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The Infinitely Divisible Compound Negative Binomial Distribution as the Sum of Laplace Distribution

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Abstract

The infinite divisibility of compound negative binomial distribution, especially as the sum of Laplace distribution has important roles in governing the mathematical model based on its characteristic function. In order to show the property of characteristic function of this compound negative binomial distribution, it is used Fourier-Stieltjes transform to have characteristic function. The characteristic function property is governed to show the continuity and quadratic form by using analytical approaches. The infinite divisibility property is obtained by introducing a function satisfied the criteria to be a characteristic function such that its convolution has the characteristic function of compound negative binomial distribution. Then it is concluded that the characteristic function of compound negative binomial distribution as the sum of Laplace distribution satisfies the property of continuity, quadratic form and infinite divisibility.

Key Words: Characteristic Function; Compound Negative Binomial; Infinitely Divisible Distribution; Laplace Distribution.

Mathematical Subject Classification: 60E07; 60E05; 60E10.

1. Introduction

The characteristic function is referred from Lukacs (1992) as Fourier-Stieltjes transform, that is defined as $\phi(t)=E(\exp(itX))$ for a random variable X where i as imaginary unit and $t\in(-\infty,\infty)$. The refinement of this transformation is existence of characteristic function for every distribution. The characteristic function is also used to express the infinitely divisible distribution. The definition of infinitely divisible characteristic function can be referred from Sato (2003) or Steutel and Harn (2004), the characteristic function $\phi(t)$ is said to be infinitely divisible if for every positive integer number m, there exists a characteristic function $\phi_m(t)$ such that $\phi(t)=(\phi_m(t))^m$. The existence of characteristic function $\phi_m(t)$ could be examined by using necessary and sufficient condition for a function to be characteristic function by Bochner's theorem, see Lukacs (1992), that is the criteria of $\phi_m(0)=1$ and the function $\phi_m(t)$ non-negative definite, where the condition of non-negative definite is satisfied by continuity and quadratic form of the function $\phi_m(t)$.

Beside definition of infinite divisibility based on the characteristic function, the infinite divisibility is constructed by the use of distribution function and random variable adopted from Artikis (1983) and Mainardi and Rogosin (2006), where distribution functions which corresponding to infinitely divisible characteristic functions are called infinitely divisible distribution. Besides, the infinite divisibility of a distribution function can be explained by using its random variable, that is for random variable X with distribution function F can be divided into m random variables such that $X=X_1+X_2+\ldots+X_m$ where the random variable $X_{i=1,2,\ldots,m}$ as independent and identically random variables with distribution function F_m, this is known as m-fold convolution of distribution function of F_m that is $F=*F_m*F_m*...*F_m$ for m times.

The most useful distribution satisfies the property of infinitely divisible is negative binomial distribution with probability distribution as follows

$$f(n|r,p) = \binom{r+n-1}{n} p^r (1-p)^n \tag{1}$$

with parameter (r,p) where $n=1,2,...$ as the number of failures before r-th success with r as the number of successes in the negative binomial experiment and p as the probability of success on an individual experiment with $p\in(0,1)$. The compound negative binomial distribution is defined in the term of

$$S = X_1 + X_2 + \cdots + X_N \tag{2}$$

where random variable N has negative binomial distribution and independent and identically random variable $X_{i=1,2,...,N}$ have certain distribution. In the case we set the random variable X_i has Laplace distribution then we said the random variable S has compound negative binomial distribution as the sum of Laplace distribution. The Laplace distribution and sometimes called the double exponential distribution with parameters (μ,b) has probability density function as follows

$$f(x|\mu,b) = \frac{1}{2b}\exp\left(-\frac{|x-\mu|}{b}\right) \tag{3}$$

for $x\in(-\infty,\infty)$ where μ is a location parameter and $b>0$ is a scale parameter or referred to as the diversity.

