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Bayesian Estimation for Xgamma Distribution Under Type-I Hybrid Censoring Scheme Using Asymmetric Loss Function

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Abstract

This article proposes the Bayes estimation of the parameter and reliability function for xgamma distribution in the presence of type-I hybrid censored observations. The Bayes estimate of the parameter has been obtained by assuming informative and non-informative priors using general entropy loss function. Obviously, censoring adds difficulties in estimation procedure; hence the Bayes estimators computed with type-I hybrid censored observation under the mentioned prior often do not assume any standard form. Therefore, Bayes estimates are computed using Tierney-Kadane approximation and Markov Chain Monte Carlo numerical technique. Further, different interval estimates namely asymptotic confidence interval, bootstrap confidence interval and highest posterior density interval along with the width of the interval and coverage probability are also discussed. The maximum likelihood estimate for the same has also been computed using non-linear maximization iterative procedure and compared with corresponding Bayes estimates using Monte Carlo simulations. The comparison of the estimators are made in terms of average loss over whole sample space and corresponding length of the interval. lastly, one medical data set has been considered for the real application of the proposed study.

Key Words: xgamma distribution; type-I hybrid censoring; point and interval estimation; Bayes computation

Mathematical Subject Classification: 62F10, 62F15, 62C10

1. Introduction

In present era, the variety of probability distributions are developed and justified on the basis of failure rate function. These distributions are generated using different methods of generalization e.g. power of the CDF method, power of survival function, Marshall-Olkin method, logarithmic transformation, DUS transformation, mixing of two distributions etc. It has been shown that the developed distributions are good alternatives to the baseline distributions. The xgamma distribution is one of such distributions which has been obtained by mixing the gamma and exponential distribution with finite proportion, introduced by Sen et al. (2016). They have discussed several properties associated with this distribution. The xgamma distribution belongs to the one parameter exponential family of distributions, hence it may used as an alternative of several one parameter distributions, namely exponential, Lindley, Rayleigh models etc. Also, the xgamma model is the finite mixture of exponential and gamma distribution; hence very flexible to analyze the data set with constant and non-constant failure rates. The probability density function (PDF) of xgamma distribution is given as;

$$f(x,\beta) = \frac{\beta^2}{1+\beta}\left(1+\frac{\beta}{2}x^2\right)e^{-\beta x} \quad ; x,\beta > 0 \tag{1}$$

and the corresponding reliability function for any mission time $t > 0$ is the probability that any electronic system survives beyond the time t, which is given by;

$$R(t,\beta) = \left(\frac{1+\beta+\beta t+0.5(\beta x t)^2}{1+\beta}\right)e^{-\beta t} \quad ; t,\beta > 0 \tag{2}$$

In analysis of time-to-event data, censoring arises due to restrictions on time, cost and for administrative convenience; thus we have incomplete lifetime associated with failure of any equipments/units/products. The observed incomplete lifetimes are termed as censored data. Several censoring schemes have been advocated and used in analysis of different probability models with an intention to obtain constructive information using incomplete data. Type-I and Type-II censoring schemes are the two most favoured censoring schemes in literature. The parametric inferences using type-I and type-II censoring schemes are extensively discussed by several researchers. The mixture of Type-I and Type-II censoring scheme is called a hybrid censoring. It was introduced by Epstein (1954) and illustrated with the truncated life test plan for exponential distribution. The advantage of this censoring scheme is that the experiment is terminated at the combination of time T and number of failures R whichever is achieved first. The hybrid censoring is also divided into two parts on the basis of T and R named as Type-I hybrid censoring scheme and Type-II hybrid censoring scheme. In Type-I hybrid censoring scheme, let n units are placed on a test, the experiment is terminated at the $\min(x_{R:n}, T)$, where $x_{R:n}$ is the time of R^{th} failure in a sample of size n. The obtained observations in Type-I hybrid censoring is random and have lifetime less or equal to T, although at the end of the experiment there will be at least one failure. For a comprehensive detail about hybrid censoring, the readers may follow Balakrishnan and Kundu (2013). The details of the parametric inference based on different probability distributions using type-I and type-II hybrid censoring schemes have been extensively studied by several researchers. For example, Gupta and Kundu (1998) described hybrid censoring through exponential failure distribution, Ebrahimi (1990) discussed the classical estimation for exponential distribution under hybrid censoring while Draper and Guttman (1987) proposes the Bayes estimation procedure for the same distribution with hybrid censoring. The estimation of the parameters of the generalized exponential distribution has been discussed by Kundu and Pradhan (2009), Dey and Pradhan (2014) considered the generalized inverted exponential distribution under hybrid censoring. Singh et al. (2014) have discussed the classical and Bayesian estimation for Marshall-Olkin exponential distribution in presence of the type-I hybrid censored data, Also, the case of type-I and type-II hybrid censored for inverse Lomax distribution is discussed by Yadav et al. (2016).

Bayesian estimation procedures play a crucial part while dealing with lifetime distributions under different censoring schemes. The Bayesian estimation of flexible Weibull-Burr XII distribution using symmetric and asymmetric loss functions under adaptive Type-II progressive censoring has been recently discussed by Kamal and Ismail (2021) using MCMC method. Chadli and Kermoune (2021) described the classical and Bayesian reliability estimation in a Rayleigh Pareto model with progressively Type-II right censored data. Talhi and Aiachi (2021) performed a Bayesian analysis of the upper truncated Zeghdoudi distribution based on type II censored data under various loss functions.

Recently, the survival estimation of xgamma distribution has been discussed by Sen et al. (2018). Recently, Yadav et al. (2018) have proposed maximum likelihood estimation and Bayes estimation procedures for the parameter and reliability characteristics of xgamma distribution using hybrid type-II censored samples. Here, the main objective of this article is to propose estimation procedures of the parameter and reliability function of xgamma using Type-I hybrid censoring schemes. The Bayes estimates are evaluated using Tierney & Kadane (TK) approximation technique and Markov Chain Monte Carlo (MCMC) technique under the asymmetric loss function. The asymptotic confidence interval (ACI), bootstrap confidence interval (BCI's) and highest posterior density (HPD) credible interval have been computed for the same schemes. The uniqueness of this study comes from the fact that, no attempt has been made for the estimation of the parameter and reliability function for xgamma using type-I hybrid censored observation using GELF so far till date. Thus, the current article aims to fill-up this gap.

The organization of the present paper is as follows; The introduction of the considered study is given in Section 1. Section 2, describes the procedure of obtaining the type-I hybrid censored data and the classical estimation procedure. Different interval estimation, namely ACI, BCIs are considered in Section 3. The Bayes estimation procedure for the parameter and reliability function using two approximation techniques along with credible intervals has been described in Section 4. The performances of the obtained estimators are investigated using Monte Carlo simulations in Section 5. The application of the proposed study using survival data is provided in Section 6. The concluding remark of the considered study is given Section 7.

2. Data and Likelihood Function

In Type-I hybrid censoring scheme, the time-to-event data is obtained by terminating the experiments at the minimum of censoring time and number of failure. Let us consider $x_1, x_2, \cdots, x_n$ items of size n are put on test and $T, x_{r:n}$ denotes the censoring time and r^{th} failure, respectively. Thus, either we may get the r failures, i.e. $x_{1:n} < x_{2:n} < \cdots < x_{r:n}$ when $x_{r:n} \leq T$ or random number of failures, say d i.e. $x_{1:n} < x_{2:n} < \cdots < x_{d:n}$ when $x_{r:n} > T$. Thus, whenever the condition of $x_{r:n} \geq T$ arises the observed number of failure will be at least one. Although, in one hand, this number is very small and directly effects the efficiency of the estimators, but fixing time is the beauty of this scheme on the other-hand. Therefore, under the mentioned censoring scheme, the combined likelihood function of xgamma distribution for the above observed data set is written as;

$$L(\beta|\underline{\mathrm{x}}) = \frac{n!}{(n-k)!}\frac{\beta^{2k}e^{-\beta[s+(n-k)c]}}{(1+\beta)^n}\left[1+\beta+\beta c+\frac{(\beta c)^2}{2}\right]^{n-k}\prod_{i=1}^{r}\left(1+\frac{\beta}{2}x_i^2\right) \tag{3}$$

where;

$$k = \begin{cases} r \;\; when \;\; x_{r:n} \leq T \\ d \;\; when \;\; x_{r:n} > T \end{cases} \tag{4}$$

and

$$c = \begin{cases} x_{r:n} \;\; when \;\; x_{r:n} \leq T \\ x_{d:n} \;\; when \;\; x_{r:n} > T \end{cases} \tag{5}$$

