The Burr X Fréchet Model for Extreme Values: Mathematical Properties, Classical Inference and Bayesian Analysis

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Abstract
In this paper, we investigate a new model based on Burr X and Fréchet distributions for extreme values and derive some of its properties. Maximum likelihood estimation along with asymptotic confidence intervals is considered for estimating the parameters of the distribution. We demonstrate empirically the flexibility of the distribution in modeling various types of real data. Furthermore, we also provide Bayes estimators and highest posterior density (HPD) intervals of the parameters of the distribution using Markov Chain Monte Carlo (MCMC) methods.

Keywords: The Fréchet distribution, Burr X Family; Moments; Order Statistics; Gibbs Algorithm; Maximum likelihood Estimation; Bayesian Estimation.

1. Introduction
The extreme value theory is a very important theory in statistics dedicated to stochastically series of independent and identical distributed random variables. Quite simply, one can say it was devoted to study the behavior of extreme values, despite these values have a very low opportunity to show up, they can end up having a very high impact to the observed system. Finance and insurance are the best fields of research to watch the significance of extreme events. The extreme value theory can be considered as a creating field of research. It began the last century as an equivalent theory to the so-called central limit theory, which is devoted to study the asymptotic distribution of the average of a sequence of random variables. The central limit theorem states that the sum and the mean of the random variables from an arbitrary distribution are normally distributed under the condition that the sample size is sufficiently large. In any case, in some practical studies we are looking for the limiting distribution of minimum or maximum values rather than the average. Assume that Y_1 , Y_2 , … , Y_n is a sequence of iid (independent and identically distributed) random variables with common cdf (cumulative distribution function) $F(y)$. One of the most interesting statistics is the sample maximum $M_n = max\{Y_1, Y_2, \ldots, Y_n\}$. One is keen on the behavior of M_n as the sample size m increases to infinity.

$$p_r\{y \geq M_n\} = p_r\{Y_1 \leq y, Y_2 \leq y, \ldots, Y_n \leq y,\}$$

$$= p_r\{Y_1 \le y\} \dots p_r\{Y_n \le y\} = F(y)^m.$$

Assume there are sequences of constants $\{a_m > 0\}$ and $\{b_m\}$ such that

$$p_r\left\{y \ge \frac{(M_m - b_m)}{a_m}\right\} \to G(y) \text{ as } m \to \infty.$$

Then if $G(y)$ is a non-degenerate distribution function, then it will belong to one of the three following fundamental types of classic extreme value family: Gumbel distribution which is Type **I**; Type **II** (Fréchet distribution); Type **III** (Weibull distribution). The extreme value theory concentrates on the behavior of the block maxima or minima.

The extreme value theory was presented firstly by Fréchet (1927) and Fisher and Tippett (1928), then took after by Von Mises (1936) and finished by Gnedenko (1943), Von Mises (1964), Kotz and Johnson (1992), among others. The so called Fréchet (‘Fr’for short) distribution is one of the important distributions in extreme value theory and it has applications ranging from accelerated life testing through earthquakes, rainfall, floods, horse racing, queues in supermarkets, wind speeds and sea waves. For more details about the Fr distribution and its applications, see Kotz and Nadarajah (2000). Moreover, applications of this distribution in various fields are given in Harlow (2002).

As of late, some new extensions of the Fréchet distribution were considered. The exponentiated Fréchet (EFr) by Nadarajah and Kotz (2003), beta Fréchet (BFr) by Nadarajah and Gupta (2004), Nadarajah and Kotz (2008) and Zaharim et al. (2009), beta Fréchet by Barreto-Souza et al. (2011) and Mubarak (2013), transmuted Fréchet (TFr) by Mahmoud and Mandouh (2013), Marshall-Olkin (MOFr)Fréchet by Krishna et.al. (2013), gamma extended Fréchet (GEFr) by da Silva et al. (2013), transmuted exponentiated Fréchet (TEFr) by Elbatal et al. (2014), transmuted Marshall-Olkin Fréchet (TMOFr) by Afify et al. (2015), transmuted exponentiated generalized Fréchet (TEGFr) by Yousof et al. (2015), Kumaraswamy Marshall-Olkin Fréchet (KMOFr) by Afify et al. (2016b), Weibull Fréchet (WFr) by Afify et al. (2016b), Kumaraswamy transmuted Marshall-Olkin Fréchet (KTMOFr) by Yousof et al. (2016), Odd Lindley Fréchet (OLFr) by Korkmaz et al. (2017), odd log-logistic Fréchet (OLLFr) by Yousof et al. (2018a), Transmuted Topp Leone Fréchet (TTLFr) by Yousof et al. (2018b), among others. Many other extensions can be found in Brito et al. (2017), Hamedani et al. (2017), Yousof et al. (2018c), Cordeiro et al. (2018), Chakraborty et al. (2018), Hamedani et al. (2017), Hamedani et al. (2018), Korkmaz et al. (2018), Alizadeh et al. (2018), Alizadeh et al. (2019), Korkmaz et al. (2019) Elbiely and Yousof (2019), Nascimento et al. (2019), Ibrahim (2019) and Hamedani et al. (2019).

The probability density function (pdf) and cumulative distribution function (cdf) of the Fr distribution are given by (for $x \ge 0$)

$$g(x\,;a\,,b) = ba^b x^{-(b+1)}\, exp\left[-\left(\frac{a}{x}\right)^b\right] \tag{1}$$

and

$$G(x,a,b) = exp\left[-\left(\frac{a}{x}\right)^b\right], \tag{2}$$

respectively, where $a > 0$ is a scale parameter and $b > 0$ is a shape parameter. Yousof et al. (2017) defined the cdf of the Burr X-G (BrX-G) family of distribution (for $x > 0$) by

$$F(x;\theta,\xi)=\left(1-exp\left\{-\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^2\right\}\right)^{\theta}. \tag{3}$$

The BrX-G density function becomes

$$f(x;\theta,\xi)=\frac{2\theta g(x;\xi)G(x;\xi)}{\overline{G}(x;\xi)^3}exp\left\{-\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^2\right\}\left(1-exp\left\{-\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^2\right\}\right)^{\theta-1}, \tag{4}$$

where $\theta>0$ is the shape parameter and $\xi=\ \xi_k=(\ \xi_1,\xi_2,\ldots)$ is a parameters vector. A random variable X with pdf (4) is denoted by $X\sim$ BX-G (θ,ξ). The cdf and the pdf of the Burr X Fr (BrXFr)

$$F(x;\theta,a,b)=\left(1-exp\left[-\left\{\frac{exp\left[-\left(\frac{a}{x}\right)^b\right]}{1-exp\left[-\left(\frac{a}{x}\right)^b\right]}\right\}^2\right]\right)^{\theta} \tag{5}$$

and

$$f(x;\theta,a,b)=\frac{2\theta ba^b x^{-(b+1)}\,exp\left(-2\left(\frac{a}{x}\right)^b-\left\{\frac{exp\left[-\left(\frac{a}{x}\right)^b\right]}{1-exp\left[-\left(\frac{a}{x}\right)^b\right]}\right\}^2\right)}{\left\{1-exp\left[-\left(\frac{a}{x}\right)^b\right]\right\}^3}$$

$$\times\left[1-exp\left(-\left\{\frac{exp\left[-\left(\frac{a}{x}\right)^b\right]}{1-exp\left[-\left(\frac{a}{x}\right)^b\right]}\right\}^2\right)\right]^{\theta-1}, \tag{6}$$

respectively. We provide a very useful linear representation for the BrXFr density function. If $|z|<1$ and $b>0$ is a real non-integer, the power series holds

$$(1-z)^{b-1}=\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(b)}{i!\Gamma(b-i)}z^i. \tag{7}$$

