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The Marshall-Olkin Extended Power Lomax Distribution with Applications

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Abstract

In this article, we define a new four-parameter model called Marshall-Olkin extended power Lomax distribution and study its properties. Limiting distributions of sample maxima and sample minima are derived. The reliability of a system when both stress and strength follows the new distribution is discussed and associated characteristics are computed for a simulated data. Finally, utilizing maximum likelihood estimation, the goodness of the distribution is tested for a real data.

Key Words: Hazard Rate Function; Marshall-Olkin Distribution; Maximum Likelihood Estimation; Power Lomax Distribution; Reliability
Mathematical Subject Classification: 60E05, 62E15, 62F10.

1. Background

Probability distributions are very useful models for characterizing variability in lifetime data. Lomax (1954) introduced an ingenious distribution namely, the Lomax distribution (also known as Pareto distribution of the second kind) which is used in business, economics, and actuarial modeling. It is very similar to the Pareto distribution and it was used by Lomax (1954) to fit in business failure data. Also, Rady et al. (2016) introduced power Lomax distribution. They discovered that the power Lomax distribution offer better fit on bladder cancer data than classical Lomax distribution. The survival function and probability density function (pdf) of power Lomax distribution are given by

$$\bar{F}(x,\theta,\lambda) = \left[1+\left(\frac{x}{\lambda}\right)^{\beta}\right]^{-\gamma},\ x,\beta,\gamma,\lambda>0 \tag{1}$$

and

$$f(x,\theta,\lambda) = \beta\gamma\lambda^{-\beta}x^{\beta-1}\left[1+\left(\frac{x}{\lambda}\right)^{\beta}\right]^{-\gamma-1},\ x,\beta,\gamma,\lambda>0 \tag{2}$$

On the other hand, Marshall and Olkin (1997) introduced a well known tool for obtaining more flexible distributions which is obtained by adding a new parameter $\alpha>0$ to an existing distribution. Let $\bar{F}(x)$ be the survival function of a continuous random variable X. Then by the technique of Marshall and Olkin (1997), we get another survival function $\bar{G}(x,\alpha)$ given by

$$\bar{G}(x,\alpha) = \frac{\alpha\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)} \quad ; -\infty<x<\infty; \alpha>0, \bar{\alpha}=1-\alpha. \tag{3}$$

The family of such distributions will be referred to as the Marshall-Olkin Extended (MOE) family. If $g(x,\alpha)$ and

$r(x,\alpha)$ are the pdf and hazard rate function (hrf) corresponding to $\bar{G}$, then we have,

$$g(x,\alpha) = \frac{\alpha f(x)}{[1-\bar{\alpha}\bar{F}(x)]^2} \quad ; -\infty < x < \infty, \alpha > 0, \bar{\alpha} = 1-\alpha \tag{4}$$

and

$$h(x,\alpha) = \frac{r(x)}{1-\bar{\alpha}\bar{F}(x)} \tag{5}$$

where $f(x)$ and $r(x)$ are the pdf and the hrf corresponding to $\bar{F}(x)$.

The rest of the paper is organized as follows. Section 2 introduces the MOE power Lomax distribution and presents features of the distribution. Section 3 focuses on limiting distribution of the sample extremes. Section 4 specifies stress-strength analysis and illustrates with respect to a simulated data. Section 5 proposes parameter estimation of the distribution by the method of maximum likelihood estimation. Section 6 deals with the application of the new distribution to a real data set. Finally, Section 7 summarizes the findings of the research.

2. MOE Power Lomax Distribution

Motivated by the advantages of Lomax distribution, we introduce and study a new distribution called MOE power Lomax (MOEPL) distribution. The model inherits desirable properties from Lomax distribution. By inserting (1) in (3), the survival function of MOEPL distribution is given by

$$\bar{G}(x,\alpha,\beta,\gamma,\lambda) = \frac{\alpha}{\left[1+\left(\frac{x}{\lambda}\right)^{\beta}\right]^{\gamma} - \bar{\alpha}} \;,\; x,\alpha,\beta,\gamma,\lambda > 0 \tag{6}$$