The characteristic function has important role to characterize the distribution. Allam (2011) used empirical characteristic function for testing the fit of certain distribution. While, the applications of negative binomial distribution and its compound have been spread on economic and finance. Panjer and Willmot (1981) have introduced the model of a finite sum of negative binomial-exponential distribution on the insurance data analysis; this is actually a model convolution of exponential distribution with number of random variables has negative binomial distribution as the compound distribution. Wang (2011) has used mix negative binomial distribution in modeling number of insurance claims and insurance data. Recently, Omair et al. (2018) have discussed the financial bivariate model based on compound negative binomial distribution, while Hanagal and Dabade (2013) have used compound negative binomial distribution to model the frailty. Furthermore, Furman (2007) has shown implicitly the distribution of convolutions of negative binomial distribution has also negative binomial distribution; this is also to indicate the infinite divisibility of negative binomial distribution by using the term of the m-fold convolution. Besides, the convolution has attracted many properties on several distributions; Devianto et al. (2015) have reviewed a convolution on generated random variable from exponential distribution with stabilizer constant and Devianto (2016) has discussed the uniform continuity of characteristic function from convoluted exponential distribution with stabilizer constant. The most recent result on exponential distribution is explained by Devianto (2018) for the property of its Levy measure to govern the class of infinite divisibility. In addition, the Laplace distribution as double exponential distribution also attracted some important results; theoretical aspect of modification of classical Laplace distribution has introduced by Mahmoudvand et al. (2015), while Yu and Zang (2005) discussed the concept of asymmetric Laplace distribution and its extension, finally Liu and Kozubowski (2018) have introduced application of folded Laplace distribution.

The previous research results have established the convolution of exponential and hypoexponential distribution property and the used of Laplace distribution as double exponential distribution, it is also obtained the infinite divisibility of negative binomial distribution by using the m-fold convolution. Furthermore, Chhaiba et al. (2016) have introduced the generalized negative binomial distribution and its infinitely divisible by using canonical representation of characteristic function contained the property of Levy measure and quasi-Levy measure. While, Inusah and Kozubowski (2006) have introduced a discrete version of Laplace distribution and the property of its infinite divisibility respect to geometric compounding. However, these recent results do not cover the characterization of compound negative binomial distribution as the sum of Laplace distribution. Then it is necessary to express the infinite divisibility of compound negative binomial distribution as the sum of Laplace distribution by using characteristic function and its some properties in the term of quadratic form and continuity.

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2. The Characteristic Function of Compound Negative Binomial Distribution as the Sum of Laplace Distribution

Let the random variable N has negative binomial distribution with parameters (r,p) as in Equation (1), then the moment generating function and characteristic function are obtained respectively as follows

$$M_N(t) = E(\exp(tN)) = \left(\frac{p}{1-(1-p)\exp(t)}\right)^r, \tag{4}$$

$$\phi_N(t) = E(\exp(itN)) = \left(\frac{p}{1-(1-p)\exp(it)}\right)^r, \tag{5}$$

The characteristic function of Laplace distribution with probability density function as in Equation (3) is obtained as the following

$$\phi_X(t) = E(\exp(itX)) = \frac{\exp(\mu it)}{b^2t^2+1}. \tag{6}$$

The compound negative binomial distribution as the sum of Laplace distribution is the sum of independent and identically random variables of Laplace distribution, where the number of random variables has negative binomial distribution. The characteristic function is stated in the following proposition.

Proposition 1. The random variable $S=X_1+X_2+\ldots+X_N$ has compound negative binomial distribution as the sum of Laplace distribution, then characteristic function is in the following form

$$\phi_S(t) = \left(\frac{p(1+b^2\,t^2)}{1+b^2t^2-(1-p)\exp(\mu it)}\right)^r. \tag{7}$$

where (r,p) as the parameters of negative binomial distribution and (μ,b) as the parameters of Laplace distribution.

Proof: By using definition of characteristic function and linearity property of expectation then it is obtained the characteristic function of compound negative binomial distribution as the sum of Laplace distribution as follows

$$\begin{aligned}\phi_S(t) &= E(\exp(itS))\\ &= E\left(E(\exp(it\textstyle\sum_{i=1}^{N}X_i\,)\,|N)\right)\\ &= E(\phi_X(t)^N)\\ &= E(\exp(N\,ln\,\phi_X(t)))\\ &= M_N(ln\,\phi_X(t))\end{aligned} \tag{8}$$

Next, substituted Equation (4) and Equation (6) in the Equation (8), then it is obtained

$$\phi_S(t) = \left(\frac{p}{1-(1-p)\phi_X(t)}\right)^r = \left(\frac{p(1+b^2\,t^2)}{1+b^2t^2-(1-p)\exp(\mu it)}\right)^r. \tag{9}$$

The Proposition 1 has shown that compound negative binomial distribution as the sum of Laplace distribution as a unique characteristic function, that is very varying from the characteristic function of the sum of Laplace distribution such as in Chhaiba et al. (2016) or Inusah and Kozubowski (2006). In addition, it is unlikely a convolution of negative binomial distribution such as in Furman (2007) or Hanagal and Dabade (2013), but this characteristic function is governed by using negative binomial compounding. This is to confirm the uniqueness of characteristic function for every difference formula of compound distribution.