Applying (7) to last term in (6) we get

$$f(x)=\frac{2\theta ba^b x^{-(b+1)}\,exp\left[-2\left(\frac{a}{x}\right)^b\right]}{\left\{1-exp\left[-\left(\frac{a}{x}\right)^b\right]\right\}^3}\sum_{i=0}^{\infty}\ exp\left(-(i+1)\left\{\frac{exp\left[-\left(\frac{a}{x}\right)^b\right]}{1-exp\left[-\left(\frac{a}{x}\right)^b\right]}\right\}^2\right). \tag{8}$$

Applying the power series to the term $exp\left(-(i+1)\left\{\frac{exp\left[-\left(\frac{a}{x}\right)^b\right]}{1-exp\left[-\left(\frac{a}{x}\right)^b\right]}\right\}^2\right)$, Equation (8) becomes

$$f(x)=2\theta ba^b x^{-(b+1)}\,exp\left[-2\left(\frac{a}{x}\right)^b\right]\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(i+1)^j\Gamma(\theta)}{i!j!\Gamma(\theta-i)}\frac{exp\left[-(2j+1)\left(\frac{a}{x}\right)^b\right]}{\left\{1-exp\left[-\left(\frac{a}{x}\right)^b\right]\right\}^{2j+3}}. \tag{9}$$

Consider the series expansion

$$(1-z)^{-b}=\sum_{k=0}^{\infty}\frac{\Gamma(b+k)}{k!\Gamma(b)}z^k,\ |z|<1,\ b>0. \tag{10}$$

Applying the expansion in (10) to (9) for the term $\left\{1-exp\left[-\left(\frac{a}{x}\right)^b\right]\right\}^{2j+3}$, equation (9) becomes

$$f(x)=\sum_{j,k=0}^{\infty}\Omega_{j,k}h_{2j+k+2}(x), \tag{11}$$

where

$$\Omega_{j,k} = \frac{2\theta(-1)^j \Gamma(\theta)\Gamma(2j+k+3)}{j!\,k!\,\Gamma(2j+3)(2j+k+2)} \sum_{i=0}^{\infty} \frac{(-1)^i (i+1)^j}{i!\,\Gamma(\theta-i)},$$

and $h_{2j+k+2}(x)$ is the Fr density with scale parameter $a(2j+k+2)^{\frac{1}{b}}$ and shape parameter b . Thus, the BrXFr density can be expressed as a double linear mixture of Fr densities. So, several of its structural properties can be obtained from Equation (11). By integrating Equation (11), the cdf of X can be given in the mixture form

$$F(x) = \sum_{j,k=0}^{\infty} \Omega_{j,k}\, H_{2j+k+2}(x), \tag{12}$$

where $H_{2j+k+2}(x)$ is the Fr cdf with scale parameter $a(2j+k+2)^{\frac{1}{b}}$ and shape parameter b . Figure 1 display some plots of the BrXFr density for selected values of θ, a, b . The density plots indicate that the BrXFr distribution can be skewed to the right with small and large values for the skewness and kurtosis measures. The plots of the BrXFr hrf for some parameter values given in Figure 2 reveal that this function can be unimodal, decreasing or increasing, depending on the parameter values.

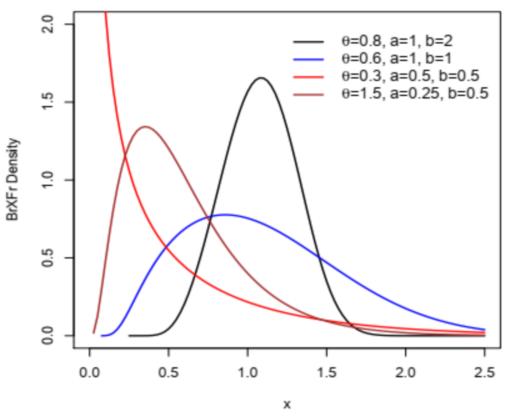

Figure 1: Density plots of BrXFr distribution.

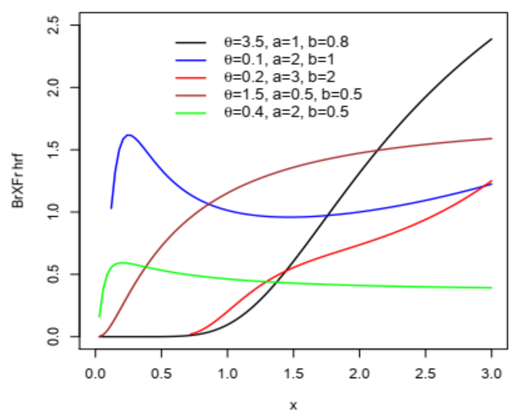

Figure 2: Hrf plots of BrXFr distribution.

The paper is unfolded as follows. In Section 2, we obtain some mathematical properties of the proposed model. In Section 3, the model parameters are estimated by using maximum likelihood method and a simulation study is performed, two applications are given to illustrate the flexibility of the proposed model. In Section 4, Bayesian estimation is performed by obtaining the posterior marginal distributions, we use the simulation method of MCMC by the Metropolis-Hastings algorithm in each step of Gibbs algorithm, cumulative mean, autocorrelation plots based on pig's data and density plots of MCMC chains of the parameters are presented. Finally, Section 7 offers some concluding remarks.

2. Properties

2.1 Probability weighted moments

The probability weighted moment (PWMs) are expectations of certain functions of a random variable and they can be defined for any random variable whose ordinary moments exist. The PWM method can generally be used for estimating parameters of a distribution whose inverse form cannot be expressed explicitly. The (s,r) th PWM of X following the BrXFr distribution, say $\rho_{s,r}$,is formally defined by

$$\rho_{s,r} = E\{X^s F(X)^r\} = \int_{-\infty}^{\infty} x^s F(x)^r f(x) dx.$$

Using (5), (6) and the last equation we can write

$$f(x)F(x)^r = \sum_{j,k=0}^{\infty} t_{j,k}\, h_{2j+k+2}(x),$$

where

$$t_{j,k} = \frac{2\theta(-1)^j \Gamma(2j+k+3)}{j!\,k!\,\Gamma(2j+3)(2j+k+2)} \sum_{i=0}^{\infty} (-1)^i (i+1)^j \binom{\theta(r+1)-1}{i}.$$

Then, the (s,r) th PWM of X can be expressed as

$$\rho_{s,r} = \sum_{j,k=0}^{\infty} \frac{t_{j,k}}{a^{-s}} (2j+k+2)^{\frac{s}{b}} \Gamma\left(1-\frac{s}{b}\right), \forall\, s < b.$$

