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The Weibull-G Poisson Family for Analyzing Lifetime Data

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Abstract

We study a new family of distributions defined by the minimum of the Poisson random number of independent identically distributed random variables having a general Weibull-G distribution (see Bourguignon et al. (2014)). Some mathematical properties of the new family including ordinary and incomplete moments, quantile and generating functions, mean deviations, order statistics, reliability and entropies are derived. Maximum likelihood estimation of the model parameters is investigated. Three special models of the new family are discussed. We perform three applications to real data sets to show the potentiality of the proposed family.

Key Words: Weibull-G family, Entropy, Generating Function, Maximum Likelihood, Order Statistic.

Mathematical Subject Classification: 62N01; 62N02; 62E10.

1. Introduction

In many applied areas such as lifetime analysis, biomedical science, reliability, engineering, social sciences, finance and insurance, there is a clear need for extended forms of the classical models, i.e., new distributions which are more flexible to capture skewness and kurtosis behavior and to improve the goodness-of-fit of the generated family. Recent developments focus on new techniques by adding shape parameters to existing distributions for building classes of more flexible distributions. However, there is a clear need for further extended distributions, which are more flexible to model lifetime data in applied areas.

Several studies have suggested that Weibull distribution, an asymmetrical distribution has limitations in fitting all types of data where the topic of reliability comes into play. For example, Drobinski et al. (2015) in their study, demonstrated that in modeling wind speed data, since the Weibull distribution is heavily relied on empirical perspective rather than physical justification, it might not be a good candidate in fitting to these types of environmental data. In their study, the authors provided due justification that the Weibull model works less efficiently for wind speed data as compared to wind components data. Basu et al. (2009) studied the usefulness of the Weibull distribution in evaluating the strength distribution for brittle materials. The authors indicated that other probability models, such as normal, log-normal works better under certain reasonable circumstances. On the other hand, the Poisson distribution has its limitation as well as demonstrated by Pak-poy (1964). It is suggested in that study that the popular Poisson model for modeling road traffic problems is not recommended in some practical situations, where the variance is much bigger than the mean. All these above merits for finding a mixture of Weibull-G family of distribution, proposed and studied by Bourguignon et al. (2014) (after adding more flexibility to Weibull model itself) with possibly a discrete probability distribution with the same support $(0,\infty)$. This is why we considered Poisson distribution with the Weibull-G type models to capture more flexibility. We hope that our proposed model will be better in capturing several patterns of the data structure to describe appropriately the associated reliability structure, in particular to those cases where the individual Weibull-G family (a specific member) or the Poisson distribution alone might not be a good model.

The well-known generators are the following: beta-G by Eugene et al. (2002), Kumaraswamy-G by Cordeiro and de Castro (2011), exponentiated generalized-G by Cordeiro et al. (2013), generalized transmuted-G by Nofal et al. (2017), transmuted exponentiated generalized-G by Yousof et al. (2015), Kumaraswamy transmuted-G by Afify et al. (2016b), transmuted geometric-G by Afify et al. (2016a), generalized odd generalized exponential family by Alizadeh et al. (2017), exponentiated Weibull-H family by Cordeiro et al. (2017), exponentiated generalized-G Poisson family by Aryal and Yousof (2017), transmuted Weibull-G family by Alizadeh et al. (2018), Marshall-Olkin generalized-G Poisson family by Korkmaz et al. (2018b) and odd Lomax-G family by Cordeiro et al. (2019). Many other useful families and new models can be cited by Brito et al. (2017), Cordeiro et al. (2018), Altun et al. (2018a-b), Gad et al. (2019), Yousof et al. (2017a-b, 2018a-d and 2019), Hamedani et al. (2017, 2018 and 2019), Korkmaz et al. (2018a), Sen et al. (2018), Korkmaz et al. (2019), Ibrahim et al. (2019), Nascimento et al. (2019) and Ibrahim (2020a-b) and Mansour et al. (2020).

We motivate our model by considering a typical system failure in a reliability context. We envision a scenario that we will encounter a data which is a mixture of discrete and continuous type. We begin by assuming the distribution of a system consisting of N independent subsystems having a zero inflated Poisson distribution. We discard the scenario that all components simultaneously will fail to work, theoretically viable but realistically not a prudent one.
Suppose $Z_1,\dots,Z_N$ be independent identically random variable (iid) with common CDF Weibull-G and N be random variable with

$$P(N=n)=\frac{1}{exp(\theta)-1}\times\frac{\theta^n}{n!}|n=1,2,\dots,\theta>0$$

and define $M_N=max(Z_1,\dots,Z_N)$ then

$$\begin{aligned}F(x)&=\sum_{n=1}^{\infty}Pr(M_N\le x|N=n)\,Pr(N=n)\\&=\sum_{n=1}^{\infty}\left\{\frac{1-exp[-\theta G(x;\alpha,\phi)]}{[exp(\theta)-1]}\right\}^n\frac{1}{exp(\theta)-1}\frac{\theta^n}{n!}\\&=\frac{1}{[exp(\theta)-1]}\left\{exp\left(\theta-\theta\,exp\left\{-\left[\frac{G(x;\phi)}{\overline{G}(x;\phi)}\right]^\alpha\right\}\right)-1\right\}.\end{aligned}\tag{1}$$

Equation (1) is called Weibull-G Poisson (WGP) distribution. Several new models can be generated by considering special distributions for $G(x;\phi)$.
The corresponding PDF of (1) reduces to

$$\begin{aligned}f(x;\theta,\alpha,\phi)=&\frac{\theta\alpha g(x;\phi)G(x;\phi)^{\alpha-1}}{[exp(\theta)-1]\overline{G}(x;\phi)^{\alpha+1}}exp\left\{-\left[\frac{G(x;\phi)}{\overline{G}(x;\phi)}\right]^\alpha\right\}\\&\times exp\left(\theta-\theta\,exp\left\{-\left[\frac{G(x;\phi)}{\overline{G}(x;\phi)}\right]^\alpha\right\}\right).\end{aligned}\tag{2}$$

The reliability function (rf) of X is given by

$$R(x;\theta,\alpha,\phi)=1-\frac{1}{[exp(\theta)-1]}\left\{exp\left(\theta-\theta\,exp\left\{-\left[\frac{G(x;\phi)}{\overline{G}(x;\phi)}\right]^\alpha\right\}\right)-1\right\},$$

where θ and α are two positive shape parameters. A random variable X with PDF (2) is denoted by $X\sim$ WGP (θ,α,ϕ).

The rest of the paper is organized as follows. In Section 2, we provide a useful mixture representation for its PDF. In Section 3, we define two special models and give some plots of their PDF's and hazard rate functions. In Section 4, we derive some of its general mathematical properties including quantile and generating functions, ordinary and incomplete moments, mean deviations, entropies, order statistics, residual and reversed residual life and stress-strength mode. Maximum likelihood estimation of the model parameters is addressed in Section 5. In Section 6, simulation results to assess the performance of the proposed maximum likelihood estimation procedure are discussed. In Section 7, we provide three applications to real data to illustrate the importance and flexibility of the new family. Finally, some concluding remarks are presented in Section 8.