The corresponding pdf is given by

$$g(x,\alpha,\beta,\gamma,\lambda) = \frac{\alpha\beta\gamma\lambda^{-\beta}x^{\beta-1}\left[1+\left(\frac{x}{\lambda}\right)^{\beta}\right]^{\gamma-1}}{\left\{\left[1+\left(\frac{x}{\lambda}\right)^{\beta}\right]^{\gamma} - \bar{\alpha}\right\}^2} \;,\; x,\alpha,\beta,\gamma,\lambda > 0 \tag{7}$$

In addition, the hrf of the MOEPL distribution becomes

$$h(x,\alpha,\beta,\gamma,\lambda) = \frac{\gamma\beta x^{\beta-1}}{x^{\beta}+\lambda^{\beta}\left\{1-\bar{\alpha}\left[1+\left(\frac{x}{\lambda}\right)^{\beta}\right]^{-\gamma}\right\}} \;,\; x,\alpha,\beta,\gamma,\lambda > 0 \tag{8}$$

The plots of pdf and hrf for selected parameters of α,β,γ and λ are shown in Figures 1. From Figure 1, we can conclude that α is a scale parameter. From Figure 2, we note that hrf of MOEPL can be monotone, non-monotone, unimodality and upside down bathtub shapes.

The quantile function of X follows MOEPL distribution, it can be expressed as

$$Q(u) = \lambda\left\{\left[(1-p)^{-1}\alpha+\bar{\alpha}\right]^{\frac{1}{\gamma}} - 1\right\}^{\frac{1}{\beta}}$$

where u is generated from the Uniform(0, 1) distribution. The r^{th} ordinary moment of X is given by

$$\mu_r^{'} = E(X^r) = r\int_0^{\infty} x^{r-1}\bar{F}(x)dx$$

Figure 1: Graphs of pdf of the MOEPL distribution for different values of α,β,γ and λ

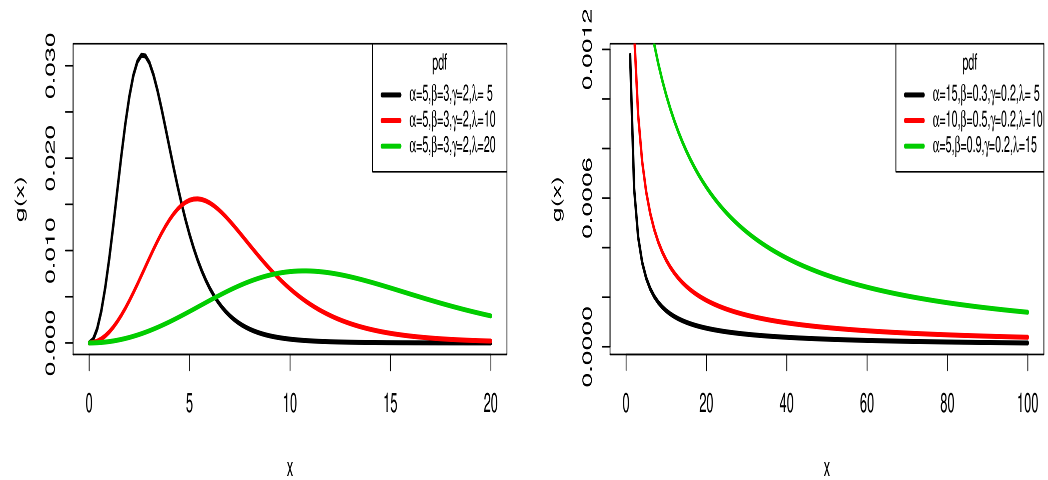
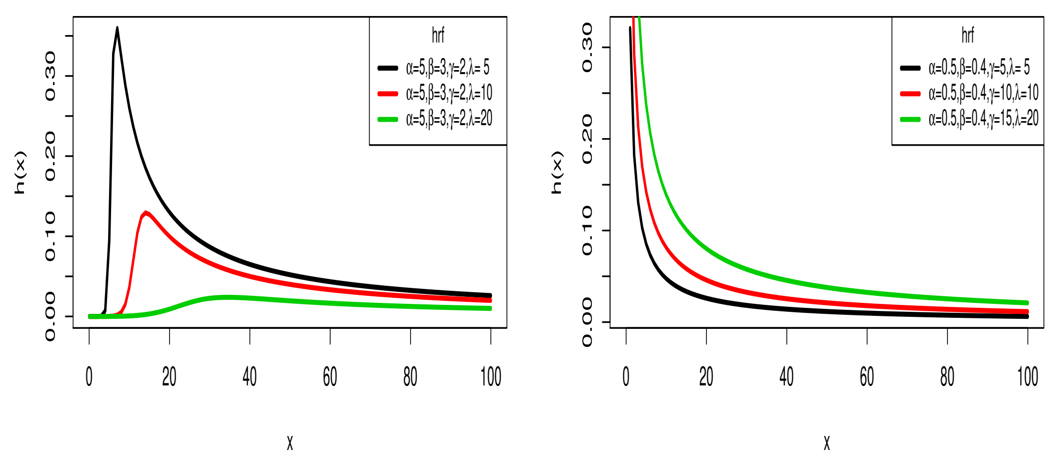


