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A New Transmuted Weibull Distribution: Properties and Application

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Abstract

In this manuscript, a new three parameter Weibull distribution namely New Transmuted Weibull distribution is proposed through a transmutation technique. Some of the mathematical properties of new model have been derived. Entropy and parameter estimation is carried out using different methods. Finally, a real life data set is incorporated to illustrate the utility of the model from practical standpoint.

Key Words: New Transmuted Weibull Distribution; Weibull Distribution; Generating Function; Entropy; Parameter Estimation.

Mathematical Subject Classification: 60E05, 62E15

1. Introduction

Weibull Distribution (WD) was introduced by Weibull (1939) while analyzing the strength data but soon realized its potential to model different types of data. Since its inception, it has been extensively used to analyze and model data emerging in multifarious fields owing to its simplicity and versatility. Khalili and Kromp (1991) studied the performance of the estimators of WD using three different approaches. Hallinan (1993) reviewed different forms of WD. Alqam et al. (2002) conducted a comparison study of the three and two parameter WD using 26 mechanical property data sets of fiber-reinforced polymeric composite materials obtained through the process of pultrusion. The cumulative distribution function (cdf) and probability density function (pdf) of WD are given by Eq. (1) and (2) respectively.

$$M(x) = 1 - e^{-\theta x^{\lambda}} \quad ; x, \theta, \lambda > 0. \tag{1}$$

$$m(x) = \theta\lambda x^{\lambda-1} e^{-\theta x^{\lambda}}. \tag{2}$$

The hazard rate function (hrf) of WD can exhibit shapes such as increasing, decreasing and constant but it cannot be exploited in situations where the hrf takes complex shapes. Consequently, a number of researchers proposed various extensions of WD every so often to enhance its flexibility. Some generalizations of WD are: Bourguignon et al. (2014) proposed the Weibull- G family of Probability Distributions, Pal et al. (2016) introduced Exponentiated Weibull Distribution, Nassar et al. (2017) introduced Alpha Power Weibull Distribution, Alizadeh et al. (2017) put forth the Transmuted Weibull-G family of Distributions, Uzma et al. (2017) introduced Transmuted Exponentiated Inverse Weibull Distribution with Applications in Medical sciences etc. For latest generalizations, readers may refer to: El-Basit et al. (2020), Mazucheli et al. (2020), Mahmood et al. (2020) etc

Lately, a new transmutation technique was introduced by Bakouch et al. (2017). A random variable X is said to follow New Transmuted distribution suggested by Bakouch et al (2017) with β as transmuted parameter if its cdf and pdf take respectively the forms given by Eq. (3) and Eq. (4).

$$G(x) = M(x) + \beta \frac{M(x)[1-M(x)]}{[1+M(x)]}; x \in R, -1 \le \beta \le 1. \quad (3)$$

$$g(x) = \left[1 - \beta + \frac{2\beta}{[1+M(x)]^2}\right] m(x). \quad (4)$$

Where $m(x)$ and $M(x)$ are respectively the pdf and cdf of base distribution.

In this paper, our aim to generalize the WD using this new transmutation technique proposed by Bakouch et al (2017). The new distribution is named New Transmuted Weibull Distribution (NTWD). The main motivation to consider NTWD is following: NTWD can exhibit more complex shapes of hrf and density function which the WD fails to depict. Also, NTWD outperforms the WD and other well-known model in terms of a real life data set. The rest of paper is organized as: The pdf, cdf, some special cases and a useful mixture representation of the pdf are discussed in section 2. Section 3 deals with the reliability measures and Section 4 is dedicated to the derivation of statistical properties of NTWD such as moments, mean, mgf, incomplete moments, mean deviation about mean and median, residual and reverse residual life moments etc. Various estimation procedures for estimating the parameters of proposed distribution are debated in section 5. In section 6, a simulation study is conducted to examine the efficiency of MLE's and a real life data set is incorporated in section 7 to illustrate the application of proposed model in real life.

2. NTWD

Upon inserting Eq. (1) in Eq. (3) and Eq. (4), we obtain the cdf of NTWD given by Eq. (5).

$$G(x) = 1 - e^{-\theta x^{\lambda}} + \frac{\beta e^{-\theta x^{\lambda}}\left(1-e^{-\theta x^{\lambda}}\right)}{2-e^{-\theta x^{\lambda}}} \quad ; x, \theta, \lambda > 0, \quad -1 \le \beta \le 1. \quad (5)$$

The corresponding pdf is given by Eq. (6).

$$g(x) = \left\{1 - \beta + \frac{2\beta}{\left[2-e^{-\theta x^{\lambda}}\right]^2}\right\} \theta\lambda x^{\lambda-1} e^{-\theta x^{\lambda}}. \quad (6)$$

The graphical overview of possible shapes of pdf of NTWD for distinct values of the parameters α, β and λ are given by Figure 1.

