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Inverse Odd Weibull Generated Family of Distributions

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Abstract

This work proposes an inverse odd Weibull (IOW) family of distributions for a lifetime distributions. Some mathematical properties of this family of distribution were derived. Survival, hazard, quantiles, reversed hazard, cumulative, odd functions, kurtosis, skewness, order statistics and entropies of this new family of distribution were examined. The parameters of the family of distributions were obtained by maximum likelihood. The behavior of the estimators were studied through simulation. The flexibility and importance of the distribution by means of real data set applications were emphasized.

Key Words: Survival function; Quantiles Function; Lifetime Distribution; Maximum likelihood; Inverse odd model

Mathematical Subject Classification: 60E05,62E10, 62E15.

1. Background

Modeling lifetime processes has received several attentions in recent time. Thus, the interest of distribution statistics has grown over the years by applying its applications to obtain the parameters of interest, improve the understanding of the processes; and to make a prediction at some locations where measurements of such processes are not available. The Weibull distribution was proposed by in Weibull (1951). This distribution has a wide range of application because of its flexibility and applicability in lifetime processes. Thus, studying a new family of probability model is motivated by increasing its capacity to model dataset that cannot be fitted properly by existing models, Anake et al.(2015). These new classes or families may add one or more parameters to already proposed models in statistical literature, Korkmaz et al.(2018). Gupta et al.(1998), Gupta and Kundu(1999), Mudholkar and Srivastava(1993), Oguntunde et al.(2013), Cordeiro et al.(2013), Oguntunde et al.(2014a, 2014b, 2014c), Oguntunde et al.(2017a), Yousof et al.(2015), Yousof et al.(2017a, 2017b, 2017c), Yousof et al.(2018), Cordeiro et al.(2017a, 2017b), Aryal et al.(2017) and Nofal et al.(2017) proposed a generator called the exponentiated classes. These classes, consist of the cumulative distribution function of positive power parameter. Al-Mofleh(2018) proposed another family of generator using the tangent function. Hassan and Al-Thobety (2012) and Hassan et al.(2015) proposed the type II inverted Weibull for both optimal design for fail-

ure step stress partially and constant-stress accelerated life tests. Hassan et al.(2018) formulated the Odds generalized exponential inverse Weibull distribution. Afify et al.(2013), Afify et al.(2016) and Afify et al.(2017) proposed the transmuted Weibull Lomax, geometric-G and the Odd exponentiated half-logistic-G families of distributions respectively. Cordeiro et al.(2010) and Rasekhi et al.(2018) proposed the Kumaraswamy Weibull geometric distribution. Manal and Fathy (2003) developed the exponentiated Weibull family of Distribution. Tojeiro et al.(2014) proposed the Weibull geometric distribution. Lee et al.(2007), Silva et al.(2010, 2013) and Khan (2015) formulated the beta Weibull and the The gamma extended Frechet distribution families of distribution. Zelibe et al.(2019) proposed the Kumaraswamy alpha power inverted exponential model. Alzaatreh et al.(2013b) proposed the Weibull-Pareto distribution. Alzaatreh et al.(2016) proposed the gamma half-Cauchy distribution. Agu and Onwukwe (2019) proposed modified laplace distribution. Pinho et al.(2015) proposed the The Harris extended exponential distribution. Harter et al. (1963) proposed the maximum likelihood of the gamma and Weibull population. Cordeiro et al.(2014) proposed McDonald Weibull model for failure data. Eliwa and El-Morshedy(2020) proposed the bivariate odd Weibull-G family. Eliwa and El-Morshedy(2019) proposed the odd flexible Weibull-H distribution. Cooray(2015) examined the moment and likelihood of the odd Weibull mode. Cooray(2006) proposed the generalization of the odd Weibull family. Jiang (2008) further examined the odd Weibull distribution. In this article, we shall correct Equations (5) and (6)of the generalized odd Weibull generated family of distribution proposed by Korkmaz et al.(2018) and proposed a better family of distribution called the inverse odd Weibull generated family of distributions.
This article is organized as follows: Section 1 introduced the model, Section 2 provided the formulation of the inverse odd Weibull family of distribution together with its maximum likelihood of its parameters. In Section 3, discussed some properties of the IOW family of distribution with a simulation studies of the kurtosis and skewness. The applications to validate the proposed distribution and results obtained were compared with existing distributions in section 4. Section 5 is the concluding remarks.
Let $u(s)$ be a probability density function for a random variable $T \in [d, h]$ for $-\infty \leq d < h < \infty$. Also, let $B[D(s)]$ be a function for the cumulative distribution function for a random variable S such that $B[D(s)]$ must satisfy the following conditions:

- $B[D(s)] \in [d, h]$,
- $B[D(s)]$ is a monotonic increasing and differentiable on the interval $[d, h]$,
- $B[D(s)]$ approaches d as $s \to -\infty$ and more also, $B[D(s)] \to d$ as $s \to \infty$.

Then by Alzaatreh et al.(2013), a T_X family of distributions can be defined as

$$F(s) = \int_{d}^{B[D(s)]} u(s)ds, \tag{1}$$

where $B[D(s)]$ is the link function on the interval $(0, \infty)$ called Gamma-G type lll, Weibull-G.
The probability density function of Equation (6) is given by

$$f(s) = \left\{\frac{d}{ds}B\Big[D(s)\Big]\right\}u\left\{B\Big[D(s)\Big]\right\}. \tag{2}$$

2. The Inverse Odd Weibull Generated Family of Distribution

In this article, we shall propose a class of distributions called the Inverse odd Weibull Generated $(''IOW'')$ family of distribution. Now, let the link function $B[D(s)]$ be defined as

$$B[D(s)] = \frac{G(s;\phi)^{\frac{1}{\alpha}}}{1 - G(s;\phi)^{\frac{1}{\alpha}}} \qquad \alpha > 0, \tag{3}$$

with the probability density function (pdf) of $u(s)$ defined as

$$u(s;\beta) = \beta s^{\beta-1}e^{-s^{\beta}} \qquad s > 0, \quad \beta > 0. \tag{4}$$