2.1. Classical Estimation

In classical set-up the MLE of the parameter is evaluated using the likelihood function, given in Equation (3). The log-likelihood function is written by;

$$\begin{aligned}\ln L(\beta|\underline{x}) =&\zeta + 2k\ln\beta - n\ln(1+\beta) - \beta\left[s+(n-k)c\right] + (n-k)\ln\left[1+\beta+\beta c+\frac{(\beta c)^2}{2}\right] \\ &+ \ln P(\beta, x_i : n)\end{aligned} \tag{6}$$

The estimate of the parameter is obtained by optimizing the above equation w.r.t. to the parameter, which yield the following likelihood equation.

$$\frac{2k}{\beta} - \frac{n}{1+\beta} - \left[s+(n-k)c\right] + \frac{(n-k)(1+c+\beta c^2)}{\left[1+\beta+\beta c+\frac{(\beta c)^2}{2}\right]} + \sum_{i=1}^{k}\frac{x_{i:n}^2}{(2+\beta x_{i:n}^2)} = 0 \tag{7}$$

The above equation clearly reflects that the analytical solution is not possible due to its mathematical complexity. Thus, any non-linear maximization technique can be used to determine the MLE of the parameter β. In particular, Optim() function of R software is used here. One major drawback with N-R method is that it can not work properly without suitable starting value; hence to overcome this difficulties hit and trial method has been used to trace the exact starting value.

Once, the MLE of the parameter β is obtained, say $\hat{\beta}_m$; the MLE of the reliability function for mission time parameter t can be computed by simply plugging the $\hat{\beta_m}$ using the invariance property of the MLE. The MLE of reliability function $R(t)$ for specified mission time $t > 0$ is given as;

$$\hat{R}(t)_m = \left(\frac{1+\hat{\beta}_m+\hat{\beta}_m t+0.5(\hat{\beta}_m\, t)^2}{1+\hat{\beta}_m}\right)e^{-\hat{\beta}_m\, t} \tag{8}$$

3. Interval estimation

3.1. Asymptotic confidence interval

Since the exact distribution of $\hat{\beta}_m$ can not be obtained explicitly, the asymptotic theory of MLE can be used to construct $100(1-\alpha)\%$ confidence interval for the parameter. Under certain regularity conditions it has been proven that $\hat{\beta}_m$ follows asymptotic normal distribution with mean $\nu = \hat{\beta}_m$ and variance $\hat{\sigma}^2 = \frac{1}{I(\hat{\beta}_m)}$, where $I(\hat{\beta}_m)$ is the Fisher information, obtained as;

$$I(\hat{\beta}_m) = -E\left[\frac{\partial^2 \ln L(\beta|\underline{x})}{\partial\beta^2}\right] \tag{9}$$

where,

$$\begin{aligned}\frac{\partial^2 \ln L(\beta|\underline{x})}{\partial\beta^2} =& -\tfrac{2k}{\beta^2} + \tfrac{n}{(1+\beta)^2} + (n-k)\left\{\frac{c^2\left[1+\beta+\beta c+\frac{(\beta c)^2}{2}\right] - (1+c+\beta c^2)^2}{\left[1+\beta+\beta c+\frac{(\beta c)^2}{2}\right]^2}\right\} \\ &- \textstyle\sum_{i=1}^{k}\dfrac{x_{i:n}^4}{(2+\beta x_{i:n}^2)^2}\end{aligned}$$

A two-sided $100(1-\alpha)\%$ asymptotic confidence interval for β is obatined as;

$$[\hat{\beta}_L, \hat{\beta}_U] \in \hat{\beta}_m \mp Z_{\frac{\alpha}{2}} \hat{\sigma}$$

where, $Z_{\frac{\alpha}{2}}$ is the upper $(1-\frac{\alpha}{2})th$ quantile of a standard normal distribution.

3.2. Bootstrap Confidence Interval

The considered article deal with the type-I hybrid censored data; hence the number of observations obtained through life testing experiments are often not very large; therefore the ACI may not be an appropriate choice. Thus, in this section, we considered an alternative procedure suggested by Efron and Tibshirani (1986) known as bootstrap method. The bootstrap method for finding confidence interval is the most efficient sampling and re-sampling procedure without the need of pivotal quantity. Most importantly, it does not suffer from the condition of having a large sample in order to perform well. Here, we discuss the different types of bootstrap confidence interval (BCIs), namely standard bootstrap $(s-boot)$, percentile boot $(p-boot)$ and students t-bootstrap $(t-boot)$. The following steps may be used to construct the 95% BCI's.

- Specify the value of sample size n, effective sample size m, model parameter β and censoring time T.
- Generate $x_1, x_2, \cdots, x_k$ i.e. $k \leq n$ ordered type-I hybrid censored sample from equation (1)
- Compute MLE $\hat{\beta}$ of β using $x_1, x_2, \cdots, x_k$.
- Again generate type-I hybrid censored bootstrap samples $x_1^*, x_2^*, \cdots, x_k^*$ from equation (1) using $\hat{\beta}$ as a population value and calculate the MLE $\hat{\beta}^*$.
- Repeat step 2-3, B times and simulate $\hat{\beta}_i^*; i = 1, 2, \cdots, B$.

3.2.1. s-boot

Let $\bar{\beta}^*$ and S^* be the sample mean and sample standard deviation of $\beta^*, i = 1, 2, \cdots, B$.

$$\bar{\beta}^* = \frac{1}{B}\sum_{i=1}^{B}\hat{\beta}_i^* \quad and \quad S^* = \sqrt{\frac{1}{B}\sum_{i=1}^{B}(\hat{\beta}_i^* - \bar{\beta}^*)^2}$$

respectively. Thus, $100(1-\alpha)\%$ s-boot confidence interval for β is given by

$$[\hat{\beta}_L^s, \hat{\beta}_U^s] \in [\hat{\beta}^* - Z_{\alpha/2}.S^*, \hat{\beta}^* + Z_{\alpha/2}.S^*]$$

3.2.2. p-boot

Let $\hat{\beta}^{*(\delta)}$ be the δ-percentile of $(\hat{\beta}_{(i)}^*;\ i = 1, 2, \cdots, B)$ and $\hat{\beta}^{*(\delta)}$ is such that

$$\frac{1}{B}\sum_{i=1}^{B} I(\hat{\beta}_{(i)}^* \leq \hat{\beta}^{*(\delta)}) = \delta \quad : 0 \leq \delta \leq 1$$

where, $I(.)$ is the indicator function. Then $100(1-\alpha)\%$ p-boot confidence interval is given by

$$\left(\hat{\beta}_L^p, \hat{\beta}_U^p\right) \in \left(\hat{\beta}^{*[B\frac{\alpha}{2}]}, \hat{\beta}^{*[B\frac{1-\alpha}{2}]}\right)$$

3.2.3. t-boot

The students t-bootstrap confidence interval is obtained by the following additional steps;

- Generate again bootstrap sample $x_1^{**}, x_2^{**}, x_k^{**}$ of size $k \leq n$ from equation (1) using $\hat{\beta}^*$.
- Compute MLE of β say $\hat{\beta}^{**}$.
- Calculate $S^{**} = \sqrt{\frac{1}{B}\sum_{i=1}^{B}(\hat{\beta}_i^{**} - \bar{\beta}^{**})^2}$ where $\bar{\beta}^{**} = \frac{1}{B}\sum_{i=1}^{B}\hat{\beta}_i^{**}$
- Compute the statistic $T = \dfrac{\hat{\beta}^{**} - \bar{\beta}^{**}}{S^{**}}$. The $100(1-\alpha)\%$ t-boot confidence interval for β is given by

$$\left(\hat{\beta}_L^p, \hat{\beta}_U^p\right) \in \left(\bar{\beta}^{**} - t^{\alpha/2}.S^{**}, \quad \bar{\beta}^{**} + t^{\alpha/2}.S^{**}\right)$$

To study the different CIs, we consider their estimated average widths and coverage probability. For each of the considered methods, the average width of the BCIs is computed based on the B different trials. The average width and coverage probability are given by

$$\mathcal{W} = \frac{\sum_{i=1}^{B}(U_i - L_i)}{B}$$

$$\mathcal{P} = \frac{\#(L \leq \beta \leq U)}{B}$$

where L_W and U_P are the $100(1-\alpha)\%$ CI based on B replicates.

