Extended Weibull Burr XII Distribution: Properties and Applications

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Abstract
A new distribution called the Weibull Generalized Burr XII distribution is introduced along with simple physical motivation. The new distribution includes fourteen sub-models, seven of them are new. The new model can be used in modeling bimodal data sets. Set of its properties are derived in details. Two applications are provided to illustrate the importance of the new model. The new model is better that other competitive models via two applications. The method of maximum likelihood is used to estimate the unknown parameters.

Keywords: Burr XII Distribution; Maximum Likelihood; Generating Function; Moments.

1. Introduction and physical motivation

A special attention has been devoted to one of twelve models introduced by Burr (1942) denoted by type XII (see Burr (1942), (1968) and (1973), Burr and Cislak (1968), Hatke (1949) and Rodriguez (1977)), the cumulative distribution function (CDF) of Burr type XII (BXII) is given as

$$G_{(a,b)}(x) = 1 - (1 + x^a)^{-b},$$

both a and b are shape parameters, when $a = 1$ the BXII model reduces to the Lomax (Lx) or Pareto type II (PaII) model and when $b = 1$ the BXII model reduces to the log-logistic (LL) model. The corresponding probability density function (PDF) is given by

$$g_{(a,b)}(x) = abx^{a-1}(1 + x^a)^{-b-1}.$$

For more detailes about the BXII model see Burr (1942, 1968 and 1973), Burr and Cislak (1968), Rodriguez (1977) and Tadikamalla (1980).

Let $g_{(a,b)}(x)$ and $G_{(a,b)}(x)$ denote the PDF and the CDF of the BXII model with parameter vector $\boldsymbol{\xi} = (a, b)$. Then the CDF of the Weibull Generalized-BXII (WGBXII) based on Yousof et al. (2018b) is defined by

$$F(x) = 1 - \exp\left\{-\left[(1 + x^a)^{\theta b} - 1\right]^{\beta}\right\}, \quad (1)$$

and its corresponding PDF is given by

$$f(x) = \beta\, \theta abx^{a-1} \frac{(1+x^a)^{b\theta-1}}{\left[(1+x^a)^{\theta b}-1\right]^{-\beta+1}} \exp\left\{-\left[(1 + x^a)^{\theta b} - 1\right]^{\beta}\right\}, \quad (2)$$

where $F(x) = F^{(\mathbf{\Theta})}_{(\mathrm{WGBXII})}(x)$, $f(x) = f^{(\mathbf{\Theta})}_{(\mathrm{WGBXII})}(x)$, $\mathbf{\Theta} = (\beta, \theta, a, b)$, $\beta > 0$ and $\theta > 0$ are two additional shape parameters. From Figure 1(a) we conclude that the the PDF of the WGBXII model can be bimodal and left skewed, from Figure 1(b) the PDF of the WGBXII model can be unimodal and right skewed. From Figure 1(c) the HRF can be bathtub, constant, unimodal, decreasing and increasing shaped.