Figure 2: Graphs of hrf of the MOEPL distribution for different values of α,β,γ and λ



Hence the r^{th} moment of MOEPL distribution, by giving substitution $y = \left[1 + \left(\frac{x}{\lambda}\right)^{\beta}\right]^{\gamma}$ is given by

$$\begin{aligned} \mu_r^{'} &= r\int_0^{\infty} x^{r-1}\frac{\alpha}{\left[1+\left(\frac{x}{\lambda}\right)^{\beta}\right]^{\gamma} - \bar{\alpha}}dx \\ &= \frac{r\alpha\lambda^r}{\beta\gamma}\sum_{i=0}^{\frac{r-\beta}{\beta}}\int_1^{\infty}\frac{(-1)^{i+\frac{r-\beta}{\beta}}\binom{\frac{r-\beta}{\beta}}{i}y^{\frac{i+1}{\gamma}-1}}{y-\bar{\alpha}}dy \end{aligned}$$

Thus $\mu_r^{'}$ cannot be expressed in a simple closed form but it can be calculated numerically.

3. Limiting Distributions of Sample Extremes

Consider the sample $X_1, X_2, \ldots, X_n$ of size n from MOEPL distribution. Let $X_{1:n} = \min(X_1, X_2, \ldots, X_n)$ be the sample minima and $X_{n:n} = \max(X_1, X_2, \ldots, X_n)$ be the sample maxima. Also note that the extreme order statistics $X_{1:n}$ and $X_{n:n}$ represents the life of series and parallel systems.

Theorem 3.1. *Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from MOEPL distribution. Then*

(i) $\lim_{n\to\infty} P(X_{1:n} \le b_n^* t) = 1 - e^{-t},\ t > 0,$ *where* $b_n^* = \lambda\left[\left(\frac{n\alpha}{n-1} + \bar{\alpha}\right)^{\frac{1}{\gamma}} - 1\right]^{\frac{1}{\beta}}$

(ii) $\lim_{n\to\infty} P(X_{n:n} \le a_n + b_n t) = e^{-t},\ t < 0,$ *where* $a_n = 0$ *and* $b_n = \lambda\left(\left[(n-1)\alpha + 1\right]^{\frac{1}{\gamma}} - 1\right)^{\frac{1}{\beta}}$.

Proof. (i) We use the following asymptotic result by Arnold et al. (1992) for $X_{1:n}$ by which

$$\lim_{n\to\infty} P(X_{1:n} \le a_n^* + b_n^* t) = 1 - e^{-t^c},\ t > 0,\ c > 0,$$

(of the Weibull type) where $a_n^* = F^{-1}(0)$ and $b_n^* = F^{-1}\left(\frac{1}{n}\right) - F^{-1}(0)$ if and only if $F^{-1}(0)$ is finite and for all $t > 0$ and $c > 0$

$$\lim_{\epsilon\to 0^+}\frac{F(F^{-1}(0) + \epsilon t)}{F(F^{-1}(0) + \epsilon)} = t^c$$

For the MOEPL distribution, we have $G^{-1}(0) = 0$ is finite and also

$$\lim_{\epsilon\to 0^+}\frac{G(\epsilon t)}{G(\epsilon)} = t^{\beta}$$

Thus we obtain that $c = \beta$, $a_n^* = 0$ and $b_n^* = \lambda\left[\left(\frac{n\alpha}{n-1} + \bar{\alpha}\right)^{\frac{1}{\gamma}} - 1\right]^{\frac{1}{\beta}}$.

