom{n}{r}.$$

3.Estimation

The log-likelihood function for Φ is given by

$$\begin{aligned}\ell_n(\Phi)&=n\,log\,2+n\,log\,c+n\,log\,\lambda+n\,log\,a\\&-\sum_{h=1}^{n}\left(\frac{\left\{1-\left[1+\left(\frac{w_h}{b}\right)\right]^{-a}\right\}^{c}}{1-\{1-[1+(b^{-1}w_h)]^{-a}\}^{c}}\right)^2\\&-n\,log\,b-(a+1)\sum_{h=1}^{n}log\left[1+\left(\frac{w_h}{b}\right)\right]\\&-3\sum_{h=1}^{n}log\left(1-\left\{1-\left[1+\left(\frac{w_h}{b}\right)\right]^{-a}\right\}^{c}\right)\\&-n\,log\,\Upsilon_{[\lambda]}+(2c-1)\sum_{h=1}^{n}log\left\{1-\left[1+\left(\frac{w_h}{b}\right)\right]^{-a}\right\}\\&-\lambda\sum_{h=1}^{n}\left\{1-exp\left[-\left(\frac{\left\{1-\left[1+\left(\frac{w_h}{b}\right)\right]^{-a}\right\}^{c}}{1-\left\{1-\left[1+\left(\frac{w_h}{b}\right)\right]^{-a}\right\}^{c}}\right)^2\right]\right\}\end{aligned}$$

The above log-likelihood function can be maximized numerically by via R (optim) or SAS (PROC NLMIXED) or Ox program (sub-routine MaxBFGS).

4.Modeling data

In this section, we provide two applications to show empirically the potentiality of the PRGLx model. In order to compare the fits of the PRGLx distribution with other competing distributions, we consider the Cramér-Von-Mises (W^*) and the Anderson-Darling (A^*) statistics. We compare the fit of the new model with competitive models namely: The ZTP Burr-X Lomax (ZTPBrXLx); the BrXLx; GLx model; the gamma Lomax (GLx) model; the beta Lomax (BLX) model and Lx model. The MLEs and the corresponding standard errors (in parentheses) of the model parameters are given in Tables 1 and 3. The statistics W^* and A^* are listed in Tables 2 and 4. The Estimated PDF, P-P plot, TTT plot, estimated HRF and and and Kaplan-Meier survival plot of the two data sets and the estimated PDF of the proposed model are displayed in Fig.s 3 and 4.

4.1Modeling failure times

This data set represents the data on failure times of 84 aircraft windshield given in application 1. From table 2 and Fig. 3, the PRGLx lifetime model is much better than the ZTP Burr-X Lx, the Burr-X Lx, gamma Lx, beta Lx, GLx and Lx models so the PRGLx model is a good alternative to these models in modeling aircraft windshield data.

Table 1: MLEs (standard errors in parentheses) for data set **I**.

Model	Estimates			
PRGLx(λ,c,a,b)	**-3.546**	**0.4132**	**2.1637**	**17.695**
	(1.991)	**(0.15210**	**(3.190)**	**(27.7)**
PBrXLx(λ,c,a,b)	-5.2837	0.2860	1.553	4.6136
	(93.989)	(0.3119)	(1.395)	(5.843)
BLx(α,β,a,b)	3.6036	118.84	33.6387	4.8307
	(0.619)	(63.715)	(9.2382)	(429.0)
BrXLx(c,a,b)	$5.24\times e^{5}$	$8.23\times e^{-1}$	$1.159\times e^{5}$	
	(0.000)	(0.10511)	(5446.8)	
GLx(c,a,b)	3.6261	26257.681	20074.51	
	(0.6236)	(99.742)	(2041.83)	
GaLx(λ,a,b)	3.5876	37029	52001	
	(0.5133)	(81.16)	(7955)	
Lx(a,b)	131789	51425		
	(296.12)	(5933.5)		

Table 2: W^* and A^* for data set **I**.

PRGLx(λ,c,a,b)	**0.0643**	**0.1504**
ZTPBrXLx(λ,c,a,b)	0.0746	0.1531
BrXLx(c,a,b)	0.0764	0.5844
BLx(α, β,a,b)	1.4084	0.1680
GLx(c,a,b)	1.7435	0.2194
GaLx(λ,a,b)	1.3667	0.1619
Lx(a,b)	1.3976	0.1665