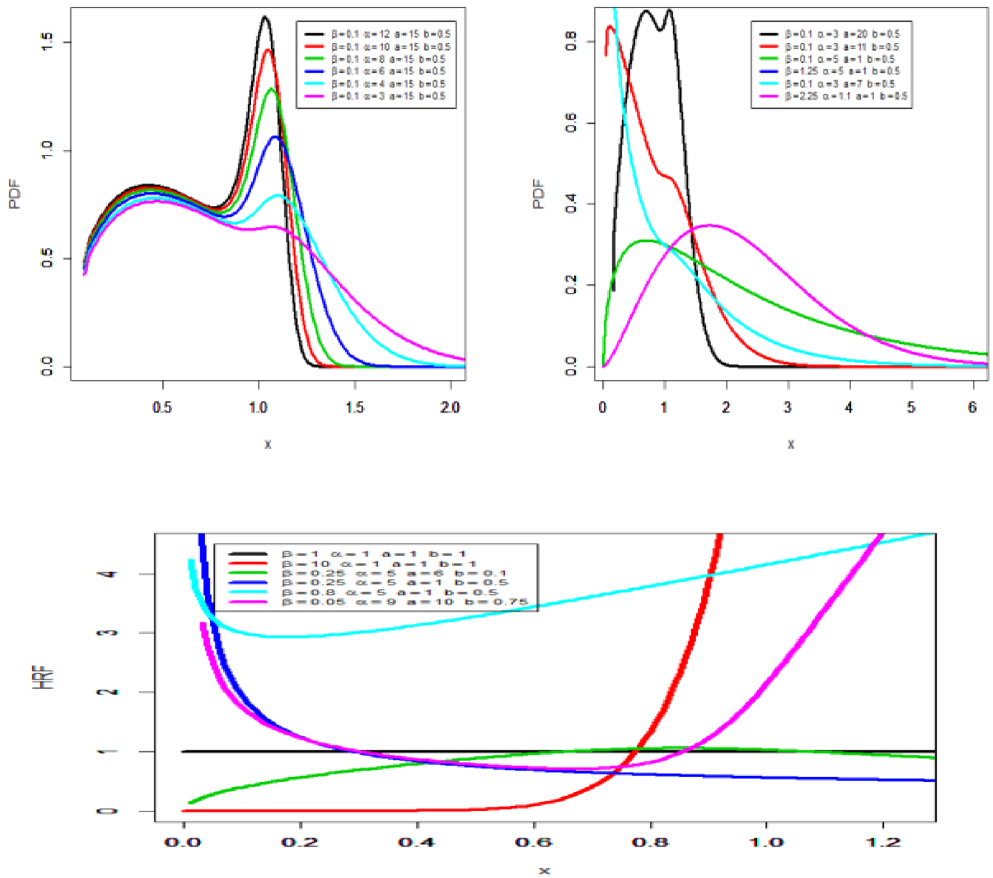


Figure 1: Plots of PDF and HRF for the new model.

Some other useful extension of the Burr XII can be found in Alizadeh et al. (2017a, b, c), Altun et al. (2017), Yousof et al. (2017), Hamedani et al. (2017), Brito et al. (2017), Aryal and Yousof (2017a, b), Merovci et al. (2017), Nasir et al. (2018), Cordeiro et al. (2018), Afify et al. (2018), Hamedani et al. (2018), Yousof et al. (2018a, b), Korkmaz et al. (2018a, b), Altun et al. (2018a, b), Alizadeh et al. (2018), Ibrahim (2019), Korkmaz et al. (2019), Hamedani et al. (2019), Nascimento et al. (2019), Alizadeh et al. (2019) and Yousof et al. (2019).

The additional parameters β and θ are sought as a manner to furnish a more flexible BXII distribution (see Figure 1). In this work, we study the WGBXII model and give a sufficient description of its mathematical properties. The new model is motivated by its important flexibility in applications (see section. 4), by means of two applications, it is noted that the WGBXII model provides better fits than nine BXII models. The PDF of the WGBXII model can be expressed as

$$f(x) = \sum_{r=0}^{\infty} v_r g_{(a,(1+r)b)}(x) \tag{3}$$

where $g_{(a,(1+r)b)}(x)$ is the BXII density with parameters a and $(1+r)b$, $\upsilon_r = -w_r$ and

$$w_r = \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k+r}}{i!r!} \binom{i\beta}{j} \binom{\theta(j-i\beta)}{k} \binom{1+k}{r}, \tag{4}$$

Similarly, the CDF (2) of X can be expressed in the mixture form

$$F(x) = \sum_{r=0}^{\infty} \upsilon_r G_{(a,(1+r)b)}(x),$$

where $G_{(a,(1+r)b)}(x)$ is the BXII CDF with parameters a and $(1+r)b$. A physical interpretation of the WGBXII distribution can be shown as follows: suppose that we have a lifetime r.v., Z, having BXII distribution. The generalized ratio

$$\left\{1-\left[1-G_{(a,b)}(x)\right]^{\theta}\right\} / \left[1-G_{(a,b)}(x)\right]^{\theta},$$

that an individual (or component) following the lifetime Z will die (fail) at time t is

$$(1+x^a)^{\theta b}-1.$$

Consider that the variability of this ratio of death is represented by the r.v. X and assume that it follows the Weibull model with shape γ. We can write

$$\Pr(Z \le x) = \Pr\left(X \le (1+x^a)^{\theta b}-1\right) = F(x),$$

which is given by (1).