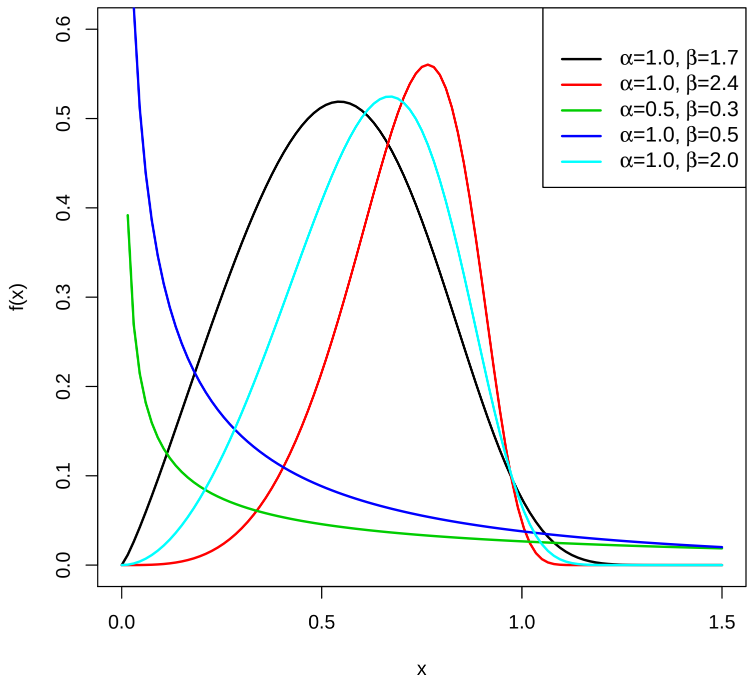

Figure 1: The pdf of the *IOW* distribution for different parameter values

Also, the cdf of Equation (4) given as

$$G(s;\phi) = 1 - e^{-s^{\beta}}. \tag{5}$$

The probability density function for $\bar{\alpha} = \frac{1}{\alpha}$ that corresponds to the *IOW* distribution is thus, expressed as

$$f_{(IOW)}(s;\alpha,\beta,\phi) = \frac{\bar{\alpha}\beta g^{\frac{1-\alpha}{\alpha}}(s;\phi)G(s;\phi)^{\frac{\beta-1}{\alpha}}}{[1-G(s;\phi)^{\frac{1}{\alpha}}]^{\beta+1}} exp\left\{-\left[\frac{G(s;\phi)^{\frac{1}{\alpha}}}{1-G(s;\phi)^{\frac{1}{\alpha}}}\right]^{\beta}\right\}, \tag{6}$$

with $g(s;\phi)$ as the baseline probability density function.

Figure 1 shows some plots for the probability density function (pdf) of the *IOW* distribution for several values of parameters. The Figure 1 plots reveal that the *IOW* density can be concave down, left skewed depending on the value of the shape and scale parameters. Then, the cumulative distribution function (cdf) of the pdf can then be expressed as

$$\begin{aligned} F_{(IOW)}(s;\alpha,\beta,\phi) &= \int_0^{\frac{G(s;\phi)^{\frac{1}{\alpha}}}{1-G(s;\phi)^{\frac{1}{\alpha}}}} \beta s^{\beta-1} e^{-s^{\beta}} \\ &= 1 - exp\{-[\frac{G(s;\phi)^{\frac{1}{\alpha}}}{1-G(s;\phi)^{\frac{1}{\alpha}}}]^{\beta}\} \\ &= 1 - exp\left\{-[1 + \frac{1}{[(-1)^{r\beta}\sum_{r=1}^{\frac{1}{\alpha}} \frac{\frac{1}{\alpha}(\frac{1}{\alpha}-1)(\frac{1}{\alpha}-2)...(\frac{1}{\alpha}-r+1)}{r!} exp(-r\beta s^{\beta})]}]\right\}, \end{aligned} \tag{7}$$

where $G(s;\phi)$ is the cdf of the parent distribution.

Figure 2 shows the plots for cumulative density function (cdf) of the *IOW* distribution for several values of parameters. Thus, for a random variable X, we defined its distribution as $X \sim IOW(\alpha,\beta,\phi)$. Now, for a stochastic system with random variable T; we can defined its cdf by $G(s;\phi)$. Thus, we can let $G(s;\phi) = G(s)$ for $\alpha > 0$, for the random variable X, (Korkmaz et al.(2018)).

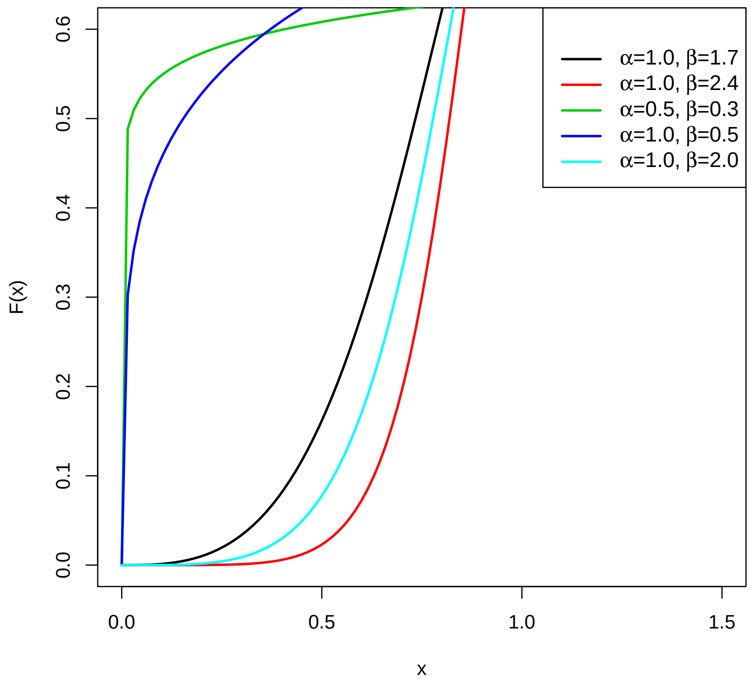

Figure 2: The cdf of the *IOW* distribution for different parameter values

2.1. Maximum Likelihood Estimator of the Inverse Odd Weibull Generated Family of Distribution

In this section, we shall estimate the parameters of the *IOW* model using maximum likelihood approach. Let $s_1, s_2, \cdots, s_n$ be a random sample of the *IOW* family of distribution with vector of parameter $\Theta = (\alpha, \beta, \phi)^u$ and ϕ is $q \times 1$ baseline vector of parameter. The likelihood function for the *IOW* distribution is given by

$$L_{IOW-G}(s) = \bar{\alpha}^n \beta^n \prod_{i=1}^{n} g^{\frac{1-\alpha}{\alpha}}(s) \frac{[G(s)]^{\frac{(\beta-1)}{\alpha}}}{[1-G(s)^{\bar{\alpha}}]^{(\beta+1)}} exp\left[-\sum_{i=1}^{n}\left[\frac{G(s)^{\bar{\alpha}}}{1-G(s)^{\bar{\alpha}}}\right]^{\beta}\right]. \tag{8}$$