2. Linear representation

In this section, we provide a useful representation for (2) using the concept of exponentiated distributions.
The WGP family density in (2) can be expressed as

$$f(x)=\frac{\theta\alpha}{[exp(\theta)-1]}\sum_{i=0}^{\infty}\frac{\theta^{i}g(x;\phi)G(x;\phi)^{\alpha-1}}{i!\,\overline{G}(x;\phi)^{\alpha+1}}exp\left\{-\left[\frac{G(x;\phi)}{\overline{G}(x;\phi)}\right]^{\alpha}\right\}\times\left(1-exp\left\{-\left[\frac{G(x;\phi)}{\overline{G}(x;\phi)}\right]^{\alpha}\right\}\right)^{i}.$$

The last equation can be expressed as

$$f(x)=\sum_{i,j,k,m=0}^{\infty}\frac{(-1)^{j+k+m}}{i!\,k!}\binom{i}{j}\binom{-[\alpha(k+1)+1]}{m}\frac{\theta^{i+1}\alpha(j+1)^{k}}{[exp(\theta)-1]}g(x;\phi)G(x;\phi)^{\alpha(k+1)+m-1}.$$

Then, the WGP density can be rewritten as

$$f(x)=\sum_{k,m=0}^{\infty}\omega_{k,m}\ \pi_{\alpha(k+1)+m}(x), \qquad (3)$$

where $\pi_{\eta}(x)=\eta g(x;\phi)G(x;\phi)^{\eta-1}$ and

$$\omega_{k,m}=\frac{\alpha(-1)^{k+m}\binom{-[\alpha(k+1)+1]}{m}}{k!\,[exp(\theta)-1][\alpha(k+1)+m]}\sum_{i,j=0}^{\infty}\frac{(-1)^{j}(j+1)^{k}\theta^{i+1}}{i!}\binom{i}{j}.$$

Equation (3) reveals that the WGP density function is a mixture of Exp-G densities. Thus, some mathematical properties of the new family can be derived from those properties of the Exp-G class. The CDF of the WGP family can also be expressed as a mixture of E-G densities. By integrating (3), we obtain the same mixture representation

$$F(x)=\sum_{k,m=0}^{\infty}\omega_{k,m}\ \Pi_{\alpha(k+1)+m}(x), \qquad (4)$$

where $\Pi_{\eta}(x)$ is the CDF of the Exp-G family with power parameter (η) .

3. Special WGP distributions

The PDF (2) allows greater flexibility of its tails and can be widely applied in many applied areas of statistics. Now, we define and study two special models of the WGP family by taking the following baseline distributions: gamma (G), log-logistic (LL) and exponentiated exponential (EE) distributions. The PDF (2) will be most tractable when the CDF $G(x;\phi)$ and the PDF $g(x;\phi)$ have simple analytic expressions.

3.1 The WG P distribution

The G distribution with positive parameters a and b has PDF and CDF (for $x>0$) given by

$$g(x)=\frac{1}{b^{a}\Gamma(a)}x^{a-1}exp\left(-\frac{x}{b}\right) \quad \text{and} \quad G(x)=\frac{1}{\Gamma(a)}\gamma\left(a,\frac{x}{b}\right),$$

respectively, where $\gamma\left(a,\frac{x}{b}\right)=\int_{0}^{x/b}t^{a-1}e^{-t}dt$ is the incomplete gamma function. Then, the PDF of the WGP distribution reduces to

$$f(x)=\frac{\theta\alpha x^{a-1}exp\left(-\frac{x}{b}\right)\Gamma(a)\left[\gamma\left(a,\frac{x}{b}\right)\right]^{\alpha-1}}{b^{a}[exp(\theta)-1]\left[\Gamma(a)-\gamma\left(a,\frac{x}{b}\right)\right]^{\alpha+1}}exp\left\{-\left[\frac{\gamma\left(a,\frac{x}{b}\right)}{\Gamma(a)-\gamma\left(a,\frac{x}{b}\right)}\right]^{\alpha}\right\}\times exp\left(\theta-\theta\,exp\left\{-\left[\frac{\gamma\left(a,\frac{x}{b}\right)}{\Gamma(a)-\gamma\left(a,\frac{x}{b}\right)}\right]^{\alpha}\right\}\right).$$

The plots in Figures 1 and 2 show some possible shapes of the density and hazard rate functions of the WGP distribution.

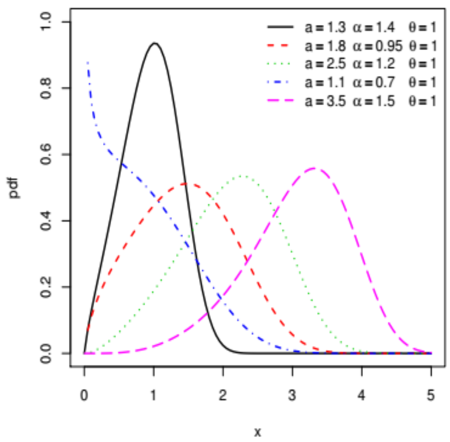


Figure 1: The WGP density plots (left panel).

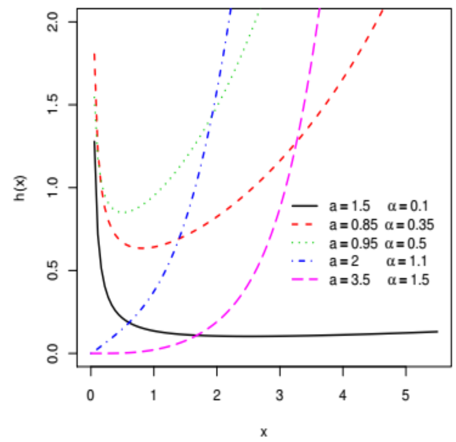


Figure 2: The WGP hrf plots (right panel).

3.2 The WLLP distribution

The LL distribution with positive parameters c and s has PDF and CDF given by $g(x)=\frac{c}{s}(\frac{x}{s})^{c-1}\left[1+(\frac{x}{s})^{c}\right]^{-2}$ (for $x>0$) and $G(x)=1-[1+(\frac{x}{s})^{c}]^{-1}$, respectively. Then, the PDF of the WLLP distribution is given by

$$f(x)=\frac{\theta\alpha c x^{c-1}\left\{1-\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-1}\right\}^{\alpha-1}}{s^{c}[exp(\theta)-1]\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-\alpha+1}}exp\left[-\left(\frac{x}{s}\right)^{c\alpha}\right]$$
$$\times exp\left\{\theta-\theta\, exp\left[-\left(\frac{x}{s}\right)^{c\alpha}\right]\right\}.$$

Plots of the density and hazard rate functions of the WLLP distribution are displayed in Figures 3 and 4 for some parameter values.

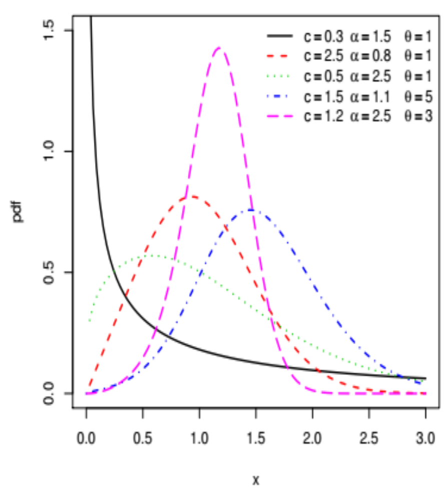


Figure 3: The WLLP density plots (left panel).

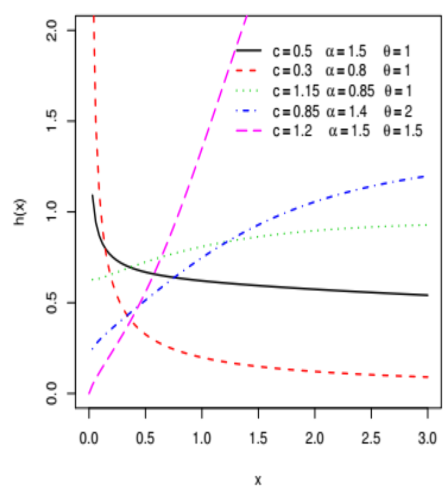


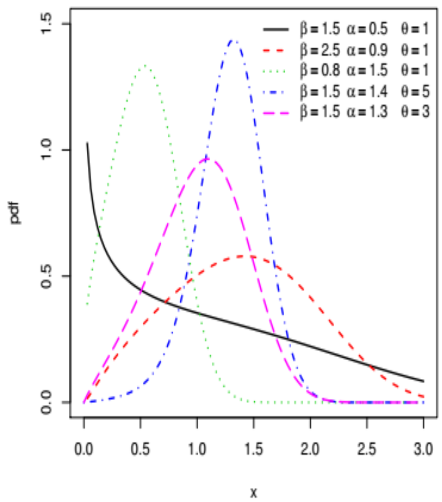
Figure 4: The WLLP hrf plots (right panel).