3. The Characteristic Function Property of Compound Negative Binomial Distribution as the Sum of Laplace Distribution

The main results of this paper are some properties of characteristic function related to compound negative binomial distribution as the Laplace distribution. First, it is introduced $\phi_{Sm}(t)$ to be a function corresponding to a random variable S_m, that is defined as

$$\phi_{S_m}(t) = \left(\frac{p(1+b^2\,t^2)}{1+b^2t^2-(1-p)\exp(\mu it)}\right)^{\frac{r}{m}}. \tag{10}$$

for integer number m where (r,p) are parameters of negative binomial distribution and (μ,b) are parameters of Laplace distribution. The following proposition is established to show the function $\phi_{Sm}(t)$ satisfied the necessary condition of Bochner's theorem to be a characteristic function.

Proposition 2. The function in Equation (10) is a characteristic function.

Proof: The function $\phi_{Sm}(t)$ is a characteristic function if it satisfies the criterion on the Bochner's theorem. Let us note the two conditions as follows

i. It is obviously $\phi_{Sm}(0)=1$,
ii. The function $\phi_{Sm}(t)$ is non-negative definite. This condition is explained by using property of continuity and the quadratic form.

First, it will be shown the function $\phi_{Sm}(t)$ is continuous by using definition of uniform continuity, that is for every $\varepsilon>0$ there exists $\delta>0$ such that $|\phi_{Sm}(t_1)-\phi_{Sm}(t_2)|<\varepsilon$ for $|t_1-t_2|<\delta$ where δ depends only on ε. The following equation is obtained by using definition of function $\phi_{Sm}(t)$ in the term

$$\begin{aligned}|\phi_{S_m}(t_1)-\phi_{S_m}(t_2)| &= \left|\left(\frac{p(1+b^2t_1^2)}{1+b^2t_1^2-(1-p)\exp(\mu i t_1)}\right)^{\frac{r}{m}}-\left(\frac{p(1+b^2t_2^2)}{1+b^2t_2^2-(1-p)\exp(\mu i t_2)}\right)^{\frac{r}{m}}\right| \\ &= p^{\frac{r}{m}}\left|\left(\frac{1}{1-\frac{(1-p)\exp(\mu i t_1)}{(1+b^2t_1^2)}}\right)^{\frac{r}{m}}-\left(\frac{(1+b^2t_2^2)}{1-\frac{(1-p)\exp(\mu i t_2)}{(1+b^2t_2^2)}}\right)^{\frac{r}{m}}\right|\end{aligned} \tag{11}$$

Then, it is defined $h=t_1-t_2$, so that for $h\to0$, it is obtained the following condition

$$|\phi_{S_m}(h+t_2)-\phi_{S_m}(t_2)| = p^{\frac{r}{m}}\left|\left(\frac{1}{1-\frac{(1-p)\exp(\mu i(h+t_2))}{(1+b^2(h+t_2)^2)}}\right)^{\frac{r}{m}}-\left(\frac{(1+b^2t_2^2)}{1-\frac{(1-p)\exp(\mu i t_2)}{(1+b^2t_2^2)}}\right)^{\frac{r}{m}}\right| \tag{12}$$

This hold for $\delta<\varepsilon$ where $|\phi_{Sm}(t_1)-\phi_{Sm}(t_2)|<\varepsilon$ for $|t_1-t_2|<\delta$ and δ depends only on ε. Then $\phi_{Sm}(t)$ is uniformly continuous. Next, it will be shown the function $\phi_{Sm}(t)$ is positively defined function with quadratic from as in the following term

$$\sum_{j=1}^{n}\sum_{l=1}^{n} c_j\,\overline{c_l}\phi_{s_m}(t_j-t_l)\geq 0 \tag{13}$$

for any complex number $c_1, c_2, ..., c_n$ and real $t_1, t_2, ..., t_n$ and $n\geq1$, hence it is obtained the following equation