Now, for $r_i = [1 - G(s,\phi)^{\frac{1}{\alpha}}]$, $\hat{\alpha} = \frac{1-\alpha}{\alpha}$ and $p_i = [\frac{G(s,\phi)^{\frac{1}{\alpha}}}{r_i}]^{\beta}$. Then, the log-likelihood function for the *IOW* can be written as

$$\begin{aligned} \ell(\Theta) =& n\log\bar{\alpha} + n\log\beta + \hat{\alpha}\sum_{i=1}^{n}\log g(s_i;\phi) + \frac{(\beta-1)}{\alpha}\sum_{i=1}^{n}\log G(s_i,\phi) \\ & -(\beta+1)\sum_{i=1}^{n}\log r_i - \sum_{i=0}^{n} p_i. \end{aligned} \tag{9}$$

Now, differentiating partially the nonlinear equation with respect to the parameters, Noting that $g'_{\gamma}(s_i,\phi) = \frac{\delta g(s_i,\phi)}{\delta\phi_{\gamma}}$; $G'(s_i,\phi) = \frac{\delta G(s_i,\phi)}{\delta\phi_{\gamma}}$; $z_i = \frac{\delta r_i}{\delta\alpha}$; $d_i = \frac{\delta p_i}{\delta\alpha}$; $t_i = \frac{\delta r_i}{\delta\phi_{\gamma}}$; and $q_i = \frac{\delta p_i}{\delta\beta}$. The vector component, say $U(\Theta) = \frac{\delta\ell}{\delta\Theta} = (\frac{\delta\ell}{\delta\alpha}, \frac{\delta\ell}{\delta\beta}, \frac{\delta\ell}{\delta\phi_{\gamma}})^u = (U_{\alpha}, U_{\beta}, U_{\phi_{\gamma}})^T$.
Thus,

$$U_{\alpha} = \frac{n}{\bar{\alpha}} + \hat{\alpha}'\sum_{i=1}^{n}\log g(s_i;\phi) - \frac{(\beta-1)}{\alpha^2}\sum_{i=1}^{n}\log G(s_i,\phi) - (\beta+1)\sum_{i=1}^{n}\frac{z_i}{r_i} - \sum_{i=0}^{n}\frac{d_i}{p_i}; \tag{10}$$

$$U_{\beta} = \frac{n}{\beta} + \frac{1}{\alpha}\sum_{i=0}^{n}\log G(s_i,\phi) - \sum_{i=0}^{n}\log r_i - \sum_{i=0}^{n} q_i; \tag{11}$$

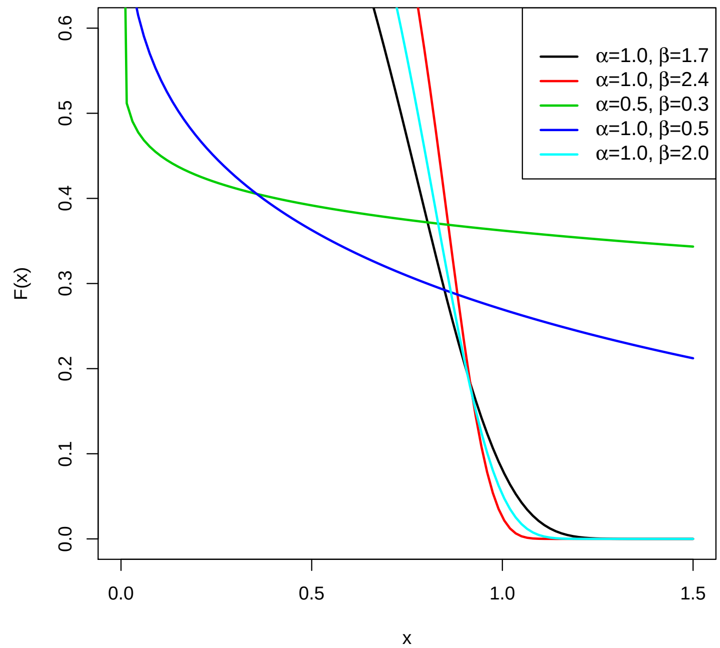

Figure 3: The Survival Function of the *IOW* Distribution for Different Parameter Values

$$U_{\phi_\gamma} = \sum_{i=0}^{n} \frac{g'(s_i,\phi)}{g(s_i,\phi)} + \frac{(\beta-1)}{\alpha}\sum_{i=0}^{n}\frac{G'(s_i,\phi)}{G(s_i,\phi)} - (\beta+1)\sum_{i=0}^{n}\frac{t_i}{r_i} - \sum_{i=0}^{n}\tau_i. \quad (12)$$

Setting $U_\alpha = U_\beta = U_{\phi_\gamma} = 0$ and solving the nonlinear equations yield the MLEs $\hat{\Theta} = (\hat{\alpha}, \hat{\beta}, \hat{\phi}^u)^u$. The Equations (10,11 and 12) are usually solved using the nonlinear quasi-Newton optimization method algorithm. This method maximize the $\ell(\Theta)$ numerically with $(q+2)\times(q+2)$ elements and a Jacobi matrix of $J(\Theta) = (\frac{\delta^2\ell}{\delta\phi_\gamma\phi_s})$. When $n \to \infty$, under standard regularity conditions, the distribution of Θ is approximately multivariate normal with mean zero and variance $J(\hat{\Theta})^{-1}$.