2.2 Residual and reversed residual life

The n th moment of the residual life, say $r_n(t) = E[(X-t)^n \mid X > t]$, $n = 1,2$,, uniquely determines $F(x)$. The n th moment of the residual life of X is given by $r_n(t) = \frac{1}{1-F(t)} \int_t^{\infty} (x-t)^n dF(x)$. Therefore,

$$r_n(t) = \frac{1}{1-F(t)} \sum_{j,k=0}^{\infty} \frac{\Omega^{*}_{j,k}}{a^{-n}} (2j+k+2)^{\frac{n}{b}} \Gamma\left(1-\frac{n}{b}, (2j+k+2)\left(\frac{a}{t}\right)^b\right), \forall\, n < b,$$

where $\Omega^{*}_{j,k} = \Omega_{j,k} \sum_{r=0}^{n} (1-t)^n$. Another interesting function is the mean residual life (MRL) function or the life expectation at age t defined by $r_1(t) = E[(X-t) \mid X > t]$, which represents the expected additional life length for a unit which is alive at age t . The MRL of X can be obtained by setting $n = 1$ in the last equation. The n th moment of the reversed residual life, say $R_n(t) = E[(t-X)^n \mid X \leq t]$ for $t > 0$ and $n = 1,2$, uniquely determines $F(x)$. We obtain $R_n(t) = \frac{1}{F(t)} \int_0^t (t-x)^n dF(x)$. Then, the n th moment of the reversed residual life of X becomes

$$R_n(t) = \frac{1}{F(t)} \sum_{j,k=0}^{\infty} \frac{\Omega^{**}_{j,k}}{a^{-n}} (2j+k+2)^{\frac{n}{b}} \Gamma\left(1-\frac{n}{b}, (2j+k+2)\left(\frac{a}{t}\right)^b\right), \forall\, n < b,$$

where $\Omega^{**}_{j,k} = \Omega_{j,k} \sum_{r=0}^{n} (-1)^r \binom{n}{r} t^{n-r}$. The so called mean inactivity time (MIT) or mean waiting time also called the mean reversed residual life function, is given by $R_1(t) = E[(t-X) \mid X \leq t]$, and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0,t)$ .The MIT of the BrXFr generator of distributions can be obtained easily by setting $n = 1$ in the above equation.

2.3 Stress-strength reliability model

Stress-strength model is the most widely approach used for reliability estimation. This model is used in many applications of physics and engineering such as strength failure and system collapse. In stress-strength modeling, $R = Pr(X_2 < X_1)$ is a measure of reliability of the system when it is subjected to random stress X_2 and has strength X_1 . The system fails if and only if the applied stress is greater than its strength and the component will function satisfactorily whenever $X_1 > X_2$. R can be considered as a measure of system performance and naturally arise in electrical and electronic systems. Other interpretation can be given as the reliability R of a system is the probability that the system is strong enough to overcome the stress imposed on it. Let X_1 and X_2 be two independent random variables with BrXFr (θ_1, a, b) and BrXFr (θ_2, a, b) distributions, respectively. Then, the reliability is defined by $R = \int_0^\infty f_1(x\,; \theta_1, a, b)F_2(x\,; \theta_2, a, b)dx$. We can write

$$R = \sum_{j,k,w,m=0}^{\infty} s_{j,k,w,m} \int_0^\infty h_{2(j+w)+k+m+4}(x)dx,$$

where

$$s_{j,k,w,m} = 4\theta_1\theta_2 \sum_{j,k,w,m=0}^{\infty} \frac{(-1)^{j+w}\Gamma(2j+k+3)\Gamma(2w+m+3)}{j!\,k!\,w!\,m!\,\Gamma(\theta_2-h)\Gamma(2j+3)\Gamma(2w+3)}$$

$$\sum_{i,h=0}^{\infty} \frac{(-1)^{i+h}(i+1)^j(h+1)^w\binom{\theta_1-1}{i}\binom{\theta_2-1}{h}}{(2w+m+2)(2j+k+2w+m+4)}.$$

Thus, the reliability can be expressed as

$$R = \sum_{j,k,w,m=0}^{\infty} s_{j,k,w,m}.$$

2.4 Order statistics and the QS order

Let $X_1, X_2, \dots, X_n$, be a random sample from the BrXFr model and let $X_{1:n}, X_{2:n}, \dots, X_{n:n}$, be the corresponding order statistics. The pdf of i^{th} order statistic, say $X_{i:n}$, can be written as

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F^{j+i-1}(x), \qquad (13)$$

where $B(\cdot,\cdot)$ is the beta function. Using (5), (6) and (13) we can write

$$f(x)F(x)^{j+i-1} = \sum_{w,k=0}^{\infty} d_{w,k}\, h_{2w+k+2}(x),$$

where

$$d_{w,k} = \frac{2\theta(-1)^w\Gamma(2w+k+3)}{w!\,k!\,\Gamma(2w+3)(2w+k+2)} \sum_{m=0}^{\infty} (-1)^m(m+1)^w\binom{\theta(j+i)-1}{m}.$$

The pdf of $X_{i:n}$ can be expressed as

$$f_{i\ :\ n}(x)=\sum_{w,k=0}^{\infty}\ \sum_{j=0}^{n-i}\ \frac{(-1)^j\binom{n-i}{j}d_{w,k}}{B(i,n-i+1)}h_{2w+k+2}(x).$$

Then, the density function of the BrXFr order statistics is a mixture of exp-G densities. Based on the last equation, we note that the properties of $X_{i\ :\ n}$ follow from those properties of Y_{2w+k+2}. For example, the moments of $X_{i\ :\ n}$ can be expressed as

$$E\left(X_{i\ :\ n}^{q}\right)=\sum_{w,k=0}^{\infty}\ \sum_{j=0}^{n-i}\ \frac{(-1)^j\binom{n-i}{j}d_{w,k}}{B(i,n-i+1)a^{-q}}(2w+k+2)^{\frac{q}{b}}\Gamma\left(1-\frac{q}{b}\right),\forall\, q<b,$$

Let X_1 and X_2 be two random variables follow BrXFr distribution with quantile spreads QS_{X_1} and QS_{X_2}, respectively. Then X_1 is called smaller than X_2 in quantile spread order, denoted as $X_1\ \leq_{QS} X_2$, if $QS_{X_1}(p)\leq QS_{X_2}(p)$ for all $p\in(0.5,1)$. The following properties of the quantile spread order can be determined. The order $\leq_{QS}$ is *location-free*, i.e., $X_1\leq_{QS} X_2$ if $(X_1+c)\leq_{QS} X_2$ for any real c. The order $\leq_{QS}$ is *dilative*, i.e, $X_1\leq_{QS} aX_1$ whenever $a\geq 1$ and $X_2\leq_{QS} aX_2$ whenever $b\geq 1$. $X_1\leq_{QS} X_2$ if, and only if $-X_1\leq_{QS}-X_2$. Assume F_{X_1} and F_{X_2} are symmetric, then $X_1\leq_{QS} X_2$ if, and only if $F_{X_1}^{-1}(p)\leq F_{X_2}^{-1}(p)$ for $p\in(0.5,1)$. The order $\leq_{QS}$ implies ordering of *the mean absolute deviation around the median*, MAD, $MAD(X_1)=E[|X-Median(X_1)|]$ and $MAD(X_2)=E[|X-Median(X_2)|]$ i.e., $X_1\leq_{QS} X_2$ implies $MAD(X_1)\leq_{QS} MAD(X_2)$.

2.5 General properties

The r th ordinary moment of X is given by $\mu'_r=E(X^r)=\sum_{j,k=0}^{\infty}\Omega_{j,k}\int_{-\infty}^{\infty} x^r\, h_{2j+k+2}(x)dx$. Then we obtain

$$\mu'_r=\sum_{j,k=0}^{\infty}\frac{\Omega_{j,k}}{a^{-r}}(2j+k+2)^{\frac{r}{b}}\Gamma\left(1-\frac{r}{b}\right),\forall\, r<b, \qquad (14)$$

setting $r=1$ in (14), we have the mean of X. The last integration can be computed numerically for most parent distributions. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. The r th central moment of X, say M_r, is $M_r=E(X-\mu)^r=\sum_{h=0}^{r}(-1)^h\binom{r}{h}(\mu'_1)^r\mu'_{r-h}$. The cumulants ($\kappa_n$) of X follow recursively from $\kappa_n=\mu'_n-\sum_{r=0}^{n-1}\binom{n-1}{r-1}\kappa_r\mu'_{n-r}$, where $\kappa_1=\mu'_1$, $\kappa_2=\mu'_2-\mu'^2_1, \kappa_3=\mu'_3-3\mu'_2\mu'_1+\mu'^3_1$, and so on. The skewness and kurtosis measures also can be calculated from the ordinary moments using well-known relationships. Here, we provide two formulae for the moment generating function (mgf) $M_X(t)=E(e^{tX})$ of X. Clearly, the first one can be derived from using (14) as

$$M_X(t)=\sum_{j,k=0}^{\infty}\Omega_{j,k}M_{2j+k+2}(t)=\sum_{j,k,r=0}^{\infty}\frac{\Omega_{j,k}}{a^{-r}}\frac{t^r}{r!}(2j+k+2)^{\frac{r}{b}}\Gamma\left(1-\frac{r}{b}\right),\forall\, r<b,$$