2. Mathematical properties

The n th ordinary moment of X is given by

$$\mu_n' = \mathbf{E}(X^n) = \int_{-\infty}^{\infty} x^n f(x) dx.$$

Then, we obtain

$$\mu_n' = \sum_{r=0}^{\infty} \upsilon_r (1+r)b\, B\left(1+\frac{n}{a}, (1+r)b-\frac{n}{a}\right)|_{(n<(1+r)ab)}, \tag{5}$$

where

$$B(a,b) = \int_0^{\infty} t^{a-1} (1+t)^{-(a+b)} dt$$

is the beta function of the second type. By setting $n=1$ in (5), we get the mean of X. The last integration is computed numerically for the new distributions (see Table 1). The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. The mean, variance, skewness and kurtosis of the WGBXII distribution are computed numerically for $\alpha = 0.5,1,2,4$ and some selected values of θ, a and β using the R software. The skewness of the WGLx distribution can range in the interval $(-2,86)$, whereas the kurtosis of the WGBXII distribution varies in the interval $(-194,5087.7)$ also the mean of X decreases as θ increases (see Table 1).

Table 1: Mean, variance, skewness and kurtosis of the WGBXII distribution.

α	θ	a	b	Mean	Variance	Skewness	Kurtosis
0.5	1	1.25	1.25	0.8548112	0.9869963	5.148905	38.25568
	1.5			0.5815021	0.2352436	4.675354	21.80937
	2			0.4589585	0.09435112	5.431293	18.57533
	2.5			0.3871006	0.04671672	7.016677	17.73847
	4			0.2772137	0.008408373	23.13982	9.333855

	4.5			0.2559157	0.004579471	42.75931	-15.71799
1	0.5	1	1	4	40	4.869908	48.96
	1			1	1	2	9
	2			0.3789361	0.09853526	1.253915	4.772774
	4			0.167348	0.01623477	0.9662979	3.691282
	10			0.06239426	0.002051677	0.8134962	3.230566
	20			0.03049457	0.0004752057	0.7647088	2.78825
	50			0.01203411	$7.266611\times e^{-5}$	0.4640694	6.717835
2	0.20	1.5	0.5	10.36408	3792.654	42.74035	5087.712
	0.25			6.151614	442.6206	17.4018	655.488
	0.5			2.271274	6.082826	4.654704	26.38992
	1			1.330393	0.07835578	86.3761	-194.1453
4	0.5	0.5	1.5	3.315232	9.121651	-2.005429	3.637416
	1			0.1824022	0.1137832	-0.0383349	0.8227737
	2			0.01555998	0.0021059	0.9741917	2.022906
	4			0.001617789	$3.960565\times e^{-5}$	1.423399	2.828123
	10			$2.976723\times e^{-5}$	$1.121697\times e^{-6}$	1.938526	3.866080

The moment generating function (MGF) $M_X(t) = \mathbf{E}(e^{t\,X})$ of X can be derived from (3) as

$$M_X(t) = \sum_{r,n=0}^{\infty} \frac{t^n}{n!} v_r (1+r) b\, B\left(1 + \frac{n}{a}, (1+r)b - \frac{n}{a}\right) |_{(n<(1+r)ab)},$$

The n th incomplete moment ($\mathbf{I}_n(t)$) of X can be expressed from (3) as

$$\mathbf{I}_n(t) = \int_{-\infty}^{t} x^n f(x) dx = \sum_{r=0}^{\infty} v_r (1+r) b\, B\left(t^a; 1 + \frac{n}{a}, (1+r)b - \frac{n}{a}\right) |_{(n<(1+r)ab)},$$

where

$$B(q; a, b) = \int_0^q t^{a-1}\,(1+t)^{-(a+b)} dt$$

is the incomplete beta function of the second type, we $n = 1$ we have 1 st incomplete moment, the main applications of the 1 st incomplete moment refer to the mean deviations and the Bonferroni and Lorenz curves which are very useful in economics, reliability, demography, insurance and medicine.

2.1 Probability weighted moments

The (s,r) th PWM of X following the WGBXII model, say $\lambda_{s,r}$, is formally defined by

$$\lambda_{s,r} = \mathbf{E}\{X^s\, F(X)^r\} = \int_{-\infty}^{\infty} x^s\, F(x)^r\, f(x)\, dx.$$

The (s,r) th PWM of X can be expressed as

$$\lambda_{s,r} = \sum_{r=0}^{\infty} v_r (1+r) b\, B\left(\frac{s}{a} + 1, (1+r)b - \frac{s}{a}\right) |_{(s<(1+r)ab)},$$

where

$$v_r = \beta\,\theta \sum_{m,i,j,k=0}^{\infty} \frac{(-1)^{m+i+j+k}(m+1)^i}{m!\,i!}(r)_k \times \binom{1+k}{r}\binom{(1+i)\beta-1}{j}\binom{\theta[-(1+i)\beta+j]-1}{k},$$

and

$$(\tau_1)_{\tau_2} = \tau_1(\tau_1-1)\ldots(1+\tau_1-\tau_2)$$

is the descending factorial and τ_2 is a positive integer.