(ii) For the maximal order statistics $X_{n:n}$ we have

$$\lim_{n\to\infty} P(X_{n:n} \le a_n + b_n t) = e^{-(-t)^l},\ t < 0,\ l > 0,$$

(of the Fréchet type) where $a_n = 0$, $b_n = F^{-1}\left(1 - \frac{1}{n}\right)$ if and only if $F^{-1}(1) = \infty$ and there exists a constant $l > 0$ such that

$$\lim_{x\to\infty}\frac{1 - F(xt)}{1 - F(x)} = t^{-l}$$

For MOEPL distribution, we have $G^{-1}(1) = \infty$ and

$$\lim_{x\to\infty}\frac{1 - G(xt)}{1 - G(x)} = t^{-\beta\gamma}$$

Thus we obtain that $l = \beta\gamma$, $a_n = 0$ and $b_n = \lambda\left(\left[(n-1)\alpha + 1\right]^{\frac{1}{\gamma}} - 1\right)^{\frac{1}{\beta}}$. □

Remarks 1: For the power Lomax distribution ($\alpha = 1$) the norming constants are $b_n^* = \lambda \left[(\frac{n}{n-1})^{\frac{1}{\gamma}} - 1\right]^{\frac{1}{\beta}}$ and $b_n = \lambda \left(n^{\frac{1}{\gamma}} - 1\right)^{\frac{1}{\beta}}$.

Remarks 2: If the limiting distributions of the random variables $(X_{1:n} - a_n^*)/b_n^*$ and $(X_{n:n} - a_n)/b_n$ are denoted by $G^*(t)$ and $G(t)$, then with regard to the fact that (Arnold et al., 1992) for any finite $i > 1$, the limiting distributions of the random variable $(X_{i:n} - a_n^*)/b_n^*$ is given by

$$\lim_{n\to\infty} P(X_{i:n} \leq a_n^* + b_n^* t) = 1 - \sum_{j=0}^{i-1} (1 - G^*(t)) \frac{[-\log(1 - G^*(t))]^j}{j!}$$

and the limiting distributions of the random variable $(X_{n-i+1:n} - a_n)/b_n$ is given by

$$\lim_{n\to\infty} P(X_{n-i+1:n} \leq a_n + b_n t) = \sum_{j=0}^{i-1} G(t) \frac{(-\log G(t))^j}{j!}.$$

From Theorem 3.1, for any finite $i > 1$ the limiting distributions of the i^{th} order statistics from the MOEPL distribution is given by

$$\begin{aligned} \lim_{n\to\infty} P(X_{1:n} \leq b_n^* t) &= 1 - \sum_{j=0}^{i-1} e^{-t^{\beta}} \frac{t^{\beta j}}{j!} \\ &= 1 - P(W < i) \end{aligned}$$

and matching limiting distributions of the $(n - i + 1)^{th}$ order statistics from the MOEPL distribution is given by

$$\begin{aligned} \lim_{n\to\infty} P(X_{n-i+1:n} \leq a_n + b_n t) &= \sum_{j=0}^{i-1} e^{-t^{-\beta\gamma}} \frac{t^{-\beta\gamma j}}{j!} \\ &= P(Z < i) \end{aligned}$$

where W and Z follows the Poisson distribution with means t^{β} and $t^{-\gamma\beta}$.

4. Stress-Strength Analysis and Estimation of Reliability

The reliability of a component in terms of the probability that the random variable X representing stress experienced by the component exceeds Y, representing the strength of the component. It has been calculated by using the equation $R = P(Y > X)$. The component fails when the stress exceeds strength, and vice versa. Several aspects of stress-strength theory has been discussed by Kotz et al. (2003). The problem when two independent random variables representing strength (X) and stress (Y) follow the same Marshall-Olkin extended distributions then the reliability of the system has been discovered by Gupta et al. (2010). Gupta et al. (2010) has showed that Marshall-Olkin extended distributions with tilt parameters α_1 and α_2 the R can be expressed as follows

$$R = \frac{\frac{\alpha_1}{\alpha_2}}{(\frac{\alpha_1}{\alpha_2} - 1)^2} \left[-\ln \frac{\alpha_1}{\alpha_2} + \frac{\alpha_1}{\alpha_2} - 1\right]$$

