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An interval type-2 fuzzy TOPSIS for multiple attribute group decision making applied to solar power systems

Iman Mohamad Sharaf

Department of Basic Sciences, Higher Technological Institute, Egypt, iman_sharaf@hotmail.com

Abstract

This article proposes a new version of the technique of order preference by similarity to an ideal solution (TOPSIS) to solve fuzzy multi-attribute group decision making (MAGDM) problems using trapezoidal interval type-2 fuzzy sets (IT2FSs). The traditional TOPSIS ranks the alternatives according to their relative degree of closeness to the ideal solutions. On the other hand, TOPSIS based on similarity measure ranks the alternatives according to their total degree of similarity to the ideal solutions. This study extends TOPSIS using similarity measure using map distance to IT2FSs. First, the similarity measure based on map distance for interval-valued fuzzy sets (IVFSs) is extended to encompass IT2FSs due to the deficiency in IT2FSs similarity measures. Then, TOPSIS using similarity measure is applied. Hence, fuzzy MAGDM problems can be handled in a more flexible intelligent manner and avoiding defuzzification with its drawbacks. An illustrative example is given to explain the approach. Then, a practical problem in assessing thermal energy storage technologies in solar power systems is solved, where the weights of the attributes and the performance of the qualitative attributes are linguistic variables modeled by IT2FSs. The reliability of two normalization techniques is examined and the impact of the theoretical and empirical reference points on the solution is investigated.

Key Words: Fuzzy multi-criteria decision making; TOPSIS; Similarity measures; Interval type-2 fuzzy sets; Solar power systems

Mathematical Subject Classification: 90B50

1. Introduction

Choosing the best option from a set of possible alternatives is one of the most challenging problems in decision-making. The decision depends on the experts' opinions and evaluations of the multiple attributes of the alternatives (Chen and Lee, 2010). Multi-attribute group decision making (MAGDM) provides an efficacious frame for preference by evaluating and ranking the multiple attributes. The challenge arises not only from the multiple conflicting attributes but also from the uncertainty and ambiguity faced when evaluating the attributes' weights and the alternatives. The fuzzy set theory proved to be a perfect tool in dealing with vagueness and imprecision.

Type-2 fuzzy sets (T2FSs) have been utilized whenever the uncertainties' level is relatively high, the system's complexity increases and type-1 fuzzy sets (T1FSs) fail to express such high complexity and uncertainty (Cheng *et al.*, 2016). For example, autonomous mobile robots that navigate in a changing dynamic environment need to cope with large amounts of uncertainties that are inherent in natural environments. T1FSs cannot fully handle such uncertainties. It is not reasonable to use an exact membership function to express something uncertain. Hence, T2FSs can handle these uncertainties by employing a fuzzy membership function producing a better performance (Hagras, 2004). They provide more parameters and more design degrees of freedom, thus reducing the effect of imprecise information. Ever since their introduction, T2FSs have been successfully applied in various areas of applications, e.g. fuzzy logic systems, neural networks, and genetic algorithms (Sepúlveda *et al.*, 2007).

While T1FSs have one membership function, T2FSs have two membership functions, a primary membership function and a secondary membership function. However, T2FSs are difficult to comprehend and utilize due to their complex structure and heavy computations (Zheng *et al.*, 2010). Therefore, simpler types are utilized namely, interval type-2 fuzzy sets (IT2FSs), and interval-valued fuzzy sets (IVFSs). These types have been intensively used due to their reduced computational requirements (Hagras, 2004).

An IT2FS is a T2FS with a secondary membership function that is equal to one. An IT2FS has two membership functions, an upper membership function and a lower membership function, each of which is a T1FS. The reference points and the membership values, or simply the heights, of the upper and the lower membership functions, are used to define an IT2FS. While IT2FSs are characterized by two heights for each membership function, IVFSs have a single height for each membership function. Hence, IVFSs are a special case of IT2FSs when the two heights are equal.

The technique of order preference by similarity to an ideal solution (TOPSIS) is a well-known method for solving MAGDM problems. TOPSIS is useful and practical in selecting and ranking the alternatives. It was initially proposed by Hwang and Yoon (1981) for real-valued data. TOPSIS method selects the alternative whose distance from the positive ideal solution is the shortest, and whose distance from the negative ideal solution is the largest. Chen (2000) extended TOPSIS to fuzzy data using T1FSs. Ashtiani *et al.* (2009) developed TOPSIS for triangular IVFSs. Chen and Lee (2010) modified the technique to handle IT2FSs. In most of the proposed traditional fuzzy TOPSIS methods the weighted ratings are defuzzified to determine the ideal solutions and to compute the closeness coefficient.

The major flaw in the traditional fuzzy TOPSIS is the loss of information due to defuzzification (Ashtiani *et al.*, 2009). Losing important information may provide wrong results (Dymova *et al.*, 2015). Some modifications were proposed to improve the performance of TOPSIS under defuzzification. Ilieva (2016) defuzzified IT2FSs into two crisp values and then computed their average value. Wu et al. (2018) employed Wu and Mendel's centroid method for IVFSs to calculate the distances between each alternative and the ideal solutions. Other modifications have been introduced to TOPSIS to avert defuzzification. Ashtiani *et al.* (2009) modified TOPSIS using triangular IVFSs. They calculated the distances between the alternatives and the ideal solutions by the normalized Euclidean distance. Rashid *et al.* (2014) modified TOPSIS using trapezoidal IVFSs. They used a heuristic expression to calculate the distance between trapezoidal IVFSs. Dymova *et al.* (2015) introduced an interval type-2 fuzzy TOPSIS using α-cuts representation to avoid the limitations and drawbacks of the existing methods. Sharaf (2018) modified TOPSIS for IVFSs by using the degree of similarity for comparison instead of the relative degree of closeness to maintain fuzziness in the preference technique. Recently, Mohamadghasemi (2020) corrected some drawbacks in the TOPSIS method proposed by Dymova *et al.* (2015) to eliminate its limitations.

In this article, TOPSIS is extended to IT2FSs using similarity measure based on map distance. First, the similarity measure based on map distance for IVFSs is extended to IT2FSs due to the deficiency of IT2FSs similarity measures. Then, TOPSIS using similarity measure is applied. By this way, the flexibility of the method is increased and defuzzification with its flaws is surpassed. An illustrative example and a practical problem in assessing thermal energy storage technologies in solar power systems are solved to illustrate the method. The reliability of two normalization techniques is examined and the impact of the theoretical and empirical reference points on the solution is investigated.

The main contribution of the article can be summarized as follows

i. Handling fuzzy MAGDM problems in a more flexible intelligent manner by utilizing IT2FSs.
ii. Extending similarity measure based on map distance for IVFSs to encompass IT2FSs.
iii. Providing a simple reliable MAGDM methodology for ranking thermal energy storage technologies (TES) in concentrated solar power systems (CSP).

The article is organized as follows. IT2FSs and their operations are defined in section 2. The extension of the similarity measure based on map distance to IT2FSs is presented in section 3. The proposed TOPSIS method is given in section 4. In section 5, a numerical example and a practical example in solar power systems are solved using the proposed method. Finally, the conclusion and discussion are given in section 6.

2. Preliminaries

Definition 2.1.1. (Dymova *et al.*, 2015): A type-2 fuzzy set is given by:

$$\widetilde{\mathrm{A}} = \int_{\forall x \in \boldsymbol{X}} \int_{\forall u \in J_x \subseteq [0,1]} \mu_{\tilde{A}}(x,u)/(x,u), \tag{1}$$

where $\mu_{\tilde{A}}(x,u)$ is a type-2 membership function and $\iint$ denotes the union over all admissible x and u.
When $\mu_{\tilde{A}}(x,u) = 1$, a T2FS reduces to an IT2FS.

Definition 2.1.2. (Kahraman *et al.*, 2014): Let $\tilde{A}^L$ and $\tilde{A}^U$ be two trapezoidal fuzzy sets. A trapezoidal IT2FS is given by:

$$\tilde{A} = \left[\tilde{A}^L, \tilde{A}^U\right] = [(a_1^L, a_2^L, a_3^L, a_4^L; w_1^L, w_2^L), (a_1^U, a_2^U, a_3^U, a_4^U;\ w_1^U, w_2^U)], \tag{2}$$

where $a_1^L, a_2^L, a_3^L, a_4^L, a_1^U, a_2^U, a_3^U, and\ a_4^U \in R$ are the reference points of the IT2FS, and $w_1^L, w_2^L, w_1^U and\ w_2^U \in [0,1]$ are the membership values. When $w_1^L = w_2^L$ and $w_1^U = w_2^U$, an IT2FS reduces to an IVFS.

The aggregation operations for the IT2FSs $\tilde{A}$ and $\tilde{B}$ are given as follows.

Definition 2.1.3. (Chen and Lee, 2010)

$$\tilde{A} \oplus \tilde{B} = \begin{bmatrix} \left(a_1^L + b_1^L, a_2^L + b_2^L, a_3^L + b_3^L, a_4^L + b_4^L; min\left(w_{1\tilde{A}}^L, w_{1\tilde{B}}^L\right), min\left(w_{2\tilde{A}}^L, w_{2\tilde{B}}^L\right)\right), \\ \left(a_1^U + b_1^U, a_2^U + b_2^U, a_3^U + b_3^U, a_4^U + b_4^U;\ min\left(w_{1\tilde{A}}^U, w_{1\tilde{B}}^U\right), min\left(w_{2\tilde{A}}^U, w_{2\tilde{B}}^U\right)\right) \end{bmatrix}. \tag{3}$$

$$\tilde{A} \otimes \tilde{B} = \begin{bmatrix} \left(a_1^L b_1^L, a_2^L b_2^L, a_3^L b_3^L, a_4^L b_4^L; min\left(w_{1\tilde{A}}^L, w_{1\tilde{B}}^L\right), min\left(w_{2\tilde{A}}^L, w_{2\tilde{B}}^L\right)\right), \\ \left(a_1^U b_1^U, a_2^U b_2^U, a_3^U b_3^U, a_4^U b_4^U;\ min\left(w_{1\tilde{A}}^U, w_{1\tilde{B}}^U\right), min\left(w_{2\tilde{A}}^U, w_{2\tilde{B}}^U\right)\right) \end{bmatrix}. \tag{4}$$

$$k.\tilde{A} = \tilde{A}.k = \begin{cases} \left[\left(k.a_1^L, k.a_2^L, k.a_3^L, k.a_4^L; w_{1\tilde{A}}^L, w_{2\tilde{A}}^L\right), \left(k.a_1^U, k.a_2^U, k.a_3^U, k.a_4^U;\ w_{1\tilde{A}}^U, w_{2\tilde{A}}^U\right)\right]; if\ k \geq 0, \\ \left[\left(k.a_4^L, k.a_3^L, k.a_2^L, k.a_1^L; w_{1\tilde{A}}^L, w_{2\tilde{A}}^L\right), \left(k.a_4^U, k.a_3^U, k.a_2^U, k.a_1^U; w_{1\tilde{A}}^U, w_{2\tilde{A}}^U\right)\right]; if\ k \leq 0, \end{cases} \tag{5}$$

where k is an arbitrary real number.

3. A similarity measure for IT2FSs

Similarity measures attracted researchers' attention due to their wide applications in clustering, case-based reasoning, and pattern recognition (Beg and Rashid, 2017). A similarity measure between two fuzzy sets, denoted by $S(\tilde{A}, \tilde{B})$, is an indication of the extent to which the fuzzy sets are similar. Similarity measures for T1FSs were comprehensively studied. A few studies proposed similarity measures for IVFSs. These similarity measures suffer from several disadvantages. They may give counter-intuitive results; they cannot find the degree of similarity between two disjoint sets, or $S(\tilde{A}, \tilde{B}) \neq 1$ even when the fuzzy sets are the same (Wu and Mendel, 2008).

To benefit from the shape and to avoid the disadvantages and complications of the previous methods, some similarity measures were developed for trapezoidal IVFSs. Chen and Chen (2008) developed a similarity measure based on the center of gravity of the lower and the upper fuzzy numbers. The similarity measure of Wei and Chen (2009) used the geometric distance, the perimeter, the height and the center of gravity. Chen and Chen (2009) introduced a similarity measure that uses the difference of the spreads and the heights of the upper fuzzy numbers, the degree of similarity and the gravities on the X-axis and the gravity on the Y-axis between IVFSs. Chen and Kao (2010) developed a similarity measure based on the standard deviation operator; while Chen (2011) proposed a similarity measure based on the quadratic mean operator. Chen et al. (2013) introduced a similarity measure based on the map distance, to overcome flaws of the previously mentioned methods, e.g. they cannot give the correct degree of similarity between two interval-valued fuzzy numbers in some cases. The results showed that their method outperforms the existing methods (Chen et al., 2013).

On the contrary, similarity measures for trapezoidal IT2FSs didn't receive attention. Due to the deficiency in similarity measures for trapezoidal IT2FSs, the similarity measure of Chen et al. (2013) based on map distance for trapezoidal IVFSs is extended to trapezoidal IT2FSs being the most convenient similarity measure.

The degree of similarity between two IT2FSs $\tilde{A}$ and $\tilde{B}$ based on map distance can be calculated as follows.

Step 1: Compute the distance values Δa_i and Δb_i.

