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Transmuted Topp-Leone Weibull Lifetime Distribution: Statistical Properties and Different Method of Estimation

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Abstract

In this work we focus on proposing a new lifetime Weibull type model called the transmuted Topp-Leone Weibull and studying its properties. We derive some new bivariate and multivariate transmuted Topp-Leone Weibull versions using "Farlie Gumbel Morgenstern (FGM) Copula", "modified FGM Copula", "Clayton Copula" and "Renyi's entropy Copula". The estimation of its unknown parameters is carried out by considering different method of estimation. The statistical performances of all methods are studied by two real data sets and a numerical Monte Carlo simulation. The Cramer-Von Mises method is the best method for modeling the carbon fibers data. The maximum likelihood method is the best method for modeling the Leukemia data, however all other methods performed well.

Key Words: Weibull Distribution, Farlie Gumbel Morgenstern; Clayton Copula; Least Square; Topp-Leone; Likelihood Estimation; Weighted Least Square; Cramer-Von-Mises; Bootstrapping.

Mathematical Subject Classification: 62N01; 62N02; 62E10.

1. Introduction and genesis

Consider the two-parameters Weibull (W) type lifetime model with probability density function (P.D.F) and cumulative function (C.D.F) are given (for $w \geq 0$) as

$$g_{a,b}(w) = ba^b x^{b-1} e^{-(aw)^b}, \tag{1}$$

and

$$G_{a,b}(w) = 1 - e^{-(aw)^b}, \tag{2}$$

Where parameter $a > 0$ dominate the shape of the model however $b > 0$ is a scale parameter. Clearly, when $a = 0$ we have the one parameter Weibull model. In this paper, we study a new model called the transmuted Topp-Leone-W (TTL-W) model. Based on the TTL-G family, we construct the new four-parameter TTL-W model. Let $g_{\underline{\boldsymbol{\varphi}}}(w) = \frac{d}{dw} G_{\underline{\boldsymbol{\varphi}}}(w)$ and $G_{\underline{\boldsymbol{\varphi}}}(w)$ denote the P.D.F and the C.D.F of the baseline model with parameter vector $\underline{\boldsymbol{\varphi}} = (a, b)$ in our case. Using the transmuted G (T-G) family pioneered by Shaw and Buckley (2007) and the Topp-Leone G (TL-G) family pioneered by Rezaei et al. (2016), Yousof et al. (2017b) defined the C.D.F of their transmuted Topp-Leone G (TTL-G) family by

$$F_{\lambda,\alpha,\underline{\boldsymbol{\varphi}}}(w) = -\lambda\left\{1 - \left[-G_{\underline{\boldsymbol{\varphi}}}(w) + 1\right]^2\right\}^{2\alpha} + (1+\lambda)\left\{1 - \left[-G_{\underline{\boldsymbol{\varphi}}}(w) + 1\right]^2\right\}^{\alpha}. \tag{3}$$

The P.D.F of the TTL-G family is given by

$$f_{\lambda,\alpha,\underline{\boldsymbol{\varphi}}}(w) = 2\alpha g_{\underline{\boldsymbol{\varphi}}}(w)\left[-G_{\underline{\boldsymbol{\varphi}}}(w)+1\right]\frac{1-2\lambda\left\{1-\left[-G_{\underline{\boldsymbol{\varphi}}}(w)\right]^{2}+1\right\}^{\alpha}+\lambda}{\left\{-\left[1-G_{\underline{\boldsymbol{\varphi}}}(w)+1\right]^{2}\right\}^{1-\alpha}}. \quad (4)$$

***Theorem* 1:**

Let T be a random variable (rv) having the well-known Exponentiated W (Exp-W) distribution with positive parameters a, b and $\boldsymbol{\delta}$. Then the P.D.F and C.D.F of T are given by $\boldsymbol{\pi}_{\boldsymbol{\delta},a,b}(t) = \boldsymbol{\delta} b a^{b} t^{b-1}\left[-e^{-(at)^{b}}+1\right]^{\boldsymbol{\delta}-1}$ $e^{-(at)^{b}}$and $\boldsymbol{\Pi}_{\boldsymbol{\delta},a,b}(t) = \left[-e^{-(at)^{b}}+1\right]^{\boldsymbol{\delta}}$ where $\boldsymbol{\pi}_{\delta,a,b}(t) = \frac{d}{dt}\boldsymbol{\Pi}_{\boldsymbol{\delta},a,b}(t)$. The qth ordinary (μ_q') and incomplete moment $(\boldsymbol{\phi}_q(t))$ of T are given by

$$\mu_q'|_{[q>-b]} = \boldsymbol{\Gamma}\left(1+\frac{q}{b}\right)\sum_{\boldsymbol{\tau}=0}^{\infty} a^{-q}\ \boldsymbol{\delta}(-1)^{\boldsymbol{\tau}}(\boldsymbol{\tau}+1)^{-(q+b)/b}\binom{\boldsymbol{\delta}-1}{\boldsymbol{\tau}},$$

and

$$\boldsymbol{\phi}_q(t)|_{[q>-b]} = \boldsymbol{\gamma}\left(1+\frac{q}{b},(at)^{b}\right)\sum_{\boldsymbol{\tau}=0}^{\infty} a^{-q}\,\boldsymbol{\delta}(-1)^{\boldsymbol{\tau}}(\boldsymbol{\tau}+1)^{-(q+b)/b}\binom{\boldsymbol{\delta}-1}{\boldsymbol{\tau}},$$

Respectively, where $\gamma(\nu,q)$ is the incomplete gamma function

$$\gamma(V,\mathbf{c})|_{(V\neq 0,-1,-2,\dots)} = \int_0^{\mathbf{c}} t^{V-1}\,exp(-t)\,dt = \frac{\mathbf{c}^{\nu}}{V}\{1F_1[V;V\mathbf{c}+1;-\mathbf{c}]\} = \sum_{\kappa=0}^{\infty}\frac{(-1)^{\kappa}}{\kappa!\,(V+\kappa)}\mathbf{c}^{V+\kappa},$$

the function $1F_1[\cdot,\cdot,\cdot]$ is a called the confluent hypergeometric function $\boldsymbol{\Gamma}(V,\mathbf{c})|_{(t>0)} = \int_{\mathbf{c}}^{\infty} t^{V-1}\,exp(-t)\,dt$ and $\boldsymbol{\Gamma}(V,\mathbf{c})+\gamma(V,\mathbf{c}) = \boldsymbol{\Gamma}(V)$. Using (2) and (3) we get the C.D.F of the TTL-W model as

$$F_{\underline{\boldsymbol{\Psi}}}(w) = (1+\lambda)\left[-e^{-2(aw)^{b}}+1\right]^{\alpha} - \lambda\left[-e^{-2(aw)^{b}}+1\right]^{2\alpha}, \quad (5)$$

where $\underline{\boldsymbol{\Psi}} = (\lambda,\alpha,a,b)$. The corresponding P.D.F is obtained by

$$f_{\underline{\boldsymbol{\Psi}}}(w) = 2\alpha b a^{b} w^{b-1} e^{-2(aw)^{b}}\frac{\left\{-2\lambda\left[1-e^{-2(aw)^{b}}\right]^{\alpha}+\lambda+1\right\}}{\left[-e^{-2(aw)^{b}}+1\right]^{1-\alpha}}. \quad (6)$$

Table 1 gives some sub models of the TTL-W model. As illustrated in Table 1, the new model generalizes fifteen sub models. The hazard rate function (H.R.F) of the TTL-W model can calculated from $f_{\underline{\boldsymbol{\Psi}}}(w)/\left[1-F_{\underline{\boldsymbol{\Psi}}}(w)\right]$. Figure 1 provides some plots of the P.D.F and H.R.F of the TTL-W model to show its flexibility. The P.D.F can be unimodal and right skewed. The H.R.F can be "upside down", "monotone-decreasing", "bathtub (**U**-shaped)", "monotone-increasing", "**J** shaped" and "constant" shapes.

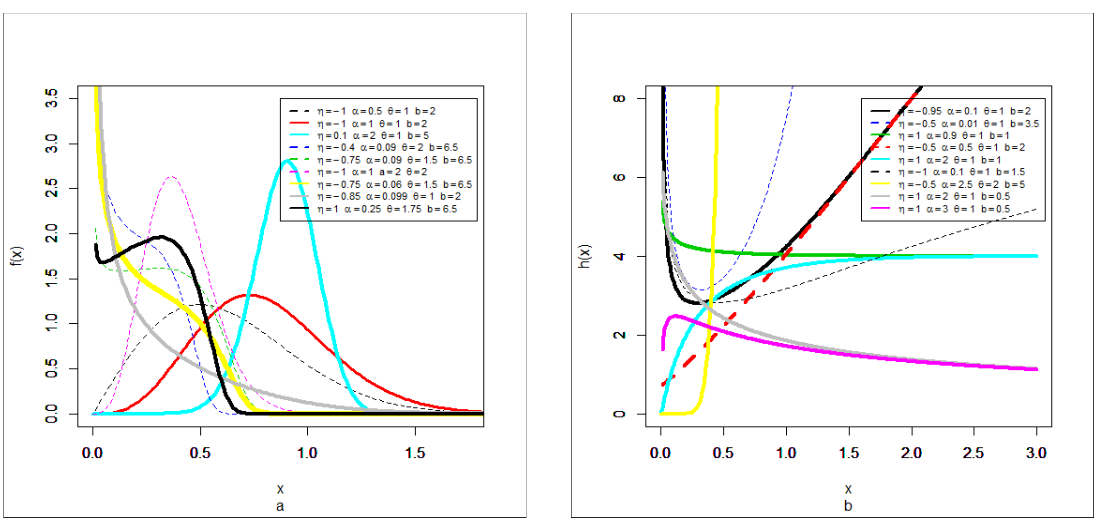

Figure 1: Plots of the TTL-W P.D.F and H.R.F for selected parameter values.