3.3 The WEEP distribution

The EE distribution with scale parameter $\lambda > 0$ and shape parameter $\beta > 0$ has PDF and CDF given by $g(x) = \beta\lambda\, exp(-\lambda x)\,[1 - exp(-\lambda x)]^{\beta-1}$ (for $x > 0$) and $G(x) = 1[-exp(-\lambda x)]^{\beta}$, respectively. Then, the WEEP density function reduces to

$$f(x) = \frac{\theta\alpha\beta\lambda[1-exp(-\lambda x)]^{\alpha\beta-1}}{\{1-[1-exp(-\lambda x)]^{\beta}\}^{\alpha+1}}\, exp\left(\frac{-[1-exp(-\lambda x)]^{\alpha\beta}}{\{1-[1-exp(-\lambda x)]^{\beta}\}^{\alpha}}\right)$$
$$\times \frac{exp(-\lambda x)}{[exp(\theta)-1]}\, exp\left[\theta - \theta\, exp\left(\frac{-[1-exp(-\lambda x)]^{\alpha\beta}}{\{1-[1-exp(-\lambda x)]^{\beta}\}^{\alpha}}\right)\right].$$

Figures 5 and 6 display some possible shapes of the density and hazard rate functions of this distribution.



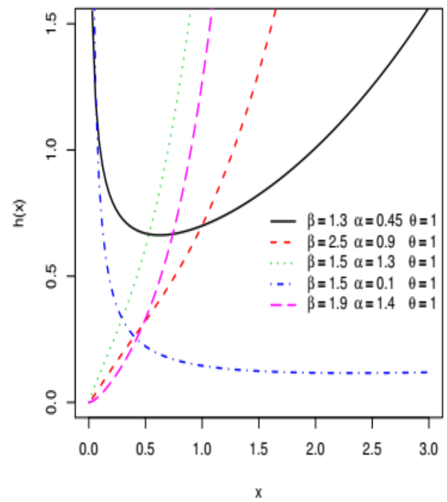


Figure 5: The WEEP density plots (left panel). Figure 6: The WEEP hrf plots (right panel).

4. Mathematical properties

In this section, we derive some general mathematical properties of the new family. Established explicit expressions to calculate statistical measures can be more efficient than computing them directly by numerical integration.

4.1 Quantile and generating functions

The quantile function (qf) of X, where $X \sim$ WGP (θ, α, ϕ), is obtained by inverting (1) to optain $Q(u) = F^{-1}$, $0 \le u \le 1$.

Simulating the WGP random variable is straightforward. If U is a uniform variate on the unit interval $(0,1)$, then the random variable $X = Q(U)$ follows (2).

For simulating from WGP if $u \sim u(0,1)$, then solution of nonlinear equation

$$x_u = G^{-1}\left[\frac{\left(-\,log\left\{1-\frac{1}{\theta}log[1+u(exp(\theta)-1)]\right\}\right)^{\frac{1}{\alpha}}}{1+\left(-\,log\left\{1-\frac{1}{\theta}log[1+u(exp(\theta)-1)]\right\}\right)^{\frac{1}{\alpha}}}\right].$$

Here, we provide two formulae for the mgf $M_X(t) = E(e^{tX})$ of X. Clearly, the first one can be derived from equation (3) as

$$M_X(t) = \sum_{k,m=0}^{\infty} \omega_{k,m}\, M_{\alpha(k+1)+m}(t),$$

where $M_k(t)$ is the mgf of Y_k. Hence, $M_X(t)$ can be determined from the exp-G generating function.

A second formula for $M_X(t)$ follows from (3) as

$$M_X(t) = \sum_{k,m=0}^{\infty} \omega_{k,m}\, \tau(t, \alpha[k+1] + m - 1),$$

where $\tau(t,k) = \int_0^1 \quad exp[tQ_G(u)]\, u^k du$ and $Q_G(u)$ is the qf corresponding to $G(x\,; \quad \phi)$, i.e., $Q_G(u) = G^{-1}(u\,;\phi)$.

4.2 Ordinary and incomplete moments

The r th moment of X, say μ_r', follows from (3) as

$$\mu_r' = E(X^r) = \sum_{k,m=0}^{\infty} \omega_{k,m}\, E\big(Y_{\alpha(k+1)+m}^r\big). \tag{5}$$

Henceforth, Y_γ denotes the Exp-G distribution with power parameter γ.

The variance, skewness, and kurtosis measures can now be calculated using the well-known relations. The n th central moment of X, say M_n, is given by

$$M_n = E(X - \mu_1')^n = \sum_{r=0}^{n} \binom{n}{r} (-\mu_1')^{n-r} E(X^r)$$

$$= \sum_{r=0}^{n} \quad \sum_{k,m=0}^{\infty} \omega_{k,m} \binom{n}{r} (\mu_r')^{n-r} E\big(Y_{\alpha(k+1)+m}^r\big).$$

The cumulants (κ_n) of X follow recursively from

$$\kappa_n = \mu_n' - \sum_{r=0}^{n-1} \quad \binom{n-1}{r-1} \kappa_r \mu_{n-r}',$$

where $\kappa_1 = \mu_1'$, $\kappa_2 = \mu_2' - \mu_1'^2, \kappa_3 = \mu_3' - 3\mu_2'\mu_1' + \mu_1'^3$, etc.

The main applications of the first incomplete moment refer to the mean deviations and the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The s th incomplete moment, say $\phi_s(t)$, of X can be expressed from (3) as

$$\phi_s(t) = \int_{-\infty}^{t} x^s\, f(x) dx = \sum_{k,m=0}^{\infty} \omega_{k,m} \int_{-\infty}^{t} x^s\, \pi_{\alpha(k+1)+m}(x) dx. \tag{6}$$

4.3 Mean Deviations

The mean deviations about the mean $[\alpha_1 = E(|X - \mu_1'|)]$ and about the median $[\alpha_2 = E(|X - M|)]$ of X are given by $\alpha_1 = 2\mu_1' F(\mu_1') - 2\phi_1(\mu_1')$ and $\alpha_2 = \mu_1' - 2\phi_1(M)$, respectively, where $\mu_1' = E(X)$, $M = Median(X) = Q(0.5)$ is the median, $F(\mu_1')$ is easily calculated from (1) and $\phi_1(t)$ is the first incomplete moment given by (6) with $s = 1$. Now, we provide two ways to determine α_1 and α_2.

First, a general Equation for $\phi_1(t)$ can be derived from (6) as

$$\phi_1(t) = \sum_{k,m=0}^{\infty} \omega_{k,m}\, V_{\alpha(k+1)+m}(x),$$

where $V_k(x) = \int_{-\infty}^{t} x\, \pi_k(x) dx$ is the first incomplete moment of the Exp-G distribution.

A second general formula for $\phi_1(t)$ is given by

$$\phi_1(t) = \sum_{k,m=0}^{\infty} \omega_{k,m}\, \upsilon_{\alpha(k+1)+m-1}(t),$$

where $\upsilon_{k-1}(t) = k \int_0^{G(t)} Q_G(u)\, u^{k-1} du$ can be computed numerically.

These equations for $\phi_1(t)$ can be applied to construct Bonferroni and Lorenz curves defined for a given probability π by $B(\pi) = \phi_1(q)/(\pi\mu_1')$ and $L(\pi) = \phi_1(q)/\mu_1'$, respectively, where $\mu_1' = E(X)$ and $q = Q(\pi)$ is the qf of X at π.