In order to estimate the expression for R it is enough to find maximum likelihood estimates (MLEs) of α_1 and α_2 . The log likelihood equation is given by

$$\begin{aligned} LL &= m\log\alpha_1 + n\log\alpha_2 + (m+n)\text{log}\beta + (m+n)\text{log}\gamma - \beta(m+n)\log\lambda \\ &+(\beta-1)\left(\sum_{i=1}^{m}\log x_i + \sum_{i=1}^{n}\log x_i\right) - 2\sum_{i=1}^{m}\left\{1-\bar{\alpha_1}\left[1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right]^{-\gamma}\right\} \\ &-2\sum_{i=1}^{n}\left\{1-\bar{\alpha_2}\left[1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right]^{-\gamma}\right\} \end{aligned}$$

Then the MLEs of α_1 and α_2 are the solutions of the non-linear equations

$$\begin{aligned} \frac{\partial LL}{\partial\alpha_1} &= \frac{m}{\alpha_1} - \sum_{i=1}^{m}\frac{2\left[1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right]^{-\gamma}}{\left\{1-\bar{\alpha_1}\left[1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right]^{-\gamma}\right\}} \\ \frac{\partial LL}{\partial\alpha_2} &= \frac{n}{\alpha_2} - \sum_{i=1}^{n}\frac{2\left[1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right]^{-\gamma}}{\left\{1-\bar{\alpha_2}\left[1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right]^{-\gamma}\right\}} \end{aligned}$$

Using the property of MLE for $m\to\infty, n\to\infty$ we have

$\sqrt{m}(\hat{\alpha_1}-\alpha_1), \sqrt{n}(\hat{\alpha_2}-\alpha_2) \xrightarrow{d} N_2(\mathbf{0}, diag\{\frac{1}{a11}, \frac{1}{a22}\}$

where $a_{11} = \lim\limits_{m,n\to\infty}\dfrac{1}{m}I_{11} = \dfrac{1}{3\alpha_1^2}$ and $a_{22} = \lim\limits_{m,n\to\infty}\dfrac{1}{n}I_{22} = \dfrac{1}{3\alpha_2^2}$

Also the Information matrix enfolds the following components

$$\begin{aligned} I_{11} &= -E\left(\frac{\partial^2 LL}{\partial\alpha_1^2}\right) \\ &= -E\left(\frac{-m}{\alpha_1^2} + 2\sum_{i=1}^{m}\frac{\left[1+\left(\frac{x}{\lambda}\right)^{\beta}\right]^{-2\gamma}}{\left\{1-\bar{\alpha}\left[1+\left(\frac{x}{\lambda}\right)^{\beta}\right]^{-\gamma}\right\}^2}\right) \\ &= \frac{m}{\alpha_1^2} - 2\alpha_1 m\int_{\alpha}^{\infty}\frac{dt}{t^4} \\ &= \frac{m}{3\alpha_1^2} \end{aligned}$$

Similarly $I_{22} = -E\left(\frac{\partial^2 LL}{\partial\alpha_1^2}\right) = -\frac{n}{3\alpha_2^2}$ and $I_{12} = I_{21} = -E\left(\frac{\partial^2 LL}{\partial\alpha_1\alpha_2}\right) = 0$

Now due to Gupta et al. (2010), R (for 95% confidence interval) is given by

$\hat{R} \mp 1.96\ \hat{\alpha}_1 b_1(\hat{\alpha}_1, \hat{\alpha}_2)\sqrt{\frac{3}{m}+\frac{3}{n}}$, where

$b_1(\alpha_1,\alpha_2) = \frac{\partial R}{\partial\alpha_1} = \frac{\alpha_2}{(\alpha_1-\alpha_2)^3}\left[-2(\alpha_1-\alpha_2) + (\alpha_1+\alpha_2)\ln\frac{\alpha_1}{\alpha_2}\right]$ and

$b_2(\alpha_1,\alpha_2) = \frac{\partial R}{\partial\alpha_2} = \frac{\alpha_1}{(\alpha_1-\alpha_2)^3}\left[2(\alpha_1-\alpha_2) - (\alpha_1+\alpha_2)\ln\frac{\alpha_1}{\alpha_2}\right] = -\frac{\alpha_1}{\alpha_2}b_1(\alpha_1,\alpha_2)$