3. Some Main Properties of the Inverse Odd Weibull Generated Family of Distribution

3.1. Survival and Hazard Rate Functions of the Inverse Odd Weibull Generated Family of Distribution

We shall provide the survival and hazard rate function of the Inverse Odd Weibull Generated Family of Distribution. Now, the probability that the random variable $x > 0$ of interest will survived beyond any specific lifetime T is given as

$$S_{(IOW)}(x) = 1 - Pr_{(IOW)}(X \leq x);$$

$$S_{(IOW)}(x) = exp\{-[\frac{G(x)^{\frac{1}{\alpha}}}{1-G(x)^{\frac{1}{\alpha}}}]^{\beta}\}. \quad (13)$$

On the other hand, the hazard rate function or failure rate is the conditional density given that for $x > 0$, has not yet occurred prior to time t is given as

$$h_{(IOW)}(x) = \frac{f_{(IOW)}(x)}{S_{(IOW)}(x)} = \frac{\alpha\beta g^{\frac{1-\alpha}{\alpha}}(x;\phi)G(x;\phi)^{\frac{\beta-1}{\alpha}}}{[1-G(x;\phi)^{\frac{1}{\alpha}}]^{\beta-1}}. \quad (14)$$

Figure 3 is the plot of the survival function for the IOW generated family of distribution, while Figure 4 is the plot of the failure rate for the inverse odd Weibull generated family of distribution for some values of parameters.

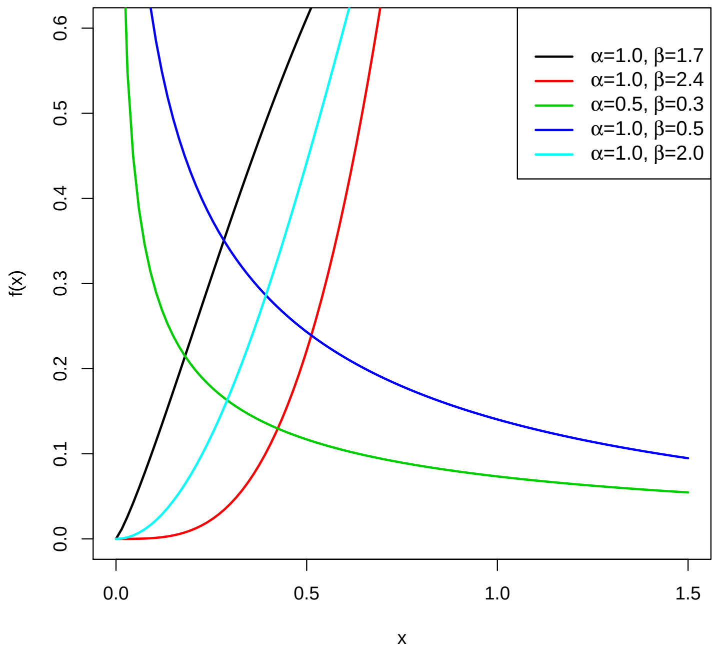

Figure 4: The Hazard Rate Functions of the *IOW* Distribution for Different Parameter Values

3.2. Quantiles Functions of the Inverse Odd Weibull Generated Family of Distribution

The quantiles function for $u \sim uniform[0,1]$ that describe the inverse of the cdf of the inverse odd Weibull generated family of distribution is given as

$$Q(u) = inf\{x \in \Re : u \leq F_{(IOW)}(x)\} = F^{-1}_{(IOW)}(u).$$

Now, let $F_{IOW}(s) = u$, then,

$$1 - u = exp[-[\frac{G(s;\phi)^{\frac{1}{\alpha}}}{1 - G(s;\phi)^{\frac{1}{\alpha}}}]^{\beta},$$

$$-\log(1-u) = [\frac{G(s;\phi)^{\frac{1}{\alpha}}}{1 - G(s;\phi)^{\frac{1}{\alpha}}}]^{\beta},$$

$$[-\log(1-u)]^{\frac{1}{\beta}} = [\frac{G(s;\phi)^{\frac{1}{\alpha}}}{1 - G(s;\phi)^{\frac{1}{\alpha}}}],$$

$$[-\log(1-u)]^{\frac{1}{\beta}} - [-\log(1-u)]^{\frac{1}{\beta}} G(s;\phi)^{\frac{1}{\alpha}} = G(s;\phi)^{\frac{1}{\alpha}},$$

$$G(s;\phi)^{\frac{1}{\alpha}} = \frac{[-\log(1-u)]^{\frac{1}{\beta}}}{1 + [-\log(1-u)]^{\frac{1}{\beta}}},$$

$$G(s;\phi) = [\frac{[-\log(1-u)]^{\frac{1}{\beta}}}{1 + [-\log(1-u)]^{\frac{1}{\beta}}}]^{\alpha},$$

but $G(s;\phi) = 1 - exp - s^{\beta}$. Then,

$$1 - e^{-s^{\beta}} = [\frac{[-\log(1-u)]^{\frac{1}{\beta}}}{1 + [-\log(1-u)]^{\frac{1}{\beta}}}]^{\alpha},$$

$$1-[\frac{[-\log(1-u)]^{\frac{1}{\beta}}}{1+[-\log(1-u)]^{\frac{1}{\beta}}}]^{\alpha}=e^{-s^{\beta}},$$

$$-\log[1-[\frac{[-\log(1-u)]^{\frac{1}{\beta}}}{1+[-\log(1-u)]^{\frac{1}{\beta}}}]^{\alpha}]=s^{\beta}$$

$$[-\log[1-[\frac{[-\log(1-u)]^{\frac{1}{\beta}}}{1+[-\log(1-u)]^{\frac{1}{\beta}}}]^{\alpha}]]^{\frac{1}{\beta}}=s.$$

Therefore, the quantile function of the Inverse Odd Weibull Generated Family of Distribution is defined as

$$Q(u)=[-\log[1-[\frac{[-\log(1-u)]^{\frac{1}{\beta}}}{1+[-\log(1-u)]^{\frac{1}{\beta}}}]^{\alpha}]]^{\frac{1}{\beta}}, \tag{15}$$

where $u=0.25, u=0.50$, and $u=0.75$, are the first three quantiles respectively (see Nzei et al.(2020)).
The median for the inverse odd Weibull generated family of distribution is obtained as

$$M=[-\log[1-[\frac{[-\log(\frac{1}{2})]^{\frac{1}{\beta}}}{1+[-\log(\frac{1}{2})]^{\frac{1}{\beta}}}]^{\alpha}]]^{\frac{1}{\beta}} \quad 0<u<1. \tag{16}$$