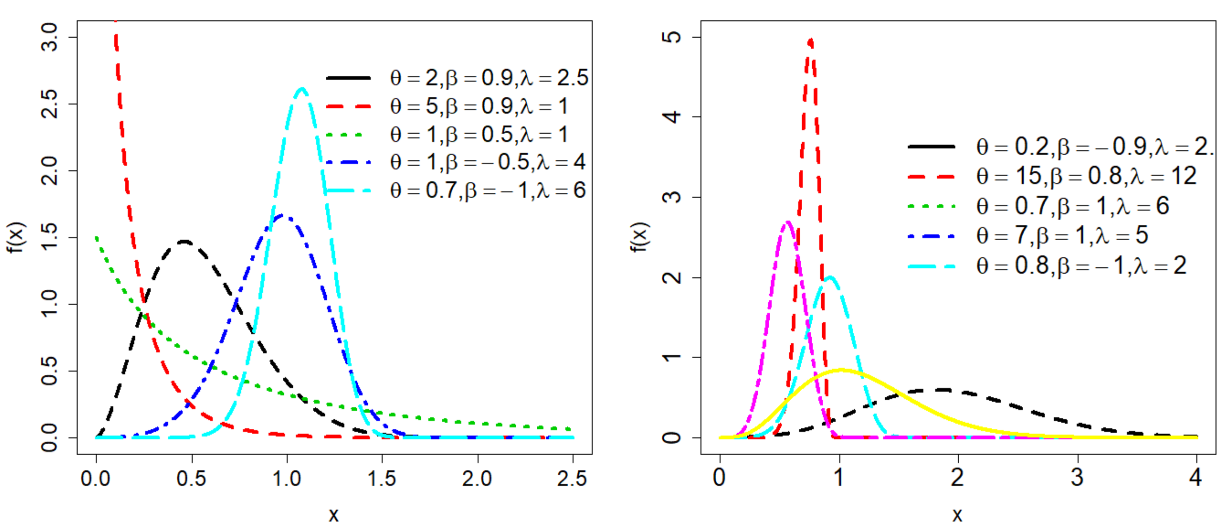

Figure 1: Plots of Pdf of NTWD.

1.1. Special Cases

In this section, the sub-models of NTWD will be discussed.

1.1.1. New Transmuted Exponential Distribution

If we put $\lambda = 1$ in Equation (5), we get new cdf of New Transmuted Exponential distribution given as

$$G(x) = 1 - e^{-\theta x} + \frac{\beta e^{-\theta x}\left(1 - e^{-\theta x}\right)}{2 - e^{-\theta x}} \quad ; x, \theta > 0, -1 \leq \beta \leq 1.$$

1.1.2. New Transmuted Rayleigh Distribution

If we put $\lambda = 2$ in Eq. (5), we get new cdf of New Transmuted Rayleigh distribution given as

$$G(x) = 1 - e^{-\theta x^2} + \frac{\beta e^{-\theta x^2}\left(1 - e^{-\theta x^2}\right)}{2 - e^{-\theta x^2}} \quad ; x, \theta > 0, \quad -1 \leq \beta \leq 1.$$

1.1.3. Exponential Distribution

If we put $\lambda = 1$and $\beta = 0$ in Eq. (5), we get new cdf of Exponential distribution given as

$$G(x) = 1 - e^{-\theta x} \quad ; x, \theta > 0.$$

1.1.4. Rayleigh Distribution

If we put $\lambda = 2$and $\beta = 0$ in Eq. (5), we get new cdf of Rayleigh distribution given as

$$G(x) = 1 - e^{-\theta x^2} \quad ; x, \theta > 0.$$

2. Useful Expansion

A useful mixture representation of pdf given in Eq. (6) can be obtained using generalized binomial expansion is given as

$$g(x) = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \theta \lambda x^{\lambda-1} b_j \,(j+1) \binom{j}{k} (-1)^k \, e^{-\theta(k+1)x^{\lambda}}.$$

The above given mixture representation is very much useful in deriving various properties of NTWD.

3. Reliability Analysis

In this section, some aspects related to reliability will be explored.

3.1. Survival Function

The survival function of NTWD is given as

$$S(x) = 1 - G(x) = e^{-\theta x^{\lambda}} - \frac{\beta e^{-\theta x^{\lambda}}\left(1 - e^{-\theta x^{\lambda}}\right)}{2 - e^{-\theta x^{\lambda}}}.$$

3.2. Hazard Rate Function (hrf)

The hrf of NTWD is given as

$$h(x)=\frac{\left\{(1+\beta)+(1-\beta)\left(1-e^{-\theta x^{\lambda}}\right)\left(3-e^{-\theta x^{\lambda}}\right)\right\}\lambda\theta x^{\lambda-1}}{\left(\left(1-\beta\left(1-e^{-\theta x^{\lambda}}\right)\right)-(1-\beta)\left(1-e^{-\theta x^{\lambda}}\right)^{2}\right)}.$$

Figure 2 displays the shape of hrf of NTWD for selected values of the parameters.