Table 1: The values of b and AMSE of the simulated estimates of R for $\beta = 9$, $\gamma = 4$ and $\lambda = 3$

	b				AMSE			
	(α_1, α_2)				(α_1, α_2)			
(m,n)	(0.5,0.8)	(0.9,1.2)	(0.8,0.5)	(1.2,0.9)	(0.5,0.8)	(0.9,1.2)	(0.8,0.5)	(1.2,0.9)
(20,20)	0.0789	0.0501	-0.0797	-0.0499	0.0079	0.0042	0.0080	0.0042
(20,25)	0.0777	0.0482	-0.0815	-0.0517	0.0075	0.0039	0.0081	0.0042
(20,30)	0.0770	0.0468	-0.0827	-0.0535	0.0073	0.0036	0.0082	0.0043
(25,20)	0.0817	0.0593	-0.0785	-0.0485	0.0082	0.0063	0.0077	0.0039
(25,25)	0.0793	0.0584	-0.0799	-0.0499	0.0076	0.0058	0.0077	0.0039
(25,30)	0.0780	0.0576	-0.0815	-0.0505	0.0073	0.0057	0.0078	0.0038
(30,20)	0.0823	0.0528	-0.0759	-0.0476	0.0082	0.0042	0.0071	0.0037
(30,25)	0.0806	0.0507	-0.0782	-0.0483	0.0077	0.0038	0.0073	0.0036
(30,30)	0.0793	0.0493	-0.0795	-0.0499	0.0075	0.0036	0.0074	0.0036

4.1. Simulation Study

Here, we conduct a simulation study to evaluate performance of estimate of R. The simulation was performed using MATLAB. We simulate $N = 10,000$ sets of X-samples and Y-samples from the MOEPL with parameters $\alpha_1, \beta, \gamma, \lambda$ and $\alpha_2, \beta, \gamma, \lambda$ respectively. We set the sample of sizes at $m = 20, 25, 30$ and $n = 20, 25, 30$. It is possible to obtain measures like average bias of the estimate (b), average mean square error of the estimate (AMSE), average confidence interval (ACI) of the estimate and coverage probability (CP) which are useful to check the validity of the estimate of R. These measures are calculated based on following equations

1) The equation of b of the simulated N estimates of R is given by:

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{R}_i - R)$$

2) ASME of the simulated N estimates of R is given by:

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{R}_i - R)^2$$

3) ACI of the asymptotic 95% confidence intervals of R is given by:

$$\frac{1}{N}\sum_{i=1}^{N} 2(1.96)\hat{\alpha}_{1i}\, b_{1i}(\hat{\alpha}\alpha_{1i}, \hat{\alpha}\alpha_{2i})\sqrt{\frac{3}{m}+\frac{3}{n}}$$

4) The CP of the N simulated confidence intervals are given by the proportion of such interval that include the parameter R.

From Table 1, we can conclude that, b is positive when $\alpha_1 < \alpha_2$ and b is negative when $\alpha_1 > \alpha_2$. Furthermore, in both cases the b decreases as the sample size increases and AMSE is almost symmetric with respect to (α_1, α_2). Besides, Table 2 specifies the symmetric property in the case of ACI. The CP is close to the nominal value in all cases and so that it is slightly greater than 0.95. From this it is obvious that numerical values of b, AMSE, ACI and CP do not show much difference for distinct parameter combinations.

Table 2: The values of ACL and CP of the simulated 95% confidence intervals of R for $\beta = 9$, $\gamma = 4$ and $\lambda = 3$