2.2 Order statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample (RS) from the WGBXII distribution and let $X_{1:n}, \ldots, X_{n:n}$ be the corresponding order statistics. The PDF of i th order statistic, say $X_{i:n}$, can be written as

$$f_{i:n}(x) = B^{-1}(i, n-i+1) f(x) \textstyle\sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F^{j+i-1}(x), \qquad (6)$$

where $B(\cdot,\cdot)$ is the beta function. Inserting (1) and (2) in equation (6) and using a power series expansion, we have

$$F^{j+i-1}(x) f(x) = \sum_{r=0}^{\infty} c_r g_{(a,(1+r)b)}(x),$$

where

$$c_r = \beta\,\theta \sum_{m,l,w,k=0}^{\infty} \sum_{j=0}^{n-i} \frac{(-1)^{m+l+w+k+j}(m+1)^l}{m!\,l!\,B(i,n-i+1)}(j+i-1)_k \times \binom{n-i}{j}\binom{1+k}{j+i-1}\binom{(1+i)\beta-1}{w}\binom{\theta[-\beta(l+1)+w]-1}{k},$$

and the PDF of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = \sum_{j=0}^{n-i} \sum_{r=0}^{\infty} \frac{(-1)^j}{B(i,n-i+1)} \binom{n-i}{j} c_r g_{(a,(1+r)b)}(x),$$

and the q th moments of $X_{i:n}$ can be expressed as

$$\mathbf{E}(X_{i:n}^q) = \sum_{r=0}^{\infty} c_r g (1+r) b\, B\left(1+\frac{q}{a}, (1+r)b-\frac{q}{a}\right)|_{(q<(1+r)ab)},$$

The n th moment of the reversed residual life, say

$$\omega_n(t) = \mathbf{E}[(t-X)^n | X \le t], \forall\ t>0 \text{ and } n = 1,2,\ldots$$

uniquely determines $F(x)$. We obtain

$$\omega_n(t) = F^{-1}(t) \int_0^t (t-x)^n dF(x).$$

Then, the n th moment of the reversed residual life of X becomes

$$\omega_n(t) = F^{-1}(t) \sum_{r=0}^{\infty} v_r^{(\omega)} (1+r) b\, B\left(t^a; (1+r)b-\frac{n}{a}, 1+\frac{n}{a}\right),$$

where

$$v_r^{(\omega)} = v_r \sum_{r=0}^{n} (-1)^r \binom{n}{r} t^{n-r}.$$

3. Parameter estimation

Consider the estimation of the unknown parameters (β, θ, a, b) of the WGBXII model from the complete data sets by the maximum likelihood (ML) method. Suppose that $x_1, \cdots, x_n$ be a RS from the WGBXII model with parameter vector $\mathbf{\Theta} = (\beta, \theta, a, b)^{\intercal}$. Then the log-likelihood function $(\ell_n(\mathbf{\Theta}))$ for $\mathbf{\Theta}$ is given by

$$\ell_n(\mathbf{\Theta}) = n\log\beta + n\log\theta + n\log a + n\log b + (b\theta - 1)\sum_{i=1}^{n} \log(1 + x_i^a)$$

$$+ (\beta - 1)\sum_{i=1}^{n} \log\left[(1 + x_i^a)^{\theta b} - 1\right] - \sum_{i=1}^{n} \left[(1 + x_i^a)^{\theta b} - 1\right]^{\beta}. \quad (7)$$

The above $\ell_n(\mathbf{\Theta})$ in (7) can be maximized numerically via SAS (PROC NLMIXED) or R (optim) or Ox program (via sub-routine MaxBFGS), among others. The components of the score vector,

$$\mathbf{U}(\mathbf{\Theta}) = \frac{\partial \ell}{\partial \mathbf{\Theta}} = \left(\frac{\partial \ell_n(\mathbf{\Theta})}{\partial \beta}, \frac{\partial \ell_n(\mathbf{\Theta})}{\partial \theta}, \frac{\partial \ell_n(\mathbf{\Theta})}{\partial a}, \frac{\partial \ell_n(\mathbf{\Theta})}{\partial b}\right)^{\intercal}$$

are easily to be derived.