For the IT2FSs $\tilde{A}$ and $\tilde{B}$, the distance values between the lower and the upper fuzzy sets are given by:

$$\Delta a_i = \left|a_i^U - a_i^L\right| \text{ and } \Delta b_i = \left|b_i^U - b_i^L\right|, \text{where } i = 1,2,3,4.$$

Step 2: Compute $S(\tilde{A}^{\Delta},\ \tilde{B}^{\Delta})$.

a) Find the standard deviations ΔS_a and ΔS_b between the upper and lower fuzzy sets.

$$\bar{a}^U = (a_1^U + a_2^U + a_3^U + a_4^U)/4,\ \bar{a}^L = (a_1^L + a_2^L + a_3^L + a_4^L)/4,$$

$$S_{\tilde{A}^U}=\sqrt{\frac{\sum_{i=1}^{4}(a_i^U-\bar{a}^U)^2}{3}},\ S_{\tilde{A}^L}=\sqrt{\frac{\sum_{i=1}^{4}(a_i^L-\bar{a}^L)^2}{3}},\ \Delta S_a=\left|S_{\tilde{A}^U}-S_{\tilde{A}^L}\right|.$$

$$\bar{b}^U=(b_1^U+b_2^U+b_3^U+b_4^U)/4,\ \bar{b}^L=(b_1^L+b_2^L+b_3^L+b_4^L)/4,$$

$$S_{\tilde{B}^U}=\sqrt{\frac{\sum_{i=1}^{4}(b_i^U-\bar{b}^U)^2}{3}},\ S_{\tilde{B}^L}=\sqrt{\frac{\sum_{i=1}^{4}(b_i^L-\bar{b}^L)^2}{3}},\Delta S_b=\left|S_{\tilde{B}^U}-S_{B^L}\right|.$$

b) Find the map distance between the upper and lower fuzzy sets.

$$T^{\Delta}=\left[\left(2-\frac{1+\max\{|\Delta a_2-\Delta a_1|,|\Delta b_2-\Delta b_1|\}}{1+\min\{|\Delta a_2-\Delta a_1|,|\Delta b_2-\Delta b_1|\}}\right)+\left(2-\frac{1+\max\{|\Delta a_4-\Delta a_3|,|\Delta b_4-\Delta b_3|\}}{1+\min\{|\Delta a_4-\Delta a_3|,|\Delta b_4-\Delta b_3|\}}\right)\right]/2.$$

c) Find $S(\tilde{A}^{\Delta},\tilde{B}^{\Delta})\in[0,1]$.

$$S(\tilde{A}^{\Delta},\tilde{B}^{\Delta})=\left[1-\frac{\sqrt{\sum_{i=1}^{4}(\Delta a_i-\Delta b_i)^2}}{2}\right]\times\left[1-\sqrt{\frac{|\Delta S_a-\Delta S_b|}{2}}\right]\times\left[1-\frac{\left|w_{1\tilde{A}}^L-w_{1\tilde{B}}^L\right|+\left|w_{2\tilde{A}}^L-w_{2\tilde{B}}^L\right|}{\left|w_{1\tilde{A}}^U+w_{1\tilde{B}}^U\right|+\left|w_{2\tilde{A}}^U+w_{2\tilde{B}}^U\right|}\right]\times T^{\Delta}. \tag{6}$$

Step 3: Compute $S(\tilde{A}^U,\ \tilde{B}^U)$.

a) Find the map distance between the upper trapezoidal fuzzy sets.

$$T^U=\left[\left(2-\frac{1+\max\{|a_2^u-a_1^u|,|b_2^u-b_1^u|\}}{1+\min\{|a_2^u-a_1^u|,|b_2^u-b_1^u|\}}\right)+\left(2-\frac{1+\max\{|a_4^u-a_3^u|,|b_4^u-b_3^u|\}}{1+\min\{|a_4^u-a_3^u|,|b_4^u-b_3^u|\}}\right)\right]/2.$$

b) Find $S(\tilde{A}^U,\tilde{B}^U)\in[0,1]$.

$$S(\tilde{A}^U,\tilde{B}^U)=\left[1-\frac{\sqrt{\sum_{i=1}^{4}(a_i^u-b_i^u)^2}}{2}\right]\times\left[1-\sqrt{\frac{\left|S_{\tilde{A}^U}-S_{\tilde{B}^U}\right|}{2}}\right]\times\left[\frac{\min\left(w_{1\tilde{A}}^U,w_{1\tilde{B}}^U\right)+\min\left(w_{2\tilde{A}}^U,w_{2\tilde{B}}^U\right)}{\max\left(w_{1\tilde{A}}^U,w_{1\tilde{B}}^U\right)+\max\left(w_{2\tilde{A}}^U,w_{2\tilde{B}}^U\right)}\right]\times T^U. \tag{7}$$

Step 4: Compute $S(\tilde{A},\tilde{B})$.

$$S(\tilde{A},\tilde{B})=\frac{S(\tilde{A}^U,\tilde{B}^U)\times\left(1+\ S(\tilde{A}^{\Delta},\tilde{B}^{\Delta})\right)}{2} \tag{8}$$

As the value of $S(\tilde{A},\tilde{B})$ increases, the degree of similarity between $\tilde{A}$ and $\tilde{B}$ increases.

The extended similarity measure satisfies the following properties:
Property 1 (Reflexivity): Two IT2FSs $\tilde{A}$ and $\tilde{B}$ are identical if and only if $S(\tilde{A},\tilde{B})=1$.
Property 2 (Symmetry): $S(\tilde{A},\tilde{B})=S(\tilde{B},\tilde{A})$.
Property 3 (Transitivity): If $\tilde{A}\le\tilde{B}\le\tilde{C}$, then $S(\tilde{A},\tilde{B})\ge S(\tilde{A},\tilde{C})$.
Property 4 (Overlap): If two IT2FSs partially overlap, $S(\tilde{A},\tilde{B})>0$.
Property 5: If $\tilde{A}$ and $\tilde{B}$ are real numbers, then $S(\tilde{A},\tilde{B})=1-|a-b|$.
The proofs of these properties are given in the appendix.

4. TOPSIS for IT2FSs

TOPSIS main idea is to select the alternative whose distance from the positive ideal solution is the minimum and its distance from the negative ideal solution is the maximum. The main aim of the proposed TOPSIS is to maintain fuzziness in information and avoid defuzzification. First, the degree of similarity between each attribute of an alternative and the ideal solution is calculated. Second, the similarity matrix is formed. Then, the total degree of similarity for all the attributes of an alternative is used for preference. The best candidate is the one corresponding to the one norm of the similarity matrix. In this section TOPSIS with similarity measure for IVFSs is extended to IT2FSs. The steps are as follows.

Consider a MAGDM in the presence of k decision-makers $D_1,D_2,\dots,D_k$, with a set of n alternatives $X=\{x_1,x_2,\dots,x_n\}$ and a set of m attributes $F=\{f_1,f_2,\dots,f_m\}$. The set of attributes is divided into two sets, the set of benefits "F_b" and the set of costs "F_c", such that $F_b\cap F_c=\emptyset$.

Step 1: Formation of the decision matrix and the average decision matrix,

$$\widetilde{\mathbf{D}}_p = \begin{matrix} & \begin{matrix} x_1 & x_2 & & x_n \end{matrix} \\ \begin{matrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{matrix} & \begin{pmatrix} \tilde{f}^p_{11} & \tilde{f}^p_{12} & \cdots & \tilde{f}^p_{1n} \\ \tilde{f}^p_{21} & \tilde{f}^p_{22} & & \tilde{f}^p_{2n} \\ & \vdots & \ddots & \vdots \\ \tilde{f}^p_{m1} & \tilde{f}^p_{m2} & \cdots & \tilde{f}^p_{mn} \end{pmatrix} \end{matrix}, \text{ and } \overline{\overline{\mathbf{D}}} = \left(\tilde{f}_{ij}\right)_{m\times n},$$

where $\tilde{f}_{ij} = \left(\frac{\tilde{f}^1_{ij} \oplus \tilde{f}^2_{ij} \oplus \ldots \oplus \tilde{f}^k_{ij}}{k}\right)$, $1 \le i \le m, 1 \le j \le n$ and $1 \le p \le k$.

Step 2: Formation of the weighting matrix and the average weighting matrix,

$$\widetilde{\mathbf{W}}_p = \begin{matrix} \begin{matrix} f_1 & f_2 & & f_m \end{matrix} \\ [\tilde{w}^p_1 \quad \tilde{w}^p_2 \quad \ldots \quad \tilde{w}^p_m] \end{matrix}, \quad \overline{\overline{\mathbf{W}}}_p = (\tilde{w}_i)_{1\times m}$$

where $\tilde{w}_i = \left(\frac{\tilde{w}^1_i \oplus \tilde{w}^2_i \oplus \ldots \oplus \tilde{w}^p_i}{k}\right)$, $1 \le i \le m$ and $1 \le p \le k$.

Step 3: Formation of the normalized average decision matrix $\overline{\overline{\mathbf{N}}} = \left(\tilde{n}_{ij}\right)_{m\times n}$.

Normalization keeps the IT2FSs $\in [0,1]$. The normalized ratings $\tilde{n}_{ij}$ is given by

$$\tilde{n}_{ij} = \left[\left(\frac{f^L_{1ij}}{f^+_{4j}}, \frac{f^L_{2ij}}{f^+_{4j}}, \frac{f^L_{3ij}}{f^+_{4j}}, \frac{f^L_{4ij}}{f^+_{4j}}; w^L_1, w^L_2\right), \left(\frac{f^U_{1ij}}{f^+_{4j}}, \frac{f^U_{2ij}}{f^+_{4j}}, \frac{f^U_{3ij}}{f^+_{4j}}, \frac{f^U_{4ij}}{f^+_{4j}}; w^U_1, w^U_2\right)\right], \text{ where } i = 1,\ldots,m,\ j \in F_b \text{ and } f^+_{4j} = \max_i f^U_{4ij}.$$

$$\tilde{n}_{ij} = \left[\left(\frac{f^-_{1j}}{f^L_{4ij}}, \frac{f^-_{1j}}{f^L_{3ij}}, \frac{f^-_{1j}}{f^L_{2ij}}, \frac{f^-_{1j}}{f^L_{1ij}}; w^L_1, w^L_2\right), \left(\frac{f^-_{1j}}{f^U_{4j}}, \frac{f^-_{1j}}{f^U_{3ij}}, \frac{f^-_{1j}}{f^U_{2ij}}, \frac{f^-_{1j}}{f^U_{1j}}; w^U_1, w^U_2\right)\right], \text{ where } i = 1,\ldots,m,\ j \in F_c \text{ and } f^-_{1j} = \min_i f^U_{1ij}.$$

Step 4: Formation of the weighted normalized decision matrix

$$\widetilde{\mathbf{D}}_w = \begin{matrix} & \begin{matrix} x_1 & x_2 & & x_n \end{matrix} \\ \begin{matrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{matrix} & \begin{pmatrix} \tilde{v}_{11} & \tilde{v}_{12} & \cdots & \tilde{v}_{1n} \\ \tilde{v}_{21} & \tilde{v}_{22} & & \tilde{v}_{2n} \\ & \vdots & \ddots & \vdots \\ \tilde{v}_{m1} & \tilde{v}_{m2} & \cdots & \tilde{v}_{mn} \end{pmatrix} \end{matrix}, \text{ where } \tilde{v}_{ij} = \tilde{w}_i \otimes \tilde{n}_{ij},\ 1 \le i \le m \text{ and } 1 \le j \le n.$$

Step 5: Set the fuzzy positive ideal solution (FPIS) $\tilde{v}^+$ and the fuzzy negative ideal solution (FNIS) $\tilde{v}^-$ as,

$$\tilde{v}^+ = [(1,1,1,1;1,1),(1,1,1,1;1,1)] \text{ and } \tilde{v}^- = [(0,0,0,0;1,1),(0,0,0,0;1,1)].$$

Step 6: Formation of the similarity matrix $\mathbf{S} = \left(S_{ij}\right)$.

Find the degree of similarity between $\tilde{v}_{ij}$ and the ideal solutions using the similarity measure based on map distance for IT2FSs.

$$S^+_{ij} = S\left(\tilde{v}_{ij}, \tilde{v}^+\right) = \frac{S(\tilde{v}^U_{ij}, \tilde{v}^{+U}) \times \left(1 + S\left(\tilde{v}^\Delta_{ij}, \tilde{v}^{+\Delta}\right)\right)}{2}, \ if\ f_i \in F_b$$

$$S^-_{ij} = S\left(\tilde{v}_{ij}, \tilde{v}^-\right) = \frac{S(\tilde{v}^U_{ij}, \tilde{v}^{-U}) \times \left(1 + S\left(\tilde{v}^\Delta_{ij}, \tilde{v}^{-\Delta}\right)\right)}{2}, \ if\ f_i \in F_c.$$

Step 7: Find the total degree of similarity of each alternative

$$S(X_j) = \sum_{i=1}^m s_{ij}, \text{ for } j = 1,\ldots,n.$$

As the value of $S(X_j)$ increases, the preference for the alternative X_j increases. The alternative corresponding to the similarity matrix one norm $\|\mathbf{S}\|_1$ is the best candidate, where

$$\|\mathbf{S}\|_1 = \max_{1\le j\le n}\left(\sum_{i=1}^m |s_{ij}|\right).$$

5. Examples

In this section, two examples are solved. The first explains the proposed TOPSIS, the second is a practical example for assessing thermal energy storage in solar power systems.

5.1. Numerical Example

This example is from Chen and Lee (2010). Since the linguistic terms used by Chen and Lee (2010) are IVFSs, these terms are redefined using IT2FSs. A company intends to buy cars and three alternatives are available. The decision-makers $D_1, D_2,$ and D_3 rate the cars using four attributes: safety (f_1), price (f_2), appearance(f_3), and performance(f_4). The safety, the appearance and the performance are the benefit attributes, while the price is the cost attribute. The set of alternatives is $X = \{\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3\}$, and the set of attributes is $F = \{f_1, f_2, f_3, f_4\}$. The decision-makers use the linguistic terms shown in Table 1. The problem's data are shown in the solution steps. For more details see Chen and Lee (2010).

Table 1: Linguistic terms and their corresponding IT2FSs for example 1.