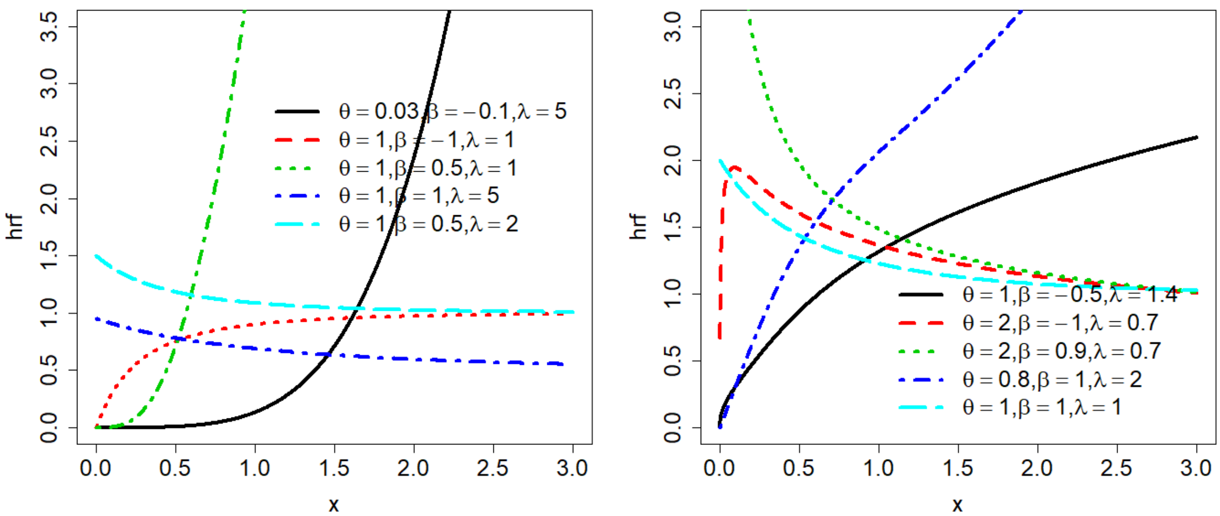

Figure 2: Hrf of NTWD

4. Statistical Properties

Some of the statistical properties of NTWD will be discussed in this section.

4.1. Moments

The rth moment about origin of NTWD can be obtained as

$$\mu_r' = \int_0^\infty x^r \sum_{j=0}^{\infty}\sum_{k=0}^{j} \theta\lambda x^{\lambda-1} b_j\,(j+1)\binom{j}{k}(-1)^k\, e^{-\theta(k+1)x^{\lambda}}dx.$$

On simplification we get

$$\mu_r' = \sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{\Gamma\left(\frac{r}{\lambda}+1\right) b_j\,(j+1)\binom{j}{k}(-1)^k}{\theta^{\frac{r}{\lambda}}(k+1)^{\frac{r}{\lambda}+1}}. \tag{7}$$

Putting r=1, we get the mean of NTWD given as

$$\mu_1' = \sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{\Gamma\left(\frac{1}{\lambda}+1\right) b_j\,(j+1)\binom{j}{k}(-1)^k}{\theta^{\frac{1}{\lambda}}(k+1)^{\frac{1}{\lambda}+1}}.$$

Also, the variance of NTWD is given as

$$\mu_2 = \left\{\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{\Gamma\left(\frac{2}{\lambda}+1\right) b_j\,(j+1)\binom{j}{k}(-1)^k}{\theta^{\frac{2}{\lambda}}(k+1)^{\frac{2}{\lambda}+1}}\right\} - \left\{\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{\Gamma\left(\frac{1}{\lambda}+1\right) b_j\,(j+1)\binom{j}{k}(-1)^k}{\theta^{\frac{1}{\lambda}}(k+1)^{\frac{1}{\lambda}+1}}\right\}^2.$$

4.2. Incomplete Moments about Origin

The n[th] incomplete moment about origin of NTWD is given as

$$\mu'_{(n)} = \sum_{j=0}^{\infty}\sum_{k=0}^{j} \frac{\gamma\left(\frac{n}{\lambda}+1, \theta(k+1)s^{\lambda}\right) b_j\,(j+1)\binom{j}{k}(-1)^k}{\theta^{\frac{r}{\lambda}}(k+1)^{\frac{r}{\lambda}+1}}.$$

4.3. Moment Generating Function (mgf)

The mgf of NTWD is given as

$$M_X(t) = \sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{r=0}^{\infty} \frac{t^r\,\Gamma\left(\frac{r}{\lambda}+1\right) b_j\,(j+1)\binom{j}{k}(-1)^k}{r!\,\theta^{\frac{r}{\lambda}}(k+1)^{\frac{r}{\lambda}+1}}.$$

4.4. Mean Deviation about Mean and Median

The mean deviation about mean for NTWD can be derived as

$$D(\mu) = 2\int_0^{\mu}(\mu - x)dx$$

$$D(\mu) = 2\mu\sum_{j=0}^{\infty}\sum_{k=0}^{j} b_j\,(j+1)\binom{j}{k}(-1)^k\left(\frac{\gamma\left(1, \theta(k+1)\mu^{\lambda}\right)}{(k+1)} - \frac{\gamma\left(\frac{1}{\lambda}+1, \theta(k+1)\mu^{\lambda}\right)}{\theta^{\frac{1}{\lambda}}(k+1)^{\frac{1}{\lambda}+1}}\right).$$