4. Applications

We provide two applications to illustrate the importance, potentiality and flexibility of the WGBXII model. For these data, we compare the WGBXII distribution, with BXII, Marshall-Olkin BXII (MOBXII), Topp Leone BXII (TLBXII), Zografos-Balakrishnan BXII (ZBBXII), Five Parameters beta BXII (FBBXII), BBXII, B exponentiated BXII (BEBXII), Five Parameters Kumaraswamy BXII (FKwBXII) and KwBXII distributions given in Afify et al. (2018), Yousof et al. (2018a, b), Altun et al. (2018a, b) and Yousof et al. (2019) . Data Set I called breaking stress data. This data set consists of 100 observations of breaking stress of carbon fibres (in Gba) given by Nichols and Padgett (2006). Data set II called leukaemia data. This real data set gives the survival times, in weeks, of 33 patients suffering from acute Myelogeneous Leukaemia (see the data sets in Appendix A). We consider the following goodness-of-fit statistics: the Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), consistent Akaike information criterion (CAIC), where

$$AIC = -2\ell(\widehat{\Theta}) + 2k,$$

$$BIC = -2\ell(\widehat{\Theta}) + k\log(n),$$

$$HQIC = -2\ell(\widehat{\Theta}) + 2k\log[\log(n)]$$

and

$$CAIC = -2\ell(\widehat{\Theta}) + 2kn/(n - k - 1),$$

where k is the number of parameters, n is the sample size, $-2\ell(\hat{\phi})$ is the maximized log-likelihood. Generally, the smaller these statistics are, the better the fit. Based on the values in Tables 2-3 and Figure 2-7 the WGBXII model provides the best fits as compared to other BXII models in the two applications with small values for BIC, AIC, CAIC and HQIC.

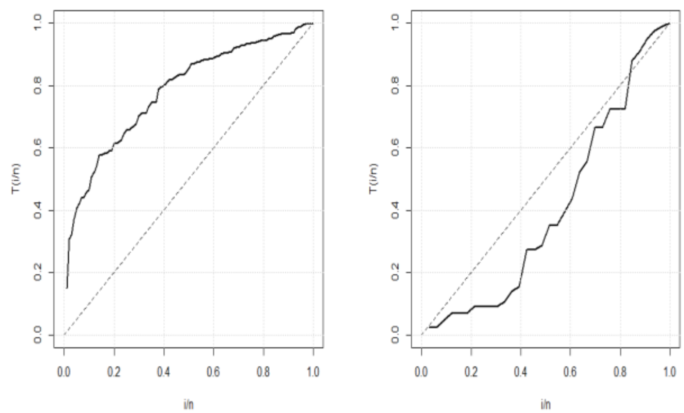
Table 2: MLEs and standard errors, confidence interval (in parentheses) with AIC, BIC, CAIC and HQIC values for the data set I.

Model	β, θ, a, b, c	AIC, BIC, CAIC, HQIC
BXII	---,---, 5.941, 0.187,---	382.94, 388.15, 383.06, 385.05
	---,---, (1.279) ,(0.044),---	
	---,---, (3.43,8.45),(0.10,0.27),---	
MOBXII	---,---, 1.192,4.834,838.73	305.78, 313.61, 306.03, 308.96
	---,---, (0.952),(4.896),(229.34)	
	---,---, 0, 3.06),(0,14.43),(389.22,1288.24)	
TLBXII	---,---, 1.350,1.061,13.728	323.52, 331.35, 323.77, 326.70
	---,---, 0.378) ,(0.384) ,(8.400)	
	---,---, (0.61, 2.09), (0.31,1.81) ,(0, 30.19)	
KwBXII	48.103 ,79.516 ,0.351 ,2.730,---	303.76, 314.20, 304.18, 308.00
	(19.348) ,(58.186) ,(0.098) ,(1.077) ,---	
	(10.18,86.03) ,(0,193.56) ,(0.16,0.54), (0.62,4.84),---	
BBXII	359.683 ,260.097 ,0.175 ,1.123 ,---	305.64, 316.06, 306.06, 309.85
	(57.941) ,(132.213),(0.013),(0.243),---	
	(246.1,473.2), (0.96,519.2), (0.14,0.20), (0.65,1.6),---	
BEBXII	0.381, 11.949, 0.937, 33.402, 1.705	305.82, 318.84, 306.46, 311.09
	(0.078), (4.635), (0.267), (6.287),(0.478)	
	(0.23,0.53) ,(2.86,21), (0.41,1.5), (21,45), (0.8,2.6)	
FKwBXII	0.542,4.223, 5.313, 0.411, 4.152	305.50, 318.55, 306.14, 310.80
	(0.137), (1.882), (2.318), (0.497), (1.995)	
	(0.3, 0.8), (0.53,7.9), (0.9,9), (0, 1.7), (0.2,8)	
ZBBXII	123.101,---,0.368, 139.247,---	302.96, 310.78, 303.21, 306.13
	(243.011), ---, (0.343), (318.546),---	
	(0, 599.40), ---,(0, 1.04), (0, 763.59),---	
WGBXII	3.736, 1.323, 0.972, 0.389	WGBXII
	(2.687), (11.114), (0.966), (3.26)	
	(0,8.9), (0,23.3), (0.2.9), (0,6.8)	