As for the second formula for $M_X(t)$, consider the Wright generalized hypergeometric function defined by

$${}_p\Psi_q\left[\begin{matrix}(\alpha_1,A_1),\dots,(\alpha_p,A_p)\\(\beta_1,B_1),\dots,(\beta_q,B_q)\end{matrix};\quad x\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\ \Gamma\left(\alpha_j+A_jn\right)}{\prod_{j=1}^{q}\ \Gamma\left(\beta_j+B_jn\right)}\frac{x^n}{n!}.$$

Then, we have

$$M(t\,;a\,,b) = {}_1\Psi_0\left[\begin{matrix}(1,-b^{-1})\\ -\end{matrix};a\,t\right]. \tag{15}$$

Combining expressions (11) and (15), we obtain the mgf of X , say $M(t)$, as

$$M(t) = \sum_{j,k=0}^{\infty} \Omega_{j,k}\,{}_1\Psi_0\left[\begin{matrix}(1,-b^{-1})\\ -\end{matrix};a\,(2j+k+2)^{\frac{1}{b}}t\right].$$

The main applications of the first incomplete moment refer to the mean deviations and the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The s th incomplete moment, say $I_s(t)$, of X can be expressed from (11), for $r < b$, as

$$I_s(t) = \sum_{j,k=0}^{\infty} \Omega_{j,k} \int_{-\infty}^{t} x^s h_{2j+k+2}(x)dx$$

$$= \sum_{j,k=0}^{\infty} \frac{\Omega_{j,k}}{a^{-s}}(2j+k+2)^{\frac{s}{b}}\Gamma\left(1-\frac{s}{b},(2j+k+2)\left(\frac{a}{t}\right)^b\right), \forall\, s < b.$$

The mean deviations about the mean $[\delta_1 = E(|X-\mu_1'|)]$ and about the median $[\delta_2 = E(|X-M|)]$ of X are given by $\delta_1 = 2\mu_1' F(\mu_1') - 2I_1(\mu_1')$ and $\delta_2 = \mu_1' - 2I_1(M)$, respectively, where $\mu_1' = E(X)$, $M = Median(X) = Q(0.5)$ is the median, $F(\mu_1')$ is easily calculated from (5) and $I_1(t)$ is the first incomplete moment given by the last Equation with $s = 1$. The general formula for $I_1(t)$ can be obtained from $I_s(t)$ as

$$I_1(t) = \sum_{j,k=0}^{\infty} \frac{\Omega_{j,k}}{a^{-1}}(2j+k+2)^{\frac{1}{b}}\Gamma\left(1-\frac{1}{b},(2j+k+2)\left(\frac{a}{t}\right)^b\right).$$

2.6 Quantile measure

The effects of the shape parameters θ and b on the skewness and kurtosis can be considered based on quantile measures. The Bowley skewness is one of the earliest skewness measures defined by

$$B = \frac{Q\left(\frac{3}{4}\right) - 2Q\left(\frac{1}{2}\right) + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)}.$$

Since only the middle two quartiles are considered and the outer two quartiles are ignored, this adds robustness to the measure. The Moors kurtosis (Moors 1988) is defined as

$$M = \frac{Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right) + Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)}.$$

Clearly, $M > 0$ and there is good concordance with the classical kurtosis measures for some distributions. These measures are less sensitive to outliers and they exist even for distributions without moments. For the standard normal distribution, these measures are 0 (Bowley) and 1.2331 (Moors).

In Figure 3, we plot the measures B and M for some parameter values. These plots indicate that both measures B and M depend on all shape parameters.

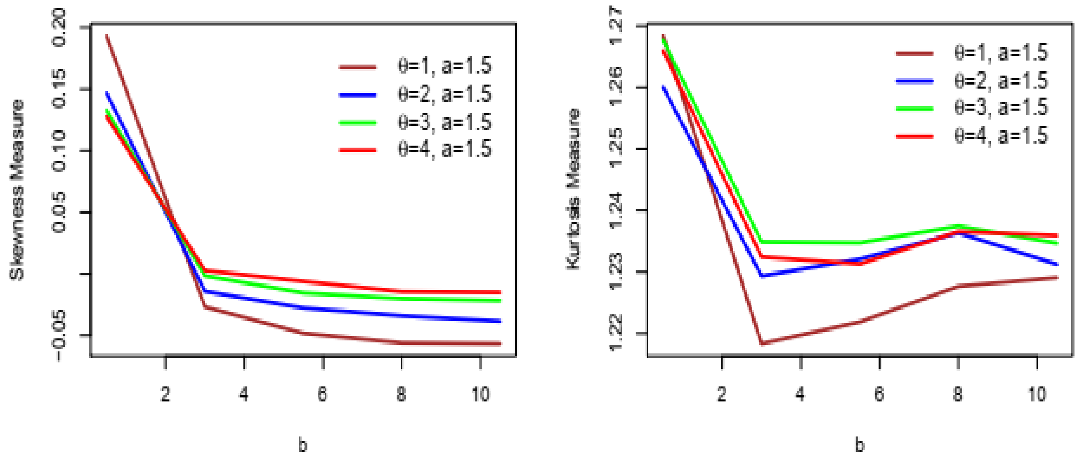

Figure 3: Plots of the skewness and kurtosis measures for the BrXFr distribution.

3. Maximum likelihood estimation and real data modelling

In this section, we introduce a procedure for maximum likelihood estimation of the BrXFr model. We also assess the performance of the maximum likelihood estimators (MLEs) with varying sample size by means of a simulation study. Using the MLEs, we model two real data sets with BrXFr distribution and other comparative distributions.

3.1 Maximum likelihood estimation

We consider the estimation of the unknown parameters of the BrXFr model from complete samples only by maximum likelihood. The MLEs of the parameters of the BrXFr (θ, a, b) model is now discussed. Let $x_1, \dots, x_n$ be a random sample of this distribution with parameter vector $\Theta = (\theta, a, b)^T$. The log-likelihood function for Θ, say $\ell = \ell(\Theta)$, is given by

$$\ell = n\,log(2) + n\,log(\theta) + n\,log(a) + n\,log(b) - (b-1)\sum_{i=1}^{n} log(x_i)$$
$$+\sum_{i=1}^{n}\left[-2\left(\frac{a}{x_i}\right)^b - \left(\frac{z_i}{1-z_i}\right)^2\right] - 3\sum_{i=1}^{n} log(1-z_i)$$
$$+(\theta-1)\sum_{i=1}^{n} log\left\{1 - exp\left[-\left(\frac{z_i}{1-z_i}\right)^2\right]\right\}, \qquad (16)$$

where $z_i = exp\left[-\left(\frac{a}{x_i}\right)^b\right]$ and the last equation can be maximized either by using the different programs like R (optim function), SAS (PROC NLMIXED) or by solving the nonlinear likelihood equations obtained by differentiating (16). The score vector elements can be easily obtained.