Also the expression for Mean deviation about mean is the following form:

$$D(M) = \mu - 2\left(\sum_{j=0}^{\infty}\sum_{k=0}^{j} \frac{\gamma\left(\frac{1}{\lambda}+1, \theta(k+1)M^{\lambda}\right) b_j\,(j+1)\binom{j}{k}(-1)^k}{\theta^{\frac{1}{\lambda}}(k+1)^{\frac{1}{\lambda}+1}}\right).$$

4.5. Residual and Reverse Residual Life Moments.

The residual life moments for NTWD can be derived as

$$M_s(t) = \sum_{l=0}^{s} t^{s-l}\binom{s}{l}(-1)^{s-l}\{E(X^l) - \mu^l(t)\}.$$

$$M_s(t) = \sum_{l=0}^{s} t^{s-l}\binom{s}{l}(-1)^{s-l}\left\{\sum_{j=0}^{\infty}\sum_{k=0}^{j} \frac{\Gamma\left(\frac{l}{\lambda}+1\right) b_j\,(j+1)\binom{j}{k}(-1)^k}{\theta^{\frac{l}{\lambda}}(k+1)^{\frac{l}{\lambda}+1}} - \sum_{j=0}^{\infty}\sum_{k=0}^{j} \frac{\gamma\left(\frac{l}{\lambda}+1, \theta(k+1)s^{\lambda}\right) b_j\,(j+1)\binom{j}{k}(-1)^k}{\theta^{\frac{l}{\lambda}}(k+1)^{\frac{l}{\lambda}+1}}\right\}.$$

$$M_s(t)=\sum_{l=0}^{s} t^{s-l}\binom{s}{l}(-1)^{s-l}\left\{\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{\Gamma\left(\frac{l}{\lambda}+1,\theta(k+1)s^{\lambda}\right)b_j\,(j+1)\binom{j}{k}(-1)^k}{\theta^{\frac{l}{\lambda}}(k+1)^{\frac{l}{\lambda}+1}}\right\}.$$

Also, the simplified expression for reverse residual life moments of NTWD is given as

$$m_s(t)=\sum_{l=0}^{s} t^{s-l}\binom{s}{l}(-1)^{l}\left\{\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{\gamma\left(\frac{l}{\lambda}+1,\theta(k+1)s^{\lambda}\right)b_j\,(j+1)\binom{j}{k}(-1)^k}{\theta^{\frac{l}{\lambda}}(k+1)^{\frac{l}{\lambda}+1}}\right\}.$$

Theorem 1. Let $X\sim NTWD(\theta,\lambda,\beta)$, then the Renyi and Mathai- Haubold entropy for NTWD is respectively given by

$$I_\delta=\frac{1}{1-\delta}\sum_{j=0}^{\infty}\sum_{l=0}^{j}\sum_{l=0}^{\infty}\binom{\delta}{j}\binom{j}{l}\binom{-2l}{i}\theta^{\delta}\lambda^{\delta-1}\beta^{j}2^{-l-i}(-1)^{j-l+i}\frac{\Gamma\big((\delta+1)(\lambda-1)+1\big)}{\big((\delta+i)\theta\big)^{\left((\delta+1)(\lambda-1)+1\right)}};\delta>0,\delta\neq 1.$$

$$g_{MH}(x)=\frac{1}{1-\delta}\left[\left\{\begin{matrix}\sum_{j=0}^{\infty}\sum_{l=0}^{j}\sum_{l=0}^{\infty}\binom{\delta}{j}\binom{j}{l}\binom{-2l}{i}\theta^{2-\delta}\lambda^{1-\delta}\beta^{j}2^{-l-i}(-1)^{j-l+i}\\ \dfrac{\Gamma\big((1-\delta)(\lambda-1)+1\big)}{((\delta+i)\theta)^{((1-\delta)(\lambda-1)+1)}}\end{matrix}\right\}-1\right];\delta>0,\delta\neq 1.$$

Proof. The Renyi entropy (Renyi, 1961) is defined as

$$I_\delta=\tfrac{1}{1-\delta}\int_0^{\infty}g^{\delta}(x)dx \qquad (8)$$

where $\delta>0,\delta\neq 1$.

Substituting Eq. (4) in Eq. (8), we get

$$I_\delta=\frac{1}{1-\delta}\int_0^{\infty}\left\{\left[1+\beta\left[\frac{2}{[1+M(x)]^2}\right]-1\right]m(x)\right\}^{\delta}dx$$

$$I_\delta=\frac{1}{1-\delta}\int_0^{\infty}\sum_{j=0}^{\infty}\binom{\delta}{j}\beta^{j}\left\{\left[\frac{2}{[1+M(x)]^2}\right]-1\right\}^{j}m^{\delta}(x)dx$$

$$I_\delta=\frac{1}{1-\delta}\int_0^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{j}\binom{\delta}{j}\binom{j}{l}\beta^{j}2^{l}(-1)^{j-l}[1+M(x)]^{-2l}m^{\delta}(x)dx$$