4. Bayesian estimation

Here, the Bayes estimation of the parameter has been discussed. The Bayes procedure is carried out with informative gamma prior and non-informative prior. The considered prior distributions are;

$$g_1(\beta) \propto \beta^{a-1}e^{-\beta b} \qquad ; \beta,\ a, b > 0 \tag{10}$$

The above defined prior is very flexible prior in the sense of assuming variety of shape of other distribution. It can be taken as non-informative prior by setting $a, b \to 0$, and resulting prior is written as

$$g_2(\beta) \propto \frac{1}{\beta} \tag{11}$$

The accuracy of the Bayes estimates is determined by specification of proper loss function. Usually, the most popular symmetric loss function is squared error loss function because it equally penalize the under as well as over estimation and seems to be inappropriate in the situation where under estimation is more serious than over estimation and vise versa. Thus, the choice of asymmetric loss function may be useful in such scenario. The most generalized asymmetric loss function is general entropy loss function (GELF), defined as;

$$L_G(\beta, \hat{\beta}) \propto \left(\frac{\hat{\beta}}{\beta}\right)^{\mu} - \mu \ln\left(\frac{\hat{\beta}}{\beta}\right) - 1 \tag{12}$$

where μ is the loss parameter which allows the different shapes of the loss function. It may be noted that when $\mu > 0$, a positive error causes more serious consequences than a negative error and vice versa. For the detailed description related to the asymmetric loss function the readers may follow Zellner (1986), Basu and Ebrahimi (1991). The loss function, defined in (12) converted to the different loss function under the following cases;

1. If $\mu = -1$, the GELF is converted to the squared error loss function (SELF).

2. If $\mu = 1$, the GELF is converted to entropy loss function (ELF).

3. if $\mu = -2$, the GELF is converted to precautionary loss function (PLF).

The Bayes estimator under GELF is given as;

$$\hat{\beta}_B = \left[E_\beta\left(\beta^{-\mu}\right)\right]^{-\frac{1}{\mu}}$$

The posterior distribution using the likelihood function defined in the Equation (3) and priors distributions defined in (10), (11) are obtained as;

$$\Pi_1(\beta|\underline{\mathrm{x}}) = \eta\frac{\beta^{2k+a-1}e^{-\beta[s+(n-k)c+b]}}{(1+\beta)^n}\,\psi(\beta,c)\prod_{i=1}^{k}\left(1+\frac{\beta}{2}x_i^2\right) \tag{13}$$

and

$$\Pi_2(\beta|\underline{\mathrm{x}}) = \eta\frac{\beta^{2k-1}e^{-\beta[s+(n-k)c]}}{(1+\beta)^n}\,\psi(\beta,c)\prod_{i=1}^{k}\left(1+\frac{\beta}{2}x_i^2\right) \tag{14}$$

Therefore, under the above mentioned loss function, the Bayes estimator for the parameter and reliability function under both prior distributions are derived as;

$$\hat{\beta}_{g_1} = \left[\eta\int_\beta \frac{\beta^{2k+a-\mu-1}e^{-\beta[s+(n-k)c+b]}}{(1+\beta)^n}\,\psi(\beta,c)\prod_{i=1}^{k}\left(1+\frac{\beta}{2}x_i^2\right)\,d\beta\right]^{-\frac{1}{\mu}} \tag{15}$$

$$\hat{\beta}_{g_2} = \left[\eta\int_\beta \frac{\beta^{2k-\mu-1}e^{-\beta[s+(n-k)c+b]}}{(1+\beta)^n}\,\psi(\beta,c)\prod_{i=1}^{k}\left(1+\frac{\beta}{2}x_i^2\right)\,d\beta\right]^{-\frac{1}{\mu}} \tag{16}$$

$$\hat{R}(t)_{g_1} = \left[\eta\int_\beta \frac{R^*(\beta,t)\beta^{2k+a-\mu-1}e^{-\beta[s+(n-k)c+b+t]}}{(1+\beta)^{n+1}}\,\psi(\beta,c)\prod_{i=1}^{k}\left(1+\frac{\beta}{2}x_i^2\right)\,d\beta\right]^{-\frac{1}{\mu}} \tag{17}$$

and

$$\hat{R}(t)_{g_2} = \left[\eta\int_\beta \frac{R^*(\beta,t)\beta^{2k-\mu-1}e^{-\beta[s+(n-k)c+t]}}{(1+\beta)^{n+1}}\,\psi(\beta,c)\prod_{i=1}^{k}\left(1+\frac{\beta}{2}x_i^2\right)\,d\beta\right]^{-\frac{1}{\mu}} \tag{18}$$

respectively.

where, η is the normalizing constant and

$$R^*(\beta,t) = 1+\beta+\beta t+0.5(\beta t)^2$$

$$\psi(\beta,c) = \left(1+\beta+\beta c+\frac{(\beta c)^2}{2}\right)^{n-k}$$

.

The posterior expectation, obtained in the Equation no. ((15) - (18)) are not in an explicit form. Therefore, the Bayes estimate of the parameter β may be obtained by using any Bayes approximation techniques. Here we have used two approximation techniques namely TK approximation technique and MCMC technique. The brief description of these approximation techniques are given in following sections.

4.1. Tierney-Kadane approximation technique

The TK approximation technique was initially introduced by Tierney and Kadane (1986) to approximate the posterior expectation which appears as the ratio of two integrals. The implementation of TK technique is straight froward. As TK suggested that the posterior expectation of any parametric function $w(\beta)$ with respect to the distribution $\Pi_1(\beta|x)$ is expressed in the following form;

$$I(x) = \frac{\int_\beta w(\beta)\, e^{l(\beta)+\tau(\beta)}\, d\beta}{\int_\beta e^{l(\beta)+\tau(\beta)}\, d\beta} \tag{19}$$

where, $l(\beta) = \ln L(\beta|\underline{\mathrm{x}})$, $\tau(\beta)$ are the logarithm of likelihood function and prior distribution, respectively. Let us define the following functions

$$\Delta(\beta) = \frac{l(\beta|\underline{\mathrm{x}}) + \ln g_1(\beta)}{n} \tag{20}$$

$$\Delta^*(\beta) = \Delta(\beta) + \frac{\ln w(\beta)}{n} \tag{21}$$

If $\hat{\beta}$ and $\hat{\beta}^*$ are the values which maximizes the Equations [(20), (21)] respectively. Then the function $I(x)$ is approximated by

$$I(x) = \sqrt{\frac{|\Sigma^*|}{|\Sigma|}} \quad exp\left[n\left\{\Delta^*(\hat{\beta}^*) - \Delta(\hat{\beta})\right\}\right] \tag{22}$$

where, $|\Sigma|$ and $|\Sigma^*|$ are the negative of inverse Hessian of $\Delta(\beta)$ and $\Delta^*(\beta)$ respectively computed at $\hat{\beta}$, $\hat{\beta}^*$.

Now in our considered case the Bayes estimator of the parameter β is obtained using the above log-likelihood function and prior distribution.

$$\Delta(\beta) = \frac{l(\beta|\underline{\mathrm{x}})}{n} + \frac{(a-1)*\ln\beta - b\beta}{n} \tag{23}$$

$$\frac{\partial\Delta}{\partial\beta} = \frac{1}{n}\left\{\frac{\partial l}{\partial\beta} + \frac{a-1}{\beta} - b\right\}, \quad \frac{\partial^2\Delta}{\partial\beta^2} = \frac{1}{n}\left\{\frac{\partial^2 l}{\partial\beta^2} - \frac{a-1}{\beta^2}\right\}, |\Sigma| = -E\left[\frac{\partial^2\Delta}{\partial\beta^2}\right]^{-1}_{\hat{\beta}}$$

Hence, after simplification,

$$\begin{aligned}\frac{\partial^2\Delta}{\partial\beta^2} = \frac{1}{n}&\left[-\frac{2k}{\beta^2} + \frac{n}{(1+\beta)^2} + (n-k)\left\{\frac{c^2\left[1+\beta+\beta c+\frac{(\beta c)^2}{2}\right] - (1+c+\beta c^2)^2}{\left[1+\beta+\beta c+\frac{(\beta c)^2}{2}\right]^2}\right\}\right] \\ &- \sum_{i=1}^{k}\frac{x_{i:n}^4}{n(2+\beta x_{i:n}^2)^2} - \frac{a-1}{n\beta^2}\end{aligned} \tag{24}$$