Hence, the 25^{th} percentile and the 75^{th} percentile are given as

$$Q_1=[-\log[1-[\frac{[-\log(\frac{3}{4})]^{\frac{1}{\beta}}}{1+[-\log(\frac{3}{4})]^{\frac{1}{\beta}}}]^{\alpha}]]^{\frac{1}{\beta}} \quad 0<u<1, \tag{17}$$

and

$$Q_3=[-\log[1-[\frac{[-\log(\frac{1}{4})]^{\frac{1}{\beta}}}{1+[-\log(\frac{1}{4})]^{\frac{1}{\beta}}}]^{\alpha}]]^{\frac{1}{\beta}} \quad 0<u<1. \tag{18}$$

3.3. Kurtosis and Skewness of the Inverse Odd Weibull Generated Family of Distribution

The kurtosis coefficient measures the heavy tailed or thin tailed of the data, while the skewness measures the symmetry nature of the distribution. The Moors' (1988) kurtosis for the IOW family of distribution is given by

$$K=\frac{Q(\frac{7}{8})-Q(\frac{5}{8})-Q(\frac{3}{8})+Q(\frac{1}{8})}{Q(\frac{6}{8})-Q(\frac{2}{8})}. \tag{19}$$

The Bowley's (1939) skewness for the IOW family of distribution is as follows

$$S=\frac{Q(\frac{3}{4})-2Q(\frac{1}{2})+Q(\frac{1}{4})}{Q(\frac{3}{4})-Q(\frac{1}{4})}. \tag{20}$$

Figures 5 and 6 show the Bowley's skewness and the Moors' kurtosis for the IOW model.

Table 1 shows the simulation for different values of parameters for kurtosis and skewness of the IOW family of distribution. The kurtosis and skewness in Table 1 increases as the parameters increases. The median, the first and third quartiles have irregular values as the values of the parameters increases.

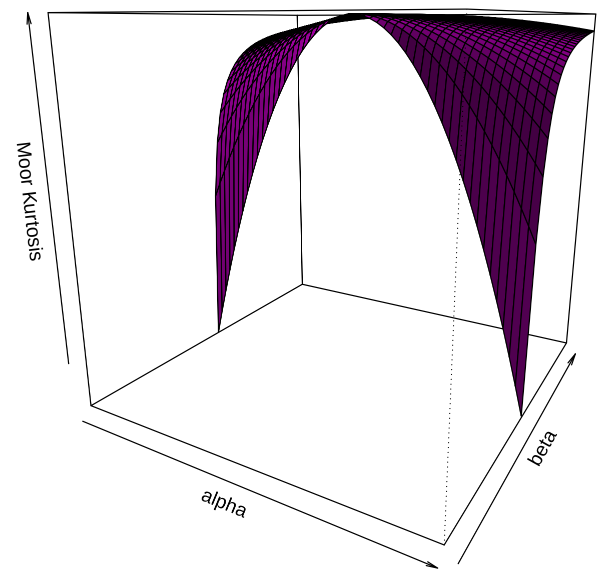

Figure 5: The Plots of the kurtosis for the *IOW* distribution

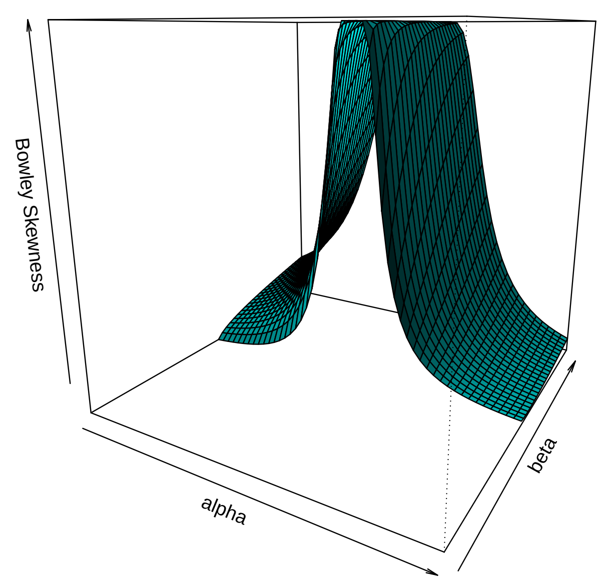

Figure 6: The Plots of the skewness for the *IOW* distribution

Table 1: Kurtosis and Skewness of the IOW Family of Distribution

Parameters β	α	Kurtosis	Skewness	Median	25^{th}	75^{th}
0.5	2	3.37715	0.923086	0.01238538	3.43333e-05	0.321199600
	5	10.96644	0.9984929	1.300555e-05	6.806615e-12	0.017259040
	8	34.62994	0.9999762	1.514099e-08	1.357338e-18	0.001271685
	15	584.0233	1.0000	2.17615e-15	0.0000	3.487949e-06
	20	4680.968	1.0000	2.820019e-20	0.0000	5.277707e-08
1.0	2	0.587385	0.2664208	0.1834364	0.05120107	0.4117219
	5	1.720512	0.6757537	0.01156546	0.00055673	0.06846023
	8	3.098746	0.8800111	0.000789255	6.206325e-06	0.01305822
	15	9.473921	0.9895068	1.520413e-06	1.724099e-10	0.0002897574
	20	20.23337	0.9981761	1.748296e-08	9.592327e-14	1.917048e-05
1.5	2	0.06069753	0.0228482	0.3581327	0.2105999	0.5125649
	5	0.567609	0.2560765	0.06476073	0.01880715	0.142351
	8	1.148213	0.4824974	0.01243047	0.001731131	0.04308101
	15	2.697187	0.8089838	0.0002671704	6.635123e-06	0.002734522
	20	4.189216	0.9106184	1.720747e-05	1.246806e-07	0.0003823684
2.0	2	-0.09883353	-0.05708644	0.4808086	0.3605138	0.5881106
	5	0.156377	0.06922978	0.1398003	0.07210623	0.2175644
	8	0.4774112	0.2156471	0.04262068	0.01485517	0.0856537
	15	1.29324	0.5151923	0.002692584	0.000373444	0.009940702
	20	1.921226	0.6704989	0.0003745916	2.689268e-05	0.00213735
5.0	2	-0.2264367	-0.1245243	0.766156	0.733985	0.791202
	5	-0.2035211	-0.1119634	0.4829469	0.4387371	0.5182538
	8	-0.1642355	-0.09029601	0.3108447	0.2669671	0.3474546
	15	-0.05883125	-0.0340803	0.1117593	0.08404231	0.1376493
	20	0.02076863	0.006404327	0.05383239	0.03681265	0.07107154
7.0	2	-0.2307446	-0.1267604	0.8296708	0.8119627	0.8433945
	5	-0.2231748	-0.1226319	0.5992527	0.5711622	0.6212062
	8	-0.2047296	-0.1122596	0.4394404	0.4072773	0.4651111
	15	-0.1539223	-0.08407185	0.2139213	0.1855394	0.2379011
	20	-0.1161175	-0.06361941	0.1279395	0.1058233	0.1474099
10	2	-0.2301873	-0.1263226	0.8790953	0.8699506	0.8861887
	5	-0.2293141	-0.1259152	0.7016092	0.6855413	0.7140833
	8	-0.2211255	-0.1212714	0.5660098	0.5456443	0.58197
	15	-0.1971436	-0.1076952	0.3438809	0.321078	0.3622498
	20	-0.1794216	-0.097783	0.2409221	0.2198604	0.2582318