3.2 Performance of MLEs

In this subsection, to assess the performance of the MLEs, a simulation study is performed using the statistical software R. We present some simulations for different sample sizes. Simulating random variables from well-defined probability distributions has been discussed in the computational statistics literature, e.g. the inverse transformation method, the rejection and acceptance sampling technique, etc. An ideal technique for simulating from the BrXFr distribution is the inversion method. We can simulate random variable X by

- Set n and $\Theta = (\theta, a, b)$.
- Simulate $u \sim U(0,1)$.
- Using inverse cdf method, generate x from

$$x = a\left(-ln\frac{\sqrt{-ln(1-u^{1/\theta})}}{1+\sqrt{-ln(1-u^{1/\theta})}}\right)^{-1/b}.$$

- Repeat the above steps, n times and obtain $x_1, x_2, \dots, x_n$ from BrXFr (θ, a, b) .

We use $\theta = 2, a = 2$ and $b = 0.5$ for the parameter values in the simulation study. For selected combination of θ, a and b we generate samples of sizes n = 50, 100, 150, 200, 250, 300, 350, 400, 450 and 500 from the BrXFr distribution. The number of Monte Carlo replications is 2000 times and the evaluation of the estimates is performed based on the bias and mean squared errors (MSEs) which are calculated using the R package from the Monte Carlo replications. The empirical results are given in Figures 4, 5 and 6. It can be observed that as sample size increases the biases and MSEs decreases. Therefore, the maximum likelihood method works very well to estimate the model parameters of the BrXFr distribution.

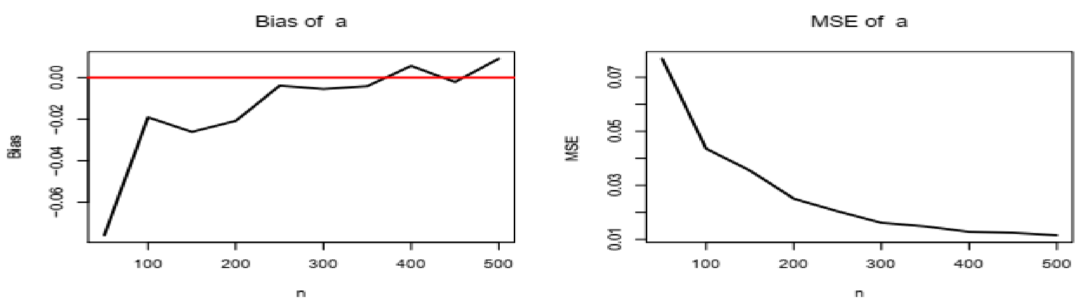

Figure 4: Bias (left) and MSE (right) of the a parameter.

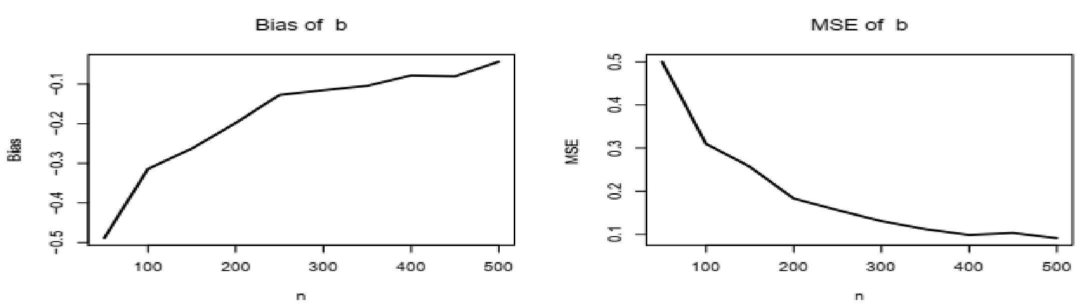

Figure 5: Bias (left) and MSE (right) of the b parameter.

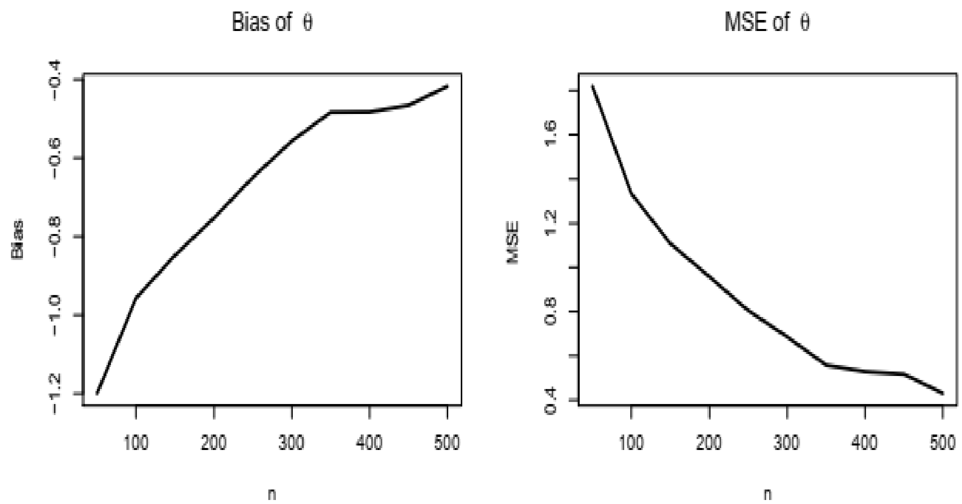

Figure 6: Bias (left) and MSE (right) of the θ parameter.

3.3 Real data modelling

In this subsection, we provide two applications to real data to illustrate the importance of the BrXFr distribution presented in Section 1. Indeed, we prove the flexibility of the new distribution by means of three real data sets. We compared the fits of the BrXFr distribution with other models such as Fréchet (Fr), Kumaraswamy Fréchet (KFr), exponentiated Fréchet (EFr), beta Fréchet (BFr), transmuted Fréchet (TFr), Marshal-Olkin Fréchet (MOFr) and McDonald Fréchet (McFr) distributions given by:

- KFr : $f(x;\alpha,\theta,a,b) = \alpha\theta b a^{b} x^{-(b+1)}\, exp\left[-\alpha\left(\frac{a}{x}\right)^{b}\right]\left\{1 - exp\left[-\alpha\left(\frac{a}{x}\right)^{b}\right]\right\}^{\theta-1}$;

- EFr : $f(x;\alpha,a,b) = \alpha b a^{b} x^{-(b+1)}\, exp\left[-\left(\frac{a}{x}\right)^{b}\right]\left\{1 - exp\left[-\left(\frac{a}{x}\right)^{b}\right]\right\}^{\alpha-1}$;

- BFr : $f(x;\alpha,\theta,a,b) = \frac{b a^{b}}{B(\alpha,\theta)} x^{-(b+1)}\, exp\left[-\alpha\left(\frac{a}{x}\right)^{b}\right]\left\{1 - exp\left[-\left(\frac{a}{x}\right)^{b}\right]\right\}^{\theta-1}$;

- TFr : $f(x;\theta,a,b) = b a^{b} x^{-(b+1)}\, exp\left[-\left(\frac{a}{x}\right)^{b}\right]\left\{1 + \theta - 2\theta\, exp\left[-\left(\frac{a}{x}\right)^{b}\right]\right\}$;

- MOFr: $f(x;\alpha,a,b) = \alpha b a^{b} x^{-(b+1)}\, exp\left[-\left(\frac{a}{x}\right)^{b}\right]\left\{\alpha + (1-\alpha)\, exp\left[-\left(\frac{a}{x}\right)^{b}\right]\right\}^{-2}$;

- McFr: $f(x;\alpha,\theta,\gamma,a,b) = \frac{\gamma b a^{b} x^{-(b+1)}}{B(\alpha,\theta)} exp\left[-\left(\frac{a}{x}\right)^{b}\right]\left(exp\left[-\left(\frac{a}{x}\right)^{b}\right]\right)^{\alpha\gamma-1}$

$$\times\left(1-\left(exp\left[-\left(\frac{a}{x}\right)^{b}\right]\right)^{\gamma}\right)^{\theta-1}.$$

The parameters of the above densities are all positive real numbers except for the TFr distributions for which $|\beta| \leq 1$.