To compute the approximate Bayes estimates of β using GELF, we take $w(\beta) = \beta^{-\mu}$, then the function $\Delta^*(\beta)$ will be;

$$\Delta^*(\beta) = \Delta(\beta) - \frac{\mu \ln \beta}{n} \tag{25}$$

$$\frac{\partial \Delta *}{\partial \beta} = \frac{\partial \Delta}{\partial \beta} - \frac{\mu}{n\beta}, \quad \frac{\partial^2 \Delta^*}{\partial \beta^2} = \frac{\partial^2 \Delta}{\partial \beta^2} + \frac{\mu}{n\beta^2}$$

using (4.13), we have;

$$\begin{aligned}\frac{\partial^2 \Delta^*}{\partial \beta^2} = \frac{1}{n}&\left[-\frac{2k}{\beta^2} + \frac{n}{(1+\beta)^2} + (n-k)\left\{\frac{c^2\left[1+\beta+\beta c+\frac{(\beta c)^2}{2}\right] - (1+c+\beta c^2)^2}{\left[1+\beta+\beta c+\frac{(\beta c)^2}{2}\right]^2}\right\}\right] \\ &- \sum_{i=1}^{k} \frac{x_{i:n}^4}{n(2+\beta x_{i:n}^2)^2} - \frac{a}{n\beta^2}\end{aligned} \tag{26}$$

$$|\Sigma_\beta^*| = -E\left[\frac{\partial^2 \Delta^*}{\partial \beta^2}\right]^{-1}_{\hat{\beta}^*}$$

Thus, the desired Bayes estimate of β under GELF is obtained as;

$$\hat{\beta}_{TK} = \left(\sqrt{\frac{|\Sigma_\beta^*|}{|\Sigma|}}\; exp\left[n\left\{\Delta^*(\hat{\beta}^*) - \Delta(\hat{\beta})\right\}\right]\right)^{-\frac{1}{\mu}} \tag{27}$$

Now, the Bayes estimate of reliability function $R(t)$ is obtained by taking;

$$w_R(\beta) = \left[\left(\frac{1+\beta+\beta t+0.5(\beta t)^2}{1+\beta}\right) e^{-\beta t}\right]^{-\mu}$$

Therefore, the function $\Delta_R^*(\beta)$ for reliability function is computed as;

$$\Delta_R^*(\beta) = \Delta(\beta) + \frac{\ln w_R(\beta)}{n} \tag{28}$$

then;

$$\begin{aligned}\frac{\partial^2 \Delta_R^*}{\partial \beta^2} = \frac{1}{n}&\left[-\frac{2k}{\beta^2} + \frac{n}{(1+\beta)^2} + (n-k)\left\{\frac{c^2\left[1+\beta+\beta c+\frac{(\beta c)^2}{2}\right] - (1+c+\beta c^2)^2}{\left[1+\beta+\beta c+\frac{(\beta c)^2}{2}\right]^2}\right\}\right] \\ &- \sum_{i=1}^{k} \frac{x_{i:n}^4}{n(2+\beta x_{i:n}^2)^2} - \frac{t^2}{n(1+\beta+\beta\, t+0.5\,\beta^2\, t^2)} - \frac{(1+t+t^2\,\beta)^2}{n(1+\beta+\beta\, t+0.5\,\beta^2\, t^2)^2}\end{aligned} \tag{29}$$

$$|\Sigma_R^*| = -E\left[\frac{\partial^2 \Delta_R^*}{\partial \beta^2}\right]^{-1}_{\hat{\beta}^*}$$

Thus, the desired Bayes estimate of $R(t)$ under GELF is given by;

$$\hat{R}(t)_{TK} = \left(\sqrt{\frac{|\Sigma_R^*|}{|\Sigma|}}\; exp\left[n\left\{\Delta_R^*(\hat{\beta}^*) - \Delta(\hat{\beta})\right\}\right]\right)^{-\frac{1}{\mu}} \tag{30}$$

Similarly, we can find the Baye estimates of the parameter and reliability function under non-informative prior.

4.2. Markov Chain Monte Carlo technique

The T-K approximation technique is quite straight forward and easy to implement specially up to two parameters. But at the same time, we can not construct the interval estimate using this technique. Therefore, here, MCMC method has been used to overcome the situation. MCMC technique is one of the best and efficient Bayes computational techniques to obtain the Bayes estimates of any parametric function based on generated posterior samples. Further, the 95% highest posterior density (HPD) credible intervals of the parameter can be easily constructed using generated sequences of posterior sample. The parametric inferences using MCMC techniques have been extensively discussed by several authors. The application of the MCMC technique in different scenario may be seen in Hastings (1970), Geman and Geman (1984), Smith and Roberts (1993), Upadhyay et al. (2001) etc. Hence, to implement the MCMC technique, the full conditional posterior densities for β is given as;

$$\Pi_1(\beta|\underline{x}) \propto \frac{\beta^{2k+a-1}e^{-\beta[s+(n-k)c+b]}}{(1+\beta)^n}\left[1+\beta+\beta c+\frac{(\beta c)^2}{2}\right]^{n-k}\prod_{i=1}^{r}\left(1+\frac{\beta}{2}x_i^2\right) \tag{31}$$

The following steps is used to draw the posterior samples from the above full conditional distribution;

- set the initial values of β say β_0
- set j=1
- generate posterior sample for β from (31) using normal distribution as a proposal density.
- repeat step 2, for all $j = 1, 2, 3, \cdots, M$ and obtained $\beta_1, \beta_2, \cdots, \beta_M$. After getting the posterior samples the Bayes estimate of the parameters, reliability function and hazard function under SELF are the mean of the corresponding posterior samples. Therefore we have,

$$\hat{\beta}_{mc} \approx E(\beta|\underline{x}) = \left(\frac{1}{M}\sum_{j=1}^{M}\beta_j^{-\mu}\right)^{-\frac{1}{\mu}}$$

$$\hat{R}(t)_{mc} \approx \left(\frac{1}{M}\sum_{j=1}^{M}R(t)_j^{-\mu}\right)^{-\frac{1}{\mu}}$$

4.2.1. Credible/HPD interval

An interval based on posterior distribution, known as credible interval within which a parameter falls with some particular probability. However, the direct evaluation of credible interval through posterior distribution is quite difficult due to the explicit expression of posterior density. Therefore, Chen and Shao (1999) algorithm has been used to construct the $100(1-\alpha)\%$ credible intervals for β based on MCMC samples. For this purpose, order the generated MCMC samples, $\beta_1, \beta_2, ..., \beta_M$ as $\beta_1 < \beta_2 < ... < \beta_M$. Then $100(1-\alpha)\%$ credible intervals of β is

$$(\beta_1, \beta_{[M(1-\alpha)+1]}), \cdots, (\beta_{[M\alpha]}, \beta_M)$$

Here $[x]$ denotes the greatest integer less than or equal to x. Then, the HPD credible interval is that interval which has the shortest length among all possible intervals. The similar algorithm may be used to obtain the estimate using non-informative prior.

5. Simulation Study

In this section, the performances of the MLE and Bayes estimator of the parameter β and reliability function $R(t)$ has been investigated using Monte Carlo simulations. The comparison between the estimators are made in terms of mean square error (MSE) and length of the intervals based on 3000 replications. The MLEs and Bayes estimates are evaluated for different variation of sample size (n), effective sample size (k) and censoring time (T). In particular, we took $n|k$ as $20|16$, $30|24$, $40|32$ and $50|40$ and $T \in (0.85, 1.0, 1.25)$ for arbitrarily chosen $\beta = 2$. The MLE of the parameter is obtained using non-linear maximization technique and the estimate of the reliability function is obtained using invariance property of MLE. The Bayes estimators are derived under gamma prior using GELF. Since, the posterior expectation takes the form of ratio of two integrals; hence two Bayes approximation techniques, namely TK and MCMC techniques have been used to approximate the ratio of integrals into a finite value. The choice of prior parameters are taken as $a = 4, b = 2$. The Bayes estimate under non-informative prior may obtained by assuming $a = b \to 0$. For GELF, four choices $(-2, -1, 1, 2)$ of loss parameter μ are taken. The negative values correspond to the seriousness of under estimation and positive values correspond to the seriousness of over estimation. Further, $100(1-\alpha)\%$ ACI, BCI and credible HPD interval estimates of the parameter have been constructed for the same variation of censoring parameters. After performing the comprehensive simulation study, average estimates (first row), the MSEs (second row) are recorded and reported in Tables 2-4. Table 5 represents the average width $(\mathcal{W})$/coverage probability $(\mathcal{P})$ obtained under different methods. From this extensive simulation study, the following points have been noticed;

- The Bayes estimators obtained under non-informative prior behave more or less similar to the MLE. Although, the Bayes estimates obtained under informative prior are more efficient. This indicates that the Bayesian procedure with appropriate prior information provides more accurate estimates of the parameters.
- The average MSEs of the Bayes estimates obtained by T-K approximation method are larger than the estimates obtained via. MCMC.
- with the variation of loss parameter μ, the following trend is noticed

  $$MSE_{\hat{\beta}}(\mu = 2) < MSE_{\hat{\beta}}(\mu = 1) < MSE_{\hat{\beta}}(\mu = -1) < MSE_{\hat{\beta}}(\mu = -2)$$

  for the parameter β in both informative and non-informative prior information cases. However, in case of reliability function $R(t)$, the following pattern has been observed.

  $$MSE_{\hat{R}(t)}(\mu = -2) < MSE_{\hat{R}(t)}(\mu = -1) < MSE_{\hat{R}(t)}(\mu = 1) < MSE_{\hat{R}(t)}(\mu = 2)$$

  Also, among the MSEs of the estimators obtained via T-K and MCMC approximation techniques along with the variation of loss parameter μ the differences are more or less same.