3.4. Cumulative Hazard Function of the Inverse Odd Weibull Generated Family of Distribution

The cumulative hazard function is obtained as the integral of the hazard function as

$$H_{(IOW_G)}(t) = \int_0^t h(u)du = -\ln S(t) = [\frac{G(s;\phi)^{\frac{1}{\alpha}}}{1-G(s;\phi)^{\frac{1}{\alpha}}}]^{\beta}. \tag{21}$$

This can be also be translated as

$$S(t) = e^{-[\frac{G(s;\phi)^{\frac{1}{\alpha}}}{1-G(s;\phi)^{\frac{1}{\alpha}}}]^{\beta}}, \tag{22}$$

and

$$f(t) = h(t)e^{-[\frac{G(s;\phi)^{\frac{1}{\alpha}}}{1-G(s;\phi)^{\frac{1}{\alpha}}}]^{\beta}}. \tag{23}$$

3.5. Reversed Hazard Function of the Inverse Odd Weibull Generated Family of Distribution

This is calculated as the ratio of the pdf to the cdf of the inverse odd Weibull generated family of distribution . It is given as

$$r(s) = \frac{\frac{\alpha\beta g^{\frac{1-\alpha}{\alpha}}(s;\phi)G(s;\phi)^{\frac{\beta-1}{\alpha}}}{[1-G(s;\phi)^{\frac{1}{\alpha}}]^{\beta+1}} exp[-[\frac{G(s;\phi)^{\frac{1}{\alpha}}}{1-G(s;\phi)^{\frac{1}{\alpha}}}]^{\beta}]}{1-exp[-[\frac{G(s;\phi)^{\frac{1}{\alpha}}}{1-G(s;\phi)^{\frac{1}{\alpha}}}]^{\beta}]}. \tag{24}$$

3.6. The Odds Function of the Inverse Odd Weibull Generated Family of Distribution

The Odd function is given as

$$O(s) = exp[\frac{G(s;\phi)^{\frac{1}{\alpha}}}{1-G(s;\phi)^{\frac{1}{\alpha}}}]^{\beta} - 1 \tag{25}$$

3.7. The k^{th} Order Statistics for the Inverse Odd Weibull Generated Family of Distribution

The k^{th} Order Statistics for n variables for the family of distribution is given as

$$\begin{aligned} g_k(y_k) =& \frac{n!}{(k-1)!(n-k)!}\{1-exp\{-[\frac{G(s;\phi)^{\frac{1}{\alpha}}}{1-G(s;\phi)^{\frac{1}{\alpha}}}]^{\beta}\}\}^{k-1} \\ &\times\{\frac{\alpha\beta g^{\frac{1-\alpha}{\alpha}}(s;\phi)G(s;\phi)^{\frac{\beta-1}{\alpha}}}{[1-G(s;\phi)^{\frac{1}{\alpha}}]^{\beta+1}} exp\{-[\frac{G(s;\phi)^{\frac{1}{\alpha}}}{1-G(s;\phi)^{\frac{1}{\alpha}}}]^{\beta}\}\}\{exp\{-[\frac{G(s;\phi)^{\frac{1}{\alpha}}}{1-G(s;\phi)^{\frac{1}{\alpha}}}]^{\beta}\}\}^{n-k}. \end{aligned} \tag{26}$$

The distribution of the median for $k = m + 1$ and $n = 2m + 1$, when n is odd is given as

$$\begin{aligned} g_{m+1}(y_{m+1}) =& \frac{(2m+1)!}{(m)!(m)!}\{1-exp\{-[\frac{G(s;\phi)^{\frac{1}{\alpha}}}{1-G(s;\phi)^{\frac{1}{\alpha}}}]^{\beta}\}\}^{m} \\ &\times\{\frac{\alpha\beta g^{\frac{1-\alpha}{\alpha}}(s;\phi)G(s;\phi)^{\frac{\beta-1}{\alpha}}}{[1-G(s;\phi)^{\frac{1}{\alpha}}]^{\beta+1}} exp\{-[\frac{G(s;\phi)^{\frac{1}{\alpha}}}{1-G(s;\phi)^{\frac{1}{\alpha}}}]^{\beta}\}\}\{exp\{-[\frac{G(s;\phi)^{\frac{1}{\alpha}}}{1-G(s;\phi)^{\frac{1}{\alpha}}}]^{\beta}\}\}^{m}. \end{aligned} \tag{27}$$

The distribution of the median for $k = m + 1$ and $n = 2m$, when n is even is given as

$$\begin{aligned} g_{m+1}(y_{m+1}) =& \frac{(2m)!}{(m)!(m-1)!}\{1-exp\{-[\frac{G(s;\phi)^{\frac{1}{\alpha}}}{1-G(s;\phi)^{\frac{1}{\alpha}}}]^{\beta}\}\}^{m} \\ &\times\{\frac{\alpha\beta g^{\frac{1-\alpha}{\alpha}}(s;\phi)G(s;\phi)^{\frac{\beta-1}{\alpha}}}{[1-G(s;\phi)^{\frac{1}{\alpha}}]^{\beta+1}} exp\{-[\frac{G(s;\phi)^{\frac{1}{\alpha}}}{1-G(s;\phi)^{\frac{1}{\alpha}}}]^{\beta}\}\}\{exp\{-[\frac{G(s;\phi)^{\frac{1}{\alpha}}}{1-G(s;\phi)^{\frac{1}{\alpha}}}]^{\beta}\}\}^{m-1}. \end{aligned} \tag{28}$$