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Table 1: Mean, variance, skewness and kurtosis for the WEEP distribution

θ	α	β	Mean	Variance	Skewness	Kurtosis
0.5	0.5	0.5	0.01598	0.00464	8.34499	104.5005
		1.5	0.10004	0.05089	3.38828	17.8778
		2	0.14287	0.08365	2.79435	12.58904
		5	0.34026	0.30518	1.66057	5.29040
	1.5	0.5	0.00927	0.00175	6.65653	6.65652
		1.5	0.053683	0.03151	3.76445	17.89543
		2	0.07273	0.05369	3.45308	14.90687
		5	0.14842	0.19721	2.92195	10.38314
	2	0.5	0.00572	0.00104	7.71279	74.99725
		1.5	0.03177	0.02002	4.93748	28.53828
		2	0.04252	0.03401	4.64573	24.90873
		5	0.08420	0.12229	4.15565	19.28276
1.5	0.5	0.5	0.01627	0.00541	8.34159	101.6934
		1.5	0.08903	0.05313	3.70740	20.24034
		2	0.12384	0.08556	3.14089	14.75641
		5	0.27772	0.29740	2.06815	6.86562
	1.5	0.5	0.00822	0.00183	7.28082	67.63469
		1.5	0.04222	0.02818	4.51647	24.56415
		2	0.05619	0.04668	4.20113	21.01491
		5	0.11058	0.16112	3.64668	15.36836
	2	0.5	0.00484	0.00103	8.69822	91.90487
		1.5	0.02419	0.01693	5.96136	40.4247
		2	0.03189	0.02802	5.65369	35.90259
		5	0.06129	0.09533	5.11985	28.57023
	5	0.5	0.00012	2.56279	47.00557	2423.183
		1.5	0.00056	0.00044	39.3412	1614.199
		2	0.00072	0.00071	38.53685	1536.394
		5	0.00131	0.00229	37.18356	1407.508
2	0.5	0.5	0.01609	0.00571	8.42407	102.3361
		1.5	0.08265	0.05320	3.91368	21.94791
		2	0.11351	0.08477	3.35794	16.28804
		5	0.24729	0.28686	2.30769	7.97881
	1.5	0.5	0.00769	0.00185	7.62109	72.694
		1.5	0.03722	0.02642	4.93981	28.7914
		2	0.04908	0.04316	4.62583	24.96535
		5	0.09477	0.14434	4.06486	18.7226

Table 1: Mean, variance, skewness and kurtosis for the WEEP distribution (Continuing)

θ	α	β	Mean	Variance	Skewness	Kurtosis
	2	0.5	0.00444	0.00102	9.22235	101.5581
		1.5	0.02102	0.01549	6.54351	48.06931
		2	0.02749	0.02531	6.23262	43.10474
		5	0.05197	0.08367	5.68358	34.85865
	5	0.5	0.00011	0.00002	51.59404	2895.111
		1.5	0.00047	0.00038	43.70736	1988.029
		2	0.00059	0.00061	42.85088	1896.793
		5	0.00108	0.00191	41.39155	1743.229
5	0.5	0.5	0.01249	0.00626	9.84363	129.5283
		1.5	0.04625	0.04359	5.75658	41.74523
		2	0.05952	0.06515	5.23424	34.06376
		5	0.11131	0.18621	4.24021	21.55607
	1.5	0.5	0.00512	0.00179	9.84951	111.4646
		1.5	0.01827	0.01763	7.82813	66.54218
		2	0.02289	0.02681	7.577204	61.81638
		5	0.03965	0.07564	7.100170	53.25236
	2	0.5	0.00274	0.00089	12.43983	171.2925
		1.5	0.00982	0.00947	10.48454	116.2765
		2	0.01228	0.01446	10.23989	110.1829
		5	0.02111	0.04092	9.77559	98.95934
	5	0.5	0.0000 5	0.00001	78.65482	6389.314
		1.5	0.00018	0.00017	74.58790	5670.93
		2	0.00023	0.00027	73.88042	5545.666
		5	0.00038	0.00076	72.56972	5315.87

4.4 Entropies

The Rényi entropy of a random variable X represents a measure of variation of the uncertainty. The Rényi entropy is defined by

$$I_\delta(X) = (1-\delta)^{-1} \log\left(\int_{-\infty}^{\infty} f(x)^\delta \, dx\right), \ \delta > 0 \text{ and } \delta \neq 1.$$

Then, we can write

$$f(x)^\delta = \left(\frac{\theta\alpha}{e^\theta - 1}\right)^\delta exp\left\{-\delta\left[\frac{G(x)}{\overline{G}(x)}\right]^\alpha\right\} exp\left(\theta\delta - \theta\delta\, exp\left\{-\left[\frac{G(x)}{\overline{G}(x)}\right]^\alpha\right\}\right)$$

$$= \sum_{i,j=0}^{\infty} \frac{(-1)^j \theta^{\delta+i} \alpha^\delta \delta^i g(x)^\delta G(x)^{\delta\alpha-\delta}}{i!\,(e^\theta-1)^\delta \overline{G}(x)^{\delta\alpha+\delta}} \binom{i}{j} exp\left\{-(j+\delta)\left[\frac{G(x)}{\overline{G}(x)}\right]^\alpha\right\}.$$

After some algebra, we have

$$f(x)^\delta = \sum_{k,m=0}^{\infty} \tau_{k,m}\, g(x)^\delta G(x)^{\alpha k+\delta(\alpha-1)+m},$$

where

$$\tau_{k,m} = \frac{(-1)^{k+m}\alpha^\delta}{k!\,(e^\theta-1)^\delta} \binom{-\alpha k - \delta(\alpha+1)}{m} \sum_{i,j=0}^{\infty} \frac{(-1)^j \delta^i (j+\delta)^k \theta^{\delta+i}}{i!} \binom{i}{j}.$$

Then, the Rényi entropy can be expressed as

$$I_\delta(X) = (1-\delta)^{-1} \log\left[\sum_{k,m=0}^{\infty} \tau_{k,m} \int_{-\infty}^{\infty} g(x)^\delta G(x;\phi)^{\alpha k+\delta(\alpha-1)+m} dx\right].$$

The δ -entropy, say $H_\delta(X)$, can be obtained as

$$H_\delta(X) = \frac{1}{\delta-1} \log\left\{1 - \left[\sum_{k,m=0}^{\infty} \tau_{k,m} \int_{-\infty}^{\infty} g(x)^\delta G(x;\phi)^{\alpha k+\delta(\alpha-1)+m} dx\right]\right\}.$$

The Shannon entropy of a random variable X , say SI , is defined by

$$SI = E\{-[\log f(X)]\}.$$

The Shannon entropy is a special case of the Rényi entropy when $\delta \uparrow 1$ and it follows by taking the limit of $I_\delta(X)$ as δ tends to 1.

4.5 Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Let $X_1, \dots, X_n$ be a random sample from the WGP family of distributions. The PDF of i th order statistic, say $X_{i:n}$, can be written as

$$f_{i:n}(x) = \frac{f(x)}{B(i,n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F(x)^{j+i-1}. \quad (7)$$

Using (1), (2) and (7) we get

$$f(x)F(x)^{j+i-1} = \sum_{k,m=0}^{\infty} t_{k,m} \quad \pi_{\alpha(k+1)+m}(x), \quad (8)$$

where $\pi_k(x)$ is the Exp-G density with power parameter η and

$$t_{k,m} = \frac{\alpha(-1)^{k+m}}{k!\,(e^\theta - 1)^{j+i}} \sum_{l=0}^{j+i-1} \sum_{h,w=0}^{\infty} \frac{(-1)^{l+w}}{h!} \theta^{h+1} (j+i-l)^h (w+1)^k \times \binom{r}{l}\binom{h}{w}\binom{-[\alpha(k+1)+1]}{m}.$$

Substituting (8) in equation (7), the PDF of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = \sum_{k,m=0}^{\infty} \sum_{j=0}^{n-i} \frac{(-1)^j t_{k,m}}{B(i,n-i+1)} \binom{n-i}{j} \pi_{\alpha(k+1)+m}(x).$$

Then, the density function of the WGP order statistics is a mixture of Exp-G densities. Based on the last Equation, we note that the properties of $X_{i:n}$ follow from those properties of $Y_{\alpha(1+k)+m}$. For example, the moments of $X_{i:n}$ can be expressed as

$$E\left(X_{i:n}^q\right) = \sum_{k,m=0}^{\infty} \sum_{j=0}^{n-i} \frac{(-1)^j t_{k,m}}{B(i,n-i+1)} \binom{n-i}{j} E\left(Y_{\alpha(k+1)+m}^q\right).$$

4.6 Stress-strength model

Stress-strength model is the most widely approach used for reliability estimation. This model is used in many applications of physics and engineering such as strength failure and system collapse. In stress-strength modeling, say $R(X_1, X_2 | X_1 > X_2) = Pr(|X_1 > X_2)$, is a measure of reliability of the system when it is subjected to random stress X_2 and has strength X_1.