- The MSEs of estimators decrease when the values of censoring parameters i.e. $n|k, T$ increases for fixed β.
- Bayes estimates of the β has least MSEs as compared to the MLE.
- It is also noticed that the Bayes HPD credible intervals obtained via. MCMC samples in both informative and non-informative prior cases have smaller length with relatively high coverage probabilities than ACI and BCIs.
- The length of ACI is higher than the BCIs, whereas in case of BCIs, boot-p provides better result than boot-s and boot-t. Although, the coverage probability obtained in ACI is relatively higher than the others and approaching to the nominal values with the increasing sample sizes (n, k) and censoring time T.
- The length of the interval evaluated for different (k, T) are decreasing for increasing percentage of k, T.

6. Real data example

In this section, the applicability of the proposed study has been demonstrated through a survival data set. The data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli. At first the considered survival data set was reported by Bjerkedal (1960). The applicability and suitability for the xgamma model based on the considered data set has been illustrated by Sen et al. (2018) among the most popular distributions namely exponential, gamma, Weibull, Lognormal and Lindley distributions using different model selection criteria such as, AIC, BIC etc. Hence, the same data set has been taken for the illustration purpose for this study. The survival times (in days) are as follows:

10, 33, 44, 56, 59, 72, 74, 77, 92, 93, 96, 100, 100, 102, 105, 107, 107, 108, 108, 108, 109, 112, 113, 115, 116, 120, 121, 122, 122, 124, 130, 134, 136, 139, 144, 146, 153, 159, 160, 163, 163, 168, 171, 172, 176, 183, 195, 196, 197, 202, 213, 215, 216, 222, 230, 231, 240, 245, 251, 253, 254, 254, 278, 293, 327, 342, 347, 361, 402, 432, 458, 555

Summary of the above data set is given below; it is noticed that the mean survival time of 72 guinea pigs infected by virulent tubercle bacilli decease is 176.8 days with standard deviation 103.45days.

Min.	1st Quartile	Median	3rd Quartile	Mean	Max	Sk	Kurtosis	Sd
10.0	108.0	149.5	224.0	176.8	555.0	1.3418	4.9910	103.45

Also, the skewness (Sk) of the data is positive, which indicate the considered data set is appropriate for xgamma model. The same may be also noticed in estimated kernel and estimated density plots (see Figure 1).

The maximum likelihood estimate and Bayes estimate of the parameter and reliability function are computed based on above data set using different type-I hybrid censored data. The type-I hybrid censored data set is obtained for different variations of k and T, i.e. $n = 72$, $k[T] \in (20[80, 110], 40[150, 250], 60[100, 300])$. The summary of the observed data under different above censoring schemes are presented through box plot, see in Figure 2.

The required numerical evaluations for the above considered schemes are carried out using R 3.1.1 software version. The computed MLE and Bayes estimates of the parameter and reliability function along with the confidence/credible intervals are presented in Table 1 and Table 2 respectively. The reliability estimate is

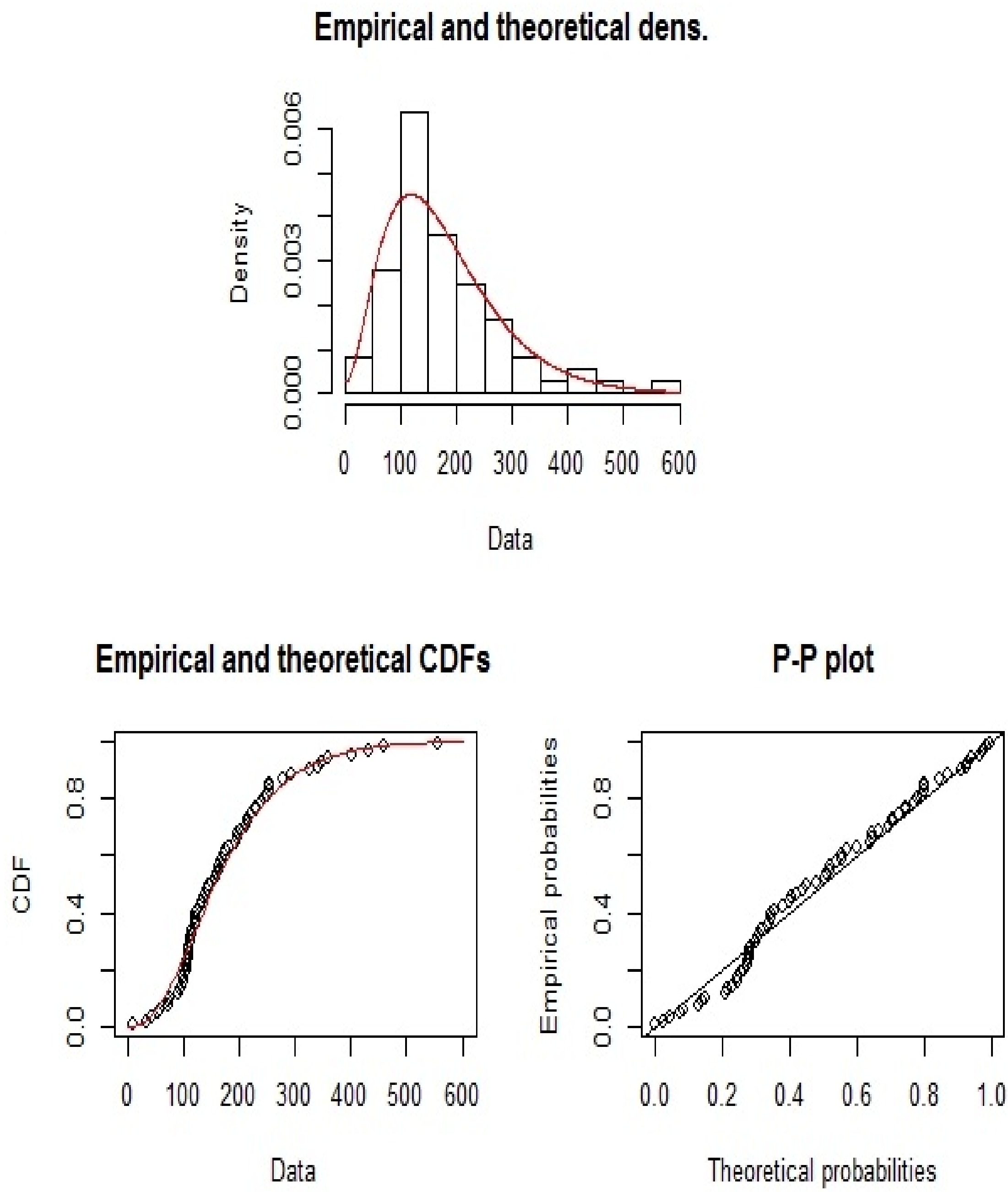

Figure 1: Estimated density plot, ECDF plot and PP plot for the considered data real data set.