Table 3: MLEs and standard errors, confidence interval (in parentheses) with AIC, BIC, CAIC and HQIC values for the data set II.

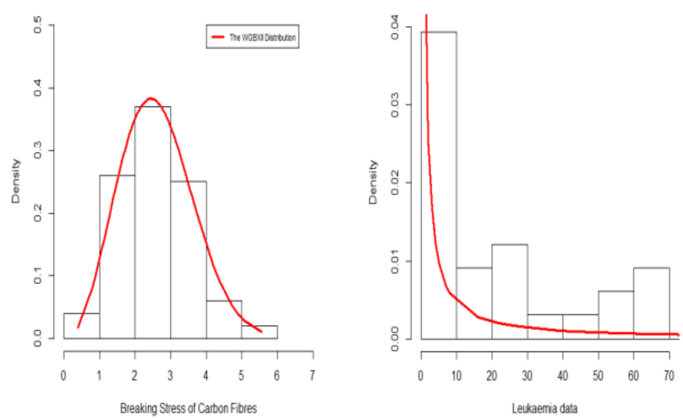
Model	β, θ, a, b, c	AIC, BIC, CAIC, HQIC
BXII	---,---, 58.711,0.006,---	328.20, 331.19, 328.60, 329.19
	---,---, (42.382), (0.004),---	
	---,---, (0, 141.78), (0, 0.01),---	
MOBXII	---,---, 11.838, 0.078, 12.251	315.54, 320.01, 316.37, 317.04
	---,---, (4.368), (0.013), (7.770)	
	---,---, (0, 141.78), (0, 0.01), (0, 27.48)	
TLBXII	---,---,0.281, 1.882 ,50.215	316.26, 320.73, 317.09, 317.76
	---,---, (0.288), (2.402), (176.50)	
	---,---, (0, 0.85), (0, 6.59), (0, 396.16)	
KwBXII	9.201, 36.428, 0.242,0.941,---	317.36, 323.30, 318.79, 319.34
	(10.060), (35.650), (0.167), (1.045),---	
	(0, 28.912), (0, 106.30), (0, 0.57), (0, 2.99),---	
BBXII	96.104, 52.121, 0.104, 1.227,---	316.46, 322.45, 317.89, 318.47
	(41.201), (33.490), (0.023), (0.326),---	
	(15.4,176.8),(0, 117.8), (0.6, 0.15), (0.59,1.9),---	
BEBXII	0.087, 5.007, 1.561, 31.270, 0.318	317.58, 325.06, 319.80, 320.09
	(0.077), (3.851), (0.012), (12.940), (0.034)	
	(0, 0.3), (0, 12.6), (1.5, 1.6), (5.9, 56.6), (0.3,0.4)	
FBBXII	15.194, 32.048, 0.233, 0.581, 21.855	317.86, 325.34, 320.08, 320.36
	(11.58), (9.867), (0.091), (0.067), (35.548)	