The system fails if and only if the applied stress is greater than its strength and the component will function satisfactorily whenever $X_1 > X_2$. $R(X_1, X_2 | X_1 > X_2)$ can be considered as a measure of system performance and naturally arise in electrical and electronic systems. Further, the reliability of the system is the probability that the system is strong enough to overcome the stress imposed on it.

Let X_1 and X_2 be two independent random variables have WGP $(\theta_1, \alpha_1, \phi)$ and WGP $(\theta_1, \alpha_1, \phi)$ distributions .The PDF of X_1 and the CDF of X_2 can be written from equations (1) and (2), respectively as

$$f_1(\theta_1, \alpha_1, \phi) = \sum_{k,m=0}^{\infty} \frac{\alpha_1(-1)^{k+m}}{k!\,(e^{\theta_1} - 1)} \binom{-\alpha_1(k+1)-1}{m} \times \sum_{i,j=0}^{\infty} \binom{i}{j} g(x;\phi) G(x;\phi)^{\alpha_1(k+1)+m-1}$$

and

$$F_2(\theta_2, \alpha_2, \phi) = \sum_{l,w=0}^{\infty} \frac{\alpha_2(-1)^{l+w}}{l!\,(e^{\theta_2} - 1)[\alpha_2(l+1)+w]} \binom{-\alpha_2(l+1)-1}{w}$$

$$\times \sum_{h,d=0}^{\infty} \frac{(-1)^d {\theta_2}^{h+1}}{h!\,(d+1)^{-l}} \binom{h}{d} G(x\,;\phi)^{\alpha_2(l+1)+w}.$$

Then, $R(X_1, X_2|X_1 > X_2)$ is given by

$$R(X_1, X_2|X_1 > X_2) = \int_0^{\infty} f_1(\theta_1, \alpha_1, \phi) F_2(\theta_2, \alpha_2, \phi)\, dx$$
$$= \sum_{k,m,l,w=0}^{\infty} d_{k,m,l,w},$$

where

$$d_{k,m,l,w} = \frac{\alpha_1 \alpha_2 (-1)^{k+m+l+w}}{k!\, l!\, (e^{\theta_1} - 1)(e^{\theta_2} - 1)[\alpha_2(l+1) + w]}$$
$$\times \binom{-\alpha_2(l+1) - 1}{w} \binom{-\alpha_1(k+1) - 1}{m}$$
$$\times \sum_{i,j,h,d=0}^{\infty} \frac{{\theta_1}^{i+1} {\theta_2}^{h+1} (-1)^{j+d} (j+1)^k (d+1)^l}{i!\, h!\, [\alpha_1(k+1) + \alpha_2(l+1) + m + w]} \binom{i}{j} \binom{h}{d}.$$

5. Estimation

Several approaches for parameter estimation were proposed in the literature but the maximum likelihood method is the most commonly employed. Here, we consider the estimation of the unknown parameters of the new family from complete samples only by maximum likelihood. Let $x_1, \dots, x_n$ be a random sample from the WGP family with parameters α, θ and ϕ . Let $\mathbf{\Theta}$ be the $p \times 1$ parameter vector. To obtain the MLE of Θ , the log-likelihood function, $\ell = \ell(\mathbf{\Theta})$, is given by

$$\ell = n \log \theta + n \log \alpha - n \log\left(e^{\theta} - 1\right) + (\alpha - 1) \sum_{i=0}^{n} \log G\,(x_i\,;\phi)$$
$$+ \sum_{i=0}^{n} \log g\,(x_i\,;\phi) - (\alpha + 1) \sum_{i=0}^{n} \log \overline{G}\,(x_i\,;\phi) - \sum_{i=0}^{n} p_i + \theta \sum_{i=0}^{n} q_i,$$

where $p_i = \left[G(x_i\,;\phi)/\overline{G}(x_i\,;\phi)\right]^{\alpha}$ and $q_i = 1 - exp(-p_i)$.

The components of the score vector are

$$U_{\theta} = \frac{n}{\theta} - \frac{n e^{\theta}}{(e^{\theta} - 1)} + \sum_{i=0}^{n} q_i,$$
$$U_{\alpha} = \frac{n}{\alpha} + \sum_{i=0}^{n} \log G\,(x_i\,;\phi) - \sum_{i=0}^{n} \log \overline{G}\,(x\,;\phi)$$
$$- \sum_{i=0}^{n} p_i \log\left[\frac{G(x_i\,;\phi)}{\overline{G}(x_i\,;\phi)}\right] + \theta \sum_{i=0}^{n} p_i\, e^{-p_i} \log\left[\frac{G(x_i\,;\phi)}{\overline{G}(x_i\,;\phi)}\right]$$

and

$$U_{\phi_k} = \sum_{i=0}^{n} \frac{g'(x_i\,;\phi)}{g(x_i\,;\phi)} + (\alpha - 1) \sum_{i=0}^{n} \frac{G'(x_i\,;\phi)}{G(x_i\,;\phi)} - \alpha \sum_{i=0}^{n} \frac{G'(x_i\,;\phi)}{\overline{G}(x_i\,;\phi)^2} \left[\frac{G(x_i\,;\phi)}{\overline{G}(x_i\,;\phi)}\right]^{\alpha-1}$$
$$+ (\alpha + 1) \sum_{i=0}^{n} \frac{G'(x_i\,;\phi)}{\overline{G}(x\,;\phi)} + \alpha\theta \sum_{i=0}^{n} e^{-p_i} \frac{G'(x_i\,;\phi)}{\overline{G}(x_i\,;\phi)^2} \left[\frac{G(x_i\,;\phi)}{\overline{G}(x_i\,;\phi)}\right]^{\alpha-1},$$

where $g'(x_i\,;\phi) = \partial g(x_i\,;\phi)/\partial \phi_k$ and $G'(x_i\,;\phi) = \partial G(x_i\,;\phi)/\partial \phi_k$.

Setting the nonlinear system of equations $U_{\theta} = U_{\alpha} = 0$ and $U_{\phi_k} = 0$ and solving them simultaneously yields the MLEs. For doing this, it is usually more convenient to adopt nonlinear optimization methods such as the quasi-Newton algorithm to maximize ℓ numerically. For interval estimation of the parameters, we obtain the $p \times p$ observed information matrix $J(\mathbf{\Theta}) = \{\frac{\partial^2 \ell}{\partial r \partial s}\}$ (for $r, s = \theta, \alpha, \phi$), whose elements can be computed numerically. Under standard regularity conditions when $n \to \infty$, the distribution of $\widehat{\mathbf{\Theta}}$ can be approximated by a multivariate normal distribution to obtain confidence intervals for the parameters. Here, $\boldsymbol{J}(\widehat{\mathbf{\Theta}})$, is the total observed information matrix evaluated at $\widehat{\mathbf{\Theta}}$. The elements of $J(\mathbf{\Theta})$ are given in the Appendix A.

6. Simulation study

In this section, we evaluate the performance of the MLEs by using Monte Carlo simulation for different sample sizes and different parameter values. We choose PWEE model for this purpose. The simulation study is repeated 10,000 times each with sample sizes $n = 25,50,75,100,200,400$ and parameter combinations
I: $\beta = 0.5$, $\alpha = 0.5$, $\lambda = 1$, and **II:** $\beta = 0.5$, $\alpha = 1.5$, $\lambda = 2$.

Table 2 presents the average bias (Bias), Mean Square Error (MSE), Coverage Probability (CP), average lower bound (LB) and average upper bound (UB) values of the parameters β , α and λ for different sample sizes. From the results, we can verify that the Bias and MSEs decreases as the sample size n increases. The CP of the confidence intervals are quite close to the nominal level of 95%. Therefore, the MLEs and their asymptotic results can be used for estimating and constructing confidence intervals even for reasonably small sample sizes.