Table 1: Sub models of the TTL-W model.

	λ	α	a	b	Reduced model
1	0	α	a	b	ExpTL-W
2	0	α	a	2	ExpTL-Rayleigh
3	0	α	a	1	ExpTL-exponential
4	0	1	a	b	TL-W
5	0	1	a	2	TL-Rayleigh
6	0	1	a	1	TL-exponential
7	0	1	1	b	TL-W
8	0	1	1	2	TL-Rayleigh
9	0	1	1	1	TL-exponential
10	λ	1	a	b	T-W
11	λ	1	a	2	T-Rayleigh
12	λ	1	a	1	T-exponential
13	λ	1	1	b	T-W
14	λ	1	1	2	T- Rayleigh
15	λ	1	1	1	T-exponential

The C.D.F of the TTL-W model in (5) can be expressed as

$$F_{\underline{\Psi}}(w) = (1+\lambda)\sum_{\tau=0}^{\infty}(-1)^{\tau}\binom{\alpha}{\tau}e^{-2\tau(aw)^{b}} - \lambda\sum_{\tau=0}^{\infty}(-1)^{\tau}\binom{\alpha}{\tau}e^{-2\tau(aw)^{b}}, \quad (7)$$

which can be rewritten as

$$F_{\underline{\Psi}}(w) = (1+\lambda)\sum_{\tau=0}^{\infty}\sum_{k=0}^{2\tau}(-1)^{\tau+k}e^{-k(axw}\binom{\alpha}{\tau}\binom{2\tau}{k} - \lambda\sum_{\tau=0}^{\infty}\sum_{k=0}^{2\tau}(-1)^{\tau+k}e^{-k(aw)^{b}}\binom{2\alpha}{\tau}\binom{2\tau}{k}.$$

Then, after some algebra we have $F_{\underline{\Psi}}(w) = \sum_{\tau=0}^{\infty}\sum_{k=0}^{2\tau}\xi_{h,k}\, e^{-k(aw)^{b}}$, then

$$F_{\underline{\Psi}}(w) = \sum_{\tau=0}^{\infty}\sum_{k=0}^{2\tau}\xi_{\tau,k}\,\Pi_{k}(w), \quad (8)$$

where $\xi_{\tau,k} = (-1)^{\tau+k}\left[\binom{\alpha}{\tau} + \lambda\binom{\alpha}{\tau} - \lambda\binom{2\alpha}{\tau}\right]\binom{2\tau}{k}$, and $\Pi_{\delta}(w) = \left[G_{a,b}(w)\right]^{\delta}$ is the C.D.F of the Exp-W distribution with power parameter δ. Then

$$f_{\underline{\Psi}}(w) = \sum_{\tau=0}^{\infty}\sum_{k=0}^{2\tau}\xi_{\tau,k}\;\pi_{k}(w), \quad (9)$$

where $\pi_{\delta}(w) = \delta g_{a,b}(w)\left[G_{a,b}(w)\right]^{\delta-1}$ is the P.D.F of the Exp-W distribution with power parameter δ. Some useful extension of the Weibull model is recently developed by Yousof. etal., (2015) (transmuted exponentiated generalized W.), Aryal. etal., (2017) (Topp Leone generated W.), Aryal. and Yousof. (2017) (exponentiated. generalized. W. Poisson), Merovci. Et al., (2017) (exponentiated transmuted W.), Yousof. et al., (2017a) (new four-parameter W.), Brito et al., (2017) (Topp-Leone Odd Log-Logistic W.), Hamedani. et al., (2018) (Type I general exponential W.), Alizadeh. etal. (2017) (generalized. odd generalized. exponential W.), Almamy. et al. (2018) (odd Lindley W.), Cordeiro. et al., (2018) (Burr XII W.), Yousof. et al., (2018) (Burr-Hatke W.), Korkmaz et al., (2019) (Odd Power Lindley W.), among others.

2. Simple type Copula

2.2 Bivariate TTL-W (BvTTL-W) type using FGM Copula

Consider the joint C.D.F (J.C.D.F) of the FGM family (Gumbel (1960), Morgenstern (1956), Farlie, D. J. G. (1960), Gumbel (1961), Johnson and Kotz (1977) and Johnson and Kotz (1975)), then $C_{\mathcal{B}}(q, p) = qp(1 + \mathcal{B}\overline{q}\overline{p})$, where the continuous marginal function $q \in [0,1]$, $p \in [0,1]$, $\mathcal{B} \in [-1,1]$ is a dependence parameter and for every $C_{\mathcal{B}}(q, 0) =$

$C_{\mathcal{B}}(0,\boldsymbol{p})=0\,|_{(\boldsymbol{q},\boldsymbol{p}\in(0,1))}$ "grounded minimum" and $C_{\mathcal{B}}(\boldsymbol{q},1)=\boldsymbol{q}$ and $C_{\mathcal{B}}(1,\boldsymbol{p})=\boldsymbol{p}$ "grounded maximum". Then, setting $\overline{\boldsymbol{q}}=\overline{\boldsymbol{q}}_{\underline{\boldsymbol{\Psi}}_1}|_{\underline{\boldsymbol{\Psi}}_1>0}$, and $\overline{\boldsymbol{p}}=\overline{\boldsymbol{p}}_{\underline{\boldsymbol{\Psi}}_2}|_{\underline{\boldsymbol{\Psi}}_2>0}$. Where $C_{\mathcal{B}}(\boldsymbol{q},\boldsymbol{p})=F(y_1,y_2)=C(F_{\underline{\boldsymbol{\Psi}}_1}(y_1),F_{\underline{\boldsymbol{\Psi}}_2}(y_2))$. The J.P.D.F can be derived from $c_{\mathcal{B}}(\boldsymbol{q},\boldsymbol{p})=1+\mathcal{B}\boldsymbol{q}^*\boldsymbol{p}^*|_{(\boldsymbol{q}^*=1-2\boldsymbol{q}\text{ and }\boldsymbol{p}^*=1-2\boldsymbol{p})}$ or from $f(y_1,y_2)=f_{\underline{\boldsymbol{\Psi}}_1}(y_1)f_{\underline{\boldsymbol{\Psi}}_2}(y_2)c(F_{\underline{\boldsymbol{\Psi}}_1}(y_1),F_{\underline{\boldsymbol{\Psi}}_2}(y_2))$.

2.3 BvTTL-W type via modified FGM Copula

Consider the following modified FGM copula defined as (Rodriguez-Lallena and Ubeda-Flores (2004)) where $C_{\mathcal{B}}(\boldsymbol{q},\boldsymbol{p})=\boldsymbol{q}\boldsymbol{p}+\mathcal{B}\widetilde{\boldsymbol{\phi}(\boldsymbol{q})}\widetilde{\boldsymbol{\psi}(\boldsymbol{p})}$, $\widetilde{\boldsymbol{\phi}(\boldsymbol{q})}=\boldsymbol{q}\overline{\boldsymbol{\phi}(\boldsymbol{q})}$ and $\widetilde{\boldsymbol{\psi}(\boldsymbol{p})}=\boldsymbol{p}\overline{\boldsymbol{\psi}(\boldsymbol{p})}$. Here $\boldsymbol{\phi}(\boldsymbol{q})$ and $\boldsymbol{\psi}(\boldsymbol{p})$ are two absolutely continuous functions on (0,1) where $\boldsymbol{\phi}(0)=\boldsymbol{\phi}(1)=\boldsymbol{\psi}(0)=\boldsymbol{\psi}(1)=0$. Let

$$a=inf\left\{\frac{\partial}{\partial\boldsymbol{q}}\widetilde{\boldsymbol{\phi}(\boldsymbol{q})}:\mathcal{H}_1\right\}<0, b=sup\left\{\frac{\partial}{\partial\boldsymbol{q}}\widetilde{\boldsymbol{\phi}(\boldsymbol{q})}:\mathcal{H}_1\right\}<0,$$

$$c=inf\left\{\frac{\partial}{\partial\boldsymbol{p}}\widetilde{\boldsymbol{\psi}(\boldsymbol{p})}:\mathcal{H}_2\right\}>0, d=sup\left\{\frac{\partial}{\partial\boldsymbol{p}}\widetilde{\boldsymbol{\psi}(\boldsymbol{p})}:\mathcal{H}_2\right\}>0.$$

Then $1\le min(ab,cd)$ where

$$\boldsymbol{q}\frac{\partial}{\partial\boldsymbol{q}}\boldsymbol{\phi}(\boldsymbol{q})=\frac{\partial}{\partial\boldsymbol{q}}\widetilde{\boldsymbol{\phi}(\boldsymbol{q})}-\boldsymbol{\phi}(\boldsymbol{q}),$$

$$\mathcal{H}_1=\left\{\boldsymbol{q}\in(0,1):\frac{\partial}{\partial\boldsymbol{q}}\widetilde{\boldsymbol{\phi}(\boldsymbol{q})}\quad\text{exists}\right\},$$

and

$$\mathcal{H}_2=\left\{\boldsymbol{p}\in(0,1):\frac{\partial}{\partial\boldsymbol{p}}\widetilde{\boldsymbol{\psi}(\boldsymbol{p})}\quad\text{exists}\right\}.$$

2.3.1 BvTTL-W-FGM (Type-I) model

Here, we consider the following functional form for both $\boldsymbol{\phi}(\boldsymbol{q})$ and $\boldsymbol{\psi}(\boldsymbol{p})$ where $\widetilde{\boldsymbol{\phi}(\boldsymbol{q})}=\boldsymbol{q}\left[1-F_{\underline{\boldsymbol{\Psi}}_1}(\boldsymbol{q})\right]|_{\underline{\boldsymbol{\Psi}}_1>0}$, and $\widetilde{\boldsymbol{\psi}(\boldsymbol{p})}=\boldsymbol{p}\left[1-F_{\underline{\boldsymbol{\Psi}}_2}(\boldsymbol{p})\right]|_{\underline{\boldsymbol{\Psi}}_2>0}$. Then using $C_{\mathcal{B}}(\boldsymbol{q},\boldsymbol{p})=\boldsymbol{q}\boldsymbol{p}+\mathcal{B}\widetilde{\boldsymbol{\phi}(\boldsymbol{q})}\widetilde{\boldsymbol{\psi}(\boldsymbol{p})}$, the BvTTL-W-FGM **(**Type**-I)** can be derived.