The first data set (Wingo data) represents a complete sample from a clinical trial describe a relief time (in hours) for 50 arthritic patients given by Wingo (1983) is selected. The data set is: 0.70, 0.84, 0.58, 0.50, 0.55, 0.82, 0.59, 0.71, 0.72, 0.61, 0.62, 0.49, 0.54, 0.36, 0.36, 0.71, 0.35, 0.64, 0.84, 0.55, 0.59, 0.29, 0.75, 0.46, 0.46, 0.60, 0.60, 0.36, 0.52, 0.68, 0.80, 0.55, 0.84, 0.34, 0.34, 0.70, 0.49, 0.56, 0.71, 0.61, 0.57, 0.73, 0.75, 0.44, 0.44, 0.81, 0.80, 0.87, 0.29, 0.50.

The second data set from Bjerkedal (1960) consists of 72 survival times for guinea pigs injected with different doses of tubercle bacilli. The data set is: 12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376.

To evaluate performance of considered models on the three data, the MLEs of the parameters for the considered models are calculated and five goodness-of-fit statistics are used to compare the new distribution. The measures of goodness of fit including the Akaike information criterion (AIC), Bayesian information criterion (BIC), Anderson-Darling (A^*), Cramér-von Mises (W^*) and Kolmogrov-Smirnov (K-S) statistics are computed to compare the fitted models. The statistics A^* and W^* are described in details in Chen, G. and Balakrishnan (1995).

In general, the smaller the values of these statistics, the better the fit to the data. The required computations are carried out in the R-language.

The numerical values of the model selection statistics AIC , BIC , A^* , W^* and K-S are listed in Tables 1 and 3. Tables 2 and 4 list the MLEs and their corresponding standard errors (in parentheses) of the model parameters.

We note from the figures in Table 1 that the BrXFr model has the lowest values of the AIC , BIC , A^* , W^* and K-S statistics (for the first data set) as compared to their submodels, suggesting that the BrXFr model provide the best fit. The histogram of the first data and the estimated pdfs and cdfs of the BrXFr model and its sub-models are displayed in Figure 7.

Similarly, it is also evident from Table 3 that the BrXFr gives the lowest values the AIC , BIC , A^* , W^* and K-S statistics (for the second data set) as compared to their sub-models, and therefore these models can be chosen as the best ones. The histogram of the second data and estimated pdfs and cdfs of the BrXFr distribution and its sub-models are displayed in Figure 8.

In addition, it is clear from Table 5 that the BrXFr gives the lowest values the AIC , BIC , A^*, W^* and K-S statistics (for the third data set) as compared to their sub-models, and therefore these models can be chosen as the best ones. The histogram of the second data and estimated pdfs and cdfs of the BrXFr distribution and its sub-models are displayed in Figure 6.

Table 1: The statistics AIC, BIC, W^*, A^*, K-S and K-S p-value for the Wingo data set.

Distribution	AIC	BIC	W^*	A^*	K-S	p-value (K-S)
BrXFr	-38.544	-32.808	0.0514	0.4193	0.0979	0.7235
Fr	-19.701	-15.877	0.3233	2.0301	0.1506	0.2066
KFr	-32.553	-24.905	0.0680	0.5641	0.1053	0.6352
EFr	-17.701	-11.965	0.3233	2.0301	0.1506	0.2064
BFr	-34.395	-26.747	0.0557	0.4711	0.0981	0.7193
TFr	-20.105	-14.369	0.2823	1.8152	0.1370	0.3045
MOFr	-29.889	-24.153	0.1120	0.8560	0.1062	0.6252
McFr	-34.815	-25.255	0.0529	0.4300	0.0980	0.7198

Table 2: MLEs and their standard errors (in parentheses) for the Wingo data set.

Distribution	α	θ	a	b	γ
Fr	-	-	0.4859	3.2078	-
	-	-	(0.0227)	(0.3263)	-
KFr	0.2953	122.39	21.470	0.7917	-
	(0.1881)	(140.51)	(12.846)	(0.1490)	-
EFr	0.9047	-	0.5013	3.2077	-
	(18.784)	-	(3.2444)	(0.3263)	-
BFr	0.3746	78.718	2.6048	1.2430	-
	(0.4318)	(48.446)	(4.7654)	(0.6754)	-
TFr	-0.5816	-	0.4400	3.4974	-
	(0.2787)	-	(0.0290)	(0.3527)	-
MOFr	52.825	-	0.2897	5.7456	-
	(96.683)	-	(0.0783)	(0.7440)	-
McFr	0.1963	474.97	33.731	1.3661	0.0397
	(0.1245)	(1006.6)	(10.601)	(0.2151)	(0.0202)
BrXFr	-	0.3519	0.6433	2.2707	-
	-	(0.1904)	(0.0669)	(0.5062)	-

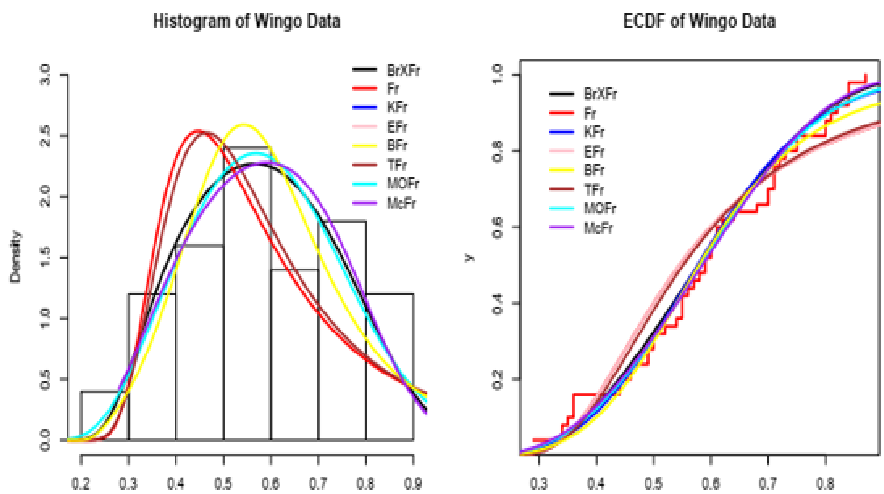

Figure 7: Histogram (left) and cdf (right) of the wingo data.

Table 3: The statistics AIC, BIC, W^*, A^*, K-S and K-S p-value for the survival times data set.

Distribution	AIC	BIC	W^*	A^*	K-S	p-value (K-S)
BrXFr	786.24	793.07	0.1400	0.7652	0.0990	0.4806
Fr	795.29	799.85	0.2148	1.2834	0.1518	0.0721
KFr	788.66	797.77	0.1417	0.7716	0.1057	0.3964
EFr	797.29	804.12	0.2147	1.2829	0.1520	0.0716
BFr	788.59	797.69	0.1366	0.7531	0.1025	0.4350
TFr	804.77	815.25	0.2568	1.3586	0.1818	0.0474
MOFr	787.41	794.24	0.1531	0.7742	0.0995	0.4643
McFr	790.54	801.93	0.1483	0.7663	0.1037	0.4205

Table 4: MLEs and their standard errors (in parentheses) for the survival times data set.