Linguistic terms	IT2FSs
Very Low (VL)	[(0.01,0.02,0.03,0.04;0.5,0.6)(0,0.02,0.03,0.05;0.9,1)]
Low(L)	[(0.1,0.15,0.2,0.25;0.5,0.6)(0.05,0.15,0.2,0.3;0.9,1)]
Medium Low (ML)	[(0.2,0.25,0.3,0.35;0.5,0.6)(0.15,0.25,0.3,0.4;0.9,1)]
Medium (M)	[(0.35,0.4,0.45,0.5;0.5,0.6)(0.3,0.4,0.45,0.55;0.9,1)]
Medium High (MH)	[(0.55,0.6,0.65,0.7;0.5,0.6)(0.5,0.6,0.65,0.75;0.9,1)]
High (H)	[(0.75,0.8,0.85,0.9;0.5,0.6)(0.7,0.8,0.85,0.95;0.9,1)]
Very High (VH)	[(0.875,0.9,0.95,1.0;0.5,0.6)(0.85,0.9,0.95,1.0;0.9,1)]

Step 1: a) Formation of the decision matrices and the average decision matrix.

$$\widetilde{\mathrm{D}}_1 = \begin{matrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{matrix} \overset{\begin{matrix} \mathrm{x}_1 & \mathrm{x}_2 & \mathrm{x}_3 \end{matrix}}{\begin{pmatrix} \mathrm{MH} & \mathrm{H} & \mathrm{VH} \\ \mathrm{H} & \mathrm{MH} & \mathrm{VH} \\ \mathrm{VH} & \mathrm{H} & \mathrm{M} \\ \mathrm{VH} & \mathrm{H} & \mathrm{H} \end{pmatrix}}, \widetilde{\mathrm{D}}_2 = \begin{matrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{matrix} \overset{\begin{matrix} \mathrm{x}_1 & \mathrm{x}_2 & \mathrm{x}_3 \end{matrix}}{\begin{pmatrix} \mathrm{H} & \mathrm{MH} & \mathrm{H} \\ \mathrm{VH} & \mathrm{H} & \mathrm{VH} \\ \mathrm{H} & \mathrm{VH} & \mathrm{MH} \\ \mathrm{H} & \mathrm{VH} & \mathrm{VH} \end{pmatrix}}, \widetilde{\mathrm{D}}_3 = \begin{matrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{matrix} \overset{\begin{matrix} \mathrm{x}_1 & \mathrm{x}_2 & \mathrm{x}_3 \end{matrix}}{\begin{pmatrix} \mathrm{MH} & \mathrm{H} & \mathrm{MH} \\ \mathrm{H} & \mathrm{VH} & \mathrm{H} \\ \mathrm{H} & \mathrm{VH} & \mathrm{MH} \\ \mathrm{H} & \mathrm{H} & \mathrm{VH} \end{pmatrix}}.$$

$$\bar{\widetilde{\mathrm{D}}} = \begin{matrix} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \end{matrix} \begin{pmatrix} \mathrm{x}_1 & \mathrm{x}_2 & \mathrm{x}_3 \\ \begin{bmatrix} (0.62,0.67,0.72,0.77; 0.5,0.6) \\ (0.57,0.67,0.72,0.82; 0.9,1) \end{bmatrix} & \begin{bmatrix} (0.68,0.73,0.78,0.83; 0.5,0.6) \\ (0.63,0.73,0.78,0.88; 0.9,1) \end{bmatrix} & \begin{bmatrix} (0.73,0.77,0.82,0.87; 0.5,0.6) \\ (0.68,0.77,0.82,0.90; 0.9,1) \end{bmatrix} \\ \begin{bmatrix} (0.79,0.83,0.88,0.93; 0.5,0.6) \\ (0.75,0.83,0.88,0.97; 0.9,1) \end{bmatrix} & \begin{bmatrix} (0.73,0.77,0.82,0.87; 0.5,0.6) \\ (0.68,0.77,0.82,0.90; 0.9,1) \end{bmatrix} & \begin{bmatrix} (0.83,0.87,0.92,0.97; 0.5,0.6) \\ (0.80,0.87,0.92,0.98; 0.9,1) \end{bmatrix} \\ \begin{bmatrix} (0.79,0.83,0.88,0.93; 0.5,0.6) \\ (0.75,0.83,0.88,0.97; 0.9,1) \end{bmatrix} & \begin{bmatrix} (0.83,0.87,0.92,0.97; 0.5,0.6) \\ (0.80,0.87,0.92,0.98; 0.9,1) \end{bmatrix} & \begin{bmatrix} (0.48,0.53,0.58,0.63; 0.5,0.6) \\ (0.43,0.53,0.58,0.68; 0.9,1) \end{bmatrix} \\ \begin{bmatrix} (0.79,0.83,0.88,0.93; 0.5,0.6) \\ (0.75,0.83,0.88,0.97; 0.9,1) \end{bmatrix} & \begin{bmatrix} (0.79,0.83,0.88,0.93; 0.5,0.6) \\ (0.75,0.83,0.88,0.97; 0.9,1) \end{bmatrix} & \begin{bmatrix} (0.83,0.87,0.92,0.97; 0.5,0.6) \\ (0.80,0.87,0.92,0.98; 0.9,1) \end{bmatrix} \end{pmatrix}$$

Step 2: a) Formation of the weighting matrices and the average weighting matrix.

$$\widetilde{\mathrm{W}}_1 = \overset{\begin{matrix} f_1 & f_2 & f_3 & f_4 \end{matrix}}{(\mathrm{VH} \quad \mathrm{H} \quad \mathrm{M} \quad \mathrm{VH})}, \widetilde{\mathrm{W}}_2 = \overset{\begin{matrix} f_1 & f_2 & f_3 & f_4 \end{matrix}}{(\mathrm{H} \quad \mathrm{VH} \quad \mathrm{MH} \quad \mathrm{H})}, \widetilde{\mathrm{W}}_3 = \overset{\begin{matrix} f_1 & f_2 & f_3 & f_4 \end{matrix}}{(\mathrm{VH} \quad \mathrm{VH} \quad \mathrm{MH} \quad \mathrm{H})}.$$

$$\bar{\widetilde{\mathrm{W}}} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ \begin{bmatrix} (0.83,0.87,0.92,0.97; 0.5,0.6) \\ (0.80,0.87,0.92,0.98; 0.9,1) \end{bmatrix} & \begin{bmatrix} (0.83,0.87,0.92,0.97; 0.5,0.6) \\ (0.80,0.87,0.92,0.98; 0.9,1) \end{bmatrix} & \begin{bmatrix} (0.48,0.53,0.58,0.63; 0.5,0.6) \\ (0.43,0.53,0.58,0.68; 0.9,1) \end{bmatrix} & \begin{bmatrix} (0.79,0.83,0.88,0.93; 0.5,0.6) \\ (0.75,0.83,0.88,0.97; 0.9,1) \end{bmatrix} \end{pmatrix},$$

Step 3: The IT2FSs$\in [0,1]$, hence normalization is not required.

Step 4: Formation of the weighted decision matrix.

$$\widetilde{\mathrm{D}}_{\mathrm{w}} = \begin{matrix} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \end{matrix} \begin{pmatrix} \mathrm{x}_1 & \mathrm{x}_2 & \mathrm{x}_3 \\ \begin{bmatrix} (0.51,0.58,0.66,0.74; 0.5,0.6) \\ (0.46,0.58,0.66,0.80; 0.9,1) \end{bmatrix} & \begin{bmatrix} (0.57,0.64,0.72,0.81; 0.5,0.6) \\ (0.51,0.64,0.72,0.87; 0.9,1) \end{bmatrix} & \begin{bmatrix} (0.60,0.66,0.75,0.84; 0.5,0.6) \\ (0.55,0.66,0.75,0.89; 0.9,1) \end{bmatrix} \\ \begin{bmatrix} (0.66,0.72,0.81,0.90; 0.5,0.6) \\ (0.60,0.72,0.81,0.95; 0.9,1) \end{bmatrix} & \begin{bmatrix} (0.60,0.66,0.75,0.84; 0.5,0.6) \\ (0.55,0.66,0.75,0.89; 0.9,1) \end{bmatrix} & \begin{bmatrix} (0.69,0.75,0.84,0.93; 0.5,0.6) \\ (0.64,0.75,0.84,0.97; 0.9,1) \end{bmatrix} \\ \begin{bmatrix} (0.38,0.44,0.52,0.59; 0.5,0.6) \\ (0.33,0.44,0.52,0.66; 0.9,1) \end{bmatrix} & \begin{bmatrix} (0.40,0.46,0.53,0.61; 0.5,0.6) \\ (0.35,0.46,0.53,0.67; 0.9,1) \end{bmatrix} & \begin{bmatrix} (0.23,0.28,0.34,0.40; 0.5,0.6) \\ (0.19,0.28,0.34,0.47; 0.9,1) \end{bmatrix} \\ \begin{bmatrix} (0.63,0.69,0.78,0.87; 0.5,0.6) \\ (0.56,0.69,0.78,0.93; 0.9,1) \end{bmatrix} & \begin{bmatrix} (0.63,0.69,0.78,0.87; 0.5,0.6) \\ (0.56,0.69,0.78,0.93; 0.9,1) \end{bmatrix} & \begin{bmatrix} (0.66,0.72,0.81,0.91; 0.5,0.6) \\ (0.60,0.72,0.81,0.95; 0.9,1) \end{bmatrix} \end{pmatrix},$$

Step 5: Set the FPIS and the FNIS as $\tilde{v}^+ = [(1,1,1,1;1,1)(1,1,1,1;1,1)]$ and $\tilde{v}^- = [(0,0,0,0;1,1),(0,0,0,0;1,1)]$.

Step 6: Formation of the similarity matrix (S_{ij}).

$$\begin{array}{r} \\ \text{Safety} \\ \text{Price} \\ \text{Appearence} \\ \text{Performance} \end{array}\begin{array}{c} \begin{array}{ccc} \mathrm{x}_1 & \mathrm{x}_2 & \mathrm{x}_3 \end{array} \\ \begin{pmatrix} S_{11}^{+} & S_{12}^{+} & S_{13}^{+} \\ S_{21}^{-} & S_{22}^{-} & S_{23}^{-} \\ S_{31}^{+} & S_{32}^{+} & S_{33}^{+} \\ S_{41}^{+} & S_{42}^{+} & S_{43}^{+} \end{pmatrix} \end{array} = \begin{array}{r} \\ \text{Safety} \\ \text{Price} \\ \text{Appearence} \\ \text{Performance} \end{array}\begin{array}{c} \begin{array}{ccc} \mathrm{x}_1 & \mathrm{x}_2 & \mathrm{x}_3 \end{array} \\ \begin{pmatrix} 0.2868 & 0.3085 & 0.3340 \\ 0.1051 & 0.1355 & 0.0957 \\ 0.2259 & 0.2390 & 0.1593 \\ 0.3287 & 0.3287 & 0.3541 \end{pmatrix} \end{array}.$$

Step 7: Find the total degree of similarity of each alternative to the ideal solutions.

$$S(X_1) = 0.9465,\ S(X_2) = 1.0117 \text{ and } S(X_3) = 0.9431.$$

From the results, $S(X_2) > S(X_1) > S(X_3)$, the ranking is $X_2 > X_1 > X_3$. Then, the best alternative is X_2. According to Chen and Lee (2010), the preferred order of the alternatives is $X_2 > X_1 > X_3$. Hence the result of the proposed TOPSIS coincides with the results of Chen and Lee (2010).

5.2. Practical Example

The commercial utilization of solar thermal power and the construction of large-scale industrial plants started in the 1980s. The variability of energy production due to night-time and cloudy weather is one of these systems' limitations (Cavallaro, 2010). To overcome this limitation, solar power plants are coupled with thermal energy storage (TES) that guarantees energy supply even in the absence of solar radiation. TES is the key component for a power plant to improve its dispatchability. TES has several advantages compared with other storage technologies, e.g. mechanical or chemical. They are lower in costs and higher in efficiency (Kuravi *et al.*, 2013).

Solar energy is converted into electricity employing a concentrated solar power (CSP) plant composed of four elements: a concentrator, a high-temperature solar receiver, a fluid transport system and a power generation bloc. It is estimated that the CSP will contribute up to 11% of the global electricity production in the year 2050 (Pelay *et al.*, 2017).

The first part (the concentrator) is a mirror designed to intercept solar radiation and concentrate it in a focal point. The second and third parts are the receiver and the heat exchanger that are linked to the solar concentrator. The heat transfer fluid (HTF) that circulates inside the heat exchanger absorbs the heat generated by solar radiation. The fourth part is another heat exchanger that transfers the accumulated thermal energy to another fluid, usually steam that drives a turbine or generator set-up (Cavallaro, 2010).

TES is integrated with the system in three possible ways. A two tank indirect system in which two HTF are used. The first HTF collects the solar thermal energy from the solar receivers in the solar field. The second HTF acts as the primary heat storage medium that is kept in two separate tanks having different temperatures. A two tank direct system, in which a single HTF is used in the solar field and the primary storage media. A Single tank thermocline system, in which a single HTF and a single insulated thermocline tank are utilized (Alva et al., 2016).

The working temperature of mineral oil lies between 290˚C and 390˚C. Yet, it is highly inflammable. Also, in case of any accidental leakage from the plant pollution may occur. The working temperature of molten salt (a mixture of sodium nitrate and potassium nitrate) lies between 290 ˚C and 550˚C. They are non-polluting and not inflammable. Molten salt has several advantages. It increases energy performance and reduces electricity production costs. It is non-toxic and eco-compatible. It keeps the temperature of steam close to the temperature required by the Rankine cycle turbines to work at high efficiency. The main disadvantage is that molten salt solidifies at relatively high temperatures, between 120 ˚C and 221 ˚C. Thus, it is essential to maintain them in the liquid state in the pipes (Cavallaro, 2010).

The following example is adapted from Cavallaro (2010). The linguistic terms assigned to the performance of the attributes and their weights are redefined using IT2FSs (while keeping them in the given ranges as possible) which are more flexible in dealing with uncertainties and model them with greater accuracy. In this example, seven TES are tested and compared. These examined systems are as follows.