2.3.2 BvTTL-W-FGM (Type-II) model:

Consider the following functional form for both $\boldsymbol{\phi}(\boldsymbol{q})$ and $\boldsymbol{\psi}(\boldsymbol{p})$ which satisfy all the conditions stated earlier where

$$\boldsymbol{\phi}(\boldsymbol{q})|_{(\mathcal{B}_1>0)}=\boldsymbol{q}^{\mathcal{B}_1}(1-\boldsymbol{q})^{1-\mathcal{B}_1}\text{ and }\boldsymbol{\psi}(\boldsymbol{p})|_{(\mathcal{B}_2>0)}=\boldsymbol{p}^{\mathcal{B}_2}(1-\boldsymbol{p})^{1-\mathcal{B}_2}.$$

The corresponding bivariate copula (henceforth, BvTTL-W-FGM **(**Type**-II)** copula) can be derived from

$$C_{\mathcal{B},\mathcal{B}_1,\mathcal{B}_2}(\boldsymbol{q},\boldsymbol{p})=\boldsymbol{q}\boldsymbol{p}[1+\mathcal{B}\boldsymbol{q}^{\mathcal{B}_1}\boldsymbol{p}^{\mathcal{B}_2}(1-\boldsymbol{q})^{1-\mathcal{B}_1}(1-\boldsymbol{p})^{1-\mathcal{B}_2}].$$

2.3.3 BvTTL-W-FGM (Type-III) model:

Consider the following functional form for both $\boldsymbol{\phi}(\boldsymbol{q})$ and $\boldsymbol{\psi}(\boldsymbol{q})$ which satisfy all the conditions stated earlier where $\boldsymbol{\phi}(\boldsymbol{q})=\boldsymbol{q}[log(1+\overline{\boldsymbol{q}})]$ and $\boldsymbol{\psi}(\boldsymbol{p})=\boldsymbol{p}[log(1+\overline{\boldsymbol{p}})]$. In this case, one can also derive a closed form expression for the associated C.D.F of the BvTTL-W-FGM (Type-**III**).

2.3.4 BvTTL-W-FGM (Type-IV) model:

due to Ghosh and Ray (2016) the C.D.F of the BvTTL-W-FGM (Type-**IV**) model can be derived from $C(\boldsymbol{q},\boldsymbol{p})=\boldsymbol{q}F^{-1}(\boldsymbol{p})+\boldsymbol{p}F^{-1}(\boldsymbol{q})-F^{-1}(\boldsymbol{q})F^{-1}(\boldsymbol{p})$ where $F^{-1}(\boldsymbol{q})$ and $F^{-1}(\boldsymbol{p})$ are derived before.

2.3 BvTTL-W type via **Clayton Copula**

The Clayton Copula can be considered as $C(\boldsymbol{q}_1,\boldsymbol{q}_2)=(\boldsymbol{q}_1^{-\mathcal{B}}+\boldsymbol{q}_2^{-\mathcal{B}}-1)^{-\frac{1}{\mathcal{B}}}|_{\mathcal{B}\in[0,\infty]}$. Let us assume that $Z\sim$ TTL-W ($\underline{\boldsymbol{\Psi}}_1$) and $T\sim$ TTL-W ($\underline{\boldsymbol{\Psi}}_2$). Then, setting $\boldsymbol{q}_1=\boldsymbol{q}(z)|_{\underline{\boldsymbol{\Psi}}_1>0}$ and $\boldsymbol{q}_2=\boldsymbol{q}(t)|_{\underline{\boldsymbol{\Psi}}_2>0}$. Then, the BvTTL-W type distribution can be derived from $F(z,t)=C(F_{\underline{\boldsymbol{\Psi}}_1}(z),F_{\underline{\boldsymbol{\Psi}}_2}(t))$.

2.4 BvTTL-W type via Renyi's entropy

Consider theorem of Pougaza and Djafari (2011) where $C(\boldsymbol{q},\boldsymbol{p})=z_2\boldsymbol{q}+z_1\boldsymbol{p}-z_1z_2$. The associated BvTTL-W will be $C(z_1,z_2)|_{(\alpha=\alpha_1=\alpha_2)}=C(F_{\underline{\boldsymbol{\Psi}}_1}(z_1),F_{\underline{\boldsymbol{\Psi}}_2}(z_1))$. The M-dimensional extension can be derived from $C(\boldsymbol{q}_i)=\left[\sum_{i=1}^{M}\boldsymbol{q}_i^{-\mathcal{B}}+1-M\right]^{-\frac{1}{\mathcal{B}}}$.

3. Statistical properties

3.1 Moments

The M^{th} ordinary moment of W is given by $\mu_M' = \mathbf{E}(W^M) = \int_{-\infty}^{\infty} w^M f_{\underline{\Psi}}(w)dw$. Then, using Theorem 1 we obtain

$$\mu_M'|_{[M>-b]} = \mathbf{\Gamma}\left(1+\frac{M}{b}\right)\sum_{m,\boldsymbol{\tau}=0}^{\infty}\sum_{k=0}^{2\boldsymbol{\tau}} \boldsymbol{\xi}_{m,\boldsymbol{\tau},k}^{(M,k)}, \quad (10)$$

where $\boldsymbol{\xi}_{m,\boldsymbol{\tau},k}^{(M,k)} = \boldsymbol{\xi}_{\boldsymbol{\tau},k} k(-1)^m a^{-M}(m+1)^{-(M+b)/b}\binom{-1+k}{m}$. Setting $M=1$ in (10), we have the mean of X as $\mu_1'|_{[M>-b]} = \mathbf{\Gamma}\left(1+\frac{1}{b}\right)\sum_{m,\boldsymbol{\tau}=0}^{\infty}\sum_{k=0}^{2\boldsymbol{\tau}} \boldsymbol{\xi}_{m,\boldsymbol{\tau},k}^{(1,k)}$ where $\boldsymbol{\xi}_{m,\boldsymbol{\tau},k}^{(1,k)} = \boldsymbol{\xi}_{\boldsymbol{\tau},k} k(-1)^m a^{-1}(m+1)^{-(1+b)/b}\binom{k-1}{m}$. The moment generating function (MGF) can be derived from equation (9) and Theorem 1 as

$$M_W(t)|_{[r>-b]} = \mathbf{\Gamma}\left(1+\frac{r}{b}\right)\sum_{m,r,\boldsymbol{\tau}=0}^{\infty}\sum_{k=0}^{2\boldsymbol{\tau}} \mathbf{Q}_{m,\boldsymbol{\tau},h,k}^{(r,k)},$$

where $\mathbf{Q}_{m,r,\boldsymbol{\tau},k}^{(r,k)} = \boldsymbol{\xi}_{\boldsymbol{\tau},k} k(-1)^m t^r \mathbf{\Gamma}^{-1}(1+r)a^{-r}(m+1)^{-(r+b)/b}\binom{-1+k}{m}$. The skewness $(\text{sk}(W))$ and kurtosis $(\text{ku}(W))$ measures can be calculated from the ordinary moments using well-known relationships (see Table 2). The mean (μ_1'), variance $(V(W))$, $\text{sk}(W)$ and ku (W) of the proposed distribution are computed numerically for some selected values of parameter λ, α, a and b using the R software. The $\text{sk}(W)$ of the proposed distribution can range in the interval $(-0.68,\ 5.1)$ and this means that the new model can be left and right skewed, whereas the ku(X) of the proposed distribution varies in the interval $(-398.1,\ 54.1)$.

Table 2: μ_1', V(X), sk(X) and ku(X) of the proposed distribution.