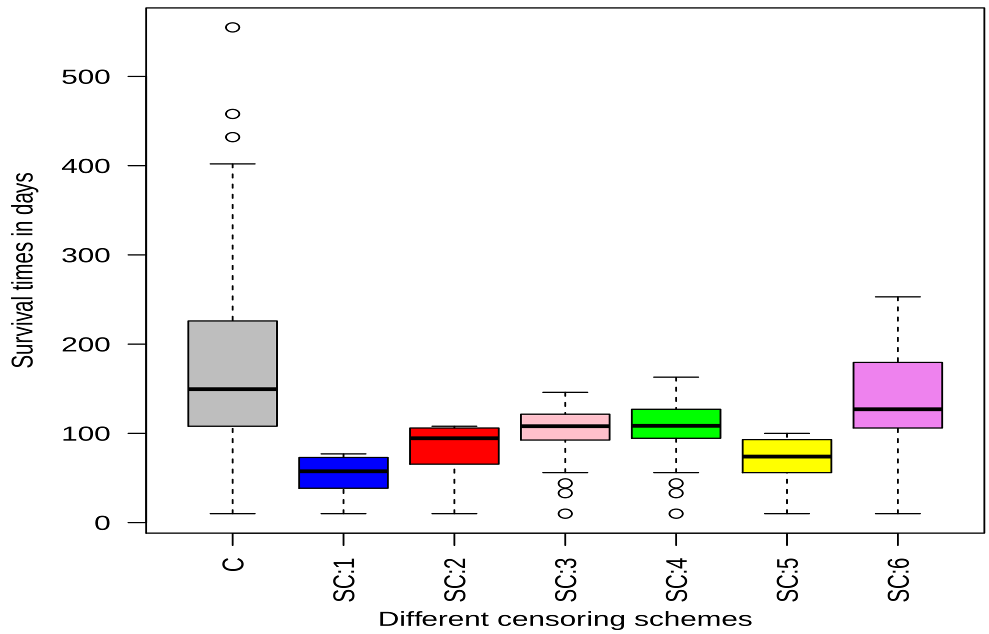

Figure 2: Summary of the data set for complete sample (C), and different censoring schemes.

evaluated for $t = 75\ days$ where the actual reliability is $R(75) = 0.5542$. In MCMC technique 10000 posterior samples are generated using normal proposal, also the convergence of the chain has been investigated and observed that MCMC chains are well mixed and converges to their stationary distributions, and approximately normally distributed. The choice of hyper parameters in case of real data set is taken as $a = 0.00001, \quad b = 0.00001$. Further, the width of the ACI, BCIs, and Bayes interval are also reported for the same design of censoring schemes. It is also to be noted that the length of the HPD credible interval (Bayes interval) is smaller than the length of ACI and BCIs (s-boot, p-boot & t-boot). However, the length of the ACI and BCIs are very much close to each other and the length of the interval evaluated for different (k, T) are decreasing for increasing percentages of k, T, see Table 2. The similar result is observed by extensive Monte Carlo simulation study performed in previous section.

7. Conclusion

In this article, the classical and Bayes estimation procedures for the parameter and reliability function of xgamma distribution have been discussed based on type-I hybrid censored samples. It is obvious that censoring adds complexity in estimation procedures, hence the estimators under classical and Bayesian setup are not obtained in nice closed form. Thus, the MLE of the parameter is obtained using $optim()$ function in R statistical software, the MLE of reliability function is computed using invariance property. The Bayes estimators are obtained with gamma prior under GELF and appeared in the form of ratio of two integrals, hence T-K and MCMC techniques have been used to obtain the estimates of the parameter and reliability function. The obtained estimators are investigated by conducting simulation study in terms of their MSEs. It is observed that the Bayes estimators obtained with informative prior for both parameter and reliability function have smaller MSEs as compared to the MLEs of the same; while in case of non-informative prior the performances of all considered estimators are more or less same for all considered variations of the censoring parameters. Further, the interval based on MLE and MCMC samples are constructed and noticed

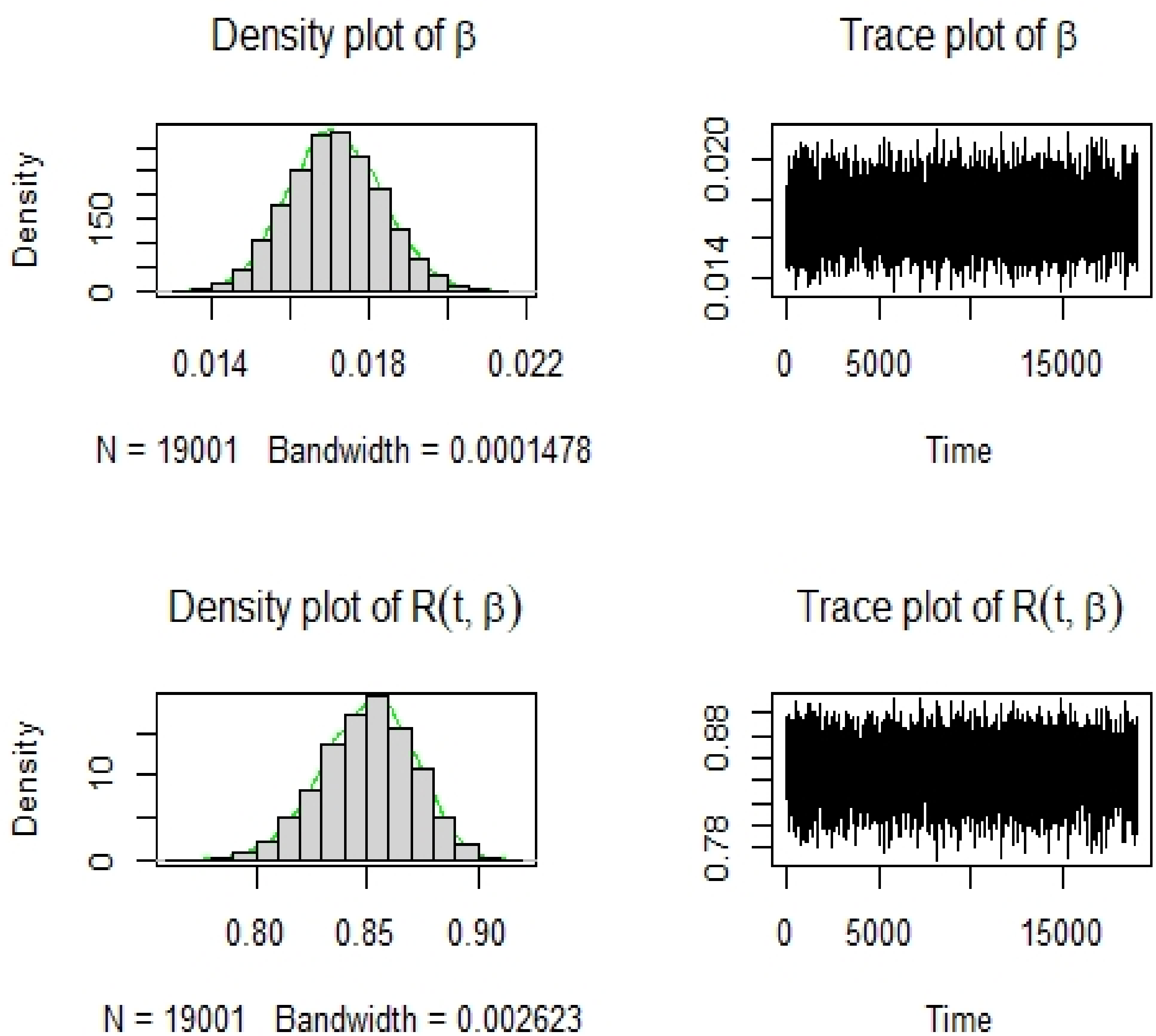

Figure 3: Posterior density and trace plots of the parameter and reliability function based on generated posterior samples for real data set.

Table 1: Real data estimates for the parameter and reliability function for the different censoring parametric combination.

n, k	T	$\hat{\beta}_{ML}$	T-K technique ($\hat{\beta}_{TK}$)				MCMC technique ($\hat{\beta}_{MC}$)			
			$\mu=-2$	$\mu=-1$	$\mu=1$	$\mu=2$	$\mu=-2$	$\mu=-1$	$\mu=1$	$\mu=2$
72, 20	80	0.0144	0.0152	0.0150	0.0147	0.0145	0.0152	0.0150	0.0147	0.0145
	110	0.0159	0.0164	0.0163	0.0161	0.0160	0.0163	0.0162	0.0160	0.0160
72, 40	150	0.0172	0.0175	0.0175	0.0173	0.0173	0.0175	0.0175	0.0173	0.0173
	250	0.0170	0.0173	0.0173	0.0171	0.0171	0.0173	0.0172	0.0171	0.0170
72, 60	100	0.0141	0.0146	0.0145	0.0143	0.0142	0.0146	0.0145	0.0143	0.0142
	300	0.0171	0.0174	0.0173	0.0172	0.0172	0.0174	0.0173	0.0172	0.0172
n, k	T	$\hat{R}(t)_{ML}$	T-K technique ($\hat{R}(t)_{TK}$)				MCMC technique ($\hat{R}(t)_{MC}$)			
			$\mu=-2$	$\mu=-1$	$\mu=1$	$\mu=2$	$\mu=-2$	$\mu=-1$	$\mu=1$	$\mu=2$
72, 20	80	0.8965	0.8970	0.8963	0.8949	0.8942	0.8959	0.8953	0.8940	0.8933
	110	0.8715	0.8715	0.8710	0.8700	0.8695	0.8717	0.8712	0.8704	0.8699
72, 40	150	0.8494	0.8493	0.8489	0.8481	0.8477	0.8500	0.8497	0.8489	0.8485
	250	0.8530	0.8529	0.8525	0.8518	0.8514	0.8534	0.8530	0.8523	0.8520
72, 60	100	0.9013	0.9012	0.9008	0.8999	0.8995	0.9017	0.9014	0.9006	0.9003
	300	0.8506	0.8505	0.8502	0.8496	0.8493	0.8510	0.8508	0.8502	0.8499