- x_1. VP-1 NS: uses Therminol VP-1 as a HTF with no heat storage. Therminol VP-1 is characterized by a low freezing point (12 ºC) and is stable up to 400 ºC.
- x_2. VP-1 TT: Therminol VP-1 serves as a HTF and for heat storage with two tanks, hot and cold, at a pressure of 66 bar.
- x_3. VP-1 TC: uses Therminol VP-1 in a single thermocline tank at a pressure of 66 bar.
- x_4. MS-TT 450: uses molten salt for heat transfer and heat storage with two tanks, hot and cold, with a maximum attainable temperature of 450˚C.
- x_5. MS-TC 450: uses molten salt in a thermocline tank with a maximum attainable temperature of 450˚C.
- x_6. MS-TT 500: it is similar to MS-TT 450 but with a maximum attainable temperature of 500˚C.

x_7. MS-TC 500: it is similar to MS-TC 450 but with a maximum attainable temperature of 500˚C.

Ten attributes are chosen, related to technical, economical, and environmental reasons. The attributes can be summarized as follows.

f_1. Investment costs (M US$): the costs of purchasing mechanical equipment, engineering services, technological installations, drilling and any other required construction work.

f_2. Operating and maintenance costs (k US$/year): the costs of materials, transportation, wages, and other costs related to the plant.

f_3. Levelized electricity cost (US$/MW h): the industrial production cost of the electricity generated by the plant.

f_4. Levelized electricity cost reduction (%): the ability of each plant to reduce the industrial cost of production.

f_5 .Thermal storage cost (M US$): the cost of building the heat storage system including the steel tanks, the fluid circulation system and pumps.

f_6. Electricity production (GW h): the level of electricity production of the plant.

f_7. State of knowledge of technology: the reliability of the utilized technology.

f_8. Environmental risk: the risk to the environment arising from accidental leakage of the HTF from tanks and hydraulic plants.

f_9. Land use (m^2): the area occupied by the plant.

f_{10}. Freezing point: the assessment of the freezing level of the HTF. The high freezing point of the molten salt leads to complications related to freezing protection.

Seven attributes are quantitative $\{f_1, f_2, f_3, f_4, f_5, f_6, f_9\}$, while the rest three attributes $\{f_7, f_8, f_{10}\}$ are qualitative. The quantitative data is from Kearney et al. (2004), the qualitative data is from Cavallaro (2010). The level of uncertainty for the qualitative attributes is higher than that of the quantitative since quantitative attributes are easier to measure. IT2FSs are used to represent the linguistic assessment of the performance of the alternatives for the qualitative attributes and the weights of these attributes. The attributes $\{f_1, f_2, f_3, f_5, f_8, f_{10}\}$ are cost attributes, the attributes $\{f_4, f_6, f_7\}$ are benefit attributes.

Due to the different units of measurements and scales, the crisp data are normalized. Several normalization techniques can be applied. According to Brauers and Zavadskas (2006), the technique of Van Delft and Nijkamp (1977) is a robust normalization technique. Hence, this technique is chosen to transform the quantitative values of the attributes to the interval $[0,1]$. The dimensionless performance for the attributes is given by

$$x_{ij}^N = \frac{x_{ij}}{\sqrt{\sum_{j=1}^{m} x_{ij}^2}}, x_{ij} \in F_b \tag{9}$$

$$x_{ij}^N = 1 - \frac{x_{ij}}{\sqrt{\sum_{j=1}^{m} x_{ij}^2}}, x_{ij} \in F_c, \tag{10}$$

where x_{ij} represents the quantitative performance of the alternative "j" for attribute " i". This transformation change cost attributes to benefit attributes, and the similarity is measured with the positive ideal solution. The IT2FSs corresponding to the linguistic assessments are given in Table 2.

Table 2: Linguistic terms and their corresponding IT2FSs for example 2.

Linguistic terms	IT2FSs
Very Low (VL)	[(0.05,0.1,0.2,0.3;0.7,0.8)(0.0,0.1,0.2,0.35;0.9,1)]
Low (L)	[(0.25,0.3,0.4,0.5;0.7,0.8)(0.2,0.3,0.4,0.55;0.9,1)]
Medium (M)	[(0.35,0.4,0.5,0.6;0.7,0.8)(0.3,0.4,0.5,0.65;0.9,1)]
High (H)	[(0.45,0.5,0.6,0.7;0.7,0.8)(0.4,0.5,0.6,0.75;0.9,1)]
Very High (VH)	[(0.55,0.7,0.8,0.85;0.7,0.8)(0.5,0.7,0.8,1.0;0.9,1)]

The steps of interval type-2 fuzzy TOPSIS are given as follows

Step 1: Formation of the decision matrix (f_{ij}).

$$\widetilde{\mathbf{D}} = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_9 \\ f_{10} \end{array} \begin{array}{ccccccc} \mathrm{x}_1 & \mathrm{x}_2 & \mathrm{x}_3 & \mathrm{x}_4 & \mathrm{x}_5 & \mathrm{x}_6 & \mathrm{x}_7 \\ 110{,}291 & 175{,}251 & 169{,}546 & 171{,}405 & 159{,}556 & 164{,}583 & 156{,}158 \\ 3583 & \mathrm{H}4088 & 4088 & 4282 & 4282 & 4282 & 4282 \\ 139.7 & 131.5 & 128.1 & 119.9 & 113.9 & 115.1 & 111.0 \\ 0 & 5.9 & 8.3 & 14.2 & 18.5 & 17.6 & 20.6 \\ 0 & 21{,}330 & 15{,}897 & 19{,}674 & 8390 & 14{,}141 & 6117 \\ 107.5 & 169.2 & 169.1 & 183.9 & 182.9 & 185.7 & 184.4 \\ \mathrm{H} & \mathrm{H} & \mathrm{L} & \mathrm{M} & \mathrm{VL} & \mathrm{M} & \mathrm{VL} \\ \mathrm{H} & \mathrm{H} & \mathrm{H} & \mathrm{L} & \mathrm{L} & \mathrm{L} & \mathrm{L} \\ 270{,}320 & 427{,}280 & 427{,}280 & 425{,}100 & 425{,}100 & 425{,}100 & 425{,}100 \\ \mathrm{L} & \mathrm{L} & \mathrm{L} & \mathrm{H} & \mathrm{H} & \mathrm{H} & \mathrm{H} \end{array}.$$

Step 2: Formation of the weighting matrix.

$$\widetilde{\mathbf{W}} = \begin{array}{cccccccccc} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 & f_9 & f_{10} \\ (\mathrm{VH} & \mathrm{M} & \mathrm{H} & \mathrm{VH} & \mathrm{H} & \mathrm{H} & \mathrm{VH} & \mathrm{H} & \mathrm{M} & \mathrm{M}). \end{array}$$

Step 3: Normalize the quantitative data. The qualitative data does not need normalization, IT2FSs $\in [0,1]$. The normalized data is given in **Table 3.**

Table 3: Normalized quantitative data for example 2.

Attribute	x_1	x_2	x_3	x_4	x_5	x_6	x_7
Investment costs	0.7385	0.5845	0.5981	0.5937	0.6217	0.6098	0.6298
Operating and maintenance costs	0.6724	0.6262	0.6262	0.6085	0.6085	0.6085	0.6085
Levelized electricity cost	0.5712	0.5964	0.6068	0.6320	0.6504	0.6467	0.6593
Levelized electricity Cost reduction	0.0000	0.1587	0.2233	0.3820	0.4977	0.4735	0.5542
Thermal storage cost	1.0000	0.4304	0.5755	0.4747	0.7760	0.6224	0.8367
Electricity production	0.2377	0.3741	0.3739	0.4066	0.4044	0.4106	0.4077
Land Use	0.7491	0.6035	0.6035	0.6055	0.6055	0.6055	0.6055

Step 4: Formation of the weighted decision matrix

$$\widetilde{\boldsymbol{D}}_{w} = \begin{array}{c} \\ f_1 \\ \\ f_2 \\ \\ f_3 \\ \\ f_4 \\ \\ f_5 \\ \\ f_6 \\ \\ f_7 \\ \\ f_8 \\ \\ f_9 \\ \\ f_{10} \\ \\ \end{array} \begin{pmatrix} \mathrm{x}_1 & \mathrm{x}_2 & \mathrm{x}_3 \\
\begin{bmatrix} (0.41, 0.52, 0.59, 0.62; 0.7, 0.8) \\ (0.37, 0.52, 0.59, 0.74; 0.9, 1) \end{bmatrix} & \begin{bmatrix} (0.32, 0.41, 0.47, 0.50; 0.7, 0.8) \\ (0.29, 0.41, 0.47, 0.58; 0.9, 1) \end{bmatrix} & \begin{bmatrix} (0.33, 0.42, 0.48, 0.51; 0.7, 0.8) \\ (0.3, 0.42, 0.48, 0.60; 0.9, 1) \end{bmatrix} \\
\begin{bmatrix} (0.24, 0.27, 0.34, 0.40; 0.7, 0.8) \\ (0.20, 0.27, 0.34, 0.44; 0.9, 1) \end{bmatrix} & \begin{bmatrix} (0.22, 0.25, 0.31, 0.39; 0.7, 0.8) \\ (0.19, 0.25, 0.31, 0.41; 0.9, 1) \end{bmatrix} & \begin{bmatrix} (0.22, 0.25, 0.31, 0.39; 0.7, 0.8) \\ (0.9, 0.25, 0.31, 0.41; 0.9, 1) \end{bmatrix} \\
\begin{bmatrix} (0.26, 0.29, 0.34, 0.40; 0.7, 0.8) \\ (0.23, 0.29, 0.34, 0.43; 0.9, 1) \end{bmatrix} & \begin{bmatrix} (0.27, 0.30, 0.36, 0.42; 0.7, 0.8) \\ (0.24, 0.30, 0.36, 0.45; 0.9, 1) \end{bmatrix} & \begin{bmatrix} (0.27, 0.30, 0.36, 0.43; 0.7, 0.8) \\ (0.24, 0.30, 0.36, 0.46; 0.9, 1) \end{bmatrix} \\
\begin{bmatrix} (0.00, 0.00, 0.00, 0.00; 0.7, 0.8) \\ (0.00, 0.00, 0.00, 0.00; 0.9, 1) \end{bmatrix} & \begin{bmatrix} (0.09, 0.11, 0.13, 0.14; 0.7, 0.8) \\ (0.08, 0.11, 0.13, 0.16; 0.9, 1) \end{bmatrix} & \begin{bmatrix} (0.13, 0.16, 0.18, 0.19; 0.7, 0.8) \\ (0.11, 0.16, 0.18, 0.22; 0.9, 1) \end{bmatrix} \\
\begin{bmatrix} (0.45, 0.50, 0.60\ 0.70; 0.7, 0.8) \\ (0.40, 0.50, 0.60, 0.75; 0.9, 1) \end{bmatrix} & \begin{bmatrix} (0.19, 0.22, 0.26, 0.30; 0.7, 0.8) \\ (0.17, 0.22, 0.26, 0.33; 0.9, 1) \end{bmatrix} & \begin{bmatrix} (0.26, 0.29, 0.35, 0.40; 0.7\ 0.8) \\ (0.23, 0.29, 0.35, 0.43; 0.9, 1) \end{bmatrix} \\
\begin{bmatrix} (0.11, 0.12, 0.14, 0.17; 0.7, 0.8) \\ (0.10, 0.12, 0.14, 0.18; 0.9, 1) \end{bmatrix} & \begin{bmatrix} (0.17, 0.19, 0.22, 0.26; 0.7, 0.8) \\ (0.15, 0.19, 0.22, 0.28; 0.9, 1) \end{bmatrix} & \begin{bmatrix} (0.17, 0.19, 0.22, 0.26; 0.7, 0.8) \\ (0.15, 0.19, 0.22, 0.28; 0.9, 1) \end{bmatrix} \\
\begin{bmatrix} (0.25, 0.35, 0.48, 0.60; 0.7, 0.8) \\ (0.20, 0.35, 0.48, 0.75; 0.9, 1) \end{bmatrix} & \begin{bmatrix} (0.25, 0.35, 0.48, 0.60; 0.7, 0.8) \\ (0.20, 0.35, 0.48, 0.75; 0.9, 1) \end{bmatrix} & \begin{bmatrix} (0.14, 0.21, 0.32, 0.43; 0.7, 0.8) \\ (0.10, 0.32, 0.55, 0.90; 0.9, 1) \end{bmatrix} \\
\begin{bmatrix} (0.20, 0.25, 0.36, 0.49; 0.7, 0.8) \\ (0.16, 0.25, 0.36, 0.56; 0.9, 1) \end{bmatrix} & \begin{bmatrix} (0.20, 0.25, 0.36, 0.49; 0.7, 0.8) \\ (0.16, 0.25, 0.36, 0.56; 0.9, 1) \end{bmatrix} & \begin{bmatrix} (0.20, 0.25, 0.36, 0.49; 0.7, 0.8) \\ (0.16, 0.25, 0.36, 0.56; 0.9, 1) \end{bmatrix} \\
\begin{bmatrix} (0.26, 0.32, 0.37, 0.45; 0.7\ 0.8) \\ (0.22, 0.32, 0.37, 0.49; 0.9, 1) \end{bmatrix} & \begin{bmatrix} (0.21, 0.24, 0.30, 0.36; 0.7, 0.8) \\ (0.18, 0.24, 0.30, 0.39; 0.9, 1) \end{bmatrix} & \begin{bmatrix} (0.21, 0.24, 0.30, 0.36; 0.7, 0.8) \\ (0.18, 0.24, 0.30, 0.39; 0.9, 1) \end{bmatrix} \\
\begin{bmatrix} (0.09, 0.12, 0.20, 0.30; 0.7, 0.8) \\ (0.06, 0.12, 0.20, 0.36; 0.9, 1) \end{bmatrix} & \begin{bmatrix} (0.09, 0.12, 0.20, 0.30; 0.7, 0.8) \\ (0.06, 0.12, 0.20, 0.36; 0.9, 1) \end{bmatrix} & \begin{bmatrix} (0.09, 0.12, 0.20, 0.30; 0.7, 0.8) \\ (0.06, 0.12, 0.20, 0.36; 0.9, 1) \end{bmatrix}
\end{pmatrix}$$