$$\begin{aligned}\sum_{j=1}^{n}\sum_{l=1}^{n} c_j\,\overline{c_l}\phi_{s_m}(t_j-t_l) &= \sum_{j=1}^{n}\sum_{l=1}^{n} c_j\,\overline{c_l}\left(\frac{p\left(1+b^2(t_j-t_l)^2\right)}{1+b^2(t_j-t_l)^2-(1-p)\exp\left(\mu i(t_j-t_l)\right)}\right)^{\frac{r}{m}} \\ &= p^{\frac{r}{m}}\sum_{j=1}^{n}\sum_{l=1}^{n} c_j\,\overline{c_l}\left(\frac{1}{1-\frac{(1-p)\exp\left(\mu i(t_j-t_l)\right)}{\left(1+b^2(t_j-t_l)^2\right)}}\right)^{\frac{r}{m}}\end{aligned} \tag{14}$$

We rearrange the Equation (14) by setting

$$\left|\frac{(1-p)\exp\left(\mu i(t_j-t_l)\right)}{\left(1+b^2(t_j-t_l)^2\right)}\right|\leq 1, \tag{15}$$

and it is defined geometric series

$$\frac{1}{1-\frac{(1-p)\exp\left(\mu i(t_j-t_l)\right)}{\left(1+b^2(t_j-t_l)^2\right)}}=\sum_{k=0}^{\infty}\left(\frac{(1-p)\exp\left(\mu i(t_j-t_l)\right)}{\left(1+b^2(t_j-t_l)^2\right)}\right)^k \tag{16}$$

for $p\in(0,1)$. Next, We use the conditions of Inequality (15) and Equation (16) in the Equation (14), to obtain the following condition

$$\begin{aligned}\sum_{j=1}^{n}\sum_{l=1}^{n} c_j\,\overline{c_l}\,\phi_{s_m}(t_j-t_l) &= p^{\frac{r}{m}}\sum_{j=1}^{n}\sum_{l=1}^{n} c_j\,\overline{c_l}\left(\sum_{k=0}^{\infty}\left(\frac{(1-p)\exp\left(\mu i(t_j-t_l)\right)}{\left(1+b^2\left(t_j-t_l\right)^2\right)}\right)^{k}\right)^{\frac{r}{m}} \\ &= p^{\frac{r}{m}}\left(\sum_{j=1}^{n} c_j\left(\sum_{k=0}^{\infty}\left(\frac{(1-p)\exp\left(\mu i t_j\right)}{\left(1+b^2t_j^2\right)}\right)^{k}\right)^{\frac{r}{m}}\right)\overline{\left(\sum_{l=1}^{n} c_l\left(\sum_{k=0}^{\infty}\left(\frac{(1-p)\exp\left(\mu \imath t_l\right)}{\left(1+b^2t_l^2\right)}\right)^{k}\right)^{\frac{r}{m}}\right)} \\ &= p^{\frac{r}{m}}\left|\sum_{j=1}^{n} c_j\left(\sum_{k=0}^{\infty}\left(\frac{(1-p)\exp\left(\mu i t_j\right)}{\left(1+b^2t_j^2\right)}\right)^{k}\right)^{\frac{r}{m}}\right|^2 \geq 0\end{aligned} \tag{17}$$

The two conditions on the Bochner's theorem are satisfied, then the function $\phi_{Sm}(t)$ is a characteristic function. The random variable S_m has characteristic function as in Equation (10), then the characteristic function $\phi_{Sm}(t)$ is corresponding to compound negative binomial distribution as the sum of Laplace distribution with the parameter $(r/m,p)$ for a variational of negative binomial distribution and (μ,b) for Laplace distribution.

The representation of characteristic function of Equation (10) was established the existence of a new type of compound negative binomial distribution as the sum of Laplace distribution. This new characteristic function has a specific form that differs from a convolution of generated exponential and hypoexponential random variables such as in Devianto et al. (2015) or Devianto (2016, 2018). These new types of compound negative binomial distribution have exposed the specific property on its characteristic function.

The following proposition is stated the infinite divisibility of compound negative binomial distribution as the sum of Laplace distribution. This result gives the strong understanding of the characteristic function property of compound negative binomial distribution.