$$I_\delta=\frac{1}{1-\delta}\int_0^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{j}\binom{\delta}{j}\binom{j}{l}\beta^{j}2^{l}(-1)^{j-l}\left[2-e^{-\theta x^{\lambda}}\right]^{-2l}(\theta\lambda x^{\lambda-1}e^{-\theta x^{\lambda}})^{\delta}dx$$

$$I_\delta=\frac{1}{1-\delta}\int_0^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{j}\sum_{l=0}^{\infty}\binom{\delta}{j}\binom{j}{l}\binom{-2l}{i}\beta^{j}2^{-l-i}(-1)^{j-l+i}e^{-i\theta x^{\lambda}}(\theta\lambda x^{\lambda-1}e^{-\theta x^{\lambda}})^{\delta}dx$$

$$I_\delta = \frac{1}{1-\delta}\sum_{j=0}^{\infty}\sum_{l=0}^{j}\sum_{l=0}^{\infty}\binom{\delta}{j}\binom{j}{l}\binom{-2l}{i}\theta^\delta\lambda^{\delta-1}\beta^j 2^{-l-i}(-1)^{j-l+i}\frac{\Gamma\big((\delta+1)(\lambda-1)+1\big)}{((\delta+i)\theta)^{((\delta+1)(\lambda-1)+1)}},$$

which is required expression of Renyi Entropy for NTWD. Also, the Mathai- Haubold entropy (Mathai and Haubold, 2006) is defined as

$$g_{MH}(x) = \frac{1}{1-\delta}\left\{\int_0^\infty g^{2-\delta}(x)dx - 1\right\}; \delta > 0, \delta \neq 1.$$

Using Eq. (4) we get

$$g_{MH}(x) = \frac{1}{1-\delta}\left\{\int_0^\infty \left\{\left[1+\beta\left[\frac{2}{[1+M(x)]^2}\right]-1\right]m(x)\right\}^{2-\delta}dx - 1\right\},$$

$$g_{MH}(x) = \frac{1}{1-\delta}\left\{\int_0^\infty \sum_{j=0}^{\infty}\binom{2-\delta}{j}\beta^j\left\{\left[\frac{2}{[1+M(x)]^2}\right]-1\right\}^j m^{2-\delta}(x)dx - 1\right\},$$

$$g_{MH}(x) = \frac{1}{1-\delta}\left\{\left[\int_0^\infty \sum_{j=0}^{\infty}\sum_{l=0}^{j}\binom{2-\delta}{j}\binom{j}{l}\beta^j 2^l(-1)^{j-l}[1+M(x)]^{-2l}m^{2-\delta}(x)dx\right] - 1\right\},$$

$$g_{MH}(x) = \frac{1}{1-\delta}\left\{\left[\int_0^\infty \sum_{j=0}^{\infty}\sum_{l=0}^{j}\binom{2-\delta}{j}\binom{j}{l}\beta^j 2^l(-1)^{j-l}\left[-e^{-\theta x^\lambda}\right]^{-2l}\left(\theta\lambda x^{\lambda-1}e^{-\theta x^\lambda}\right)^{2-\delta}dx\right] - 1\right\},$$

$$g_{MH}(x) = \frac{1}{1-\delta}\left[\left\{\sum_{j=0}^{\infty}\sum_{l=0}^{j}\sum_{l=0}^{\infty}\binom{\delta}{j}\binom{j}{l}\binom{-2l}{i}\theta^{2-\delta}\lambda^{1-\delta}\beta^j 2^{-l-i}(-1)^{j-l+i}\frac{\Gamma\big((1-\delta)(\lambda-1)+1\big)}{((\delta+i)\theta)^{((1-\delta)(\lambda-1)+1)}}\right\} - 1\right],$$

which is the required expression.

4.6. L-Moments

The L-moments for NTWD can be derived as

$$E(X_{i:n}^r) = \int_0^\infty x^r f_{i:n}(x)dx. \tag{9}$$

We have

$$f_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!}G^{i-1}(x)[1-G(x)]^{n-i}g(x).$$

$$f_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!}\sum_{u=0}^{n-i}\binom{n-i}{u}(-1)^u G^{u+i-1}(x)g(x).$$

$$f_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!}\sum_{u=0}^{n-i}\binom{n-i}{u}(-1)^u\left\{\left[1-\beta+\frac{2\beta}{[1+M(x)]}\right]M(x)\right\}^{u+i-1}g(x)$$

$$f_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!}\sum_{u=0}^{n-i}\sum_{v=0}^{u+i-1}\binom{n-i}{u}\binom{u+i-1}{v}(-1)^u(1-\beta)^{u+i-1-v}(2\beta)^v\big(1+M(x)\big)^{-v}M^{u+i-1}(x)\,g(x).$$

$$f_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!}\sum_{u=0}^{n-i}\sum_{v=0}^{u+i-1}\sum_{w=0}^{\infty}\binom{n-i}{u}\binom{u+i-1}{v}\binom{-v}{w}(-1)^u \times (1-\beta)^{u+i-1-v}(2\beta)^v M^{w+u+i-1}(x)\,g(x).$$

$$f_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!}\sum_{u=0}^{n-i}\sum_{v=0}^{u+i-1}\sum_{w=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{w+u+i-1}\binom{u+i-1}{v}\binom{n-i}{u}\binom{-v}{w}\binom{j}{k} \times \theta\lambda b_j\,(j+1)(-1)^{k+u+l}\binom{w+u+i-1}{l}(1-\beta)^{u+i-1-v}(2\beta)^v x^{\lambda-1}e^{-\theta(l+k+1)x^{\lambda}}.$$