3.8. The Entropies

The measure of the uncertainty of the random variable X is either measure as Renyi entropy is defined as

$$I_R(\theta) = \frac{1}{1-\theta}\log[\int_0^{\infty} f^{\theta}_{IOW_G}(s)ds], \quad for\theta > 0 \ \ and \ \ \theta \neq 1. \tag{29}$$

Also, the Shannon entropy of the random variable S is defined as

$$E[-log[f(S)]] = -\log\alpha - \log\beta - E[\log[g^{\frac{1-\alpha}{\alpha}}(s;\phi)]] - \frac{(\beta-1)}{\alpha}E[\log[G(s;\phi)]] \\ + (\beta+1)E[\log[1-G(s;\phi)^{\frac{1}{\alpha}}]] + E[[\frac{G(s;\phi)}{1-G(s;\phi)}]^{\beta}]. \tag{30}$$

4. Real Life Application

A real life dataset is applied to the proposed model to test the performance of the model based on statistic
Several criteria were used to determine the distribution for the best fit: Akaike Information Criteria (AIC), Consistent Akaike Information Criteria (CAIC), Bayesian Information Criteria (BIC), and Hannan and Quinn Information Criteria (HQIC). The Anderson Darling (A) statistic, Cramér–von Mises statistic the value (W), the Kolmogorov Smirnov (KS) statistic, and the p value were also provided.

4.1. Carbon Data

Our first set of data is from Nichols and Padgett(2006) and Eghwerido et al.(2020a). It consists of 100 observations taken on breaking stress of carbon fibers (in Gba). The dataset are as follow: 3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 3.27, 2.87, 1.47, 3.11,4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.9, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53,2.67, 2.93, 3.22, 3.39, 2.81, 4.2, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59,2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19,1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69,1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.7, 2.03, 1.8, 1.57, 1.08, 2.03, 1.61, 2.12,1.89, 2.88, 2.82, 2.05, 3.65.
The descriptive statistics of the carbon fibers dataset are showed in Table 2. We observed that the carbon fibers dataset

Table 2: Descriptive statistics for the Carbon Fibers dataset to 2 decimal points

Mean	Median	Mode	St.D	IQR	Variance	Skewness	Kurtosis	$25^{th}P.$	$75^{th}P.$
2.62	2.70	"2.17""3.68"	1.01	1.38	1.03	0.37	3.10	1.84	3.22

has mode of 2.17 and 3.68 and is slightly positive skewness because the value is close to zero. The mean of the carbon fibers dataset values is smaller than the median. Thus, the carbon fibers dataset distribution is left-skewed. Also, the carbon fibers dataset is positive excess kurtosis. This indicates that the carbon fibers dataset is a fat-tailed distribution, and is said to be leptokurtic.
Table 3. is the goodness-of-fit and the performance rating of the *IOW* distribution using several test statistics for the carbon fibers dataset. Table 4 is the test statistic for the different distributions examined.

4.2. Glass Fiber Data

The data on 1.5 cm strengths of glass fibres were obtained by workers at the UK National Physical Laboratory was also used to compare the performance of the *IOW* distribution as used by Smith and Naylor (1987), Bourguinon et al.(2014), Efe-Eyefia et al.(2020), Eghwerido et al.(2019), Eghwerido et al.(2020b), Eghwerido et al.(2020c) Eghwerido et al.(2020d), Eghwerido et al.(2020e), Eghwerido et al.(2020f), Merovci et al.(2016), Oguntunde et al.(2017a) and Oguntunde et al.(2017b). The observations are as follows: 0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28, 1.29, 1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50, 1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63, 1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68, 1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82, 1.84, 1.84, 2.00, 2.01, 2.24.
The descriptive statistics of the glass fibers dataset are showed in Table 5. Table 6 is the measure of comparison for the various distribution under consideration. A plot of some distributions against the empirical histogram of the glass fiber data is as shown in Figure 7. This is to demonstrate the performance of the *IOW* distribution. Also, a plot for the empirical cdf of the competing *IOW* distribution of the glass fiber data is shown in Figure 6. The plots in Figures 7 and

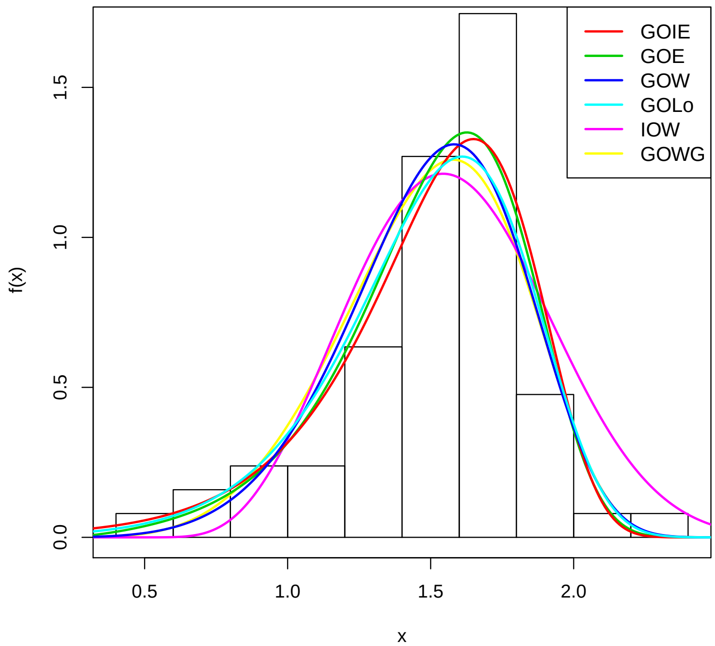

Figure 7: A Plot of Distributions with the Empirical Histogram of the Glass Fiber data