Proposition 3. The compound negative binomial distribution as the sum of Laplace distribution is infinitely divisible distribution.

Proof: It will be shown compound negative binomial distribution as the sum of Laplace distribution is infinitely divisible distribution by using the characteristic function $\phi_S(t)$ and $\phi_{Sm}(t)$ such that it satisfies $\phi_S(t)=(\phi_{Sm}(t))^m$. We can set the following equation

$$\left(\phi_{S_m}(t)\right)^m = \left(\left(\frac{p(1+b^2\,t^2)}{1+b^2t^2-(1-p)\exp(\mu i t)}\right)^{\frac{r}{m}}\right)^m = \left(\frac{p(1+b^2\,t^2)}{1+b^2t^2-(1-p)\exp(\mu i t)}\right)^r = \phi_S(t) \tag{18}$$

Because it satisfies $\phi_S(t)=(\phi_{Sm}(t))^m$ for any positive integer number $m\geq 2$, so that the characteristic function of compound negative binomial distribution as the sum of Laplace distribution is an infinitely divisible distribution.

The random variable S_m has corresponding characteristic function $\phi_{Sm}(t)$ of compound negative binomial distribution as the sum of Laplace distribution with the parameter $(r/m,p)$ for negative binomial distribution with generalization for selected rational number r/m and parameters (μ,b) for Laplace distribution. This result confirmed the compound negative binomial distribution as the sum of Laplace distribution is also infinitely divisible, where the property of infinite divisibility also satisfies for negative binomial distribution such as explaining in Furman (2007) and for Laplace distribution and its geometric compounding in Inusah and Kozubowski (2006).

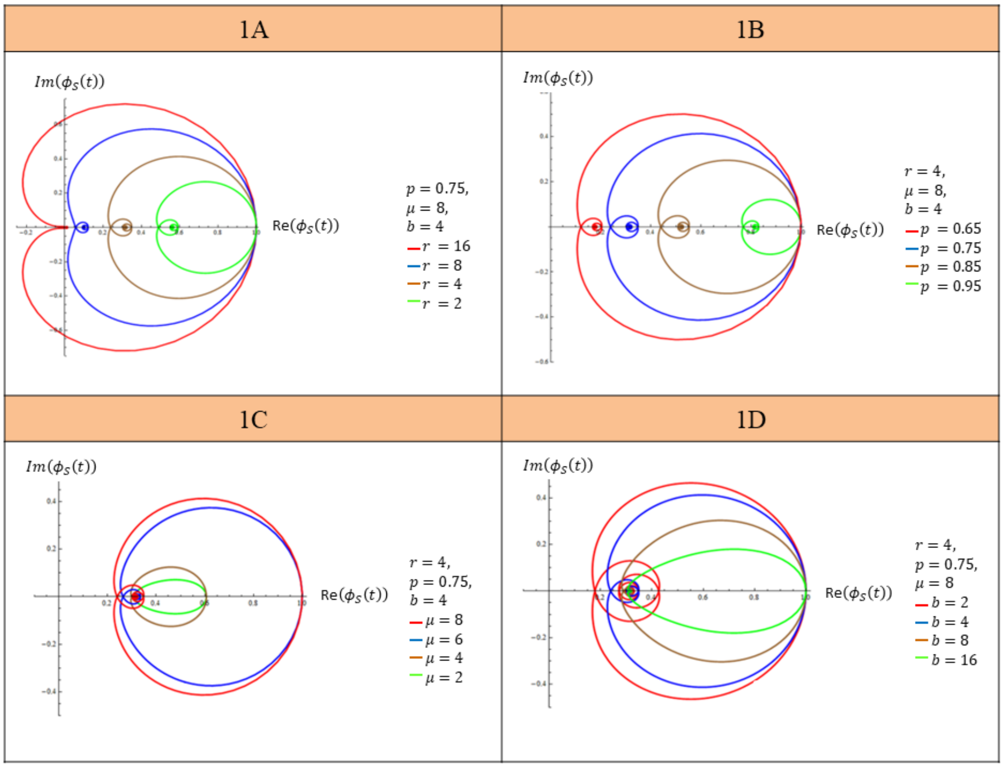

Figure 1: Parametric curves of characteristic function from compound negative binomial distribution as the sum of Laplace distribution with different values of parameter.