Distribution	α	θ	a	b	γ
Fr	-	-	54.199	1.4147	-
	-	-	(4.788)	(0.117)	-
KFr	223.15	5.944	0.1049	0.7031	-
	(192.21)	(5.663)	(0.1323)	(0.2397)	-
EFr	13.013	-	8.831	1.414	-
	(73.103)	-	(35.086)	(0.1172)	-
BFr	107.372	27.618	0.3143	0.2705	-
	(40.840)	(10.153)	(0.3603)	(0.1224)	-
TFr	-0.7166	-	1.2656	4.7121	-
	(0.2616)	-	(0.0579)	(0.3657)	-
MOFr	61.944	-	14.194	2.481	-
	(103.99)	-	(8.225)	(0.2753)	-
McFr	62.118	33.933	7.164	0.2442	0.7716

	(43.329)	(16.702)	(11.600)	(1.871)	(0.3272)
BrXFr	-	0.2075	15.012	1.6173	-
	-	(0.0716)	(18.888)	(3.4146)	-

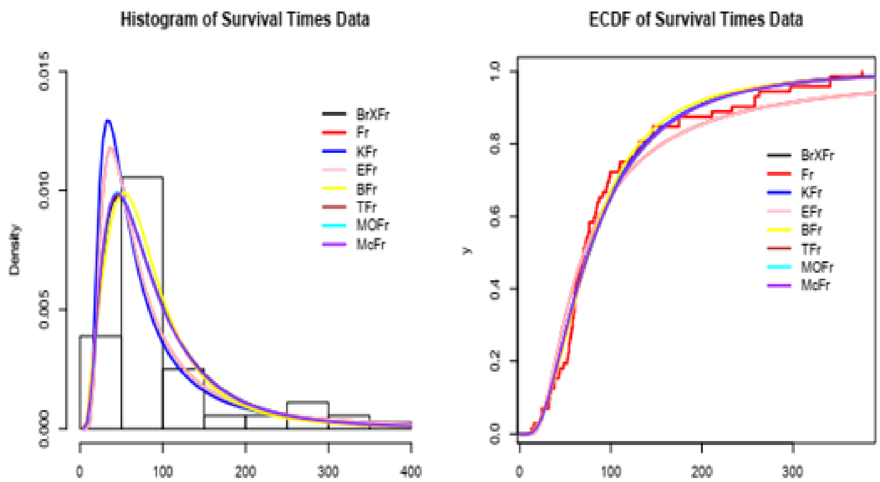

Figure 8: Histogram (left) and cdf (right) of the survival times data.

4. Bayesian estimation

In this section, we use Bayesian procedures to construct the estimators of the unknown parameters of BrXFr distribution. There are many situations where maximum likelihood estimator does not converge, especially with higher dimension models. In such cases, the use of Bayesian methods is sought. At first sight, Bayesian methods seem to be very complex as the estimators involve intractable integrals. However, the advanced MCMC techniques make possible to apply Bayesian methods even in higher dimension models. Under Bayesian estimation, we are updating the likelihoods with prior knowledge explore the posterior probabilities of the parameters. Here we assume the gamma priors of the parameters (θ, a, b) of the following forms

$$\pi_1(\theta) \sim Gamma(c_1, d_1),$$

$$\pi_2(a) \sim Gamma(c_2, d_2)$$

and

$$\pi_3(b) \sim Gamma(c_3, d_3),$$

where, $Gamma(c, d)$ stands for gamma distribution with shape parameter c and scale parameter d . It is further assumed that the parameters are to be independently distributed. The joint prior distribution is given by

$$\pi(\theta, a, b) = \frac{d_1^{c_1}}{\Gamma(c_1)} \frac{d_2^{c_2}}{\Gamma(c_2)} \frac{d_3^{c_3}}{\Gamma(c_3)} \theta^{c_1-1} a^{c_2-1} b^{c_3-1} \, exp\big(-(\theta d_1 + a d_2 + b d_3)\big).$$

The posterior distribution of the parameters can be defined easily.

It is not easy to calculate Bayes estimates and so the numerical approximation techniques are needed. Therefore, we propose the use of MCMC techniques namely Gibbs sampler and Metropolis Hastings (MH) algorithm. Since the conditional posteriors of the parameters cannot be obtained in any standard forms, we, therefore used a hybrid MCMC strategy for drawing samples from the joint posterior of the parameters. To implement the Gibbs algorithm, the full conditional posteriors of α, β and θ can be defined easily.

The simulation algorithm, we followed is given by

1) Provide initial values, say $\theta_{(0)}$, $a_{(0)}$ and $b_{(0)}$, then at i th stage,
2) Using MH algorithm, Generate
$$a_{(0)} \sim \pi_1\left(a_{(i-1)} | b_{(i-1)}, \theta_{(i-1)}\right)$$
3) Using MH algorithm, Generate
$$b_{(0)} \sim \pi_1\left(b_{(i-1)} | a_{(i-1)}, \theta_{(i-1)}\right)$$
4) Using MH algorithm, Generate
$$\theta_{(0)} \sim \pi_1\left(\theta_{(i-1)} | a_{(i-1)}, b_{(i-1)}\right)$$
5) Repeat steps $2-4$, $M(=50000)$ times to get the samples of size M from the corresponding posteriors of interest.
6) Obtain the Bayes estimates of θ, a and b using the following formulae
$$\hat{a}_{\text{Bayes}} = \frac{1}{M-M_0}\sum_{j=1}^{M-M_0} a_j$$
$$\hat{b}_{\text{Bayes}} = \frac{1}{M-M_0}\sum_{j=1}^{M-M_0} b_j$$
and
$$\hat{\theta}_{\text{Bayes}} = \frac{1}{M-M_0}\sum_{j=1}^{M-M_0} \theta_j$$
respectively, where $M_0(\approx 2000)$ is the burn-in period of the generated Markov chains.
7) Obtain the $100\times(1-\psi)\%$ HPD credible intervals for a , b and θ by applying the methodology of Chen et al. (1999). The HPD credible intervals for a , b and θ are
$\left(a_{(j^*)}, a_{(j^*+[(1-\psi)M])}\right)$, $\left(b_{(j^*)}, b_{(j^*+[(1-\psi)M])}\right)$ and $\left(\theta_{(j^*)}, \theta_{(j^*+[(1-\psi)M])}\right)$
respectively. The j^* is chosen such that
$$a_{(j^*+[(1-\psi)M])} - a_{(j^*)} = \min_{1\le j\le M-[(1-\psi)M]}\left(a_{(j+[(1-\psi)M])} - a_{(j)}\right),$$
$$b_{(j^*+[(1-\psi)M])} - b_{(j^*)} = \min_{1\le j\le M-[(1-\psi)M]}\left(b_{(j+[(1-\psi)M])} - b_{(j)}\right)$$
and
$$\theta_{(j^*+[(1-\psi)M])} - \theta_{(j^*)} = \min_{1\le j\le M-[(1-\psi)M]}\left(\theta_{(j+[(1-\psi)M])} - \theta_{(j)}\right).$$
Here, $[x]$ denotes the largest integer less than or equal to x .

Note that there have been several attempts made to suggesting the proposal density for the target posterior in the implementation of MH algorithm. By parametrizing the posterior on the entire real line, Gelfand (1990) and Upadhyay (2001) have suggested to use the normal approximation of the posterior as a proposal candidate in MH algorithm. Alternatively, it is also realistic to have the thought of using the truncated normal distribution without parametrizing the original parameters. Therefore, we propose the use of the truncated normal distribution as the proposal kernel to the target posterior.