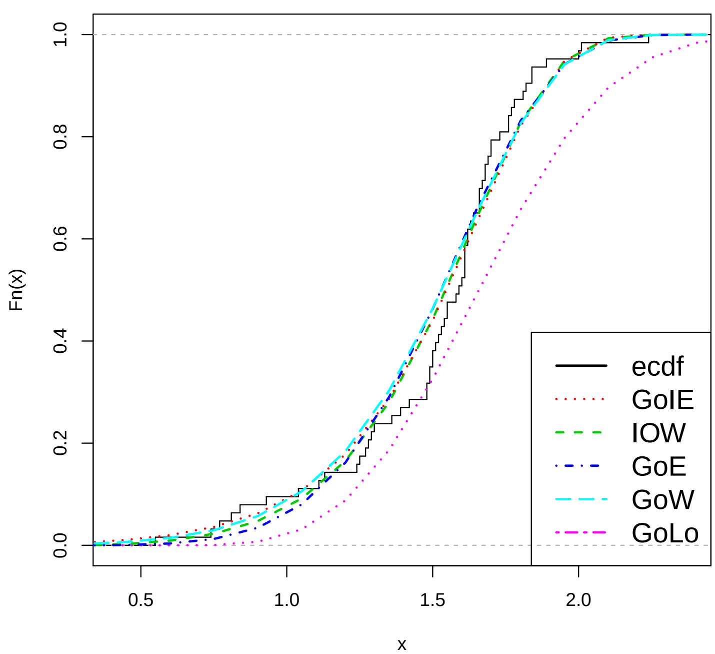

Figure 8: A Plot of Empirical cdf of the distributions of the Glass Fiber data

Table 3: Performance Rating of the IOW distribution with Carbon Fibers Dataset.

Distribution	Parameter	AIC	CAIC	BIC	HQIC	W	A
IOW	$\hat{\alpha} = 4085.1225855$ $\hat{\beta} = -0.2288471$	-2029.215	-2029.092	-2024.005	-2027.107	0.3537269	1.933555
GOWG-G	$\hat{\alpha} = 0.3513662$ $\hat{\beta} = -1.8754485$	301.2179	301.3416	306.4283	303.3266	0.2250905	1.234457
GoW	$\hat{\alpha} = 2.2593908$ $\hat{\beta} = -0.2016595$ $\hat{a} = 0.2649778$ $\hat{b} = 2.9807809$	290.5644	290.9854	300.985	294.7818	0.06475948	0.3833602
GoLo	$\hat{\alpha} = 0.009063479$ $\hat{\beta} = 5.065619209$ $\hat{a} = 1.984851851$ $\hat{b} = 0.647117443$	292.8646	293.2857	303.2853	297.0821	0.06109876	0.4763095
GoE	$\hat{\alpha} = 0.08423018$ $\hat{\beta} = 0.86595286$ $\hat{a} = 0.91342723$	304.2500	304.5000	312.0656	307.4131	0.15973880	1.260799
GoIE	$\hat{\alpha} = 0.01879975$ $\hat{\beta} = 3.10656725$ $\hat{a} = 0.63392041$	289.8412	290.0912	297.6567	293.0043	0.06032237	0.4457551

Table 4: Test statistic for the IOW distribution with Carbon Fibers Dataset

Distributions	KS	P Value	Log-likelihood
IOW	0.8711368	0	-1016.608
GOWG	0.1329093	0.05843415	148.609
GoW	0.0632502	0.8185524	141.2822
GoIE	0.06232949	0.8320187	141.9206
GoE	0.09621573	0.312806	149.125
GoLo	0.06365319	0.8125448	142.4323

8 show the *IOW* distribution is more suitable for the glass data than the other competing distributions. Table 7 shows the Kolmogorov-Smirnov Test, p-value and the log-Likelihood test of the various distributions under consideration for glass fiber dataset.

4.3. Discussion

The performance of the models are determined by the value that corresponds to the lowest Akaike Information Criteria (AIC) or the highest Log-likelihood value is regarded as the *best* model. In the two real life cases considered, the *IOW* distribution has the lowest AIC value with -2029.215 and -54.64913 respectively. Also, the *IOW* has the highest value of Log-likelihood of 1016.608and 29.32457 respectively. Hence, it is regarded as a better model for the data used.

Table 5: Descriptive statistics for the Glass Fibers dataset to 2 decimal points

Mean	Median	Mode	St.D	IQR	Variance	Skewness	Kurtosis	$25^{th}P.$	$75^{th}P.$
1.51	1.59	1.61	0.32	0.31	0.11	-0.81	0.80	1.38	1.69

Table 6: Performance Rating of the IOW-G distribution with Glass Fibers Dataset

Distribution	Parameter	AIC	CAIC	BIC	HQIC	W	A
IOW	$\hat{\alpha} = 2.300551$ $\hat{\beta} = -2.579208$	-54.64913	-54.44913	-50.36286	-52.96332	0.56319	3.047042
GOWG-G	$\hat{\alpha} = 4.499297$ $\hat{\beta} = 1.373304$	35.36342	35.56342	39.64969	37.04923	0.2540335	1.397502
GoW	$\hat{\alpha} = 0.224534626$ $\hat{\beta} = 0.009237687$ $\hat{a} = 0.797323628$ $\hat{b} = 5.617617029$	38.37693	39.06658	46.94946	41.74855	0.2329559	1.283159
GoLo	$\hat{\alpha} = 0.004592168$ $\hat{\beta} = 8.179090952$ $\hat{a} = 0.506999372$ $\hat{b} = 1.515829086$	37.00548	37.69513	45.57802	40.3771	0.1685148	0.9461908
GoE	$\hat{\alpha} = -0.004768848$ $\hat{\beta} = -1.810999364$ $\hat{a} = -1.987714978$	35.6353	36.04208	42.06471	38.16402	0.144475	0.8424945
GoIE	$\hat{\alpha} = 0.2030548$ $\hat{\beta} = 11.5541435$ $\hat{a} = 2.0003185$	34.08187	34.48865	40.51127	36.61059	0.1644603	0.9253032

Table 7: Test statistic for the IOW-G distribution with Glass Fibers Dataset

Distributions	KS	p Value	Log-likelihood
IOW	0.2464499	0.0009493034	29.32457
GOWG	0.1669343	0.05971754	15.68171
GoW	0.1520247	0.108711	15.18846
GoIE	0.1327289	0.2169969	14.04093
GoE	0.1313551	0.2271038	14.81765
GoLo	0.1542228	0.09987383	14.50274

5. Concluding Remarks

The inverse odd Weibull generated family distribution has been successfully derived; expressions for its basic statistical properties which include the cumulative hazard function, reversed hazard function, and quantile, median, hazard function, odds function and the order statistics distribution have been successfully established. The shape of the distribution could be inverted bathtub or decreasing (depending on the value of the parameters). An application to a two real life data shows that the inverse odd Weibull general family of distribution is a strong and better competitor for other families of distributions.

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