x_4	x_5	x_6	x_7
[(0.33, 0.42, 0.48, 0.50; 0.7, 0.8) (0.30, 0.42, 0.48, 0.60; 0.9, 1)]	[(0.34, 0.44, 0.50, 0.53; 0.7, 0.8) (0.31, 0.44, 0.50, 0.62; 0.9, 1)]	[(0.34, 0.43, 0.49, 0.52; 0.7, 0.8) (0.31, 0.43, 0.49, 0.61; 0.9, 1)]	[(0.35, 0.44, 0.50, 0.54; 0.7, 0.8) (0.31, 0.44, 0.50, 0.63; 0.9, 1)]
[(0.21, 0.24, 0.30, 0.37; 0.7, 0.8) (0.18, 0.24, 0.30, 0.4; 0.9, 1)]	[(0.21, 0.24, 0.30, 0.37; 0.7, 0.8) (0.18, 0.24, 0.30, 0.40; 0.9, 1)]	[(0.21, 0.24, 0.30, 0.37; 0.7, 0.8) (0.18, 0.24, 0.30, 0.40; 0.9, 1)]	[(0.21, 0.24, 0.30, 0.37; 0.7, 0.8) (0.18, 0.24, 0.30, 0.4; 0.9, 1)]
[(0.28, 0.32, 0.38, 0.44; 0.7, 0.8) (0.25, 0.32, 0.38, 0.47; 0.9, 1)]	[(0.29, 0.33, 0.39, 0.46; 0.7, 0.8) (0.26, 0.33, 0.39, 0.49; 0.9, 1)]	[(0.29, 0.32, 0.39, 0.46; 0.7, 0.8) (0.26, 0.32, 0.39, 0.49; 0.9 1)]	[(0.30, 0.33, 0.40, 0.46; 0.7, 0.8) (0.26, 0.33, 0.40, 0.50; 0.9, 1)]
[(0.21, 0.27, 0.31, 0.32; 0.7, 0.8) (0.19, 0.27, 0.31, 0.38; 0.9, 1)]	[(0.27, 0.35, 0.40, 0.42; 0.7, 0.8) (0.25, 0.35, 0.40, 0.50; 0.9, 1)]	[(0.26, 0.33, 0.38, 0.40; 0.7, 0.8) (0.24, 0.33, 0.38, 0.47; 0.9, 1)]	[(0.30, 0.39, 0.44, 0.47; 0.7, 0.8) (0.28, 0.39, 0.44, 0.55; 0.9, 1)]
[(0.21, 0.24, 0.28, 0.33; 0.7, 0.8) (0.19, 0.24, 0.28, 0.36; 0.9, 1)]	[(0.35, 0.39, 0.47, 0.54; 0.7, 0.8) (0.31,0.39,0.47,0.58; 0.9,1)]	[(0.28, 0.31, 0.37, 0.44; 0.7, 0.8) (0.25, 0.31, 0.37, 0.47; 0.9, 1)]	[(0.38, 0.42, 0.50, 0.59; 0.7, 0.8) (0.33, 0.42, 0.50, 0.63; 0.9, 1)]
[(0.18, 0.20, 0.24, 0.28; 0.7, 0.8) (0.16, 0.20, 0.24, 0.31; 0.9, 1)]	[(0.18, 0.20, 0.24, 0.28; 0.7,0.8) (0.16, 0.20, 0.24, 0.30; 0.9, 1)]	[(0.18, 0.21, 0.25, 0.29; 0.7 0.8) (0.16, 0.21, 0.25, 0.31; 0.9, 1)]	[(0.18, 0.20, 0.24, 0.29; 0.7 0.8) (0.16, 0.20, 0.24, 0.31; 0.9, 1)]
[(0.19, 0.28, 0.40, 0.51; 0.7,0.8) (0.15, 0.28, 0.40, 0.65; 0.9,1)]	[(0.03, 0.07, 0.16, 0.26; 0.7,0.8) (0.00, 0.07, 0.16, 0.35; 0.9,1)]	[(0.19, 0.28, 0.40, 0.51; 0.7,0.8) (0.15, 0.28, 0.40, 0.65; 0.9,1)]	[(0.03, 0.07, 0.16, 0.26; 0.7,0.8) (0.00, 0.07, 0.16, 0.35; 0.9,1)]
[(0.11,0.15, 0.24, 0.35; 0.7, 0.8) (0.08, 0.15, 0.24, 0.41; 0.9,1)]	[(0.11,0.15, 0.24, 0.35; 0.7, 0.8) (0.08, 0.15, 0.24, 0.41; 0.9,1)]	[(0.11,0.15, 0.24, 0.35; 0.7, 0.8) (0.08, 0.15, 0.24, 0.41; 0.9,1)]	[(0.11,0.15, 0.24, 0.35; 0.7, 0.8) (0.08, 0.15, 0.24, 0.41; 0.9,1)]
[(0.21, 0.24, 0.30, 0.36; 0.7, 0.8) (0.18, 0.24, 0.30, 0.39; 0.9, 1)]	[(0.21, 0.24, 0.30, 0.36; 0.7, 0.8) (0.18, 0.24, 0.30, 0.39; 0.9, 1)]	[(0.21, 0.24, 0.30, 0.36; 0.7, 0.8) (0.18, 0.24, 0.30, 0.39; 0.9, 1)]	[(0.21, 0.24, 0.30, 0.36; 0.7, 0.8) (0.18, 0.24, 0.30, 0.39; 0.9, 1)]
[(0.16, 0.20, 0.30, 0.42; 0.7,0.8) (0.12, 0.20, 0.30, 0.49; 0.9,1)]	[(0.16, 0.20, 0.30, 0.42; 0.7,0.8) (0.12, 0.20, 0.30, 0.49; 0.9,1)]	[(0.16, 0.20, 0.30, 0.42; 0.7,0.8) (0.12, 0.20, 0.30, 0.49; 0.9,1)]	[(0.16, 0.20, 0.30, 0.42; 0.7,0.8) (0.12, 0.20, 0.30, 0.49; 0.9,1)]

Step 5: Set $\tilde{v}^+ = [(1,1,1,1;1,1)(1,1,1,1;1,1)]$ and $\tilde{v}^- = [(0,0,0,0;1,1)(0,0,0,0;1,1)]$.

Step 6: Formation of the similarity matrix (S_{ij}).

$$\mathbf{S} = \begin{matrix} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_9 \\ f_{10} \end{matrix} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ S_{11}^+ & S_{12}^+ & S_{13}^+ & S_{14}^+ & S_{15}^+ & S_{16}^+ & S_{17}^+ \\ S_{21}^+ & S_{22}^+ & S_{23}^+ & S_{24}^+ & S_{25}^+ & S_{26}^+ & S_{27}^+ \\ S_{31}^+ & S_{32}^+ & S_{33}^+ & S_{34}^+ & S_{35}^+ & S_{36}^+ & S_{37}^+ \\ S_{41}^+ & S_{42}^+ & S_{43}^+ & S_{44}^+ & S_{45}^+ & S_{46}^+ & S_{47}^+ \\ S_{51}^+ & S_{52}^+ & S_{53}^+ & S_{54}^+ & S_{55}^+ & S_{56}^+ & S_{57}^+ \\ S_{61}^+ & S_{62}^+ & S_{63}^+ & S_{44}^+ & S_{65}^+ & S_{66}^+ & S_{67}^+ \\ S_{71}^+ & S_{72}^+ & S_{73}^+ & S_{74}^+ & S_{75}^+ & S_{76}^+ & S_{77}^+ \\ S_{81}^- & S_{82}^- & S_{83}^- & S_{84}^- & S_{85}^- & S_{86}^- & S_{87}^- \\ S_{91}^+ & S_{92}^+ & S_{93}^+ & S_{94}^+ & S_{95}^+ & S_{96}^+ & S_{97}^+ \\ S_{01}^- & S_{02}^- & S_{03}^- & S_{04}^- & S_{05}^- & S_{06}^- & S_{07}^- \end{matrix}$$

$$\mathbf{S} = \begin{matrix} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_9 \\ f_{10} \end{matrix} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0.2546 & 0.2257 & 0.2287 & 0.2287 & 0.238 & 0.2337 & 0.2354 \\ 0.1783 & 0.1548 & 0.1548 & 0.1663 & 0.1663 & 0.1663 & 0.1663 \\ 0.1936 & 0.1998 & 0.2022 & 0.2081 & 0.2123 & 0.2115 & 0.2143 \\ 0.0000 & 0.0847 & 0.1127 & 0.1706 & 0.2044 & 0.1979 & 0.2186 \\ 0.2773 & 0.1562 & 0.1964 & 0.1685 & 0.2389 & 0.2059 & 0.2505 \\ 0.0956 & 0.1397 & 0.1396 & 0.1493 & 0.1487 & 0.1505 & 0.1496 \\ 0.1592 & 0.1592 & 0.1209 & 0.1419 & 0.0669 & 0.1419 & 0.0669 \\ 0.3029 & 0.3029 & 0.3029 & 0.3857 & 0.3857 & 0.3857 & 0.3857 \\ 0.1917 & 0.1654 & 0.1654 & 0.1657 & 0.1657 & 0.1657 & 0.1657 \\ 0.4219 & 0.4219 & 0.4219 & 0.3430 & 0.3430 & 0.3430 & 0.3430 \end{matrix}.$$

Step 7: Find the total degree of similarity of each alternative to the ideal solution.

$S(x_1) = 2.0759$, $S(x_2) = 2.0250$, $S(x_3) = 2.0585$, $S(x_4) = 2.1269$, $S(x_5) = 2.1657$, $S(x_6) = 2.1996$, and $S(x_7) = 2.1961$.

The results show that the ranking is as follows

$\mathrm{MS-TT\ 500 > MS-TC\ 500 > MS-TC\ 450 > MS-TT\ 450 > VP-1NS > VP-1TC > VP-1TT}$.

Then, the best TES is $\mathrm{MS-TT\ 500}$.

The final ranking according to Cavallaro (2010) is
$\mathrm{MS-TT\ 500 > MS-TC\ 500 > VP-1NS > MS-TC\ 450 > MS-TT\ 450 > VP-1TT > VP-1TC}$.

Comparing the results in both techniques, the best TES is $\mathrm{MS-TT\ 500}$, molten salt for heat transfer and heat storage with two tanks with a maximum temperature of 500˚C, followed by $\mathrm{MS-TC\ 500}$, molten salt placed in a thermocline tank with a maximum temperature of 500˚C. The worst TES are $\mathrm{VP-1TC}$ and $\mathrm{VP-1TT}$ in both techniques but in a different order. Regarding the moderately-performing TES, despite $\mathrm{MS-TC\ 450}$ is better than $\mathrm{MS-TT\ 450}$ in the two methods, their position to $\mathrm{VP-1NS}$ is different. While $\mathrm{VPI-NS}$ is worse than $\mathrm{MS-TC\ 450}$ and $\mathrm{MS-TT\ 450}$ in the proposed technique, it's better than them in Cavallaro (2010).

The proposed technique is implemented by normalizing the crisp quantitative variables first, then multiplying them by fuzzy weights. The other alternative is to multiply by fuzzy weights then normalize. The proposed technique is resolved by multiplying by fuzzy weights then normalizing to investigate which normalization method is more appropriate.

For a weighted performance rating of an alternative "j" for the " i^{th}" attribute, the formula given in **Step 3** is used. It turned out that this implementation fails in solving this problem since the rating of the first alternative for the fifth attribute "thermal storage cost" is "zero" which results in a division by zero. Therefore, it can be concluded that when handling crisp quantitative data in a fuzzy environment, normalization of the crisp ratings of the alternatives for the attributes should be carried out before forming the weighted decision matrix to avoid division by zero in case of a zero rating.

In this implementation, theoretical reference points were utilized. Alternatively, the empirical reference points could be utilized. Hence, the proposed technique is resolved using the empirical reference point which can be obtained as follows.

$$v_B^+ = v_C^- = \begin{bmatrix} \left(\max_j x_{1ij}^L, \max_j x_{2ij}^L, \max_j x_{3ij}^L; \max_j x_{4ij}^L; \max_j w_{1ij}^L, \max_j w_{2ij}^L\right) \\ \left(\max_j x_{1ij}^U, \max_j x_{2ij}^U, \max_j x_{3ij}^U; \max_j x_{4ij}^U; \max_j w_{1ij}^U, \max_j w_{2ij}^U\right) \end{bmatrix},$$

$$v_B^- = v_C^+ \begin{bmatrix} \left(\min_j x_{1ij}^L, \min_j x_{2ij}^L, \min_j x_{3ij}^L; \min_j x_{4ij}^L; \min_j w_{1ij}^L, \min_j w_{2ij}^L\right) \\ \left(\min_j x_{1ij}^U, \min_j x_{2ij}^U, \min_j x_{3ij}^U; \min_j x_{4ij}^U; \min_j w_{1ij}^U, \min_j w_{2ij}^U\right) \end{bmatrix}, \text{where } j = 1, \dots, n.$$

The results using empirical reference point are

$$S(x_1) = 7.1191, S(x_2) = 6.2927, S(x_3) = 6.1319, S(x_4) = 6.4291, S(x_5) = 6.5599, S(x_6) = 6.7244, \text{and } S(x_7) = 6.8262.$$

Then, the ranking is

$$\mathrm{VP-1NS > MS-TC\ 500 > MS-TT500 > MS-TC\ 450 > MS-TT\ 450 > VP-1TT > VP-1TC}.$$

This ranking is quite different from the previous. $\mathrm{VP-1NS}$ occupies first place, $\mathrm{MS-TT\ 500}$ recedes to third place, $\mathrm{MS-TC\ 500}$ retains second place. The rank of TES using molten salt at 450 ºC is the same with respect to each other while $\mathrm{VP-1TT}$ and $\mathrm{VP-1TC}$ are still the worst.