Table 2: Monte Carlo simulation results: Bias, MSE, CP, LB and UB.

	n	Bias	MSE	CP	LB	UB	n	Bias	MSE	CP	LB	UB
			I						II			
β	25	0.821	1.491	0.94	0.133	2.272	25	0.525	1.164	0.93	0.138	2.989
	50	0.553	0.165	0.94	0.738	1.838	50	1.169	1.006	0.98	0.761	2.100
	75	0.414	0.093	0.94	0.407	1.214	75	0.817	0.592	0.98	0.228	1.561
	100	0.320	0.047	0.95	0.254	1.857	100	0.598	0.046	0.90	0.542	1.727
	200	0.182	0.017	0.95	0.139	1.293	200	0.264	0.029	0.92	0.300	1.463
	400	0.088	0.005	0.93	0.241	0.943	400	0.098	0.009	0.92	0.482	0.765
α	25	0.009	0.252	0.79	0.519	1.534	25	0.094	0.400	0.92	0.651	1.838
	50	-0.017	0.152	0.78	0.218	1.176	50	0.175	0.526	0.94	0.492	1.830
	75	-0.029	0.098	0.79	0.112	1.019	75	0.147	0.404	0.94	0.286	1.540
	100	-0.032	0.073	0.82	0.088	0.949	100	0.146	0.098	0.94	0.248	1.458
	200	-0.034	0.032	0.84	0.149	0.797	200	0.092	0.055	0.91	0.168	1.109
	400	-0.023	0.006	0.86	0.248	0.716	400	0.070	0.005	0.91	0.230	0.922
λ	25	1.703	1.394	0.92	0.333	4.727	25	0.640	1.973	0.93	0.069	1.292
	50	1.222	0.256	0.92	1.719	2.142	50	0.316	0.964	0.90	0.574	1.057
	75	0.942	0.103	0.93	0.178	1.913	75	0.251	0.698	0.93	0.407	2.666
	100	0.764	0.092	0.94	0.774	1.249	100	0.179	0.511	0.89	0.376	2.402
	200	0.448	0.063	0.93	0.319	1.004	200	0.063	0.249	0.91	0.362	1.942
	400	0.233	0.012	0.92	0.322	1.009	400	0.013	0.040	0.92	0.426	1.658

7. Applications

In this section, we consider three applications to three real data sets to illustrate the flexibility of the new family of distribution. We also analyzed the hazard rates of these three data sets. In order to identify the shapes of data, we consider the graphical method based on total time on test (TTT) transformed, introduced by Barlow and Campo (1975). The empirical illustration of TTT transform is given by Aarset (1987).

The first data set presents increasing-shaped (unimodal) hazard function while the second and third data sets present upside-down bathtub shaped hazard function. From Figure 3(a), the TTT plot for the data set 1 shows that hazard function $\tau(x)$ is concave giving an indication of increasing shape, while in Figures 4(c) and 5(e), TTT-plot for the data sets 2 and 3 show that the hazard rate function is first concave and then convex, giving an indication of upside-down bathtub shape. Hence, the WGP family could be in principle an appropriate model for fitting these data sets.

The Figures 8, 10 and 12, we consider kernel density estimation (a non- parametric approach) with Gaussian Filter. Let $X_1, X_2, \ldots, X_n$ be an independently identically distributed (IID) random vector of variables which follows an unknown distribution f. The kernel density estimator is given by

$$\hat{f}_h(x) = \frac{1}{n}\sum_{i=1}^{n} K_h(x - x_i) = \frac{1}{nh}\sum_{i=1}^{n} K\left[\frac{x - x_j}{h}\right]$$

where $K(.)$ is the kernel function usually symmetric and $\int_{-\infty}^{\infty} K(x)dx = 1$, and $h > 0$ is a smoothing parameter, also known as bandwidth.

The MLEs are calculated and the goodness-of-fit statistics including the log-likelihood function evaluated at the MLEs, Akaike information criterion (AIC), Kolmogorov-Smirnov (K-S) and its P-value are determined to compare the fitted models. The required computations are carried out in the R-language.

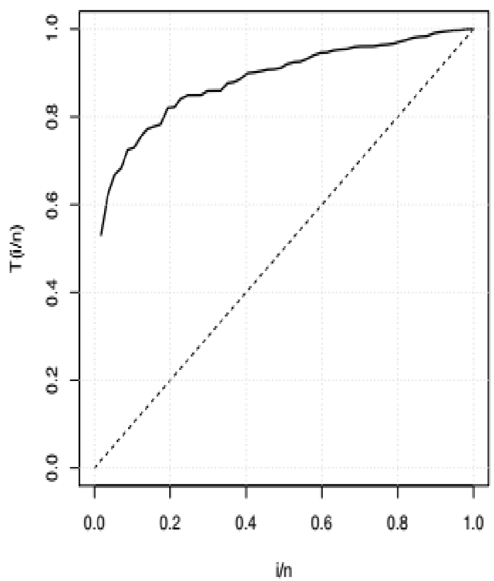

Figure 7: TTT plot for data set 1.

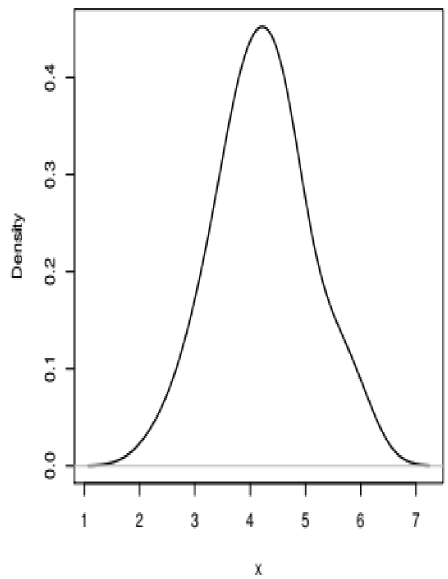

Figure 8: Gaussian kernel density estimation for data set 1.

The first data set (Crowder et. al [21]) refers to the failure stresses of single carbon fibers (length 1mm). The data are: 2.247, 2.64, 2.842, 2.908, 3.099, 3.126, 3.245, 3.328, 3.355, 3.383, 3.572, 3.581, 3.681, 3.726, 3.727, 3.728, 3.783, 3.785, 3.786, 3.896, 3.912, 3.964, 4.05, 4.063, 4.082, 4.111, 4.118, 4.141, 4.216, 4.251, 4.262, 4.326, 4.402, 4.457, 4.466, 4.519, 4.542, 4.555, 4.614, 4.632, 4.634, 4.636, 4.678, 4.698, 4.738, 4.832, 4.924, 5.043, 5.099, 5.134, 5.359, 5.473, 5.571, 5.684, 5.721, 5.998, 6.06. A summary of these data is:

$$n = 57,\ \bar{x} = 4.2350,\ s = 0.8352,\ skewness = 0.0710,\ kurtosis = 2.7098.$$

Based on the figures in table 1, we conclude that all the models provide the adequate fit, whereas that EW and GEE provides the best fit followed by WEEP and WLLP. The summary statistics and figure indicate that the first data set is approximately symmetric. This indicates that the new family of distributions has the ability to fit data set with symmetric shape. The P-P plot given in Figure 13 also supports the results of Table 3.

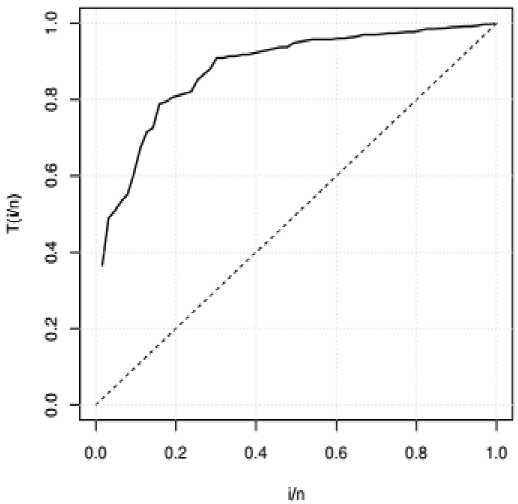

Figure 9: TTT plot for data set 2.