	ACL				CL			
	(α_1, α_2)				(α_1, α_2)			
(m,n)	(0.5,0.8)	(0.9,1.2)	(0.8,0.5)	(1.2,0.9)	(0.5,0.8)	(0.9,1.2)	(0.8,0.5)	(1.2,0.9)
(20,20)	0.3557	0.3557	0.3557	0.3556	0.9805	0.9878	0.9881	0.9877
(20,25)	0.3377	0.3375	0.3373	0.3376	0.9784	0.9883	0.9845	0.9880
(20,30)	0.3251	0.3248	0.3250	0. 0.3226	0.9705	0.9884	0.9794	0.9854
(25,20)	0.3377	0.3344	0.3376	0.3375	0.9733	0.9593	0.9761	0.9879
(25,25)	0.3186	0.3152	0.3185	0.3212	0.9709	0.9520	0.9698	0.9861
(25,30)	0.3051	0.3019	0.3051	0.3051	0.9728	0.9472	0.9660	0.9857
(30,20)	0.3250	0.3050	0.3250	0.3250	0.9680	0.9864	0.9795	0.9884
(30,25)	0.3051	0.3023	0.3082	0.3044	0.9672	0.9855	0.9716	0.9874
(30,30)	0.2911	0.2910	0.2899	0.2910	0.9652	0.9867	0.9744	0.9868

5. Estimation

MLEs are important point estimators in statistical inference. We estimate the MLEs of the model parameters from complete samples. Let $X_1, X_2, ..., X_n$ is a random sample of size n from the MOEPL distribution with parameters α, β, γ and λ. Let $\Phi = (\alpha, \beta, \gamma, \lambda)^T$ be the $p \times 1$ parameter vector.

For determining the MLEs of α, β, γ and λ, the log-likelihood function is given as

$$\begin{aligned}\log l(\Phi) &= n\log(\alpha\beta\gamma) - \beta\log\gamma - (\gamma+1)\sum_{i=1}^{n}\log\left[1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right] \\ &\quad +(\beta-1)\sum_{i=1}^{n}\log x_i - 2\sum_{i=1}^{n}\log\left\{1-\bar{\alpha}\left[1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right]^{-\gamma}\right\}\end{aligned}$$

The components of the score vector,
$\mathbf{U}(\Phi) = \frac{\partial \log l}{\partial \Phi} = \left(U_\alpha = \frac{\partial \log l}{\partial \alpha}, U_\beta = \frac{\partial \log l}{\partial \beta}, U_\gamma = \frac{\partial \log l}{\partial \gamma}, U_\lambda = \frac{\partial \log l}{\partial \lambda}\right)^T$ are given by

$$U_\alpha = \frac{n}{\alpha} - \sum_{i=1}^{n}\frac{2}{\left[1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right]^{\gamma} - \bar{\alpha}}$$

$$U_\beta = \frac{n}{\beta} + \sum_{i=1}^{n} log x_i - 2\sum_{i=1}^{n}\frac{(\gamma+1)(\frac{x_i}{\lambda})^{\beta}\log\frac{x_i}{\lambda}}{\left[1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right]} - 2\sum_{i=1}^{n}\frac{\vec{\alpha}\gamma\left[1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right]^{-\gamma-1}(\frac{x_i}{\lambda})^{\beta}\log\frac{x_i}{\lambda}}{1-\bar{\alpha}\left[1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right]^{-\gamma}}$$

$$U_\gamma = \frac{n}{\gamma} - \sum_{i=1}^{n}\log\left[1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right] - 2\sum_{i=1}^{n}\frac{\bar{\alpha}\left[1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right]^{-\gamma}\log\left[1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right]}{1-\bar{\alpha}\left[1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right]^{-\gamma}}$$

$$U_\lambda = -\frac{n\beta}{\lambda} + (\gamma+1)\sum_{i=1}^{n}\frac{\beta\lambda^{-\beta-1}x_i}{\left[1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right]} - 2\sum_{i=1}^{n}\frac{\bar{\alpha}\gamma x_i^{\beta}\beta\left[1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right]^{-\gamma-1}}{\lambda^{\beta}\left\{1-\bar{\alpha}\left[1+\left(\frac{x_i}{\lambda}\right)^{\beta}\right]^{-\gamma}\right\}}$$

Setting the nonlinear system of equations $U_\alpha = 0$, $U_\beta = 0$, $U_\gamma = 0$ and $U_\lambda = 0$ and solving them simultaneously yields the MLEs of $\hat{\Phi}$.

6. Data Analysis

This section concentrates on application of the proposed model to a real life data set. We compare the performance of the MOEPL distribution with the other generalized Lomax model on a real data set already in the literature. The data set represents the life of fatigue fracture of Kevlar 373/epoxy that are subject to constant pressure at the 90% stress level until all had failed. For previous studies on the data sets, see, Barlow et al. (1984) and Andrews and Herzberg (1985). The data are:

0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960.