Using the empirical reference points gives VP-1NS the priority in six attributes out of ten being the reference point itself. For example, for the seventh attribute "state of knowledge of technology", the difference in the similarity measures of VP-1NS and $\mathrm{MS-TT\ 500}$ using the theoretical reference point is $\Delta S = 0.0923$, which increases to $\Delta S = 0.5347$ when using the empirical reference point. Then, using the empirical reference points is more discriminating since it increases the difference in the ranking metric of the alternatives. On the other hand, when using the theoretical reference point the alternatives are equally treated being measured from the same reference point. Therefore, further investigations are required on the impact of using theoretical and empirical reference points on the solution of reference point techniques, e.g. TOPSIS and VIKOR, when using IT2FSs.

From the previous, we have two candidates $\mathrm{VP-1NS}$ and $\mathrm{MS-TT\ 500}$ the alternatives in first place using the theoretical and empirical reference points. Taking a closer look at the two alternatives and the evaluation attributes for further investigations. It is observed that $\mathrm{VP-1NS}$ utilize Therminol VP-1 as a HTF with no heat storage. Meanwhile, $\mathrm{MS-TT\ 500}$ utilize molten salts for dual functions, i.e. heat transfer and heat storage. Hence, the two alternatives are examined using the proposed TOPSIS after eliminating the fifth attribute "thermal storage cost" which gives $\mathrm{VP-1NS}$ an advantage over $\mathrm{MS-TT\ 500}$ when using the empirical reference point. The results revealed that $\mathrm{MS-TT\ 500}$ is ranked first employing the theoretical and the empirical reference point. It can be concluded that $\mathrm{VP}-1\mathrm{NS}$ is ranked first using the empirical reference point due to the lack of one feature "thermal energy storage" since it has a very high vapor pressure (> MPa at 400 ºC) which makes it practically difficult to store in any significant quantity. Accordingly, $\mathrm{MS}-\mathrm{TT}\ 500$ is the best option that combines heat transfer and heat storage.

In this particular example, the alternatives are close in the ranking metric. In Cavallaro (2010) the difference between the closeness coefficients of the first and final rank is 0.095. In the proposed TOPSIS the difference between the total degrees of similarity of the first and final rank is 0.1746. Here, qualitative attributes play a crucial role since they might be the decisive attributes in ranking. The assessment of the experts involved in the decision-making process and their weights is a vital process that must be handled carefully.

6. Conclusion and discussion

In this article, TOPSIS using similarity measure was extended to encompass IT2FSs. First, the similarity measure based on map distance for IVFSs was extended to IT2FSs. Then, the solution steps follow. The similarity matrix is formed and the alternative corresponding to the one norm of the similarity matrix is the best choice. Consequently, the flexibility and effectiveness of handling MAGDM problems are increased, and defuzzification with its flaws is avoided. Two examples were given to illustrate the MAGDM process of the proposed method. In the two examples, linguistic variables are expressed as IT2FSs. In the first example, the data are all qualitative. The result of the proposed TOPSIS coincides with the results from which the example is adopted. In the second example, the weights of the attributes are qualitative, while the ratings of the attributes are both crisp quantitative and qualitative. The top-ranked TES coincides with that of Cavallaro (2010) with slight differences in the ranking of the other alternatives.

The study investigated two important aspects in the implementation of TOPSIS using IT2FSs, the normalization of the crisp qualitative variables and the ideal solutions utilized. It turned out that normalization of the crisp ratings of the alternatives for the attributes should be carried out before forming the weighted decision matrix to avoid division by zero in in case of a zero rating. Regarding the ideal solutions, the results revealed that the solution might be affected by the utilized ideal solutions, which is a limitation of the method. Using the empirical reference points is more discriminating since it increases the difference between the ranking metric of the alternatives. Meanwhile, the theoretical reference points treat the alternative equally being measured from the same reference point. Despite the result of the theoretical reference points appears to be more reliable, still further investigations are required on the impact of using theoretical and empirical reference points on the solution of reference point techniques, e.g. TOPSIS and VIKOR, using IT2FSs.

The study also revealed that when the alternatives are close in the quantitative ratings, the qualitative ratings play a crucial role since they might be the decisive attributes in ranking. The selection of the experts involved in the decision making process and their weights is a vital process that must be handled carefully.

From the previous, future research would study the effect of theoretical and empirical reference points on the solution of reference point techniques using IT2FSs and which one is more reliable. Similar studies can be performed for other types of fuzzy sets, e.g. intuitionistic (IFSs) and Pythagorean fuzzy sets (PFSs).

Finally, Cavallaro et al. (2019) evaluated CSP using TOPSIS in an intuitionistic fuzzy environment. In the study, all the attributes including investment costs, levelized cost of energy, potential reduction of costs, and land use were evaluated using linguistic terms. Future research in evaluating CSP would focus on the following aspects.

- Which is more accurate, evaluating all the attributes using qualitative data or using both qualitative and fuzzy quantitative data?
- Which is more reliable in defining qualitative data: IT2FSs, IFSs, or both depending on the nature of attributes, e.g. IT2FSs for costs and IFSs for environmental risk?
- In the case of different types of fuzzy data, to what extent information fusion can be effective and successful?

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Appendix

The proposed similarity measure satisfies the following properties.

Let $\tilde{A}$ and $\tilde{B}$ be IT2FSs given by $\tilde{A} = [(a_1^L, a_2^L, a_3^L, a_4^L; w_{1\tilde{A}}^L, w_{2\tilde{A}}^L)(a_1^U, a_2^U, a_3^U, a_4^U; w_{1\tilde{A}}^U, w_{2\tilde{A}}^U)]$ and $\tilde{B} = [(b_1^L, b_2^L, b_3^L, b_4^L; w_{1\tilde{B}}^L, w_{2\tilde{B}}^L)(b_1^U, b_2^U, b_3^U, b_4^U; w_{1\tilde{B}}^U, w_{2\tilde{B}}^U)]$

Property 1(Reflexivity) . $\tilde{A}$ and $\tilde{B}$ are identical if and only if $S(\tilde{A}, \tilde{B}) = 1$.

Proof. (i) Let $\tilde{A}$ and $\tilde{B}$ be identical, then $a_1^L = b_1^L, a_2^L = b_2^L, a_3^L = b_3^L, a_4^L = b_4^L, a_1^U = b_1^U, a_2^U = b_2^U, a_3^U = b_3^U, a_4^U = b_4^U, w_{1\tilde{A}}^L = w_{1\tilde{B}}^L, w_{2B}^L = w_{2\tilde{A}}^L, w_{1\tilde{A}}^U = w_{1B}^U \; and \; w_{2\tilde{A}}^U = w_{2B}^U$.

Step 1: Calculating the distance values Δa_i and Δb_i.

$\Delta a_i = |a_i^U - a_i^L|$ and $\Delta b_i = |b_i^U - b_i^L|$,where $i = 1,2,3,4$.

Since $a_i^U = b_i^U$, and $a_i^L = b_i^L$, then $\Delta a_i = \Delta b_i$.

Step 2: Calculating the degree of similarity $S(\tilde{A}^\Delta, \tilde{B}^\Delta)$.

a) Calculating the standard deviations ΔS_a and ΔS_b .

$\bar{a}^U = (a_1^U + a_2^U + a_3^U + a_4^U)/4 = (b_1^U + b_2^U + b_3^U + b_4^U)/4 = \bar{b}^U$,

$\bar{a}^L = (a_1^L + a_2^L + a_3^L + a_4^L)/4 = (b_1^L + b_2^L + b_3^L + b_4^L)/4 = \bar{b}^L$,

$$S_{\tilde{A}^U} = \sqrt{\frac{\sum_{i=1}^4 (a_i^U - \bar{a}^U)^2}{3}} = \sqrt{\frac{\sum_{i=1}^4 (b_i^U - \bar{b}^U)^2}{3}} = S_{\tilde{B}^U}, \quad S_{\tilde{A}^L} = \sqrt{\frac{\sum_{i=1}^4 (a_i^L - \bar{a}^L)^2}{3}} = \sqrt{\frac{\sum_{i=1}^4 (b_i^L - \bar{b}^L)^2}{3}} = S_{\tilde{B}^L}.$$

$\Delta S_a = |S_{\tilde{A}^U} - S_{\tilde{A}^L}| = |S_{\tilde{B}^U} - S_{B^L}| = \Delta S_b$.

b) Calculating the map distance between the upper and lower fuzzy sets.

$$T^\Delta = \left[\left(2 - \frac{1+\max\{|\Delta a_2 - \Delta a_1|, |\Delta b_2 - \Delta b_1|\}}{1+\min\{|\Delta a_2 - \Delta a_1|, |\Delta b_2 - \Delta b_1|\}}\right) + \left(2 - \frac{1+\max\{|\Delta a_4 - \Delta a_3|, |\Delta b_4 - \Delta b_3|\}}{1+\min\{|\Delta a_4 - \Delta a_3|, |\Delta b_4 - \Delta b_3|\}}\right)\right]/2.$$

Since $|\Delta a_2 - \Delta a_1| = |\Delta b_2 - \Delta b_1|$ and $|\Delta a_4 - \Delta a_3| = |\Delta b_4 - \Delta b_3|$,

Then $\frac{1+\max\{|\Delta a_2 - \Delta a_1|, |\Delta b_2 - \Delta b_1|\}}{1+\min\{|\Delta a_2 - \Delta a_1|, |\Delta b_2 - \Delta b_1|\}} = 1$ and $\frac{1+\max\{|\Delta a_4 - \Delta a_3|, |\Delta b_4 - \Delta b_3|\}}{1+\min\{|\Delta a_4 - \Delta a_3|, |\Delta b_4 - \Delta b_3|\}} = 1$.

$$T^\Delta = [(2-1) + (2-1)]/2 = 1.$$

c) Calculating $S(\tilde{A}^\Delta, \tilde{B}^\Delta)$.

$$S(\tilde{A}^\Delta, \tilde{B}^\Delta) = \left[1 - \frac{\sqrt{\sum_{i=1}^4 (\Delta a_i - \Delta b_i)^2}}{2}\right] \times \left[1 - \sqrt{\frac{|\Delta S_a - \Delta S_b|}{2}}\right] \times \left[1 - \frac{|w_{1\tilde{A}}^L - w_{1\tilde{B}}^L| + |w_{2\tilde{A}}^L - w_{2\tilde{B}}^L|}{|w_{1\tilde{A}}^U + w_{1\tilde{B}}^U| + |w_{2\tilde{A}}^U + w_{2\tilde{B}}^U|}\right] \times T^\Delta,$$

$$S(\tilde{A}^\Delta, \tilde{B}^\Delta) = \left[1 - \frac{0}{2}\right] \times \left[1 - \sqrt{\frac{0}{2}}\right] \times \left[1 - \frac{0}{2|w_{1\tilde{A}}^U| + 2|w_{2\tilde{A}}^U|}\right] \times 1 = 1.$$

Step 3: Calculating $S(\tilde{A}^U, \tilde{B}^U)$.

a) Calculating the map distance between the upper trapezoidal fuzzy sets.

Since $|a_2^u - a_1^u| = |b_2^u - b_1^u|$ and $|a_4^u - a_3^u| = |b_4^u - b_3^u|$,

then $\max\{|a_2^u - a_1^u|, |b_2^u - b_1^u|\} = \min\{|a_2^u - a_1^u|, |b_2^u - b_1^u|\}$ and

$\max\{|a_4^u - a_3^u|, |b_4^u - b_3^u|\} = \min\{|a_4^u - a_3^u|, |b_4^u - b_3^u|\}$.

$$T^U = \left[\left(2 - \frac{1+\max\{|a_2^u - a_1^u|, |b_2^u - b_1^u|\}}{1+\min\{|a_2^u - a_1^u|, |b_2^u - b_1^u|\}}\right) + \left(2 - \frac{1+\max\{|a_4^u - a_3^u|, |b_4^u - b_3^u|\}}{1+\min\{|a_4^u - a_3^u|, |b_4^u - b_3^u|\}}\right)\right]/2 = [(2-1) + (2-1)]/2 = 1$$

.

b) Calculating the degree of similarity $S(\tilde{A}^U, \tilde{B}^U)$.

Since $w_{1\tilde{A}}^U = w_{1\tilde{B}}^U$ and $w_{2\tilde{A}}^U = w_{2\tilde{B}}^U$, then $\min(w_{1\tilde{A}}^U, w_{1\tilde{B}}^U) = \max(w_{1\tilde{A}}^U, w_{1\tilde{B}}^U)$ and $\min(w_{2\tilde{A}}^U, w_{2\tilde{B}}^U) = \max(w_{2\tilde{A}}^U, w_{2\tilde{B}}^U)$.

$$S(\tilde{A}^U, \tilde{B}^U) = \left[1 - \frac{\sqrt{\sum_{i=1}^4 (a_i^u - b_i^u)^2}}{2}\right] \times \left[1 - \sqrt{\frac{|S_{\tilde{A}^U} - S_{\tilde{B}^U}|}{2}}\right] \times \left[\frac{\min(w_{1\tilde{A}}^U, w_{1\tilde{B}}^U) + \min(w_{2\tilde{A}}^U, w_{2\tilde{B}}^U)}{\max(w_{1\tilde{A}}^U, w_{1\tilde{B}}^U) + \max(w_{2\tilde{A}}^U, w_{2\tilde{B}}^U)}\right] \times T^U$$

$$= \left[1 - \frac{0}{2}\right] \times \left[1 - \sqrt{\frac{0}{2}}\right] \times \left[\frac{w_{1\tilde{A}}^U + w_{2\tilde{A}}^U}{w_{1\tilde{A}}^U + w_{2\tilde{A}}^U}\right] \times 1 = 1.$$

Step 4: Calculating the degree of similarity $S(\tilde{A}, \tilde{B})$.

$$S(\tilde{A},\tilde{B})=\frac{S(\tilde{A}^U,\tilde{B}^U)\times\left(1+S(\tilde{A}^\Delta,\tilde{B}^\Delta)\right)}{2}=\frac{1\times(1+1)}{2}=1. \quad \square$$

(ii) Let $S(\tilde{A},\tilde{B})=1$.