λ	α	a	b	μ_1'	V(W)	sk(W)	ku(W)
1.5	1	0.5	0.5	1.750000	8.562500	5.119910	54.08711
			1	1.500000	1.250000	1.609969	7.080000
			5	1.805566	0.080897	−0.167734	3.000812
			10	1.894169	0.023256	−0.425684	3.341781
			20	1.944749	0.006287	−0.566375	3.638961
			50	1.977314	0.001057	−0.655667	3.870117
			75	1.984787	0.000475	−0.675816	3.903296
5	3	0.1	3	10.42449	2.905149	0.2661283	3.117302
		0.5		2.084897	0.116206	0.2661283	3.117302
		2		0.521224	0.007263	0.2661268	3.117339
		5		0.208490	0.001162	0.2661283	3.117302
		10		0.104245	0.000290	0.2661283	3.117434
		20		0.052122	7.2628×10^5	0.2661245	−398.058
10	1	5	5	0.180557	0.00080897	−0.167733	3.000813
	10			0.222872	0.00022656	0.2859269	3.150747
	20			0.231334	0.00017336	0.3825359	3.251737
	50			0.240914	0.00012849	0.4839039	3.394393

3.2 Incomplete moments

The Mth incomplete moment can be expressed as $\boldsymbol{\phi}_M(\mathbf{z}) = \int_{-\infty}^{\mathbf{z}} w^M f_{\underline{\Psi}}(w)dx$. Then, using Theorem 1 we have

$$\boldsymbol{\phi}_M(\mathbf{z})|_{[M>-b]} = \boldsymbol{\gamma}\left(1+\frac{M}{b},(a\mathbf{z})^b\right)\sum_{m,\boldsymbol{\tau}=0}^{\infty}\sum_{k=0}^{2\boldsymbol{\tau}} \emptyset_{m,\boldsymbol{\tau},k}^{(M,k)}, \quad (11)$$

where $\emptyset_{m,\boldsymbol{\tau},k}^{(M,k)} = \boldsymbol{\xi}_{\boldsymbol{\tau},k} k(-1)^m a^{-M}(m+1)^{-(M+b)/b}\binom{-1+k}{m}$. A general equation for $\boldsymbol{\phi}_1(t)$ can be derived from (11) as $\boldsymbol{\phi}_1(\mathbf{z})| = \boldsymbol{\gamma}\left(1+\frac{1}{b},(a\mathbf{z})^b\right)\sum_{m,\boldsymbol{\tau}=0}^{\infty}\sum_{k=0}^{2\boldsymbol{\tau}}\emptyset_{m,\boldsymbol{\tau},k}^{(1,k)}$ where $\emptyset_{m,\boldsymbol{\tau},k}^{(1,k)} = \boldsymbol{\xi}_{h,k} k(-1)^m a^{-1}(m+1)^{-(1+b)/b}\binom{-1+k}{m}$ and $\boldsymbol{\phi}_1(z)$ is the first incomplete moment of the new distribution.

3.3 Probability weighted moments (PWMs)

The $(M,\mathbf{z})$th PWM of W is formally defined by $\boldsymbol{\rho}_{M,\mathbf{z}} = \mathbf{E}\left(F_{\underline{\Psi}}(w)^{\mathbf{z}}W^M\right)$. Using equations (5) and (6), we can write

$$f_{\underline{\Psi}}(w)F_{\underline{\Psi}}(w)^{\mathbf{z}} = \sum_{m=0}^{r}\sum_{\boldsymbol{\tau}=0}^{\infty}\sum_{k=0}^{2\boldsymbol{\tau}+1}\mathbf{c}_{m,\boldsymbol{\tau},k}\,(k+1)ba^b t^{b-1}e^{-(at)^b}\left[-e^{-(at)^b}+1\right]^k,$$

where

$$\mathbf{c}_{m,\boldsymbol{\tau},k} = \frac{2}{1+k}\alpha\lambda^m(1+\lambda)^{\mathbf{z}-m}(-1)^{m+j+k}\binom{r}{m}\binom{1+2j}{k}$$
$$\times\left[\binom{-1+(\mathbf{z}+m+1)\alpha}{j}+\lambda\binom{-1+(\mathbf{z}+m+1)\alpha}{j}-2\lambda\binom{-1+(\mathbf{z}+m+2)\alpha}{j}\right].$$

Then, using Theorem 1 the $(M,\mathbf{z})$th PWM of X can be expressed as

$$\boldsymbol{\rho}_{M,r}|_{[M>-b]} = \boldsymbol{\Gamma}\left(1+\frac{M}{b}\right)\sum_{h,m=0}^{\mathbf{z}}\sum_{\boldsymbol{\tau}=0}^{\infty}\sum_{k=0}^{2\boldsymbol{\tau}+1}\mathbf{C}_{\boldsymbol{\tau},m,h,k}^{(M,1+k)},$$

where $\mathbf{C}_{\boldsymbol{\tau},m,h,k}^{(M,1+k)} = (1+k)\mathbf{c}_{m,\boldsymbol{\tau},k}\frac{(-1)^{\boldsymbol{\tau}}}{h!a^s(\boldsymbol{\tau}+1)^{(M+b)/b}}\binom{k}{h}$.

3.4 moments of residual life (MRL) and reversed residual life (MRRL)

The M^{th} MRL, say $\mathbf{MRL}_M(\mathbf{z})\ |_{[M=1,2,\dots\ and\ W>\mathbf{z}]} = \mathbf{E}[(W-\mathbf{z})^M]$, then

$$\mathbf{MRL}_M(\mathbf{z})\ |_{[M=1,2,\dots\ and\ X>\mathbf{z}]} = \frac{\int_{\mathbf{z}}^{\infty} f_{\underline{\Psi}}(w)\,(w-\mathbf{z})^M dw}{1-F_{\underline{\Psi}}(\mathbf{z})},$$

therefore

$$\mathbf{MRL}_M(\mathbf{z})|_{[M>-b]} = \frac{\boldsymbol{\Gamma}\left(1+\frac{M}{b},(a\mathbf{z})^b\right)}{1-F(t)}\sum_{m,\boldsymbol{\tau}=0}^{\infty}\sum_{k=0}^{2\boldsymbol{\tau}}\mathbf{w}_{\boldsymbol{\tau},k}^{*}\quad a^{-M}\frac{k(-1)^m}{(m+1)^{(M+b)/b}}\binom{-1+k}{m},$$

where $\mathbf{w}_{\boldsymbol{\tau},k}^{*} = \boldsymbol{\xi}_{\boldsymbol{\tau},k}\sum_{r=0}^{M}\binom{M}{r}(-\mathbf{z})^{M-r}$. The M^{th} MRRL, say

$$\mathbf{MRRL}_M(\mathbf{z})|_{[\mathbf{z}>0,W\le\mathbf{z}\text{ and }M=1,2,\dots]} = \mathbf{E}[(\mathbf{z}-W)^M],$$

then

$\mathbf{MRRL}_M(\mathbf{z})|_{[\mathbf{z}>0,X\le\mathbf{z}\text{ and }M=1,2,\dots]} = \frac{1}{F_{\underline{\Psi}}(\mathbf{z})}\int_0^{\mathbf{z}}(\mathbf{z}-w)^M f_{\underline{\Psi}}(w)dw$. Then, the n^{th} MRRL of W becomes

$$\mathbf{MRRL}_M(\mathbf{z})|_{[M>-b]} = \frac{\boldsymbol{\gamma}\left(1+\frac{M}{b},(a\mathbf{z})^b\right)}{F(\mathbf{z})}\sum_{m,\boldsymbol{\tau}=0}^{\infty}\sum_{k=0}^{2\boldsymbol{\tau}}\mathbf{w}_{\boldsymbol{\tau},k}^{**}\quad a^{-M}\frac{k(-1)^m}{(m+1)^{(M+b)/b}}\binom{-1+k}{m},$$

where $\mathbf{w}_{\boldsymbol{\tau},k}^{**} = \boldsymbol{\xi}_{\boldsymbol{\tau},k}\sum_{r=0}^{M}(-1)^r\binom{M}{r}\mathbf{z}^{M-r}$.

3.5 Stress-strength model

Let W_1 and W_2 be two independent rvs have TTL-W (λ_1,α,a,b) and TTL-W (λ_2,α,a,b) distributions. Then, the reliability is defined by $\mathbf{R}_{(W_1>W_2)} = \int_0^{\infty}F_{\lambda_1,\alpha,a,b}(w)\ f_{\lambda_2,\alpha,a,b}(w)dw$. Then, we can write

$$\mathbf{R}_{(W_1>W_2)} = \sum_{\boldsymbol{\tau},w=0}^{\infty}\sum_{k=0}^{2\boldsymbol{\tau}}\sum_{m=0}^{2w}\boldsymbol{\Omega}_{\boldsymbol{\tau},w,k,m}.$$

where

$$\boldsymbol{\Omega}_{\boldsymbol{\tau},w,k,m} = \frac{k}{k+m}(-1)^{\boldsymbol{\tau}+k+w+m}\binom{2h}{k}\left[\binom{\alpha}{\boldsymbol{\tau}}+\lambda\binom{\alpha}{\boldsymbol{\tau}}-\lambda\binom{2\alpha}{\boldsymbol{\tau}}\right]\binom{2w}{m}\left[\binom{\alpha}{w}+\lambda\binom{\alpha}{w}-\lambda\binom{2\alpha}{w}\right].$$