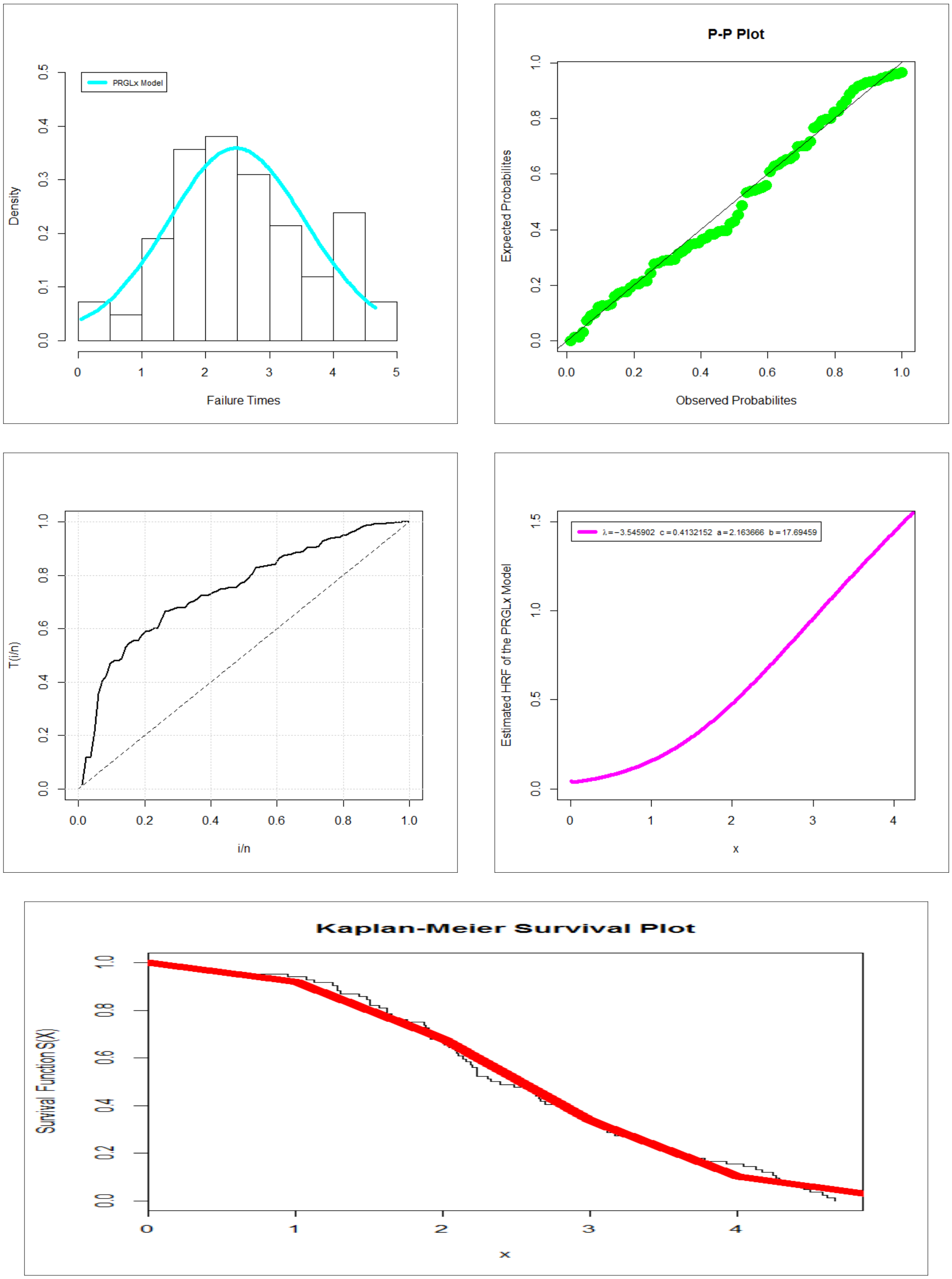

Fig. 3: Estimated-PDF, P-P plot, TTT plot, estimated-HRF
and Kaplan-Meier plot for data **I**.

4.2Modeling service times

This real data set represents the data on service times of 63 aircraft windshield given in Murthy et al. (2004). From table 4 and Fig. 4, the PRGLx lifetime model is much better than the ZTP Burr-X Lx, the Burr-X Lx, the gamma Lx, beta Lx, GLx and Lx models so the new PRGLx model is a good alternative to these models in modeling the service times data.

Table 3: MLEs (standard errors in parentheses) for data set II.

Model	Estimates			
PRGLx(λ,c,a,b)	**-1.1903**	**0.468**	**1.1224**	**7.361**
	(1.7684)	**(0.1636)**	**(1.322)**	**(8.742)**
ZTPBrXLx(λ,c,a,b)	-1.4557	0.4652	1.3517	3.9449
	(2.303)	(0.215)	(91.499)	(6.440)
BLx(α,β,a,b)	1.9218	169.58	31.259	4.9685
	(0.3185)	(339.2)	(316.8)	(50.53)
BrXLx(c,a,b)	0.6467	0.5987	1.6211	
	(0.0475)	(0.390)	(0.959)	
GLx(c,a,b)	1.9145	32881.9	22971.2	
	(0.3483)	(162.22)	(3209.5)	

Table 4: W^* and A^* for data set **I I**.