Since $S(\tilde{A}^\Delta,\tilde{B}^\Delta)\in[0,1]$ and $S(\tilde{A}^U,\tilde{B}^U)\in[0,1]$,
then $S(\tilde{A},\tilde{B})=1$ implies that $S(\tilde{A}^U,\tilde{B}^U)=1$ and $S(\tilde{A}^\Delta,\tilde{B}^\Delta)=1$.
$S(\tilde{A}^U,\tilde{B}^U)=1$,

$$\left[1-\frac{\sqrt{\sum_{i=1}^4(a_i^u-b_i^u)^2}}{2}\right]\times\left[1-\sqrt{\frac{\left|S_{\tilde{A}^U}-S_{\tilde{B}^U}\right|}{2}}\right]\times\left[\frac{\min\left(w_{1\tilde{A}}^U,w_{1\tilde{B}}^U\right)+\min\left(w_{2\tilde{A}}^U,w_{2\tilde{B}}^U\right)}{\max\left(w_{1\tilde{A}}^U,w_{1\tilde{B}}^U\right)+\max\left(w_{2\tilde{A}}^U,w_{2\tilde{B}}^U\right)}\right]\times T^U=1.$$

Since the product of the four terms is equal to one, and each term has a maximum value of one, then each term of the four terms cannot be less than one, i.e. must be equal to one.

Then, $\sqrt{\sum_{i=1}^4(a_i^u-b_i^u)^2}=0\Rightarrow a_i^u=b_i^u$, $S_{\tilde{A}^U}=S_{\tilde{B}^U}$ *and* $T^U=1$.

$$\frac{\min\left(w_{1\tilde{A}}^U,w_{1\tilde{B}}^U\right)+\min\left(w_{2\tilde{A}}^U,w_{2\tilde{B}}^U\right)}{\max\left(w_{1\tilde{A}}^U,w_{1\tilde{B}}^U\right)+\max\left(w_{2\tilde{A}}^U,w_{2\tilde{B}}^U\right)}=1\Rightarrow w_{1\tilde{A}}^U=w_{1\tilde{B}}^U \text{ and } w_{2\tilde{A}}^U=w_{2\tilde{B}}^U.$$

Then, $\tilde{A}^U$ *and* $\tilde{B}^U$ are identical.

$$S(\tilde{A}^\Delta,\tilde{B}^\Delta)=\left[1-\frac{\sqrt{\sum_{i=1}^4(\Delta a_i-\Delta b_i)^2}}{2}\right]\times\left[1-\sqrt{\frac{|\Delta S_a-\Delta S_b|}{2}}\right]\times\left[1-\frac{\left|w_{1\tilde{A}}^L-w_{1\tilde{B}}^L\right|+\left|w_{2\tilde{A}}^L-w_{2\tilde{B}}^L\right|}{\left|w_{1\tilde{A}}^U+w_{1\tilde{B}}^U\right|+\left|w_{2\tilde{A}}^U+w_{2\tilde{B}}^U\right|}\right]\times T^\Delta=1,$$

Since the product of the four terms is equal to one, and each term has a maximum value of one, then each term of the four terms must be equal to one.

$\sqrt{\sum_{i=1}^4(\Delta a_i-\Delta b_i)^2}=0\Rightarrow\Delta a_i=\Delta b_i\Rightarrow\left|a_i^U-a_i^L\right|=\left|b_i^U-b_i^L\right|$,
since $a_i^U=b_i^U$, then $a_i^L=b_i^L$, and $\Delta S_a=\Delta S_b$ and $T^\Delta=1$ directly follows.

$$\frac{\left|w_{1\tilde{A}}^L-w_{1\tilde{B}}^L\right|+\left|w_{2\tilde{A}}^L-w_{2\tilde{B}}^L\right|}{\left|w_{1\tilde{A}}^U+w_{1\tilde{B}}^U\right|+\left|w_{2\tilde{A}}^U+w_{2\tilde{B}}^U\right|}=0\Rightarrow\left|w_{1\tilde{A}}^L-w_{1\tilde{B}}^L\right|+\left|w_{2\tilde{A}}^L-w_{2\tilde{B}}^L\right|=0\Rightarrow w_{1\tilde{A}}^L=w_{1\tilde{B}}^L \text{ and } w_{2\tilde{A}}^L=w_{2\tilde{B}}^L.$$

Since $a_i^L=b_i^L$ and $w_{1\tilde{A}}^L=w_{1\tilde{B}}^L$ *and* $w_{2\tilde{A}}^L=w_{2\tilde{B}}^L$, then $\tilde{A}^L$ *and* $\tilde{B}^L$ are identical.
Since $\tilde{A}^U$ *and* $\tilde{B}^U$ are identical and $\tilde{A}^L$ *and* $\tilde{B}^L$ are identical, then $\tilde{A}$ and $\tilde{B}$ are identical. ■

Property 2 (Symmetry). $S(\tilde{A},\tilde{B})=S(\tilde{B},\tilde{A})$.

Proof.

Since $(\Delta a_i-\Delta b_i)^2=(\Delta b_i-\Delta a_i)^2$, $|\Delta S_a-\Delta S_b|=|\Delta S_b-\Delta S_a|$, $\left|w_{1\tilde{A}}^L-w_{1\tilde{B}}^L\right|=\left|w_{1\tilde{B}}^L-w_{1\tilde{A}}^L\right|$,
$\left|w_{2\tilde{A}}^L-w_{2\tilde{B}}^L\right|=\left|w_{2\tilde{B}}^L-w_{2\tilde{A}}^L\right|$, $\left|w_{1\tilde{A}}^U+w_{1\tilde{B}}^U\right|=\left|w_{1\tilde{B}}^U+w_{1\tilde{A}}^U\right|$, $\left|w_{2\tilde{A}}^U+w_{2\tilde{B}}^U\right|=\left|w_{2\tilde{B}}^U+w_{2\tilde{A}}^U\right|$,
$\max\{|\Delta a_2-\Delta a_1|,|\Delta b_2-\Delta b_1|\}=\max\{|\Delta b_2-\Delta b_1|,|\Delta a_2-\Delta a_1|\}$,
$\min\{|\Delta a_2-\Delta a_1|,|\Delta b_2-\Delta b_1|\}=\min\{|\Delta b_2-\Delta b_1|,|\Delta a_2-\Delta a_1|\}$,
$\max\{|\Delta a_4-\Delta a_3|,|\Delta b_4-\Delta b_3|\}=\max\{|\Delta b_4-\Delta b_3|,|\Delta a_4-\Delta a_3|\}$,
$\min\{|\Delta a_4-\Delta a_3|,|\Delta b_4-\Delta b_3|\}=\min\{|\Delta b_4-\Delta b_3|,|\Delta a_4-\Delta a_3|\}$,
then,

$$S(\tilde{A}^\Delta,\tilde{B}^\Delta)=\left[1-\frac{\sqrt{\sum_{i=1}^4(\Delta a_i-\Delta b_i)^2}}{2}\right]\times\left[1-\sqrt{\frac{|\Delta S_a-\Delta S_b|}{2}}\right]\times\left[1-\frac{\left|w_{1\tilde{A}}^L-w_{1\tilde{B}}^L\right|+\left|w_{2\tilde{A}}^L-w_{2\tilde{B}}^L\right|}{\left|w_{1\tilde{A}}^U+w_{1\tilde{B}}^U\right|+\left|w_{2\tilde{A}}^U+w_{2\tilde{B}}^U\right|}\right]\times T^\Delta$$

$$=\left[1-\frac{\sqrt{\sum_{i=1}^4(\Delta b_i-\Delta a_i)^2}}{2}\right]\times\left[1-\sqrt{\frac{|\Delta S_b-\Delta S_a|}{2}}\right]\times\left[1-\frac{\left|w_{1\tilde{B}}^L-w_{1\tilde{A}}^L\right|+\left|w_{2\tilde{B}}^L-w_{2A}^L\right|}{\left|w_{1\tilde{B}}^U+w_{1\tilde{A}}^U\right|+\left|w_{2\tilde{B}}^U+w_{2\tilde{A}}^U\right|}\right]\times T^\Delta=S(\tilde{B}^\Delta,\tilde{A}^\Delta).$$

Since $(a_i^u-b_i^u)^2=(b_i^u-a_i^u)^2$, $\left|S_{\tilde{A}^U}-S_{\tilde{B}^U}\right|=\left|S_{\tilde{B}^U}-S_{\tilde{A}^U}\right|$,
$\min\left(w_{1\tilde{A}}^U,w_{1\tilde{B}}^U\right)=\min\left(w_{1\tilde{B}}^U,w_{1\tilde{A}}^U\right)$, $\min\left(w_{2\tilde{A}}^U,w_{2\tilde{B}}^U\right)=\min\left(w_{2\tilde{B}}^U,w_{2\tilde{A}}^U\right)$,
$\max\left(w_{1\tilde{A}}^U,w_{1\tilde{B}}^U\right)=\max\left(w_{1\tilde{B}}^U,w_{1\tilde{A}}^U\right)$, $\max\left(w_{2\tilde{A}}^U,w_{2\tilde{B}}^U\right)=\max\left(w_{2\tilde{B}}^U,w_{2\tilde{A}}^U\right)$,
$\max\{|a_2^u-a_1^u|,|b_2^u-b_1^u|\}=\max\{|b_2^u-b_1^u|,|a_2^u-a_1^u|\}$,
$\min\{|a_2^u-a_1^u|,|b_2^u-b_1^u|\}=\min\{|b_2^u-b_1^u|,|a_2^u-a_1^u|\}$,
$\max\{|a_4^u-a_3^u|,|b_4^u-b_3^u|\}=\max\{|b_4^u-b_3^u|,|a_4^u-a_3^u|\}$,
$\min\{|a_4^u-a_3^u|,|b_4^u-b_3^u|\}=\min\{|b_4^u-b_3^u|,|a_4^u-a_3^u|\}$, then

$$S(\tilde{A}^U,\tilde{B}^U)=\left[1-\frac{\sqrt{\sum_{i=1}^4(a_i^u-b_i^u)^2}}{2}\right]\times\left[1-\sqrt{\frac{|S_{\tilde{A}^U}-S_{\tilde{B}^U}|}{2}}\right]\times\left[\frac{\min(w_{1\tilde{A}}^U,w_{1\tilde{B}}^U)+\min(w_{2\tilde{A}}^U,w_{2\tilde{B}}^U)}{\max(w_{1\tilde{A}}^U,w_{1\tilde{B}}^U)+\max(w_{2\tilde{A}}^U,w_{2\tilde{B}}^U)}\right]\times T^U$$

$$=\left[1-\frac{\sqrt{\sum_{i=1}^4(b_i^u-a_i^u)^2}}{2}\right]\times\left[1-\sqrt{\frac{|S_B-S_{A^U}|}{2}}\right]\times\left[\frac{\min(w_{1\tilde{B}}^U,w_{1\tilde{A}}^U)+\min(w_{2\tilde{B}}^U,w_{2\tilde{A}}^U)}{\max(w_{1\tilde{B}}^U,w_{1\tilde{A}}^U)+\max(w_{2\tilde{B}}^U,w_{2\tilde{A}}^U)}\right]\times T^U=S(\tilde{B}^U,\tilde{A}^U).$$

$$S(\tilde{A},\tilde{B})=\frac{S(\tilde{A}^U,\tilde{B}^U)\times\left(1+S(\tilde{A}^\Delta,\tilde{B}^\Delta)\right)}{2}=\frac{S(\tilde{B}^U,\tilde{A}^U)\times\left(1+S(\tilde{B}^\Delta,\tilde{A}^\Delta)\right)}{2}=S(\tilde{B},\tilde{A}).$$ ■

Property 3 (Transitivity): If $\tilde{A}\le\tilde{B}\le\tilde{C}$, then $S(\tilde{A},\tilde{B})\ge S(\tilde{A},\tilde{C})$.

i.e. If three IT2FSs have the same shape, then the degree of similarity between two nearby ITFSs should be larger than the degree of similarity between two further away IT2FSs.

Proof. Suppose $\tilde{A},\tilde{B}\, and\, \tilde{C}$ have the same shape and $\tilde{A}\le\tilde{B}\le\tilde{C}$ such that

$\tilde{A}=[(a_1^L,a_2^L,a_3^L,a_4^L;w_{1\tilde{A}}^L,w_{2\tilde{A}}^L)\ (a_1^U,a_2^U,a_3^U,a_4^U;w_{1\tilde{A}}^U,w_{2\tilde{A}}^U)]$, $\tilde{B}=[(b_1^L,b_2^L,b_3^L,b_4^L;w_{1\tilde{B}}^L,w_{2\tilde{B}}^L)\ (b_1^U,b_2^U,b_3^U,b_4^U;w_{1\tilde{B}}^U,w_{2\tilde{B}}^U)]$, $b_i^L=a_i^L+d\ and\ b_i^U=a_i^U+d$ and

$\tilde{C}=[(c_1^L,c_2^L,c_3^L,c_4^L;w_{1\tilde{C}}^L,w_{2\tilde{C}}^L)(c_1^U,c_2^U,c_3^U,c_4^U;w_{1\tilde{C}}^U,w_{2\tilde{C}}^U)]$, $c_i^L=a_i^L+2d\ and\ c_i^U=a_i^U+2d$.