	(0, 37.8), (12.7,51.4), (0.05,0.4), (0.45,0.7), (0, 91.5)	
FKwBXII	14.732, 15.285, 0.293, 0.839, 0.034	317.76, 325.21, 319.98, 320.26
	(12.390), (18.868), (0.215), (0.854), (0.075)	
	(0, 39.02), (0, 52.27), (0, 0.71), (0, 2.51), (0, 0.18)	
ZBBXII	41.973,---,0.157, 44.263,---	313.86, 318.35, 314.39, 315.36
	(38.787),---,(0.082), (47.648),---	
	(0, 117.99),---,(0, 0.32,) (0, 137.65),---	
WGBXII	0.107, 1.829, 76.391, 0.469,---	WGBXII
	(0.021), (1.058), (0.00), (0.0712),---	
	(0.6,0.14), (0,3.8),---, (0.36,0.64),---	



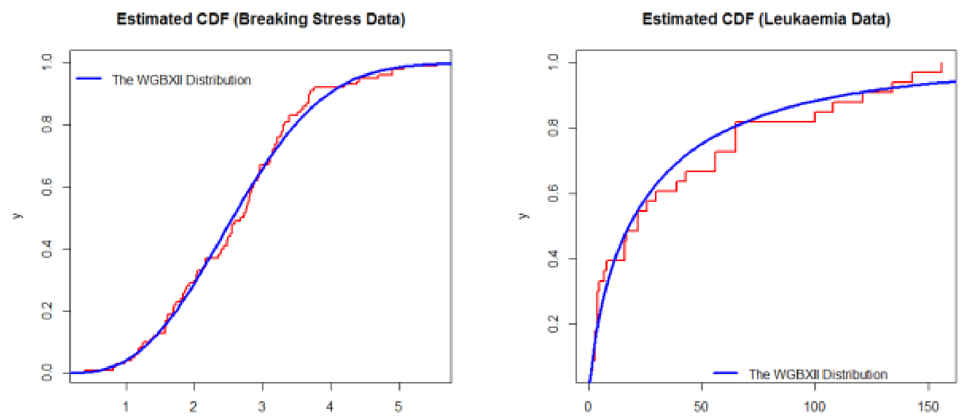
Data set I Data set II

Figure 2: TTT plots.



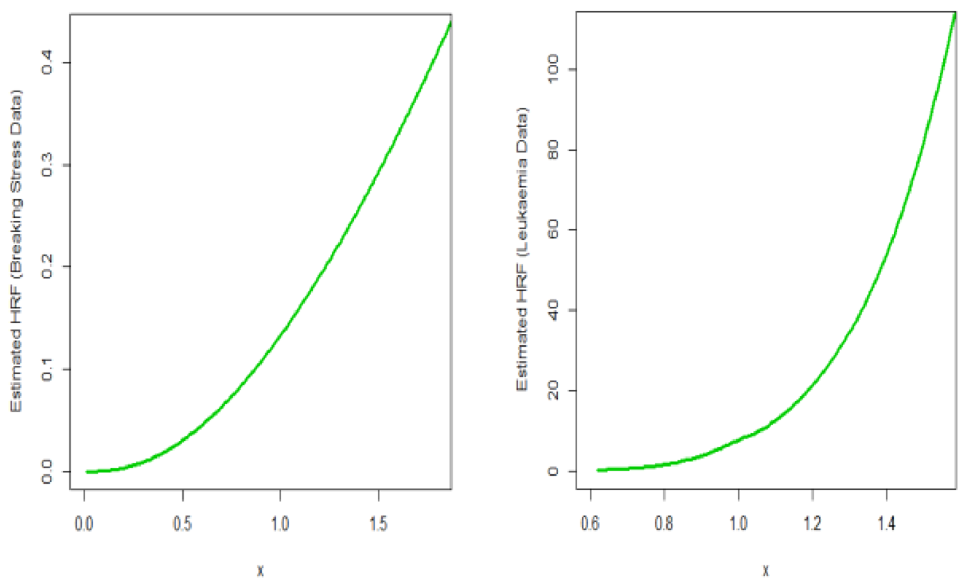
Data set I Data set II

Figure 3: Estimated PDFs.