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\sum_{u=0}^{n-i}\sum_{v=0}^{u+i-1}\sum_{w=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{w+u+i-1}\eta_{u,v,w,j,k}\theta\lambda x^{\lambda-1}\,e^{-\theta(l+k+1)x^{\lambda}} \qquad (10)$$

Where

$$\eta_{u,v,w,j,k=}\ b_j\,(j+1)(-1)^{k+u+l}\binom{n-i}{u}\binom{u+i-1}{v}\binom{-v}{w}\binom{j}{k}\times\binom{w+u+i-1}{l}(1-\beta)^{u+i-1-v}(2\beta)^v.$$

Therefore, upon substituting Eq. (10) in Eq. (9), we get

$$E(X_{i:n}^r) = \int_0^{\infty} x^r \frac{n!}{(i-1)!\,(n-i)!}\sum_{u=0}^{n-i}\sum_{v=0}^{u+i-1}\sum_{w=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{w+u+i-1}\eta_{u,v,w,j,k}\theta\lambda x^{\lambda-1} \times e^{-\theta(l+k+1)x^{\lambda}}\,dx.$$

$$E(X_{i:n}^r) = \frac{n!}{(i-1)!\,(n-i)!}\sum_{u=0}^{n-i}\sum_{v=0}^{u+i-1}\sum_{w=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{w+u+i-1}\eta_{u,v,w,j,k}\int_0^{\infty} x^r\theta\lambda x^{\lambda-1} \times e^{-\theta(l+k+1)x^{\lambda}}\,dx.$$

$$E(X_{i:n}^r) = \frac{n!}{(i-1)!\,(n-i)!}\sum_{u=0}^{n-i}\sum_{v=0}^{u+i-1}\sum_{w=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{w+u+i-1}\frac{\eta_{u,v,w,j,k}\Gamma\left(\frac{r}{\lambda}+1\right)}{\theta^{\frac{r}{\lambda}}(l+k+1)^{\frac{r}{\lambda}+1}}.$$

4.7. Stress Strength Reliability

If $X_1\sim NTWD(\theta,\lambda,\beta_1)$ and $X_2\sim NTWD(\theta,\lambda,\beta_2)$, then the stress strength reliability denoted by R for NTWD can be obtained as

$$R = \int_0^{\infty} g_1(x)G_2(x)dx$$

$$R = \sum_{j=0}^{\infty}\sum_{k=0}^{j+i+1}\sum_{i=0}^{\infty} b_j^{(\beta_1)}b_i^{(\beta_1)}\,(j+1)\binom{j}{k}\binom{j+i+1}{k}(-1)^k\int_0^{\infty}\theta\lambda x^{\lambda-1}\,e^{-\theta(k+1)x^{\lambda}}\,dx.$$

After simplification of the above equation we get

$$R = \sum_{j=0}^{\infty} \sum_{k=0}^{j+i+1} \sum_{i=0}^{\infty} b_j^{(\beta_1)} b_i^{(\beta_1)} \frac{(j+1)}{(k+1)} (j+1) \binom{j}{k} \binom{j+i+1}{k} (-1)^k.$$

5. Estimation

There are a number of methods of estimating the parameters. A few among them are discussed below:

5.1. Maximum likelihood estimation

The log likelihood function is given as

$$logl = \sum_{i=1}^{n} log\left[1-\beta+2\beta e^{2\theta x^{\lambda}}\right] + nlog\,\theta + nlog\lambda + (\lambda-1)\sum_{i=1}^{n} logx_i - \sum_{i=1}^{n} \theta x_i^{\lambda}.$$

$$logl = \sum_{i=1}^{n} log\,T(x_i,\theta,\beta,\lambda) + nlog\,\theta + nlog\lambda + (\lambda-1)\sum_{i=1}^{n} logx_i - \sum_{i=1}^{n} \theta x_i^{\lambda}, \quad (11)$$

where $T(x_{i,}\theta,\beta,\lambda) = \left[1-\beta+2\beta e^{2\theta x^{\lambda}}\right]$.