We compare the results of MOEPL distribution with following generalizations of Lomax distribution which are generalized by using different generators:

1. beta generalized Lomax (BGL) distribution (Eugene et al., 2002)
2. Exponentiated exponential Poisson Lomax (EEPL) distributions (Ristić and Nadarajah, 2013)
3. Exponentiated generalized Lomax distributions (EGL) (Cordeiro et al., 2013)
4. Exponentiated Lomax(EL) distribution (Gupta et al., 1998)
5. Exponentiated Kumaraswamy Lomax (EKumL) distributions (Lemonte et al., 2013)
6. Gamma uniform Lomax (GUL) distributions (Torabi and Montazeri, 2012)
7. Weibull Lomax (WL) distributions (Alzaatreh et al., 2013)

We apply these distributions to the above data set and estimate the parameters by the maximum likelihood method. We also calculate various measures like values of $-$log-likelihood ($-$logL), AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), the values of the Kolmogorov-Smirnov (K-S) statistic and the corresponding p-values.
The results of comparison are summarized in Table 3. From these results we can observe that MOEPL distribution provide smallest $-$logL, AIC, BIC and K-S statistics values and highest p-value as compare to other distributions. On the basis of the results obtained, it is concluded that the MOEPL distribution is very suitable for this data set than the other generalize Lomax distributions.

7. Conclusions

In this article, we introduce a new generalization of the Lomax distribution which can be quite flexible in analyzing continuous data in different areas. It is proved to be an important alternative model to other existing generalizations of Lomax distributions. We provide the asymptotic distributions of the extreme values. Additionally stress-strength analysis is carried out and the validity of the estimate of reliability so obtained is studied through simulation studies. As expected, when $\alpha_1 < \alpha_2$ the bias is positive and when $\alpha_1 > \alpha_2$ the bias is negative. Also the absolute bias and MSEs decreases as sample size increases and the length of the confidence interval is also symmetric with respect to α_1, α_2 and decreases as the sample size increases. The applicability of the model is verified by applying to a real data set. The MOEPL distribution provides better fit than other considered extended Lomax distributions.

Table 3: Estimated values, $-$logL, AIC, BIC, K-S statistics and p-value for data set

Distribution	Estimates	$-$logL	AIC	BIC	K-S	p-value
MOEPL(α, β,γ,λ)	$\hat{\alpha}$=**36.1174** $\hat{\beta}$= **0.8599** $\hat{\theta}$= **4.9302** $\hat{\lambda}$= **1.4578**	**120.0236**	**248.0471**	**257.3701**	**0.0711**	**0.8112**
BGL(a,b,r,s)	$\hat{a}$=1.7303 $\hat{b}$=5.6522 $\hat{r}$= 0.0325 $\hat{s}$= 4.7672	122.2238	252.4477	261.7706	0.09241574	0.5056
EEPL(a,b,r,s)	$\hat{a}$=0.9279 $\hat{b}$=3.9078 $\hat{r}$= 0.1588 $\hat{s}$= 7.5082	122.2238	248.7956	258.1185	0.08077	0.6739
EGL(a,b,r,s)	$\hat{a}$=5.6646 $\hat{b}$=1.7711 $\hat{r}$= 0.0231 $\hat{s}$= 5.6646	122.2889	252.5777	261.9007	0.0905	0.5317
EL(a,r,s)	$\hat{a}$= 1.8383 $\hat{r}$= 0.0452 $\hat{s}$= 17.2241	122.3927	250.7855	257.7777	0.0886	0.5587
EKumL(a,b,c,r,s)	$\hat{a}$=2.1141 $\hat{b}$=9.4417 $\hat{c}$=0.7465 $\hat{r}$= 0.1557 $\hat{s}$= 1.2897	121.7866	253.5733	265.227	0.0917	0.5149
GUL(a,r,s)	$\hat{a}$= 1.6068 $\hat{r}$= 0.7700 $\hat{s}$= 1.0378	122.2416	250.4833	257.4755	0.0994	0.4140
WL(a,b,r,s)	$\hat{a}$= 1.5794 $\hat{b}$= 0.5398 $\hat{r}$= 0.1856 $\hat{s}$= 1.6692	121.8335	251.6671	260.99	0.09207	0.5104

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