We now provide Bayes estimates and HPD credible intervals of the parameters of BrXFr distribution using real data sets. Under MCMC algorithm, truncated normal distribution with mean $\hat{\Theta}$ and variance $J(\hat{\Theta})^{-1}$ is taking to be the proposal density for posterior sample generation. We generate 50000 samples for the posteriors using algorithm given above. Although for any initial values chains converge to their stationary distributions, there exits high autocorrelation in the chains. To reduce the autocorrelation, the chains are thinned and every 10th value is taken for the estimation. Second and third rows of Figure 9 show autocorrelation plots based on pig's data for the complete and thinned chains respectively.

We observe that after thinning the chains, autocorrelation reduces significantly. It is also worth to note from the figure that chains start converging after 2000 iterations. We discard first 2000 iterations as a burn-in period of the chains which is enough to remove the effect of the initial values.

The MLEs and Bayes estimates along with the confidence intervals are presented in Table 5 for both the data sets. Since we do not have prior information about the parameters, we took hyper-parameters as $c_1 = d_1 = c_2 = d_2 = c_3 = d_3 = 0.001$. That results in a vague/flat gamma prior distribution of the parameters.

We can compare both the estimation methods on the bases of standard errors of the estimators and width of the confidence intervals. We can observe the following; standard errors of Bayes estimators are smaller than that of the MLEs; the width of the HPD intervals is smaller than the width of the asymptotic confidence intervals. We may also note that in some cases, the lower limit of the asymptotic confidence interval is negative which is not realistic. In that case, Bayes HPD intervals are recommended.

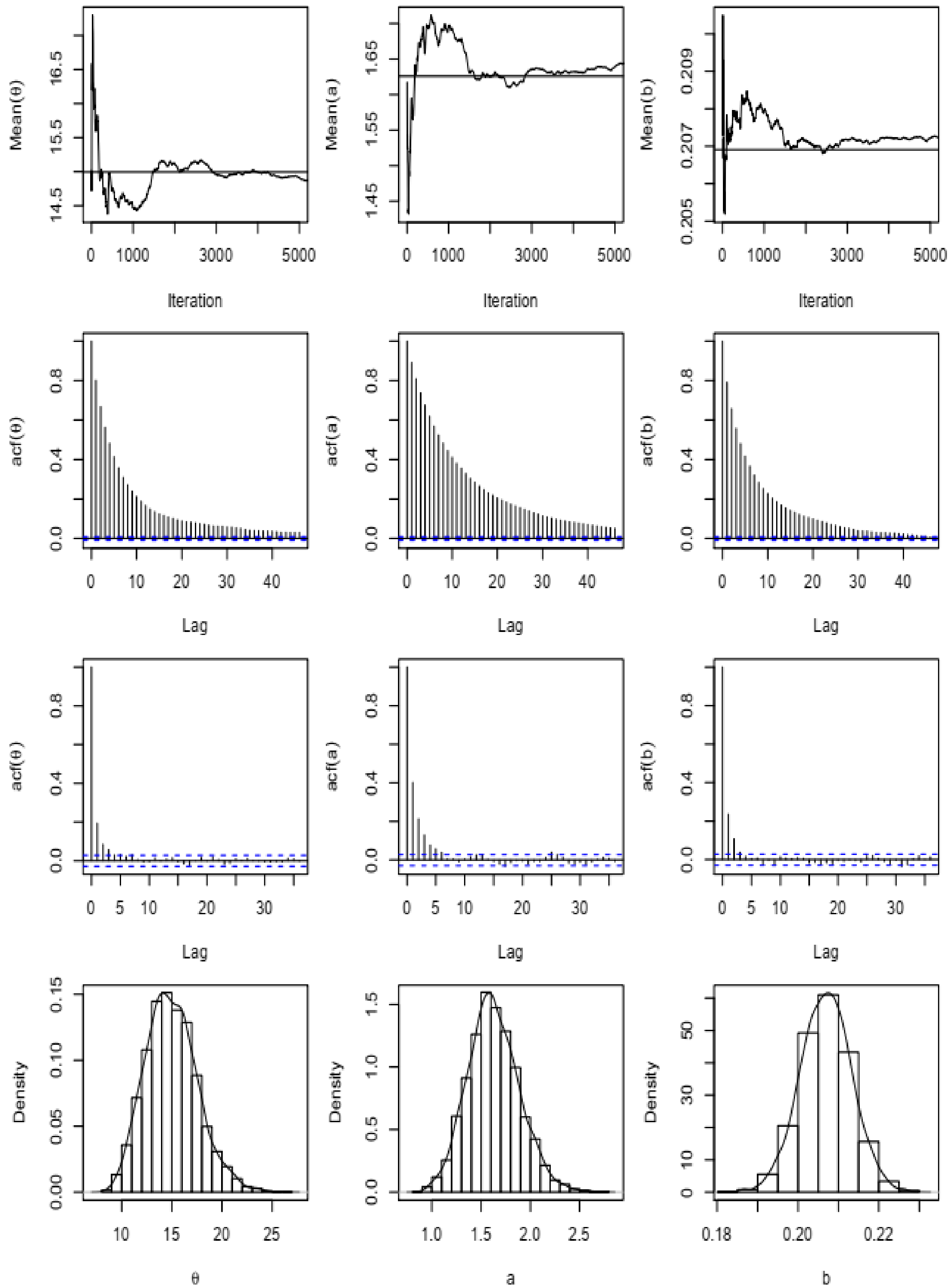

Figure 9: Cumulative mean, acf and density plots of MCMC chains of the parameters.

Table 5: Classical and Bayesian estimation of the parameters Θ=(θ,a,b) based on real data sets.

		Classical estimation					Bayes estimation				
		$\widehat{\Theta}_{\text{MLE}}$	SE	Θ_L	Θ_U	Width	$\widehat{\Theta}_{\text{MLE}}$	SE	Θ_L	Θ_U	Width
Wingo	θ	0.3519	0.1905	0.2811	0.4233	0.1422	0.3506	0.0319	0.2847	0.4100	0.1253
	a	0.6433	0.0670	0.6345	0.6521	0.0176	0.6434	0.0034	0.6369	0.6503	0.0133
	b	2.2703	0.5066	1.7669	2.7729	1.0060	2.2620	0.1971	1.8593	2.6363	0.7770
Pigs	θ	15.048	20.423	-802.477	832.573	1635.050	14.985	2.599	10.100	20.180	10.080
	a	1.6113	3.6724	-24.8218	28.0445	52.8663	1.6269	0.2627	1.1280	2.1452	1.0172
	b	0.2074	0.0771	0.1958	0.2191	0.0233	0.2069	0.0064	0.1937	0.2186	0.0249

5. Conclusions

In this paper we introduce a new four-parameter lifetime model called the Burr X Fréchet Model (BrXFr). We derive some of its structural properties including ordinary and incomplete moments, moment of residual and reversed residual life functions, quantile and generating functions, probability weighted moments and order statistics and their moments. The new density function can be expressed as a linear mixture of Fr densities. The maximum likelihood method is used to estimate the model parameters. Censored maximum likelihood estimation is presented in general case of multi-censored data. Simulation results to assess the performance of the maximum likelihood estimators are discussed in case of uncensored data. We demonstrate empirically the importance and flexibility of the new model in modeling various types of data. Bayesian estimation is performed by obtaining the posterior marginal distributions of model parameters. We use the simulation method of MCMC by the Metropolis-Hastings algorithm in each step of Gibbs algorithm. The trace plots and estimated conditional posterior distributions of parameters are presented. We hope that the proposed model will attract wider applications in areas such as engineering, survival and lifetime data, economics (income inequality), meteorology, hydrology, and others.

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