3.6 Order statistics

Let $W_1, \dots, W_n$ be a random sample (R.S) of sixe (n) from the TTL-W distribution and let $W_{(1)}, \dots, W_{(n)}$ their corresponding order statistics. The P.D.F of i −th order statistic, say $W_{i\ :\ n}$, can be written as

$$f_{i\ :\ n}(w) = \sum_{h=0}^{n-i} \frac{(-1)^h \binom{-i+n}{j}}{B(i, n-i+1)} f_{\underline{\Psi}}(w) F_{\underline{\Psi}}(w)^{-1+h+i}, \tag{12}$$

where $B(.,.)$ is the beta function. Using (5) and (6) and (12), we get

$$f_{\underline{\Psi}}(w) F_{\underline{\Psi}}(w)^{-1+h+i} = \sum_{h=0}^{-1+h+i} \sum_{m=0}^{\infty} \sum_{k=0}^{2m+1} \mathbf{v}_{h,m,k}\, (k+1) b a^b w^{b-1} e^{-(at)^b} \left[-e^{-(atw}+1\right]^k,$$

where

$$\mathbf{v}_{h,m,k} = \frac{2}{1+k} \alpha \lambda^h (1+\lambda)^{h+i-h-1} (-1)^{h+m+k} \binom{-1+i+h}{h}$$
$$\times \binom{2m+1}{k} \left\{ \binom{-1+(h+i+h)\alpha}{m} + \lambda \binom{-1+(h+i+h)\alpha}{m} - 2\lambda \binom{-1+(h+i+h+1)\alpha}{m} \right\},$$

the P.D.F of $W_{i\,:\,n}$ will be

$$f_{i:n}(w) = \sum_{h=0}^{n-i} \frac{(-1)^h \binom{n-i}{j}}{B(i, n-i+1)} \sum_{\boldsymbol{\tau}=0}^{h+i-1} \sum_{m=0}^{\infty} \sum_{k=0}^{2m+1} \mathbf{v}_{\boldsymbol{\tau},m,k}\, (k+1) b a^b w^{b-1} e^{-(atw} \left[-e^{-(aw)^b}+1\right]^k,$$

the moments of $W_{i\,:\,n}$ will be

$$\mathbf{E}\left(W_{i\,:\,n}^q\right)\big|_{[r>-q]} = \mathbf{\Gamma}\left(1+\frac{q}{b}\right) \sum_{w,m=0}^{\infty} \sum_{\boldsymbol{\tau}=0}^{h+i-1} \sum_{k=0}^{2m+1} \sum_{h=0}^{n-i} \mathbf{V}_{w,m,\boldsymbol{\tau},k,j}^{(q,1+k)},$$

where

$$\mathbf{V}_{w,m,\boldsymbol{\tau},k,j}^{(q,1+k)} = \mathbf{v}_{\boldsymbol{\tau},m,k} \frac{(k+1)(-1)^{w+h} \Gamma(k+1)}{w!\, a^q\, \mathrm{B}(i, 1+n-i) \Gamma(1+k-w)} (w+1)^{-(q+b)/b} \binom{-i+n}{h}.$$

4. Estimation

4.1 Maximum likelihood estimation

Let $w_1, w_2, \dots, w_n$ be a R.S from the TTL-W with parameters λ, α, a and b. Let $(\lambda, \alpha, a, b)^T = \underline{\Psi}$ be the 4×1 parameter vector. For getting the M-L-E of $\underline{\Psi}$, we have the log-likelihood function ($\ell(\underline{\Psi})$) as

$$\ell = \ell(\underline{\Psi}) = n\, log(2) + n\, log\, \alpha + n\, log\, b + nb\, log\, a + (b-1) \sum_{i=1}^{n} log(w_i)$$
$$-2 \sum_{i=1}^{n} (\alpha w_i)^b + (\alpha-1) \sum_{i=1}^{n} log c_i + \sum_{i=1}^{n} log z_i,$$

where $c_i = 1 - e^{-2(aw_i)^b}$ and $z_i = 1 + \lambda - 2\lambda c_i^\alpha$. The score vector components are

$$\mathbf{U}_\lambda = \sum_{i=1}^{n} \frac{p_i}{z_i}, \mathbf{U}_\alpha = \frac{n}{\alpha} - \frac{2b}{\alpha^{1-b}} \sum_{i=1}^{n} x_i^b + \sum_{i=1}^{n} log c_i + \sum_{i=1}^{n} \frac{q_i}{z_i}, \mathbf{U}_a = \frac{nb}{a} + (\alpha-1) \sum_{i=1}^{n} \frac{w_i}{c_i} + \sum_{i=1}^{n} \frac{t_i}{z_i}$$

and

$$\mathbf{U}_b = \frac{n}{b} + n\, log\, a + \sum_{i=1}^{n} log(\mathbf{w}_i) - 2 \sum_{i=1}^{n} \frac{log(\alpha w_i)}{(\alpha x_i)^{-b}} + (\alpha-1) \sum_{i=1}^{n} \frac{m_i}{c_i} + \sum_{i=1}^{n} \frac{d_i}{z_i},$$

where

$$\mathbf{w}_i = 2ba^{b-1} x_i^b e^{-2(aw_i)^b}, \qquad m_i = 2e^{-2(ax_i)^b} (ax_i)^b\, log(aw_i),$$
$$p_i = 1 - 2c_i^\alpha, q_i = -2\lambda s_i^\alpha\, log c_i, t_i = -2\alpha\lambda w_i c_i^{\alpha-1} \text{ and } d_i = -2\alpha\lambda m_i c_i^{\alpha-1}.$$

Generally, there are different forms of censorship such as type I, type II, etc. Here, we will consider a general case of multi-censorship in which there are m subjects of which m_0 are known to have failed at the times $w_1, w_2, \dots, w_{n_0-1}, w_{n_0}$, n_1 are known to have failed in the interval $[s_{j-1}, s_j]$, $j = 1, 2, \dots, m_1 - 1, m_1$, m_2 survived to a time r_j, $j = 1, 2, \dots, m_2 - 1, m_2$ but not observed any longer. Where $m_0 + m_1 + m_2 = m$ and that type I censoring and type II censoring are particular cases of multi-censoring. The Log-L ($\ell_m(\underline{\Psi})$) for $\underline{\Psi}$ is

$$\ell_m(\underline{\boldsymbol{\Psi}}) = m_0 \log 2 + m_0 \log \alpha + m_0 \log b$$
$$+m_0 b \log a + (b-1)\sum_{i=1}^{m_0} \log(x_i) - 2\sum_{i=1}^{m_0}(\alpha w_i)^b + (\alpha - 1)\sum_{i=1}^{m_0} \log\left[1 - e^{-2(aw_i)^b}\right]$$
$$+\sum_{i=1}^{m_0} \log\left(1 - 2\lambda\left\{-e^{-2(aw_i)^b} + 1\right\}^{\alpha} + \lambda\right)$$
$$+\sum_{i=1}^{m_1} \log\left\{\begin{matrix}\left[(1+\lambda)\left\{-e^{-2(as_i)^b}+1\right\}^{\alpha} - \lambda\left\{-e^{-2(as_i)^b}+1\right\}^{2\alpha}\right] \\ -\left[(1+\lambda)\left\{-e^{-2(as_{i-1})^b}+1\right\}^{\alpha} - \lambda\left\{-e^{-2(as_{i-1})^b}+1\right\}^{2\alpha}\right]\end{matrix}\right\}$$
$$+\sum_{i=1}^{m_2} \log\left\{1 - \left[(1+\lambda)\left\{-e^{-2(ar_i)^b}+1\right\}^{\alpha} - \lambda\left\{-e^{-2(ar_i)^b}+1\right\}^{2\alpha}\right]\right\}.$$

For the normal equations see Appendix A.

4.2 Least square (L-S) and weighted least square (W-L-S) estimation

The theory of L-S estimation and W-L-S estimation is based on the minimization of the sum of the square of differences (SSD) of theoretical C.D.F and the empirical C.D.F. Suppose $F_{\underline{\boldsymbol{\Psi}}}(w_{i\,:\,n})$ denotes the distribution function of TTL-W distribution and if $w_1 < w_2 < \cdots < w_{n-1} < w_n$ be the n ordered rs. The least square estimates are obtained by minimizing $\mathbf{Ls}(\underline{\boldsymbol{\Psi}}) = \sum_{i=1}^{n}\left[F_{\underline{\boldsymbol{\Psi}}}(w_{i\,:\,n}) - \vartheta_{i,1+n}\right]^2$, where $\vartheta_{i,1+n} = \frac{i}{1+n}$. Now using (5) we have

$$\mathbf{Ls}(\underline{\boldsymbol{\Psi}}) = \sum_{i=1}^{n}\left\{(1+\lambda)\left[-e^{-2(aw_i)^b}+1\right]^{\alpha} - \lambda\left[-e^{-2(aw_i)^b}+1\right]^{2\alpha} - \vartheta_{i,1+n}\right\}^2.$$

The L-SE of the parameters are obtained by solving the following non-linear equations

$$\sum_{i=1}^{n}\left\{(1+\lambda)\left[-e^{-2(aw_i)^b}+1\right]^{\alpha} - \lambda\left[-e^{-2(aw_i)^b}+1\right]^{2\alpha} - \vartheta_{i,1+n}\right\} C_\lambda(w_i,\underline{\boldsymbol{\Psi}}) = 0,$$

$$\sum_{i=1}^{n}\left\{(1+\lambda)\left[-e^{-2(aw_i)^b}+1\right]^{\alpha} - \lambda\left[-e^{-2(aw_i)^b}+1\right]^{2\alpha} - \vartheta_{i,1+n}\right\} C_\alpha(w_i,\underline{\boldsymbol{\Psi}}) = 0,$$

$$\sum_{i=1}^{n}\left\{(1+\lambda)\left[-e^{-2(aw_i)^b}+1\right]^{\alpha} - \lambda\left[-e^{-2(aw_i)^b}+1\right]^{2\alpha} - \vartheta_{i,1+n}\right\} C_a(w_i,\underline{\boldsymbol{\Psi}}) = 0,$$

and

$$\sum_{i=1}^{n}\left\{(1+\lambda)\left[-e^{-2(aw_i)^b}+1\right]^{\alpha} - \lambda\left[-e^{-2(aw_i)^b}+1\right]^{2\alpha} - \vartheta_{i,1+n}\right\} C_b(w_i,\underline{\boldsymbol{\Psi}}) = 0,$$

where $C_\lambda(w_i,\underline{\boldsymbol{\Psi}}), C_\alpha(w_i,\underline{\boldsymbol{\Psi}}), C_a(w_i,\underline{\boldsymbol{\Psi}})$ and $C_b(w_i,\underline{\boldsymbol{\Psi}})$ are the values of the first derivatives of the C.D.F w.r.t. parameters of TTL-W distribution. The L-S estimates of the parameters λ, α, a, b are obtained solving the above simultaneous equations by using the numerical approximation techniques. The W-L-S estimations (W-L-SEs) are obtained by minimizing the given form of equation with respect to the parameters

$$\mathbf{WLS}(\underline{\boldsymbol{\Psi}}) = \sum_{i=1}^{n} \boldsymbol{w}_{(i)}\left[F_{\underline{\boldsymbol{\Psi}}}(w_{i\,:\,n}) - \vartheta_{i,1+n}\right]^2.$$