Upon differentiating Eq. (11) w.r.t θ, λ and β respectively, we obtain the following non-linear Eq.:

$$\frac{\partial}{\partial\theta} logl = \sum_{i=1}^{n} \frac{\frac{\partial}{\partial\theta} T(x_{i,}\theta,\beta,\lambda)}{T(x_{i,}\theta,\beta,\lambda)} + \frac{n}{\theta} - \sum_{i=1}^{n} x_i^{\lambda} . \quad (12)$$

$$\frac{\partial}{\partial\lambda} logl = \sum_{i=1}^{n} \frac{\frac{\partial}{\partial\lambda} T(x_{i,}\theta,\beta,\lambda)}{T(x_{i,}\theta,\beta,\lambda)} + \frac{n}{\lambda} + \sum_{i=1}^{n} logx_i - \sum_{i=1}^{n} \theta x_i^{\lambda} logx_{i,}. \quad (13)$$

$$\frac{\partial}{\partial\beta} logl = \sum_{i=1}^{n} \frac{\frac{\partial}{\partial\beta} T(x_{i,}\theta,\beta,\lambda)}{T(x_{i,}\theta,\beta,\lambda)}. \quad (14)$$

Since the equations are nonlinear, hence the ML estimates can be obtained by respectively equating the above given three Eq. (12) to Eq. (14) to zero and solving them using N-R technique simultaneously. Also, for the NTWD the second order derivatives exist for $\log l$ function. Thus the inverse dispersion matrix (IDM) is given as

$$\begin{pmatrix} \hat{\theta} \\ \hat{\lambda} \\ \hat{\beta} \end{pmatrix} \sim N\left[\begin{pmatrix} \theta \\ \lambda \\ \beta \end{pmatrix}, \begin{pmatrix} \hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13} \\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23} \\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33} \end{pmatrix}\right].$$

$$V^{-1} = -E\begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} \quad (15)$$

Where $V_{11} = \frac{\partial^2}{\partial\theta^2} logl,\ V_{12} = V_{21} = \frac{\partial^2}{\partial\theta\partial\lambda} logl,\ V_{13} = V_{31} = \frac{\partial^2}{\partial\theta\partial\beta} logl,$

$V_{22} = \frac{\partial^2}{\partial\lambda^2} logl,\ V_{23} = V_{32} = \frac{\partial^2}{\partial\lambda\partial\lambda} logl,$

$V_{33} = \frac{\partial^2}{\partial\beta^2} logl.$

The variance co-variance matrix of NTWD is given by Eq. (14). Upon solving the IDM the asymptotic variances and co-variances of the MLE's for $\hat{\theta}, \hat{\lambda}$ and $\hat{\beta}$ are obtained. Also the 100(1-α) % confidence interval for θ, λ and βcan be obtained as

$$\hat{\theta} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{11}}, \quad \hat{\lambda} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{22}}, \qquad \hat{\beta} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{33}} \; .$$

Where $Z_{\frac{\alpha}{2}}$ is the upper α^{th} percentile of the standard normal distribution.

5.2. Least Square method (LS)

Let $x_i;\ i = 1,2,..n.$ be the observed value of NTWD with parameters θ, λ and β, in increasing order and $X_{i:n}; i = 1,2,..n$ be the associated order satatistics. Upon minimizing the following equation, least square estimates can be obtained

$$S(\Theta) = \sum_{i=1}^{n}[G(x_i) - E(G(x_i))]^2.$$

Note that $E\big(G(x_i)\big) = \frac{i}{n+1}$.

$$S(\Theta) = \sum_{i=1}^{n}\left[\left(1 - e^{-\theta x_i^{\lambda}} + \frac{\beta e^{-\theta x_i^{\lambda}}\left(1-e^{-\theta x_i^{\lambda}}\right)}{2-e^{-\theta x_i^{\lambda}}}\right) - \frac{i}{n+1}\right]^2. \qquad (16)$$

Upon differentiating Eq. (16) w.r.t θ, λ and β respectively we obtain three non-linear equation. Equating each of the three non-linear equations obtained, to zero and solving them simultaneously we get the required LS estimates.

5.3. Minimum spacing method (MS)

Let $x_i; i = 1,2,..n.$ be the observed value of NTWD with parameters θ, λ and β, in increasing order.the geometric mean of the differences is given as

$$GM = \sqrt[n+1]{\prod_{i=1}^{n+1} D_i},$$

where $D_i = G(x_i) - G(x_{i-1})$.

Taking log on both sides we get

$$\log(GM) = \frac{1}{n+1}\sum_{i=1}^{n+1} log[G(x_i) - G(x_{i-1})].$$

Using Eq. (5), we get

$$\log(GM) = \frac{1}{n+1}\sum_{i=1}^{n+1} \ log\left[\left(1 - e^{-\theta x_i^{\lambda}} + \frac{\beta e^{-\theta x_i^{\lambda}}\left(1 - e^{-\theta x_i^{\lambda}}\right)}{2 - e^{-\theta x_i^{\lambda}}}\right) - \left(1 - e^{-\theta 1} + \frac{\beta e^{-\theta x_{i-1}^{\lambda}}\left(1 - e^{-\theta x_{i-1}^{\lambda}}\right)}{2 - e^{-\theta x_{i-1}^{\lambda}}}\right)\right]. \qquad (17)$$

We differentiating Eq. (17) w.r.t θ, λ and β respectively, equating them to zero and solving them simultaneously we get the required MS estimates of parameters.