Since $\tilde{A}\ and\ \tilde{B}$ have the same shape, then $S(\tilde{A}^\Delta,\tilde{B}^\Delta)=1$ and the degree of similarity reduces to

$$S(\tilde{A},\tilde{B})=\frac{S(\tilde{A}^U,\tilde{B}^U)\times\left(1+S(\tilde{A}^\Delta,\tilde{B}^\Delta)\right)}{2}=S(\tilde{A}^U,\tilde{B}^U).$$

$\sqrt{\sum_{i=1}^4(a_i^u-b_i^u)^2}=\sqrt{\sum_{i=1}^4(a_i^u-a_i^u-d)^2}=2d.$

Substituting $b_i^U=a_i^U+d$ and

$\bar{b}^U=(b_1^U+b_2^U+b_3^U+b_4^U)/4=(a_1^U+d+a_2^U+d+a_3^U+d+b_4^U+d)/4=\bar{a}^U+d$ in

$S_{\tilde{B}^U}=\sqrt{\frac{\sum_{i=1}^4(b_i^U-\bar{b}^U)^2}{3}}$, gives $S_{\tilde{B}^U}=\sqrt{\frac{\sum_{i=1}^4(a_i^U+d-\bar{a}^U-d)^2}{3}}=\sqrt{\frac{\sum_{i=1}^4(a_i^U-\bar{a}^U)^2}{3}}=S_{\tilde{A}^U}.$

Since $\tilde{A}\ and\ \tilde{B}$ have the same shape, then $w_{1\tilde{A}}^U=w_{1\tilde{B}}^U and\ w_{2\tilde{A}}^U=w_{2\tilde{B}}^U$ and

$$\frac{\min(w_{1\tilde{A}}^U,w_{1\tilde{B}}^U)+\min(w_{2\tilde{A}}^U,w_{2\tilde{B}}^U)}{\max(w_{1\tilde{A}}^U,w_{1\tilde{B}}^U)+\max(w_{2\tilde{A}}^U,w_{2\tilde{B}}^U)}=1.$$

Since $|b_2^u-b_1^u|=|a_2^u+d-a_1^u-d|=|a_2^u-a_1^u|$ and $|b_4^u-b_3^u|=|a_4^u+d-a_3^u-d|=|a_4^u-a_3^u|$, then $T^U=1$.

$$S(\tilde{A}^U,\tilde{B}^U)=\left[1-\frac{\sqrt{\sum_{i=1}^4(a_i^u-b_i^u)^2}}{2}\right]\times\left[1-\sqrt{\frac{|S_{\tilde{A}^U}-S_{\tilde{B}^U}|}{2}}\right]\times\left[\frac{\min(w_{1\tilde{A}}^U,w_{1\tilde{B}}^U)+\min(w_{2\tilde{A}}^U,w_{2\tilde{B}}^U)}{\max(w_{1\tilde{A}}^U,w_{1\tilde{B}}^U)+\max(w_{2\tilde{A}}^U,w_{2\tilde{B}}^U)}\right]\times T^U,$$

$$=\left[1-\frac{2d}{2}\right]\times\left[1-\sqrt{\frac{0}{2}}\right]\times 1\times 1=1-d.$$

Then, $S(\tilde{A},\tilde{B})=1-d$.

Similarly, since $\tilde{A}\ and\ \tilde{C}$ have the same shape, then $S(\tilde{A}^\Delta,\tilde{C}^\Delta)=1$ and the degree of similarity reduces to $S(\tilde{A}^U,\tilde{C}^U)$.

$\sqrt{\sum_{i=1}^4(a_i^u-c_i^u)^2}=\sqrt{\sum_{i=1}^4(a_i^u-a_i^u-2d)^2}=4d.$

Substituting $c_i^U=a_i^U+2d$ and

$\bar{c}^U=(c_1^U+c_2^U+c_3^U+c_4^U)/4=(a_1^U+2d+a_2^U+2d+a_3^U+2d+b_4^U+2d)/4=\bar{a}^U+2d$

Substituting in $S_{\tilde{C}^U}=\sqrt{\frac{\sum_{i=1}^4(c_i^U-\bar{c}^U)^2}{3}}$, gives $S_{\tilde{C}^U}=\sqrt{\frac{\sum_{i=1}^4(a_i^U+2d-\bar{a}^U-2d)^2}{3}}=\sqrt{\frac{\sum_{i=1}^4(a_i^U-\bar{a}^U)^2}{3}}=S_{\tilde{A}^U}.$

Since $\tilde{A}\ and\ \tilde{C}$ have the same shape, then $w_{1\tilde{A}}^U=w_{1\tilde{C}}^U and\ w_{2\tilde{A}}^U=w_{2\tilde{C}}^U$ and

$$\frac{\min(w_{1\tilde{A}}^U,w_{1\tilde{B}\tilde{C}}^U)+\min(w_{2\tilde{A}}^U,w_{2\tilde{C}}^U)}{\max(w_{1\tilde{A}}^U,w_{1\tilde{C}}^U)+\max(w_{2\tilde{A}}^U,w_{2\tilde{C}}^U)}=1.$$

Since $|c_2^u-c_1^u|=|a_2^u+2d-a_1^u-2d|=|a_2^u-a_1^u|$ and $|c_4^u-c_3^u|=|a_4^u+2d-a_3^u-2d|=|a_4^u-a_3^u|$, then $T^U=1$.

$$S(\tilde{A}^U,\tilde{C}^U)=\left[1-\frac{\sqrt{\sum_{i=1}^4(a_i^u-c_i^u)^2}}{2}\right]\times\left[1-\sqrt{\frac{|S_{\tilde{A}^U}-S_{\tilde{C}^U}|}{2}}\right]\times\left[\frac{\min(w_{1\tilde{A}}^U,w_{1\tilde{C}}^U)+\min(w_{2\tilde{A}}^U,w_{2\tilde{C}}^U)}{\max(w_{1\tilde{A}}^U,w_{1\tilde{C}}^U)+\max(w_{2\tilde{A}}^U,w_{2\tilde{C}}^U)}\right]\times T^U,$$

$$= \left[1 - \frac{4d}{2}\right] \times \left[1 - \sqrt{\frac{0}{2}}\right] \times 1 \times 1 = 1 - 2d.$$

Then, $S(\tilde{A}, \tilde{C}) = 1 - 2d$.
Since $S(\tilde{A}, \tilde{B}) = 1 - d$ and $S(\tilde{A}, \tilde{C}) = 1 - 2d$, then $S(\tilde{A}, \tilde{B}) > S(\tilde{A}, \tilde{C})$. ■

Property 4 (Overlap): If two IT2FSs partially overlap, $S(\tilde{A}, \tilde{B}) > 0$.

Since $S(\tilde{A}, \tilde{B}) = \frac{S(\tilde{A}^U, \tilde{B}^U) \times \left(1 + S(\tilde{A}^\Delta, \tilde{B}^\Delta)\right)}{2}$, then $S(\tilde{A}, \tilde{B}) = 0$ if $S(\tilde{A}^U, \tilde{B}^U) = 0$.

$S(\tilde{A}^U, \tilde{B}^U) = 0$ if $\frac{\sqrt{\sum_{i=1}^4 (a_i^u - b_i^u)^2}}{2} = 1$, $\sqrt{\frac{|S_{\tilde{A}^U} - S_{\tilde{B}^U}|}{2}} = 1$, or $T^U = 0$.

It can be shown that $\sqrt{\frac{|S_{\tilde{A}^U} - S_{\tilde{B}^U}|}{2}} \neq 1$ as follows.

$Since\ 0 \le a_i^u \le 1\ and\ 0 \le \bar{a}^U \le 1,\ then\ S_{\tilde{A}^U} = \sqrt{\frac{\sum_{i=1}^4 (a_i^U - \bar{a}^U)^2}{3}} < 1.2.$

Similarly, $since\ 0 \le b_i^u \le 1, and\ 0 \le \bar{b}^U \le 1,\ then\ S_{\tilde{B}^U} = \sqrt{\frac{\sum_{i=1}^4 (b_i^U - \bar{b}^U)^2}{3}} < 1.2.$

Then, $|S_{\tilde{A}^U} - S_{\tilde{B}^U}| < 1.2$ and $\sqrt{\frac{|S_{\tilde{A}^U} - S_{\tilde{B}^U}|}{2}} \neq 1$.
It can be also shown that $T^U \neq 0$.
Since $\max\{|a_2^u - a_1^u|, |b_2^u - b_1^u|\} < 1$ and $\min\{|a_2^u - a_1^u|, |b_2^u - b_1^u|\} < 1$,
then $\frac{1 + \max\{|a_2^u - a_1^u|, |b_2^u - b_1^u|\}}{1 + \min\{|a_2^u - a_1^u|, |b_2^u - b_1^u|\}} < 2$.
Similarly, since $\max\{|a_4^u - a_3^u|, |b_4^u - b_3^u|\} < 1$ and $\min\{|a_4^u - a_3^u|, |b_4^u - b_3^u|\} < 1$,
then $\frac{1 + \max\{|a_4^u - a_3^u|, |b_4^u - b_3^u|\}}{1 + \min\{|a_4^u - a_3^u|, |b_4^u - b_3^u|\}} < 2$.

$$T^U = \left[\left(2 - \frac{1 + \max\{|a_2^u - a_1^u|, |b_2^u - b_1^u|\}}{1 + \min\{|a_2^u - a_1^u|, |b_2^u - b_1^u|\}}\right) + \left(2 - \frac{1 + \max\{|a_4^u - a_3^u|, |b_4^u - b_3^u|\}}{1 + \min\{|a_4^u - a_3^u|, |b_4^u - b_3^u|\}}\right)\right] / 2 \neq 0.$$

If $\frac{\sqrt{\sum_{i=1}^4 (a_i^u - b_i^u)^2}}{2} = 1 \Rightarrow \sum_{i=1}^4 (a_i^u - b_i^u)^2 = 4$.

Since $0 \le a_i^u \le 1$ and $0 \le b_i^u \le 1$, $\sum_{i=1}^4 (a_i^u - b_i^u)^2 = 4 \Rightarrow$ either $a_i^u = 0\ and\ b_i^u = 1\ \forall i$ or $a_i^u = 1\ and\ b_i^u = 0\ \forall i$, then the IT2FSs does not overlap. ■

Property 5. If $\tilde{A}$ and $\tilde{B}$ are real numbers, then $S(\tilde{A}, \tilde{B}) = 1 - |a - b|$.

Proof. If $\tilde{A}$ is a real number, then
$a_1^L = a_2^L = a_3^L = a_4^L = a_1^U = a_2^U = a_3^U = a_4^U = a$, and $w_{1\tilde{A}}^L = w_{2\tilde{A}}^L = w_{1\tilde{A}}^U = w_{2\tilde{A}}^U = 1$.
Similarly, if $\tilde{B}$ is a real number, then
$b_1^L = b_2^L = b_3^L = b_4^L = b_1^U = b_2^U = b_3^U = b_4^U = b$ and $w_{1\tilde{B}}^L = w_{2B}^L = w_{1B}^U = w_{2B}^U = 1$.

$$T^\Delta = \left[\left(2 - \frac{1 + \max\{|\Delta a_2 - \Delta a_1|, |\Delta b_2 - \Delta b_1|\}}{1 + \min\{|\Delta a_2 - \Delta a_1|, |\Delta b_2 - \Delta b_1|\}}\right) + \left(2 - \frac{1 + \max\{|\Delta a_4 - \Delta a_3|, |\Delta b_4 - \Delta b_3|\}}{1 + \min\{|\Delta a_4 - \Delta a_3|, |\Delta b_4 - \Delta b_3|\}}\right)\right] / 2 = \left[\left(2 - \frac{1+0}{1+0}\right) + \left(2 - \frac{1+0}{1+0}\right)\right] / 2 = 1.$$

$$S(\tilde{A}^\Delta, \tilde{B}^\Delta) = \left[1 - \frac{\sqrt{\sum_{i=1}^4 (\Delta a_i - \Delta b_i)^2}}{2}\right] \times \left[1 - \sqrt{\frac{|\Delta S_a - \Delta S_b|}{2}}\right] \times \left[1 - \frac{|w_{1\tilde{A}}^L - w_{1\tilde{B}}^L| + |w_{2\tilde{A}}^L - w_{2\tilde{B}}^L|}{|w_{1\tilde{A}}^U + w_{1\tilde{B}}^U| + |w_{2\tilde{A}}^U + w_{2\tilde{B}}^U|}\right] \times T^\Delta$$

$$= \left[1 - \frac{0}{2}\right] \times \left[1 - \sqrt{\frac{0}{2}}\right] \times \left[1 - \frac{0}{4}\right] \times 1 = 1.$$

$$T^U = \left[\left(2 - \frac{1 + \max\{|a_2^u - a_1^u|, |b_2^u - b_1^u|\}}{1 + \min\{|a_2^u - a_1^u|, |b_2^u - b_1^u|\}}\right) + \left(2 - \frac{1 + \max\{|a_4^u - a_3^u|, |b_4^u - b_3^u|\}}{1 + \min\{|a_4^u - a_3^u|, |b_4^u - b_3^u|\}}\right)\right] / 2 = \left[\left(2 - \frac{1+0}{1+0}\right) + \left(2 - \frac{1+0}{1+0}\right)\right] / 2 = 1.$$

$$S(\tilde{A}^U, \tilde{B}^U) = \left[1 - \frac{\sqrt{\sum_{i=1}^4 (a_i^u - b_i^u)^2}}{2}\right] \times \left[1 - \sqrt{\frac{|S_{\tilde{A}^U} - S_{\tilde{B}^U}|}{2}}\right] \times \left[\frac{\min(w_{1\tilde{A}}^U, w_{1\tilde{B}}^U) + \min(w_{2\tilde{A}}^U, w_{2\tilde{B}}^U)}{\max(w_{1\tilde{A}}^U, w_{1\tilde{B}}^U) + \max(w_{2\tilde{A}}^U, w_{2\tilde{B}}^U)}\right] \times T^U$$

$$= \left[1 - \frac{\sqrt{\sum_{i=1}^4 (a-b)^2}}{2}\right] \times \left[1 - \sqrt{\frac{0}{2}}\right] \times \left[\frac{1+1}{1+1}\right] \times 1 = 1 - |a - b|.$$

$S(\tilde{A}, \tilde{B}) = \frac{(1 - |a-b|) \times (1 + 1)}{2} = 1 - |a - b|$. ■