PRGLx(λ,c,a,b)	**0.0312**	**0.1219**
ZTPBrXLx(λ,c,a,b)	0.0337	0.1229
BrXLx(c,a,b)	0.0876	0.5278
BLx(α,β,a,b)	1.1336	0.1872
GLx(c,a,b)	1.2331	0.2037
GaLx(λ,a,b)	1.1121	0.2038
Lx(a,b)	1.1265	0.1861

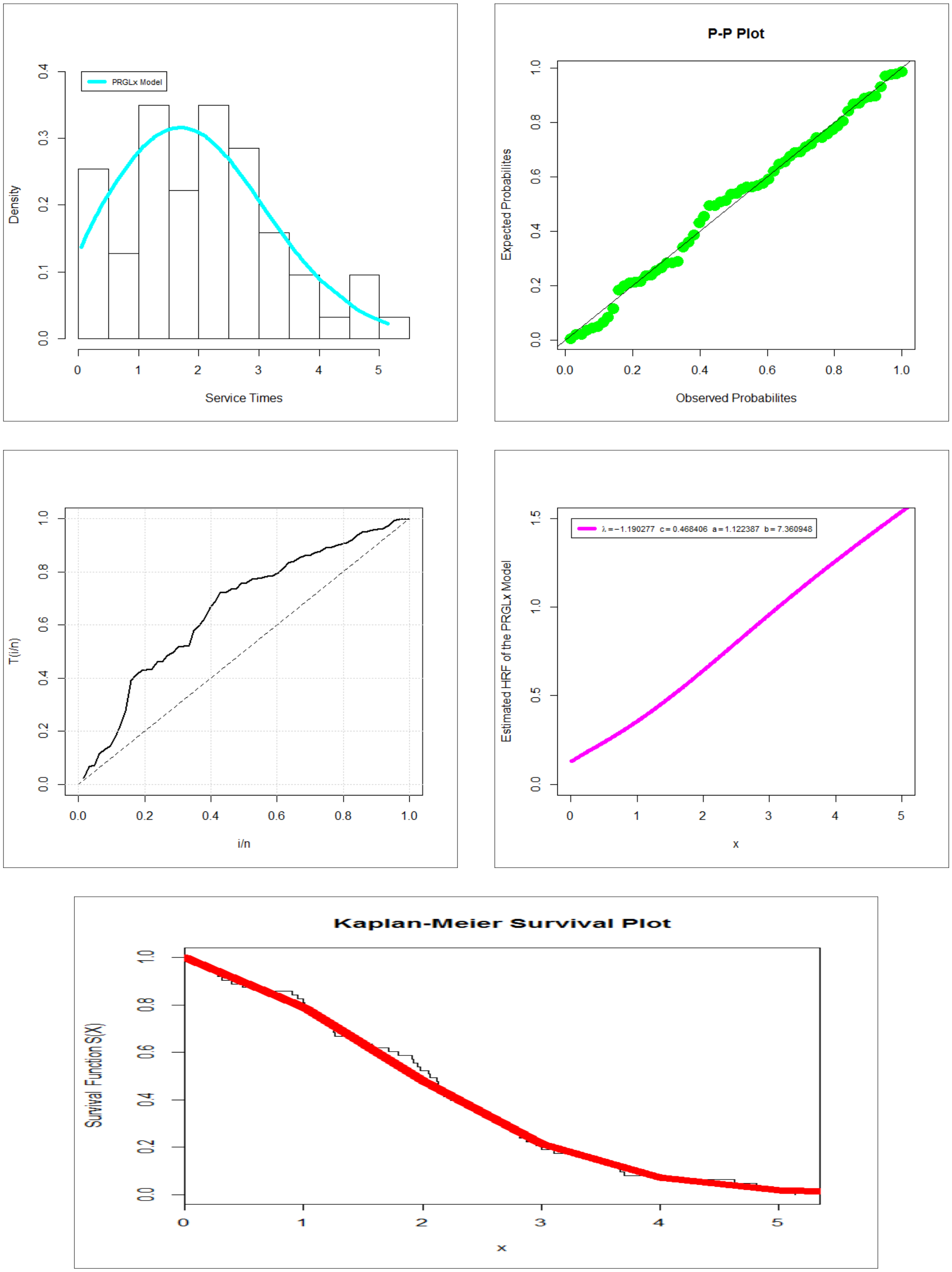

Fig. 4: Estimated-PDF, P-P plot, TTT plot, estimated-HRF
and Kaplan-Meier plot for data **II**.

5. Concluding remarks

In this paper, we introduced a new extension of the continuous Lomax distribution with a strong physical motivation. Some of its statistical properties such as moments, incomplete moments, moment generating function, quantile function, random number generation, quantile spread ordering and moment of the reversed residual life are derived. Two real data applications are provided to illustrate the importance and flexibility of the new model. The new lifetime model is much better than other competitive models such as the ZTP Burr-X Lomax, the Burr-X Lomax, the gamma Lomax, the beta Lomax, the generalized Lomax and the original Lomax model so the new model is a useful alternative to these models in modeling the failure and service times data.

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