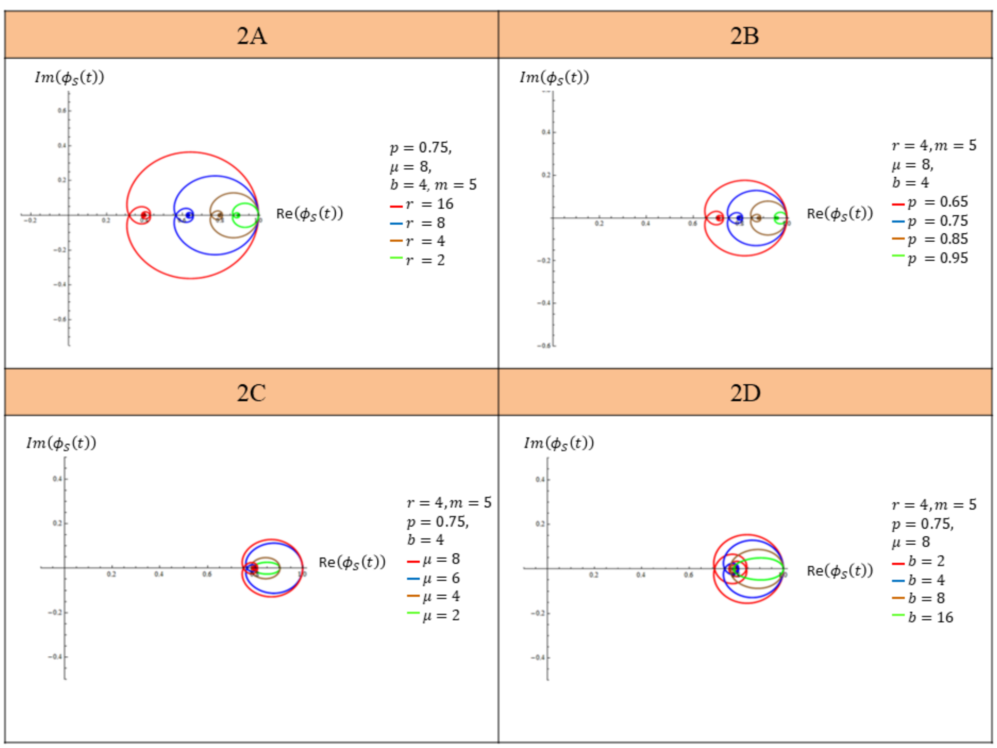

Figure 2: Parametric curves of characteristic function from *m*-division of compound negative binomial distribution as the sum of Laplace distribution with with different values of parameter.

The various graphs of parametric curves of characteristic function $\phi_S(t)$ of compound negative binomial distribution as the sum of Laplace distribution are presented in Figure 1. Furthermore, the graphs of parametric curves of characteristic function of the *m*-division of random variable S into random variable S_m with characteristic function $\phi_{Sm}(t)$ are presented in Figure 2.

The parametric curve of characteristic function is governed on Cartesian coordinate by plotting the real part on the horizontal axis and imaginary part on the vertical axis. The figure on 1A and 2A have the various parameter r, the figure on 1B and 2B have the various parameter p, the figure on 1C and 2C have various parameter μ *and* the figure on 1D and 2D have various parameter b. These parametric curves of characteristic function of compound negative binomial distribution as the sum of Laplace distribution have exhibit the smooth curves and never vanish on the complex plane, in addition the *m*-division of random variable S to be random variable S_m in Figure 2, that is characteristic function $\phi_{Sm}(t)$ from random variable S_m has small scaling comparing with parametric curves of characteristic function $\phi_S(t)$ as in Figure 1. These understandings have to confirm graphically that the characteristic function of compound negative binomial distribution as the sum of Laplace distribution is continuous, infinitely divisible and never vanish on the complex plane.

Conclusion

The compound negative binomial distribution as the sum of Laplace distribution is constructed as the sum of Laplace distribution where the number of this random variables summation has negative binomial distribution. The characteristic function of this compound distribution is obtained by using Fourier-Stieltjes transform which satisfies infinite divisibility. The division of random variable from compound negative binomial distribution as the sum of Laplace distribution has specific characteristic function which satisfies the continuity and quadratic form property. The findings on the property of infinitely divisible compound distribution have important roles in governing the mathematical model based on characteristic function in the term of its continuity and quadratic form.

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