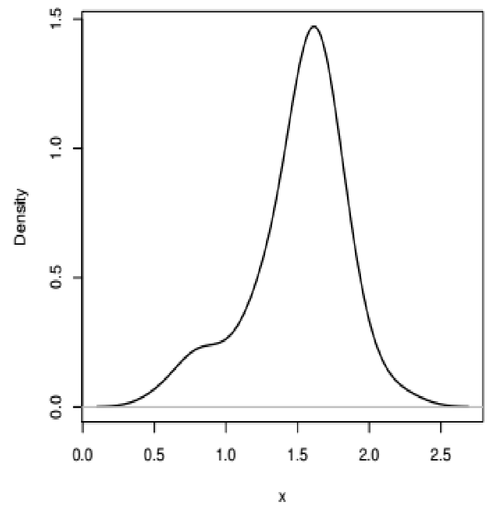

Figure 10: Gaussian kernel density estimation for data set 2.

Table 3: MLEs, their standard errors (in parentheses) and goodness of fit measures for the first data set.

Distribution	Estimates			$-\ell$	AIC	K-S	P-value
WGP(a, α, b)	18.626	0.669	0.239	71.972	149.944	0.119	0.366
	(1.614)	(0.384)	(0.242)				
WLLP(c, α, s)	0.850	5.779	4.316	71.440	148.880	0.0836	0.789
	(0.036)	(1.082)	(0.120)				
WEEP(β, α, λ)	73.665	0.718	1.090	70.368	146.736	0.0817	0.812
	(1.403)	(0.217)	(0.294)				
GEE(α, β, δ)	17.195	33.7264	0.2234	70.656	147.313	0.0609	0.975
	(42.158)	(10.8439)	(0.4127)				
EW(c, α, λ)	3.4586	2.6978	3.7682	70.049	146.098	0.059	0.982
	(1.2814)	(2.1688)	(0.7208)				

The second data set corresponds to strengths of 15 cm fibres reported by Smith and Naylor (1987). The summary statistics of the second data set are:

$$n = 46, \ \bar{x}=1.13, \ s=0.2713, \ \textit{skewness} = 0.7935 \text{ and } \textit{kurtosis} = 0.5995.$$

From figures in table 3, we conclude that WGP and WEEP models provide the adequate fit, whereas that GEE do not provide the god fit. The summary statistics and figure indicate that the first data set is approximately left skewed. This indicates that the new family of distributions can fit data set with left skewed characteristic. The P-P plot given in figure 14 also supports the results of table 3

Table 4: MLEs, their standard errors (in parentheses) and goodness of fit measures for the second data set.

Distribution	Estimates			$-\ell$	AIC	K-S	P-value
WGP(a, α, b)	0.7325	4.6132	3.5440	13.740	33.480	0.134	0.207
	(1.3366)	(5.8305)	(10.130)				
WLLP(c, α, s)	1.9295	2.7085	1.5454	14.149	34.299	0.141	0.166
	(0.5286)	(0.8715)	(0.0374)				
WEEP(β, α, λ)	0.6057	5.2402	0.2471	13.734	33.469	0.134	0.208
	(0.0320)	(0.7194)	(1.8881)				
GEE(α, β, δ)	18.814	24.6684	0.4336	24.509	55.0189	0.2186	0.0048
	(20.375)	(7.0427)	(0.3939)				
EW(c, α, λ)	7.2846	0.6712	1.7180	14.675	35.351	0.146	0.135
	(1.7069)	(0.2488)	(0.0860)				

The third data set describes the 101 stress-rupture lives of 49 kevlar epoxy strands, which were subjected to constant sustained pressure at the 90 stress level until all had failed, so that we have complete data with exact times of failure. The failure times (in hours) are given in Cooray and Ananda (2008). The summary statistics of the first data set are:

$$n = 101,\ \bar{x} = 1.0248,\ s = 1.1193,\ skewness = 3.00172 \text{ and } kurtosis = 13.7089.$$

From the figures in table 5, we verify that WGLLP, WEEP and WGP provides the best fit. A close look at the summary statistics and figure 15 indicate that the third data are right skewed. So, the proposed family has the ability to fit right skewed data. The P-P plot in figure also supports the result in table 5.

Table 5: MLEs, their standard errors (in parentheses) and goodness of fit measures for the third data set.

Distribution	Estimates			$-\ell$	AIC	K-S	P-value
WGP(a, α, b)	0.7325	4.6132	3.5440	104.279	214.558	0.081	0.509
	(1.3366)	(5.8305)	(10.130)				
WLLP(c, α, s)	1.9295	2.7085	1.5454	102.398	210.797	0.078	0.561
	(0.5286)	(0.8715)	(0.0374)				
WEEP(β, α, λ)	0.6057	5.2402	0.2471	103.936	213.873	0.079	0.544
	(0.0320)	(0.7194)	(1.8881)				
GEE(α, β, δ)	18.8149	24.6684	0.4336	103.936	213.873	0.079	0.544
	(20.3751)	(7.0427)	(0.3939)				
EW(c, α, λ)	7.2846	0.6712	1.7180	102.787	211.574	0.084	0.468
	(1.7069)	(0.2488)	(0.0860)				

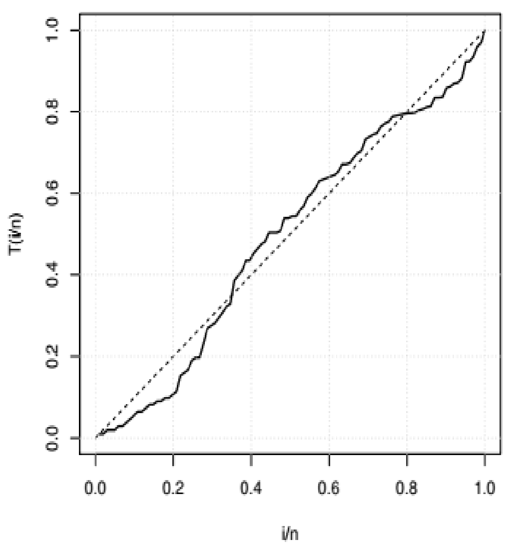

Figure 11: TTT plot for data set 3.

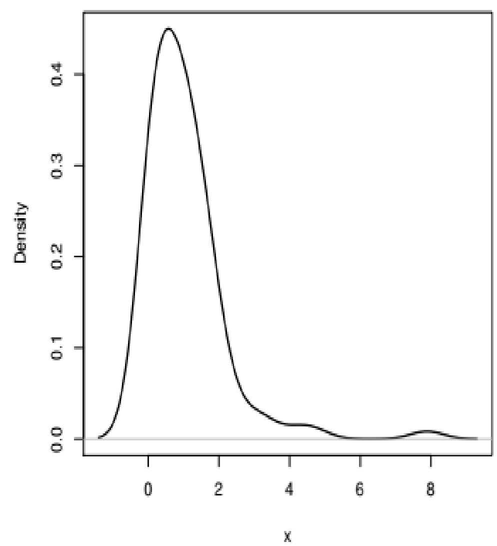

Figure 12: Gaussian kernel density estimation for data set 3.

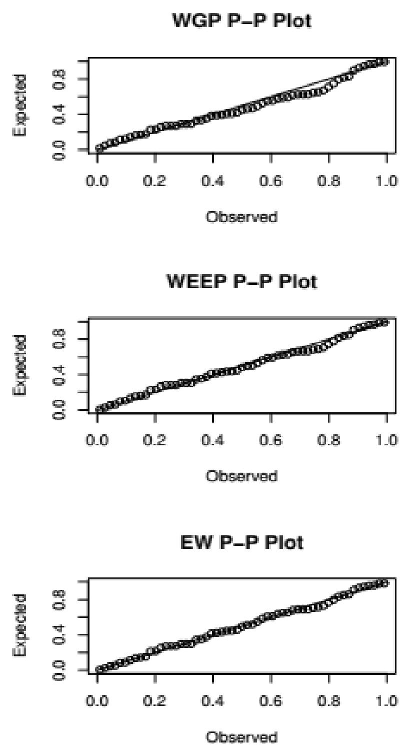

Figure 13: P-P plots of WGP, WLLP, WEEP, GEE and EW models for data set 1.