Table 2: Length ($\mathcal{L}$) of the ACI, BCIs and Bayes interval for different schemes.

n, k	T	ACI ($\mathcal{L}$)	BCI ($\mathcal{L}$)			Bayes ($\mathcal{L}$)
			s-boot	p-boot	t-boot	
20	80	0.0088	0.0091	0.0092	0.0107	0.0083
	110	0.0068	0.0073	0.0071	0.0080	0.0065
40	150	0.0058	0.0058	0.0059	0.0062	0.0055
	250	0.0055	0.0056	0.0055	0.0058	0.0053
60	100	0.0071	0.0073	0.0073	0.0086	0.0067
	300	0.0049	0.0050	0.0048	0.0053	0.0045

that the average width of HPD intervals are less as compared to the average width of asymptotic confidence interval. Finally, we believe that the methodologies discussed in the present article will be very useful for researchers, reliability practitioners and scientists in medicine where the analysis of reliability/medical data under censoring mechanism needs to be performed.

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Conflict of interest

The authors declare that they have no conflict of interests.

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Table 3: Average estimates and mean square error of the parameter using informative prior.

n, k	T	$\hat{\beta}_{ML}$	T-K technique ($\hat{\beta}_{TK}$)				MCMC technique ($\hat{\beta}_{MC}$)			
			$\mu=-2$	$\mu=-1$	$\mu=1$	$\mu=2$	$\mu=-2$	$\mu=-1$	$\mu=1$	$\mu=2$
20, 16	0.85	1.6425	1.6648	1.6246	1.5436	1.5027	1.6463	1.6088	1.5326	1.4938
		0.1763	0.1474	0.1333	0.1151	0.1110	0.1444	0.1324	0.1170	0.1139
	1.00	1.6377	1.6574	1.6213	1.5484	1.5116	1.6426	1.6089	1.5406	1.5059
		0.1608	0.1369	0.1243	0.1072	0.1028	0.1347	0.1238	0.1090	0.1053
	1.25	1.6261	1.6433	1.6120	1.5489	1.5171	1.6325	1.6034	1.5445	1.5146
		0.1371	0.1193	0.1088	0.0939	0.0895	0.1176	0.1084	0.0951	0.0911
30, 24	0.85	1.6238	1.6412	1.6141	1.5596	1.5322	1.6332	1.6081	1.5574	1.5318
		0.1182	0.1071	0.0990	0.0870	0.0833	0.1062	0.0990	0.0883	0.0849
	1.00	1.5762	1.5950	1.5710	1.5225	1.4981	1.5882	1.5659	1.5210	1.4983
		0.0933	0.0849	0.0800	0.0736	0.0722	0.0846	0.0802	0.0746	0.0734
	1.25	1.5676	1.5840	1.5631	1.5212	1.5001	1.5786	1.5594	1.5206	1.5011
		0.0815	0.0751	0.0712	0.0658	0.0645	0.0748	0.0713	0.0666	0.0653
40, 32	0.85	1.5714	1.5881	1.5679	1.5272	1.5068	1.5830	1.5643	1.5267	1.5077
		0.0781	0.0726	0.0689	0.0638	0.0626	0.0723	0.0690	0.0645	0.0633
	1.00	1.5533	1.5689	1.5508	1.5145	1.4962	1.5645	1.5479	1.5144	1.4975
		0.0647	0.0604	0.0577	0.0542	0.0534	0.0602	0.0578	0.0547	0.0540
	1.25	1.5501	1.5635	1.5478	1.5163	1.5005	1.5598	1.5454	1.5164	1.5018
		0.0570	0.0538	0.0515	0.0484	0.0476	0.0536	0.0516	0.0488	0.0487
50, 40	0.85	1.5418	1.5566	1.5405	1.5080	1.4917	1.5527	1.5378	1.5079	1.4928
		0.0558	0.0525	0.0505	0.0482	0.0471	0.0524	0.0507	0.0486	0.0482
	1.00	1.5456	1.5587	1.5441	1.5150	1.5003	1.5553	1.5419	1.5151	1.5015
		0.0507	0.0482	0.0464	0.0440	0.0434	0.0481	0.0465	0.0443	0.0437
	1.25	1.5422	1.5534	1.5408	1.5156	1.5029	1.5506	1.5390	1.5157	1.5040
		0.0433	0.0416	0.0400	0.0380	0.0374	0.0414	0.0401	0.0382	0.0377

Table 4: Average estimates of the parameter and corresponding MSEs under non-informative prior.

n, k	T	$\hat{\beta}_{ML}$	T-K technique ($\hat{\beta}_{TK}$)				MCMC technique ($\hat{\beta}_{MC}$)			
			$\mu = -2$	$\mu = -1$	$\mu = 1$	$\mu = 2$	$\mu = -2$	$\mu = -1$	$\mu = 1$	$\mu = 2$
20, 16	0.85	1.6425	1.7004	1.6542	1.5610	1.5139	1.6761	1.6336	1.5468	1.5026
		0.1763	0.2226	0.2011	0.1716	0.1638	0.2127	0.1949	0.1707	0.1646
	1.00	1.6377	1.6823	1.6415	1.5593	1.5178	1.6630	1.6254	1.5491	1.5103
		0.1608	0.1803	0.1632	0.1395	0.1329	0.1733	0.1590	0.1392	0.1339
	1.25	1.6261	1.6634	1.6287	1.5587	1.5234	1.6494	1.6174	1.5526	1.5197
		0.1371	0.1526	0.1388	0.1188	0.1126	0.1473	0.1356	0.1183	0.1130
30, 24	0.85	1.6238	1.6201	1.5907	1.5316	1.5019	1.6097	1.5827	1.5281	1.5004
		0.1182	0.1174	0.1095	0.0990	0.0965	0.1151	0.1084	0.0994	0.0973
	1.00	1.5762	1.6044	1.5784	1.5261	1.4997	1.5958	1.5720	1.5238	1.4994
		0.0933	0.1001	0.0942	0.0863	0.0845	0.0983	0.0932	0.0865	0.0849
	1.25	1.5676	1.5913	1.5690	1.5242	1.5017	1.5847	1.5643	1.5231	1.5023
		0.0815	0.0868	0.0821	0.0757	0.0740	0.0854	0.0813	0.0757	0.0743
40, 32	0.85	1.5714	1.6207	1.5989	1.5552	1.5332	1.6145	1.5946	1.5543	1.5340
		0.0781	0.0865	0.0809	0.0726	0.0699	0.0849	0.0800	0.0726	0.0701
	1.00	1.5533	1.5739	1.5548	1.5163	1.4970	1.5687	1.5512	1.5159	1.4981
		0.0647	0.0683	0.0651	0.0610	0.0601	0.0674	0.0647	0.0611	0.0602
	1.25	1.5501	1.5803	1.5631	1.5286	1.5113	1.5757	1.5600	1.5283	1.5124
		0.0570	0.0652	0.0622	0.0581	0.0569	0.0643	0.0618	0.0580	0.0570
50, 40	0.85	1.5418	1.5676	1.5511	1.5180	1.5014	1.5633	1.5482	1.5179	1.5026
		0.0558	0.0599	0.0573	0.0538	0.0528	0.0591	0.0569	0.0537	0.0529
	1.00	1.5456	1.5620	1.5468	1.5162	1.5009	1.5581	1.5442	1.5162	1.5021
		0.0507	0.0530	0.0510	0.0482	0.0476	0.0524	0.0506	0.0482	0.0476
	1.25	1.5422	1.5561	1.5430	1.5167	1.5035	1.5529	1.5409	1.5168	1.5046
		0.0433	0.0451	0.0435	0.0412	0.0405	0.0446	0.0432	0.0411	0.0405

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Table 5: Average width and coverage probability obtained through MLE, bootstrap and Bayes methods.