The W-L-SE of the parameters are obtained by solving the following non-linear equations;

$$\sum_{i=1}^{n}\left\{(1+\lambda)\left[-e^{-2(aw_i)^b}+1\right]^{\alpha} - \lambda\left[-e^{-2(aw_i)^b}+1\right]^{2\alpha} - \vartheta_{i,1+n}\right\} \boldsymbol{w}_{(i)}\, C_\lambda(w_i,\underline{\boldsymbol{\Psi}}) = 0,$$

$$\sum_{i=1}^{n}\left\{(1+\lambda)\left[-e^{-2(aw_i)^b}+1\right]^{\alpha}-\lambda\left[-e^{-2(aw_i)^b}+1\right]^{2\alpha}-\vartheta_{i,1+n}\right\}\boldsymbol{w}_{(i)}\,C_{\alpha}(w_i,\underline{\boldsymbol{\Psi}})=0,$$

$$\sum_{i=1}^{n}\left\{(1+\lambda)\left[-e^{-2(aw_i)^b}+1\right]^{\alpha}-\lambda\left[-e^{-2(aw_i)^b}+1\right]^{2\alpha}-\vartheta_{i,1+n}\right\}\boldsymbol{w}_{(i)}\,C_{a}(w_i,\underline{\boldsymbol{\Psi}})=0,$$

and

$$\sum_{i=1}^{n}\left\{(1+\lambda)\left[-e^{-2(aw_i)^b}+1\right]^{\alpha}-\lambda\left[-e^{-2(aw_i)^b}+1\right]^{2\alpha}-\vartheta_{i,1+n}\right\}\boldsymbol{w}_{(i)}\,C_{b}(w_i,\underline{\boldsymbol{\Psi}})=0,$$

where $C_{\lambda}(w_i,\underline{\boldsymbol{\Psi}}), C_{\alpha}(w_i,\underline{\boldsymbol{\Psi}}), C_{a}(w_i,\underline{\boldsymbol{\Psi}})$ and $C_{b}(w_i,\underline{\boldsymbol{\Psi}})$ are the values of first derivatives of the C.D.F of TTL-W distribution and $\boldsymbol{w}_{(i)}=[(n+1)^2(n+2)]\div[i(n-i+1)]$.

4.3 Method of Cramer-Von-Mises

The Crammer-Von Mises estimates of the parameters (see MacDonald (1971)) are obtained by minimizing the following expression w.r.t. to the parameters as

$$CVM(\lambda,\alpha,a,b)=\frac{1}{12n}+\sum_{i=1}^{n}\left[-V_{2i-1,2n}+F_{\lambda,\alpha,a,b}(w_{i\,:\,n})\right]^2,$$

where $V_{2i-1,2n}=\frac{2i-1}{2n}$, and

$$CVM(\lambda,\alpha,a,b)=\sum_{i=1}^{n}\left\{(1+\lambda)\left[-e^{-2(aw_i)^b}+1\right]^{\alpha}-\lambda\left[-e^{-2(aw_i)^b}+1\right]^{2\alpha}-V_{2i-1,2n}\right\}^2.$$

The Cramer-Von-Mises estimators (C-V-ME) of the parameters are obtained by solving the following non-linear equations

$$\sum_{i=1}^{n}\left\{(1+\lambda)\left[-e^{-2(aw_i)^b}+1\right]^{\alpha}-\lambda\left[-e^{-2(aw_i)^b}+1\right]^{2\alpha}-V_{2i-1,2n}\right\}C_{\lambda}(w_i,\underline{\boldsymbol{\Psi}})=0,$$

$$\sum_{i=1}^{n}\left\{(1+\lambda)\left[-e^{-2(aw_i)^b}+1\right]^{\alpha}-\lambda\left[-e^{-2(aw_i)^b}+1\right]^{2\alpha}-V_{2i-1,2n}\right\}C_{\alpha}(w_i,\underline{\boldsymbol{\Psi}})=0,$$

$$\sum_{i=1}^{n}\left\{(1+\lambda)\left[-e^{-2(aw_i)^b}+1\right]^{\alpha}-\lambda\left[-e^{-2(aw_i)^b}+1\right]^{2\alpha}-V_{2i-1,2n}\right\}C_{a}(w_i,\underline{\boldsymbol{\Psi}})=0,$$

and

$$\sum_{i=1}^{n}\left\{(1+\lambda)\left[-e^{-2(aw_i)^b}+1\right]^{\alpha}-\lambda\left[-e^{-2(aw_i)^b}+1\right]^{2\alpha}-V_{2i-1,2n}\right\}C_{b}(w_i,\underline{\boldsymbol{\Psi}})=0,$$

where $C_{\lambda}(w_i,\underline{\boldsymbol{\Psi}}), C_{\alpha}(w_i,\underline{\boldsymbol{\Psi}}), C_{a}(w_i,\underline{\boldsymbol{\Psi}})$ and $C_{b}(w_i,\underline{\boldsymbol{\Psi}})$ are the values of the first derivatives of the C.D.F of TTL-W distribution with respect to λ,α,a,b respectively.

5. Real data analysis

5.1 Application 1

The 1st data set consists of 100 observations of carbon fibers breaking stress given by Nicholas and Padgett (2006). The data are (0.39, 0.850, 1.08, 2.590, 2.67, 2.730, 2.74, 1.250, 1.47, 1.570, 1.61, 1.610, 1.69, 1.80, 1.840, 1.87, 1.890, 2.03, 2.03, 2.05, 2.120, 2.35, 2.41, 2.430, 2.480, 2.50, 2.53, 2.55, 2.550, 2.56, 2.790, 2.870, 2.88, 2.930, 2.950, 2.96, 2.97, 3.090, 2.81, 2.82, 2.850, 3.11, 3.110, 3.15, 3.150, 3.270, 3.28, 3.31, 3.31, 3.33, 3.39, 3.190, 3.22, 3.22, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.420, 4.70, 4.90). Table3 gives the values of estimators of α,λ,a and b , the KS test statistics and its p-value for new for the four different estimation methods.

Table 3. The values of estimators, KS, p-values for the first application.

Method	α	λ	a	b	KS	p-value
ML	0.75959	0.17308	0.25205	4.18957	0.08173	0.77000
L-S	0.43393	0.57147	0.24279	7.69659	0.05988	0.97195
W-L-S	0.54471	0.36649	0.24784	6.01142	0.06591	0.93667
C-V-M	**0.43903**	**0.59075**	**0.24421**	**7.82014**	**0.05876**	**0.97662**
Bootstrap	0.58086	0.14799	0.23655	5.92953	0.19592	0.01260

From Table 3, we conclude that the C-V-M method is the best method for modelling the carbon fibers with KS=0.05876 and biggest p-value=0.97662. However, all other methods performed well.

5.2 Application 2

Data 2nd data set (65, 56, 26, 22, 1, 1, 5, 65, 16, 22, 3, 4, 2, 3, 56, 65, 17, 7, 156, 8, 4, 3, 30, 4, 100, 134, 16, 108, 121, 4, 39, 143, 43) called leukemia data. This real data set gives the survival times, in weeks, of 33 patients suffering from acute Myelogenous Leukemia (see Feigl and Zelen (1965)).

Table 4. The values of estimators, KS, p-values for the second application.

Method	α	λ	a	b	KS	p-value
ML	**162.73188**	**0.37048**	**292.24054**	**0.10682**	**0.18919**	**0.57690**
L-S	141.25302	0.26132	121.83737	0.11267	0.22642	0.34809
W-L-S	140.45773	0.27289	133.10529	0.11413	0.19373	0.54616
C-V-M	143.59549	0.21879	127.52666	0.11336	0.22312	0.36580
Bootstrap	123.95395	0.49129	155.62680	0.11419	0.28442	0.12777

From Table 4, we conclude that the ML method is the best method for modeling the Leukemia data with KS=0.18919 and biggest p-value=0.57690. However, all other methods performed well. For comparing with existing competitive models, many real data sets can be found in Refaie (2018), Elbiely and Yousof (2018, 2019), Goual and Yousof (2019), Al-Babtain et al. (2020a,b), Goual et al. (2019, 2020), Hamedani et al. (2019), Ibrahim (2019), Ibrahim et al. (2019, 2020), Korkmaz et al. (2018), Korkmaz et al. (2017, 2019a,b), Mansour et al. (2020a,b), Yadav et al. (2020) and Ibrahim (2020a,b).