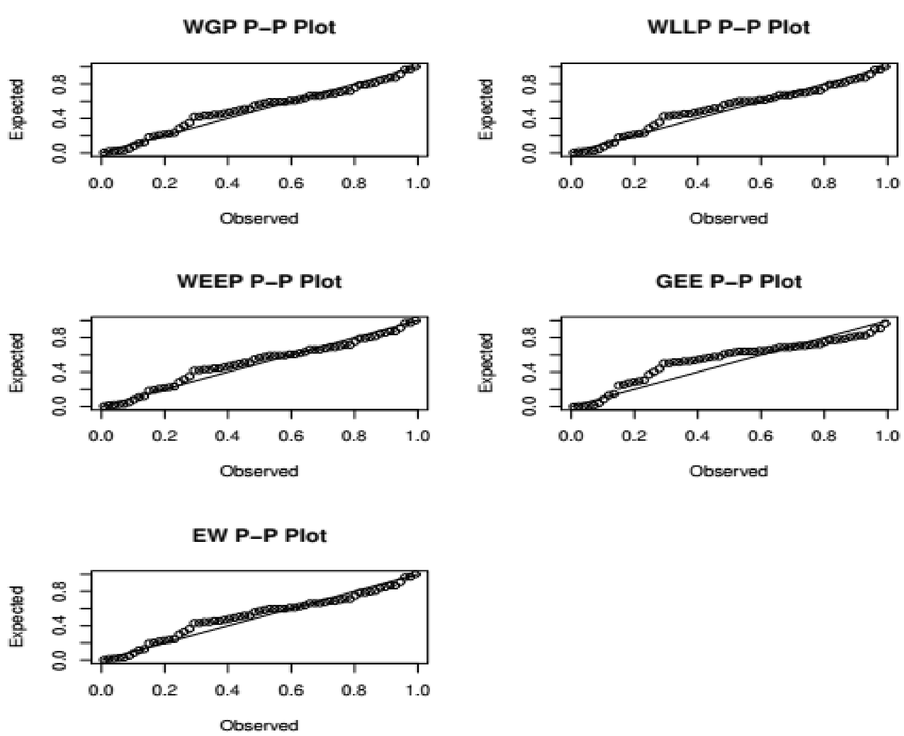

Figure 14: P-P plots of WGP, WLLP, WEEP, GEE and EW models for data set 2.

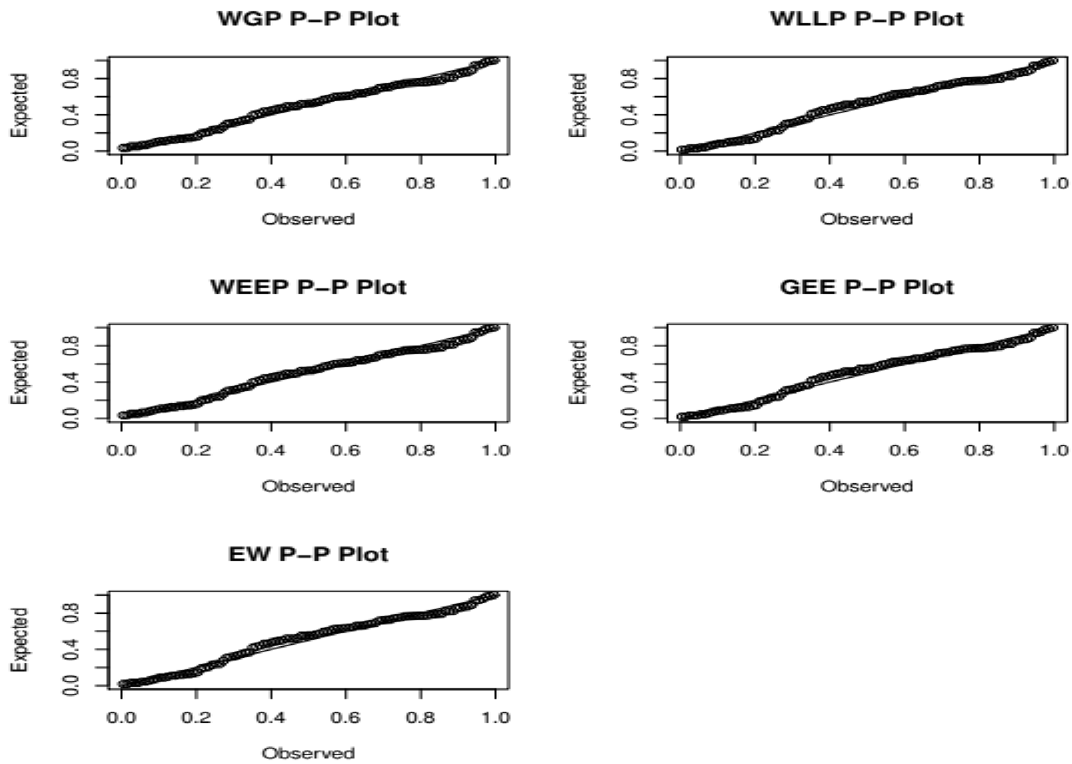

Figure 15: P-P plots of WGP, WLLP, WEEP, GEE and EW models for data set 3.

8. Conclusions

In this paper, we present a new Weibull-G Poisson (WGP) family of distributions, which extends the Weibull-G family by adding one extra shape parameter. Some mathematical properties of the new family including explicit expressions for the ordinary and incomplete moments, quantile and generating functions, mean deviations, entropies and order statistics are provided. The model parameters are estimated by maximum likelihood and the observed information matrix is determined. We perform a Monte Carlo simulation study to assess the finite sample behavior of the maximum likelihood estimators. We prove empirically by means of three real data sets that some special models of the WGP family can give better fits than other models generated from well-known families.

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Appendix A

The elements of the observed matrix $J(\Theta)$ are given below:

$$U_{\theta\theta} = \frac{-n}{\theta^2} - \frac{ne^{\theta}\left(e^{\theta}-1\right) - ne^{2\theta}}{(e^{\theta}-1)^2}, U_{\theta\alpha} = \sum_{i=0}^{n} a_i, U_{\theta\phi} = \sum_{i=0}^{n} b_i,$$

$$U_{\alpha\alpha} = \frac{-n}{\alpha^2}\sum_{i=0}^{n}[\partial d_i/\partial\alpha] + \theta\sum_{i=0}^{n}[\partial a_i/\partial\alpha],$$

$$U_{\alpha\psi} = \sum_{i=0}^{n}\frac{G'(x_i\,;\phi)}{G(x_i\phi)} + \sum_{i=0}^{n}\frac{G'(x_i\,;\phi)}{\overline{G}(x_i\,;\phi)} - \sum_{i=0}^{n}[\partial d_i/\partial\phi] + \theta\sum_{i=0}^{n}[\partial a_i/\partial\phi]$$

and

$$U_{\phi_k\phi_k} = \sum_{i=0}^{n}\frac{g(x_i\,;\phi)g''(x_i\,;\phi) - [g'(x_i\,;\phi)]^2}{g(x_i\,;\phi)^2} + (\alpha-1)\sum_{i=0}^{n}\frac{G(x_i\phi)G''(x_i\phi) - [G'(x_i\,;\phi)]^2}{G(x_i\,;\phi)^2}$$
$$+(\alpha+1)\sum_{i=0}^{n}\frac{\overline{G}(x\,;\phi)G''(x_i\,;\phi) + [G'(x_i\,;\phi)]^2}{\overline{G}(x\,;\phi)^2} - \sum_{i=0}^{n}[\partial t_i/\partial\phi] + \theta\sum_{i=0}^{n}[\partial b_i/\partial\phi].$$

where $g''(x_i\,;\phi) = \partial^2 g(x_i\,;\phi)/\partial\phi_k^2$ and $G''(x_i\,;\phi) = \partial^2 G(x_i\,;\phi)/\partial\phi_k^2$.