n, k	T	ACI		BCIs						HPD Interval			
				s-boot		p-boot		t-boot		Bayes_inf		Bayes_noinf	
		$\mathcal{W}$	$\mathcal{P}$	$\mathcal{W}$	$\mathcal{P}$	$\mathcal{W}$	$\mathcal{P}$	$\mathcal{W}$	$\mathcal{P}$	$\mathcal{W}$	$\mathcal{P}$	$\mathcal{W}$	$\mathcal{P}$
	0.85	2.8417	0.981	1.5517	0.978	1.5313	0.935	1.9644	0.991	1.3394	0.994	1.4340	0.991
20, 16	1.00	2.6417	0.975	1.4172	0.968	1.4126	0.925	1.8647	0.988	1.2715	0.989	1.3475	0.952
	1.25	2.6683	0.992	1.3371	0.975	1.3343	0.931	1.7635	0.989	1.1823	0.987	1.2435	0.99
	0.85	2.7050	0.985	1.2654	0.935	1.2608	0.925	1.5877	0.985	1.0991	0.979	1.1312	0.979
30, 24	1.00	2.5618	0.988	1.1870	0.954	1.1802	0.927	1.4835	0.987	1.0226	0.958	1.0602	0.985
	1.25	2.5753	0.992	1.1050	0.951	1.0986	0.918	1.3796	0.995	0.9501	0.991	0.9803	0.952
	0.85	2.5316	0.991	1.0680	0.956	1.0640	0.931	1.3056	0.987	0.9367	0.952	0.9780	0.953
40, 24	1.00	2.5006	0.994	0.9977	0.952	0.9926	0.925	1.2142	0.985	0.8806	0.99	0.9047	0.959
	1.25	2.4918	0.994	0.9250	0.956	0.9194	0.932	1.1219	0.988	0.8308	0.979	0.8600	0.976
	0.85	2.4941	0.994	0.9305	0.956	0.9277	0.941	1.1121	0.986	0.8198	0.985	0.8392	0.979
50, 40	1.00	2.4714	0.995	0.8819	0.94	0.8783	0.924	1.0551	0.985	0.7887	0.983	0.8054	0.985
	1.25	2.4615	0.991	0.8131	0.944	0.8087	0.928	0.9679	0.979	0.7339	0.987	0.7473	0.981

Table 6: Average estimates of the reliability function and corresponding MSEs with informative prior.

n, k	T	$\hat{R}(t)_{ML}$	T-K technique $(\hat{R}(t)_{TK})$				MCMC technique $(\hat{R}(t)_{MC})$			
			$\mu=-2$	$\mu=-1$	$\mu=1$	$\mu=2$	$\mu=-2$	$\mu=-1$	$\mu=1$	$\mu=2$
20,16	0.85	0.5186	0.5387	0.5294	0.5090	0.4978	0.5420	0.5333	0.5144	0.5042
		0.0125	0.0083	0.0090	0.0110	0.0126	0.0085	0.0091	0.0109	0.0122
	1	0.5225	0.5403	0.5320	0.5138	0.5040	0.5427	0.5349	0.5181	0.5091
		0.0115	0.0080	0.0086	0.0103	0.0116	0.0082	0.0087	0.0102	0.0113
	1.25	0.5234	0.5392	0.5318	0.5162	0.5077	0.5406	0.5338	0.5193	0.5116
		0.0101	0.0072	0.0077	0.0092	0.0102	0.0073	0.0078	0.0091	0.0100
30,24	0.85	0.5308	0.5432	0.5369	0.5235	0.5164	0.5441	0.5383	0.5259	0.5194
		0.0080	0.0061	0.0064	0.0074	0.0080	0.0062	0.0065	0.0073	0.0079
	1	0.5371	0.5478	0.5422	0.5303	0.5240	0.5482	0.5430	0.5320	0.5263
		0.0066	0.0052	0.0054	0.0061	0.0065	0.0052	0.0054	0.0060	0.0065
	1.25	0.5294	0.5396	0.5346	0.5241	0.5186	0.5399	0.5352	0.5256	0.5205
		0.0070	0.0056	0.0058	0.0066	0.0071	0.0056	0.0059	0.0066	0.0071
40,32	0.85	0.5374	0.5464	0.5416	0.5316	0.5264	0.5468	0.5424	0.5332	0.5284
		0.0056	0.0046	0.0048	0.0053	0.0056	0.0046	0.0048	0.0053	0.0056
	1	0.5370	0.5451	0.5408	0.5318	0.5272	0.5454	0.5414	0.5332	0.5289
		0.0051	0.0042	0.0044	0.0049	0.0052	0.0043	0.0044	0.0049	0.0051
	1.25	0.5402	0.5472	0.5434	0.5357	0.5317	0.5473	0.5439	0.5368	0.5331
		0.0048	0.0041	0.0043	0.0046	0.0049	0.0042	0.0043	0.0046	0.0048
50, 40	0.85	0.5399	0.5470	0.5431	0.5351	0.5310	0.5472	0.5436	0.5363	0.5325
		0.0048	0.0041	0.0042	0.0046	0.0048	0.0041	0.0043	0.0046	0.0048
	1	0.5408	0.5471	0.5437	0.5365	0.5328	0.5473	0.5441	0.5375	0.5341
		0.0045	0.0039	0.0040	0.0043	0.0045	0.0039	0.0040	0.0043	0.0045
	1.25	0.5421	0.5476	0.5446	0.5384	0.5353	0.5478	0.5450	0.5393	0.5364
		0.0036	0.0032	0.0033	0.0035	0.0037	0.0032	0.0033	0.0035	0.0036

Table 7: Average estimates of the reliability function and corresponding MSEs with non-informative prior.

n, k	T	$\hat{R}(t)_{ML}$	T-K technique $(\hat{R}(t)_{TK})$				MCMC technique $(\hat{R}(t)_{MC})$			
			$\mu=-2$	$\mu=-1$	$\mu=1$	$\mu=2$	$\mu=-2$	$\mu=-1$	$\mu=1$	$\mu=2$
20,16	0.85	0.5186	0.5349	0.5244	0.5013	0.4884	0.5388	0.5291	0.5080	0.4965
		0.0125	0.0102	0.0111	0.0139	0.0160	0.0103	0.0111	0.0135	0.0152
	1	0.5225	0.5378	0.5283	0.5077	0.4964	0.5406	0.5319	0.5131	0.5029
		0.0115	0.0093	0.0101	0.0125	0.0142	0.0094	0.0101	0.0121	0.0135
	1.25	0.5234	0.5344	0.5263	0.5087	0.4992	0.5363	0.5287	0.5127	0.5041
		0.0101	0.0085	0.0092	0.0112	0.0126	0.0086	0.0092	0.0109	0.0121
30,24	0.85	0.5308	0.5391	0.5321	0.5174	0.5095	0.5402	0.5338	0.5203	0.5132
		0.0080	0.0071	0.0076	0.0088	0.0097	0.0072	0.0076	0.0087	0.0094
	1	0.5371	0.5494	0.5433	0.5305	0.5237	0.5502	0.5447	0.5330	0.5268
		0.0066	0.0059	0.0062	0.0070	0.0075	0.0060	0.0062	0.0069	0.0074
	1.25	0.5294	0.5458	0.5406	0.5295	0.5236	0.5463	0.5414	0.5313	0.5260
		0.0070	0.0058	0.0060	0.0067	0.0072	0.0058	0.0060	0.0067	0.0077
40,32	0.85	0.5374	0.5439	0.5388	0.5281	0.5224	0.5443	0.5396	0.5298	0.5247
		0.0056	0.0054	0.0057	0.0063	0.0068	0.0055	0.0057	0.0063	0.0066
	1	0.5370	0.5427	0.5381	0.5285	0.5235	0.5430	0.5388	0.5300	0.5255
		0.0051	0.0045	0.0047	0.0053	0.0057	0.0045	0.0047	0.0052	0.0055
	1.25	0.5402	0.5456	0.5417	0.5335	0.5293	0.5458	0.5422	0.5347	0.5309
		0.0048	0.0043	0.0044	0.0048	0.0051	0.0043	0.0044	0.0048	0.0050
50, 40	0.85	0.5399	0.5456	0.5416	0.5331	0.5287	0.5458	0.5421	0.5344	0.5304
		0.0048	0.0040	0.0042	0.0046	0.0049	0.0040	0.0042	0.0045	0.0048
	1	0.5408	0.5498	0.5462	0.5388	0.5349	0.5500	0.5467	0.5399	0.5364
		0.0045	0.0044	0.0044	0.0040	0.0042	0.0044	0.0046	0.0044	0.0049
	1.25	0.5421	0.5505	0.5474	0.5410	0.5377	0.5507	0.5478	0.5420	0.5390
		0.0036	0.0033	0.0034	0.0036	0.0038	0.0034	0.0034	0.0036	0.0037