6. Monte Carlo (M-C) Simulations

A M-C simulation study is performed in this Section to compare the performance of the different estimators of the unknown parameters of the new distribution via their mean squared errors (MSEs). All the computations in this section are done by Mathcad program Version 15.0. Samples of size 1000 is generated from the new distribution where n =20, 50, 100 and 200. The average values (AVs) and MSEs of M-L-Es, L-SEs, W-L-SEs, C-V-Ms and Bootstrapping are obtained and reported in in Tables 5-8. From Tables 5-8, all the estimates have the property of consistency, i.e., the MSEs decrease as sample size n increase.

Table 5: AVs and MSEs for $n = 20$.

Parameters	M-L-E	L-S	W-L-S	C-V-M	Bootstrap
α=2.3	2.4 (0.2640)	2.3 (0.328)	2.3 (0.31)	2.4 (0.35)	2.34 (0.241)
λ=0.5	0.5 (0.371)	0.6 (0.210)	0.56 (0.21)	0.50 (0.16)	0.7 (0.83)
a=0.5	0.51 (0.0384)	0.56(0.091)	0.58 (0.152)	0.57(0.155)	0.59 (0.071)
b=0.5	0.51 (0.00648)	0.6 (0.128)	0.55 (0.081)	0.6 (0.09)	0.51 (0.007)
α=1.2	1.2 (0.0477)	1.2 (0.060)	1.2 (0.0576)	1.2 (0.057)	1.2 (0.191)
λ=0.9	0.84 (0.1127)	0.96 (0.214)	0.95 (0.197)	0.92(0.190)	0.95 (0.193)
a=1.6	1.7 (0.2864)	1.7 (0.421)	1.74 (0.429)	1.7 (0.30)	1.9 (0.50)
b=0.8	0.81 (0.0138)	0.82 (0.039)	0.81 (0.022)	0.82 (0.025)	0.79 (0.014)
α=1.9	1.54 (0.1563)	1.9 (0.055)	1.92 (0.045)	1.9 (0.048)	1.18 (0.190)
λ=2.0	1.95(0.17)	2.02 (0.195)	2 (0.16)	2.01 (0.181)	2.05 (0.19)
a=0.8	0.92 (0.0172)	0.80 (0.005)	0.80 (0.003)	0.80 (0.004)	1.87 (0.50)
b=1.6	0.1.32 (0.0872)	1.6 (0.035)	1.62 (0.032)	1.60 (0.030)	0.79 (0.014)

Table 6: AVs and MSEs for $n = 50$.

Parameters	M-L-E	L-S	W-L-S	C-V-M	Bootstrap
α=2.3	2.3 (0.10599)	2.32 (0.12)	2.3 (0.092)	2.33 (0.117)	2.03 (0.140)
λ=0.5	0.44 (0.08372)	0.52 (0.079)	0.51 (0.072)	0.51 (0.078)	0.77 (0.180)
a=0.5	0.50 (0.1553)	0.52 (0.0181)	0.52 (0.016)	0.51 (0.017)	0.68 (0.066)
b=0.5	0.504 (0.00284)	0.51 (0.011)	0.52 (0.018)	0.51(0.011)	0.46 (0.004)
α=1.2	1.21 (0.01771)	1.21 (0.023)	1.2 (0.019)	1.2 (0.023)	1.07 (0.032)
λ=0.9	0.85 (0.1098)	0.92 (0.082)	0.91 (0.069)	0.92 (0.163)	1.29 (0.503)
a=1.6	1.60 (0.1083)	1.6 (0.124)	1.6 (0.108)	1.64 (0.121)	2.08 (0.452)
b=0.8	0.81 (0.0052)	0.81 (0.0080)	0.81 (0.0066)	0.81 (0.008)	0.73 (0.010)
α=1.9	1.5 (0.1362)	1.9 (0.022)	1.91 (0.015)	1.9 (0.022)	1.44 (0.224)
λ=2.0	1.985(0.086)	2 (0.075)	1.99 (0.032)	2.01 (0.089)	1.86(0.14)
a=0.8	0.91 (0.0135)	0.80 (0.0017)	0.80 (0.00098)	0.80 (0.002)	0.95 (0.024)
b=1.6	1.33 (0.0767)	1.6 (0.014)	1.61 (0.010)	1.6 (0.013)	1.25 (0.130)

Table 7: AVs and MSEs for $n = 100$.

Parameters	M-L-E	L-S	W-L-S	C-V-M	Bootstrap
α=2.3	2.3 (0.0458)	2.3 (0.0519)	2.3 (0.0461)	2.31 (0.052)	2.01 (0.084)
λ=0.5	0.45 (0.038)	0.50 (0.035)	0.502 (0.035)	0.51 (0.04)	0.69 (0.081)
a=0.5	0.50 (0.007)	0.51 (0.0068)	0.51 (0.006)	0.51(0.007)	0.62 (0.030)
b=0.5	0.502 (0.001)	0.51 (0.0036)	0.51 (0.002)	0.51 (0.004)	0.44 (0.005)
α=1.2	1.21 (0.0083)	1.2 (0.011)	1.2 (0.0095)	1.2 (0.01)	0.62 (0.030)
λ=0.9	0.85 (0.0441)	0.90 (0.038)	0.90 (0.034)	0.91 (0.04)	0.91 (0.060)
a=1.6	1.59 (0.0513)	1.62 (0.051)	1.6 (0.0490)	1.6 (0.050)	1.7 (0.0740)
b=0.8	0.80 (0.0025)	0.80 (0.0036)	0.50 (0.001)	0.8 (0.003)	0.79 (0.003)
α=1.9	1.54 (0.1340)	1.91 (0.0104)	1.9 (0.0060)	1.9 (.0.010)	1.52 (0.150)
λ=2.0	1.92 (0.0540)	1.99 (0.036)	1.99 (0.012)	2.01 (0.040)	1.94 (0.064)
a=0.8	0.91 (0.0128)	0.79 (.0.0008)	0.80 (0.642)	0.8 (.0.008)	0.92 (0.0140)
b=1.6	1.33 (0.0753)	1.61 (0.0065)	1.61 (0.0046)	1.6 (0.006)	1.31 (0.0841)

Table 8: AVs and MSEs for $n = 200$.

Parameters	M-L-E	L-S	W-L-S	C-V-M	Bootstrap
α=2.3	2.3 (0.02526)	2.31 (0.0249)	2.3 (0.022)	2.3 (0.0252)	2.06 (0.074)
λ=0.5	0.49 (0.0212)	0.502 (0.017)	0.50 (0.017)	0.51 (0.018)	0.61 (0.026)
a=0.5	0.50 (0.0044)	0.503 (0.033)	0.50 (0.003)	0.50 (0.034)	0.68 (0.041)
b=0.5	0.50 (0.0006)	0.503 (0.001)	0.50(0.001)	0.51(0.001)	0.46 (0.0018)
α=1.2	1.2 (0.0047)	1.2 (0.0051)	1.2 (0.0046)	1.2 (0.0051)	1.06 (0.021)
λ=0.9	0.87 (0.022)	0.90 (0.0187)	0.9 (0.017)	0.91 (0.019)	1.07 (0.063)
a=1.6	1.6 (0.0298)	1.6 (0.0249)	1.6 (0.0232)	1.6 (0.031)	2.07 (0.27)
b=0.8	0.80 (0.0014)	0.80 (0.0017)	0.80(0.001)	0.80 (0.002)	0.73 (0.0064)
α=1.9	1.54 (0.1328)	1.90 (0.0051)	1.9 (0.0029)	1.90 (0.005)	1.4 (0.242)
λ=2.0	2.13(0.0164)	1.99 (0.0178)	1.99(0.005)	2 (0.0180)	0.96(0.0271)
a=0.8	0.91 (0.013)	0.80 (0.0003)	0.80(0.000)	0.80 (0.0004)	0.96 (0.027)
b=1.6	1.33 (0.075)	1.6 (0.0031)	1.6 (0.0021)	1.6 (0.0030)	1.2 (0.14)

7. Conclusions

In this paper we focus on proposing a new lifetime Weibull model called the transmuted Topp-Leone Weibull model and studying its mathematical properties. The skewness of the proposed distribution can range in the interval $(-0.68, 5.1)$ and this means that the new model can be left and right skewed, whereas the kurtosis of the proposed distribution varies in the interval $(-398.1, 54.1)$. The estimation of its unknown parameters is carried out by considering different method of estimation. The performances of all methods are studied by two real data sets and a Monte Carlo simulation.

The Cramer-Von Mises method is the best method for modeling the carbon fibers data with KS=0.05876 and biggest p-value=0.97662. The maximum likelihood method is the best method for modeling the Leukemia data with KS=0.18919 and biggest p-value=0.57690, however all other methods performed well.