5.4. Weighted least Square (WLS)

The likelihood Eq. of WLS estimates is given as

$$W(\Theta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[G(x_i)-\frac{i}{n+1}\right]^2.$$

$$W(\Theta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\left(1-e^{-\theta x_i^{\lambda}}+\frac{\beta e^{-\theta x_i^{\lambda}}\left(1-e^{-\theta x_i^{\lambda}}\right)}{2-e^{-\theta x_i^{\lambda}}}\right)-\frac{i}{n+1}\right]^2. \qquad (18)$$

We differentiating Eq. (18) w.r.t θ, λ and β respectively, equating them to zero and solving them simultaneously we get the required WLS estimates of parameters.

5.5. Cramer Von Mises (CVM)

The likelihood Eq. of CVM estimates is given as

$$C(\Theta)=\frac{1}{12n}+\sum_{i=1}^{n}\left[G(x_i)-\frac{2i-1}{2n}\right]^2.$$

$$W(\Theta)=\frac{1}{12n}+\sum_{i=1}^{n}\left[\left(1-e^{-\theta x_i^{\lambda}}+\frac{\beta e^{-\theta x_i^{\lambda}}\left(1-e^{-\theta x_i^{\lambda}}\right)}{2-e^{-\theta x_i^{\lambda}}}\right)-\frac{2i-1}{2n}\right]^2. \qquad (19)$$

Eq. (19) is differentiated w.r.t θ, λ and β respectively and equated to zero to obtain the CVM estimates.

6. Simulation Study

In this section, R software has been used to accomplish the simulation study of the behavior of ML estimates for different sample sizes. 1000 samples of size n are drawn using Eq. (6). The sample sizes considered are n=(25,50,75,300,500) for parameter combinations (θ,β,λ)=(0.2,0.7,0.5) and (θ,β,λ)=(0.2,0.7,0.5). The results obtained are displayed in Table 1. It can be concluded from the Table 1 that the MSE gets reduced as the sample size increases.

Table 1: Bias and MSE of Parameters

θ	β	λ	n	$Bias(\theta)$	$MSE(\theta)$	$Bias(\beta)$	$MSE(\beta)$	Bias(λ)	$MSE(\lambda)$
			25	2.664	7.280	0.802	0.646	1.224	1.654
			50	2.596	6.817	0.797	0.636	1.152	1.394
			75	2.595	6.784	0.785	0.629	1.126	1.314
0.2	0.7	0.5	100	2.578	6.681	0.780	0.625	1.110	1.261
			300	2.574	6.636	0.768	0.609	1.095	1.209
			500	2.568	6.601	0.707	0.562	1.092	1.199

			25	2.447	6.149	1.006	1.015	1.294	1.818
			50	2.421	5.938	0.996	0.999	1.253	1.635
			75	2.410	5.864	0.988	0.991	1.239	1.580
0.3	0.9	0.4	100	2.380	5.702	0.984	0.990	1.219	1.517
			300	2.360	5.602	0.969	0.968	1.298	1.578
			500	**2.370**	**5.623**	**0.913**	**0.905**	**1.192**	**1.427**

7. Application

A number of criterions are available in literature to access the flexibility of generalized probability models. The criterions that will be used in this paper to examine the efficiency of proposed model are Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC) and Akaike Information Criteria Corrected (AICc). The model which admits the minimum value of each of the given criterion is considered as the best fitted model for the given real life data set. The probability models used for comparison with the new model are WD, Rayleigh Distribution (RD) and Exponential Distribution (ED). The data set considered is a random sample consisting of the remission times (in months) of 128 bladder cancer patients (Lee and Wang, 2003). Table 2 present the ML estimates and values of comparison criterion for the given dataset.

Table 2: The ML Estimates and Values of Comparison Criterion for the Cancer Dataset

Model	Estimates						
	$\hat{\theta}$	$\hat{\beta}$	$\hat{\lambda}$	$-2\hat{\ell}$	AIC	BIC	AICc
NTWD	0.219 (0.047)	−0.976 (0.164)	0.829 (0.073)	412	828.51	821.99	828.70
RD	9.931 (0.438)	-	-	492	984.53	985.69	984.56
ED	0.106 (0.078)	-	-	415	830.68	832.71	831.84
WD	0.0939 (0.0190)	-	1.0477 (0.067)	415	832.17	834.49	832.20

Based on the values reported in Table 2, we can conclude that NTWD provides better fit than compared models. Thus, we can conclude that NTWD provides best fit for the given dataset. The estimated variance-covariance matrix is given as

$$j(\widehat{\Theta}) = \begin{bmatrix} 0.002273 & -0.00588 & -0.003263 \\ -0.00588 & 0.02718 & 0.008338 \\ -0.003263 & 0.008338 & 0.005361 \end{bmatrix}$$

The 95% confidence interval for the parameters θ, λ and β are given by $[0.1224, 0.3093]$, $[-1.299, -0.653]$ and $[0.6860, 0.9730]$ respectively.

8. Conclusions

In this paper, a new transmutation technique is employed to achieve a more flexible generalization of Weibull distribution. The new distribution is named New Transmuted Weibull distribution. Some important statistical properties are expounded. Different methods of estimating the parameters of the proposed distribution are discussed. The utility of the proposed model is illustrated by means of a real life dataset.

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