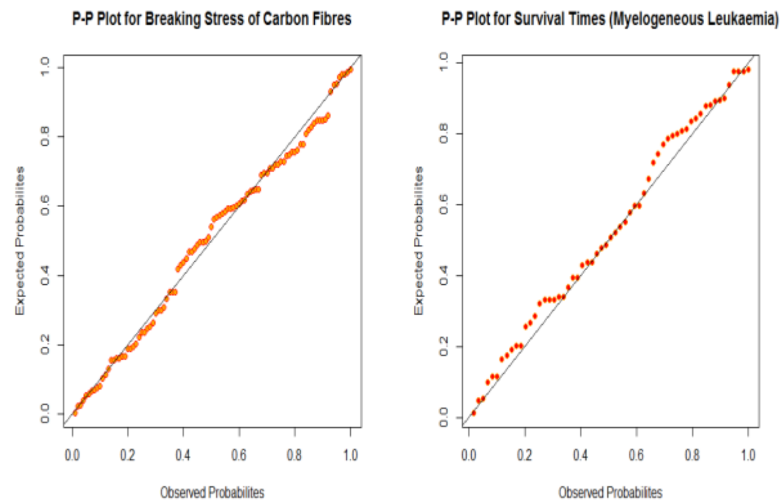
Data set **I** Data set **II**

Figure 4: Estimated CDFs.



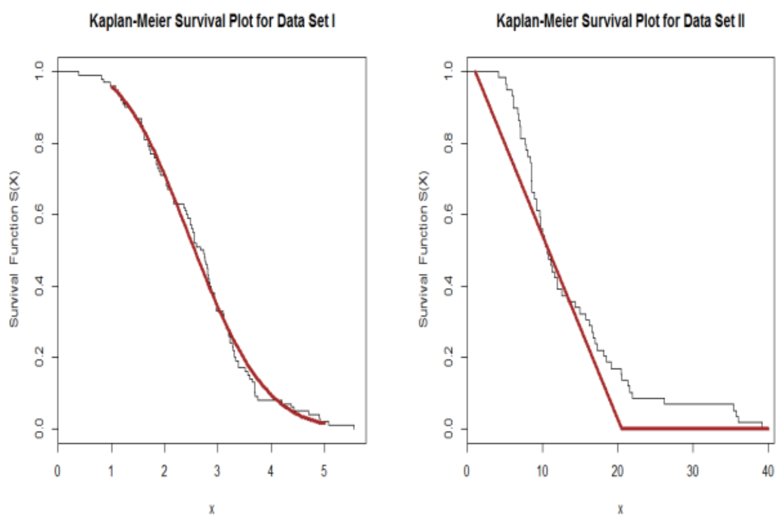
Data set **I** Data set **II**

Figure 5: Estimated HRFs.



Data set I Data set II

Figure 6: P-P plots.



Data set I Data set II

Figure 7: Kaplan-Meier Survival plots.

5.Conclusions

In this work, we introduced a new distribution called the Weibull Generalized BXII (WGBXII) distribution. We introduced a simple physical motivation for the new model. Set of its properties are derived. Two applications are provided to illustrate the importance of the new model. The new model is better that other nine competitive models via two applications. The method of maximum likelihood is used to estimate the unknown parameters. The new model provide adequate fits as compared to other related models with small values for AIC, BIC, CAIC and HQIC. The new model is much better than the standard Burr XII, beta Burr XII, Kumaraswamy Burr XII, Five parameter Kumaraswamy Burr XII , Topp Leone Burr XII, Marshall–Olkin BXII, beta exponentiated Burr XII, Five parameter beta Burr XII, and Zografos-Balakrishnan Burr XII models in modeling breaking stress and leukaemia data sets.

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Appendix A

Data Set I: {0.980, 5.56, 5.08, 0.39, 1.570, 3.19, 4.90, 2.930, 2.85, 2.77, 2.760, 1.73, 2.480, 3.680, 1.08, 3.220, 3.75, 3.220, 3.70, 2.74, 2.730, 2.50, 3.60, 3.110, 3.270, 2.87, 1.47, 3.110, 4.420, 2.40, 3.150, 2.67,3.31, 2.810, 2.56, 2.170, 4.91, 1.590, 1.18, 2.480, 2.03, 1.690, 2.43, 3.390, 3.56, 2.830, 3.68, 2.00, 3.51, 0.850, 1.61, 3.28, 2.950, 2.81, 3.15, 1.920, 1.84, 1.22, 2.170, 1.61, 2.120, 3.090, 2.970, 4.20, 2.35, 1.410, 1.59, 1.12, 1.690, 2.79, 1.89, 1.870, 3.3900, 3.33, 2.550, 3.68, 3.19, 1.710, 1.25, 4.70, 2.88, 2.96, 2.55, 2.590, 2.97, 1.57, 2.170, 4.38, 2.030, 2.82, 2.530, 3.31, 2.38, 1.36, 0.81, 1.17, 1.84, 1.80, 2.05, 3.65}.
Data Set II: {65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, 43}.