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Appendix A:

$$\frac{\partial}{\partial\lambda}\ell_m(\underline{\boldsymbol{\Psi}})=\sum_{i=1}^{m_0}\frac{1-2\left\{1-e^{-2(aw_i)^b}\right\}^{\alpha}}{1-2\lambda\left\{1-e^{-2(aw_i)^b}\right\}^{\alpha}+\lambda}+\sum_{i=1}^{m_1}\frac{\left[\mathbf{A}^{(\lambda)}(s_i)-\mathbf{A}^{(\lambda)}(s_{i-1})\right]}{\left\{\begin{bmatrix}(1+\lambda)\left\{1-e^{-2(as_i)^b}\right\}^{\alpha}\\-\lambda\left\{1-e^{-2(as_i)^b}\right\}^{2\alpha}\end{bmatrix}-\begin{bmatrix}(1+\lambda)\left\{1-e^{-2(as_{i-1})^b}\right\}^{\alpha}\\-\lambda\left\{1-e^{-2(as_{i-1})^b}\right\}^{2\alpha}\end{bmatrix}\right\}}$$

$$+\sum_{i=1}^{m_2}\frac{\mathbf{A}^{(\lambda)}(r_i)}{1-\left[(1+\lambda)\left\{1-e^{-2(ar_i)^b}\right\}^{\alpha}\right]-\lambda\left\{1-e^{-2(ar_i)^b}\right\}^{2\alpha}},$$

$$\frac{\partial\ell_m(\underline{\boldsymbol{\Psi}})}{\partial\alpha}=\frac{m_0}{\alpha}-\frac{2b}{\alpha^{1-b}}\sum_{i=1}^{m_0}x_i^b+\sum_{i=1}^{m_0}log\left\{1-e^{-2(ax_i)^b}\right\}+\sum_{i=1}^{m_0}\frac{-2\lambda\left\{1-e^{-2(aw_i)^b}\right\}^{\alpha}log\left\{1-e^{-2(aw_i)^b}\right\}}{1+\lambda-2\lambda\left\{1-e^{-2(aw_i)^b}\right\}^{\alpha}}$$

$$+\sum_{i=1}^{m_1}\frac{\left[\mathbf{B}^{(\alpha)}(s_i)-\mathbf{B}^{(\alpha)}(s_{i-1})\right]}{\left\{\begin{bmatrix}(1+\lambda)\left\{1-e^{-2(as_i)^b}\right\}^{\alpha}\\-\lambda\left\{1-e^{-2(as_i)^b}\right\}^{2\alpha}\end{bmatrix}-\begin{bmatrix}(1+\lambda)\left\{1-e^{-2(as_{i-1})^b}\right\}^{\alpha}\\-\lambda\left\{1-e^{-2(as_{i-1})^b}\right\}^{2\alpha}\end{bmatrix}\right\}}+\sum_{i=1}^{m_2}\frac{\mathbf{B}^{(\alpha)}(r_i)}{1-\begin{bmatrix}(1+\lambda)\left\{1-e^{-2(ar_i)^b}\right\}^{\alpha}\\-\lambda\left\{1-e^{-2(ar_i)^b}\right\}^{2\alpha}\end{bmatrix}},$$

$$\frac{\partial\ell_m(\underline{\boldsymbol{\Psi}})}{\partial a}=\frac{m_0 b}{a}+(\alpha-1)2ba^{b-1}\sum_{i=1}^{m_0}\frac{x_i^b e^{-2(aw_i)^b}}{1-e^{-2(aw_i)^b}}-2\alpha\lambda\sum_{i=1}^{m_0}\frac{\left[2ba^{b-1}w_i^b e^{-2(aw_i)^b}\right]\left\{1-e^{-2(aw_i)^b}\right\}^{\alpha-1}}{1+\lambda-2\lambda\left\{1-e^{-2(aw_i)^b}\right\}^{\alpha}}$$

$$+\sum_{i=1}^{m_1}\frac{\left[\mathbf{C}^{(a)}(s_i)-\mathbf{C}^{(a)}(s_{i-1})\right]}{\left\{\begin{matrix}\begin{bmatrix}(1+\lambda)\left\{1-e^{-2(as_i)^b}\right\}^{\alpha}\\-\lambda\left\{1-e^{-2(as_i)^b}\right\}^{2\alpha}\end{bmatrix}\\-\begin{bmatrix}(1+\lambda)\left\{1-e^{-2(as_{i-1})^b}\right\}^{\alpha}\\-\lambda\left\{1-e^{-2(as_{i-1})^b}\right\}^{2\alpha}\end{bmatrix}\end{matrix}\right\}}+\sum_{i=1}^{m_2}\frac{\mathbf{C}^{(a)}(r_i)}{1-\begin{bmatrix}(1+\lambda)\left\{1-e^{-2(ar_i)^b}\right\}^{\alpha}\\-\lambda\left\{1-e^{-2(ar_i)^b}\right\}^{2\alpha}\end{bmatrix}}$$

and

$$\frac{\partial\ell_m(\underline{\boldsymbol{\Psi}})}{\partial b}=\frac{m_0}{b}+n\,log\,a+\sum_{i=1}^{m_0}log(x_i)-2\sum_{i=1}^{m_0}\frac{log(\alpha w_i)}{(\alpha w_i)^{-b}}+\sum_{i=1}^{m_0}\frac{-4\alpha\lambda(aw_i)^b e^{-2(aw_i)^b}\left\{1-e^{-2(aw_i)^b}\right\}^{\alpha-1}}{1+\lambda-2\lambda\left\{1-e^{-2(aw_i)^b}\right\}^{\alpha}}$$

$$+\sum_{i=1}^{m_1}\frac{\left[\mathbf{D}^{(b)}(s_i)-\mathbf{D}^{(b)}(s_{i-1})\right]}{\left\{\begin{matrix}\begin{bmatrix}(1+\lambda)\left\{1-e^{-2(as_i)^b}\right\}^{\alpha}\\-\lambda\left\{1-e^{-2(as_i)^b}\right\}^{2\alpha}\end{bmatrix}\\-\begin{bmatrix}(1+\lambda)\left\{1-e^{-2(as_{i-1})^b}\right\}^{\alpha}\\-\lambda\left\{1-e^{-2(as_{i-1})^b}\right\}^{2\alpha}\end{bmatrix}\end{matrix}\right\}}+\sum_{i=1}^{m_2}\frac{\mathbf{D}^{(b)}(r_i)}{1-\begin{bmatrix}(1+\lambda)\left\{1-e^{-2(ar_i)^b}\right\}^{\alpha}\\-\lambda\left\{1-e^{-2(ar_i)^b}\right\}^{2\alpha}\end{bmatrix}},$$

where

$$\mathbf{A}^{(\lambda)}(s_i)=-\left\{1-e^{-2(as_i)^b}\right\}^{2\alpha}+\left\{1-e^{-2(as_i)^b}\right\}^{\alpha},$$

$$\mathbf{B}^{(\alpha)}(s_i)=log\left\{1-e^{-2(as_i)^b}\right\}\left\{1-e^{-2(as_i)^b}\right\}^{\alpha}\left(1-2\lambda\left\{1-e^{-2(as_i)^b}\right\}^{\alpha}+\lambda\right),$$

$$\mathbf{C}^{(a)}(s_i)=2\alpha b s_i(as_i)^{b-1}e^{-2(as_i)^b}\left\{1-e^{-2(as_i)^b}\right\}^{\alpha-1}\left(1-2\lambda\left\{1-e^{-2(as_i)^b}\right\}^{\alpha}+\lambda\right),$$

$$\mathbf{D}^{(b)}(s_i)=2\alpha(as_i)^b\,log(as_i)e^{-2(as_i)^b}\left\{1-e^{-2(as_i)^b}\right\}^{\alpha-1}\left(1-2\lambda\left\{1-e^{-2(as_i)^b}\right\}^{\alpha}+\lambda\right),$$

$$\mathbf{A}^{(\lambda)}(s_{i-1})=\left\{1-e^{-2(as_{i-1})^b}\right\}^{\alpha}-\left\{1-e^{-2(as_{i-1})^b}\right\}^{2\alpha},$$

$$\mathbf{B}^{(\alpha)}(s_{i-1}) = log\left\{1-e^{-2(as_{i-1})^b}\right\}\left\{1-e^{-2(as_{i-1})^b}\right\}^{\alpha}\left(1-2\lambda\left\{1-e^{-2(as_{i-1})^b}\right\}^{\alpha}+\lambda\right),$$
$$\mathbf{C}^{(a)}(s_{i-1}) = 2\alpha b s_{i-1}(as_{i-1})^{b-1}e^{-2(as_{i-1})^b}\left\{1-e^{-2(as_{i-1})^b}\right\}^{\alpha-1}\left(1-2\lambda\left\{1-e^{-2(as_{i-1})^b}\right\}^{\alpha}+\lambda\right),$$
$$\mathbf{D}^{(b)}(s_{i-1}) = 2\alpha(as_{i-1})^b\, log(\,as_{i-1})e^{-2(as_{i-1})^b}\left\{1-e^{-2(as_{i-1})^b}\right\}^{\alpha-1}\left(1-2\lambda\left\{1-e^{-2(as_{i-1})^b}\right\}^{\alpha}+\lambda\right),$$
$$\mathbf{A}^{(\lambda)}(r_i) = \left\{1-e^{-2(ar_i)^b}\right\}^{2\alpha}-\left\{1-e^{-2(ar_i)^b}\right\}^{\alpha},$$
$$\mathbf{B}^{(\alpha)}(r_i) = log\left\{1-e^{-2(ar_i)^b}\right\}\left\{1-e^{-2(ar_i)^b}\right\}^{\alpha}\left(1-2\lambda\left\{1-e^{-2(ar_i)^b}\right\}^{\alpha}+\lambda\right)$$
$$\mathbf{C}^{(a)}(r_i) = 2\alpha b r_i(ar_i)^{b-1}e^{-2(ar_i)^b}\left\{1-e^{-2(ar_i)^b}\right\}^{\alpha-1}\left(1-2\lambda\left\{1-e^{-2(ar_i)^b}\right\}^{\alpha}+\lambda\right),$$

and

$$\mathbf{D}^{(b)}(r_i) = 2\alpha(ar_i)^b\, log(\,ar_i)e^{-2(ar_i)^b}\left\{1-e^{-2(ar_i)^b}\right\}^{\alpha-1}\left(1-2\lambda\left\{1-e^{-2(ar_i)^b}\right\}^{\alpha}+\lambda\right).$$