Pak.j.stat.oper.res. Vol.17 No. 1 2021 pp 213-226 DOI: http://dx.doi.org/10.18187/pjsor.v17i1.2724

Pakistan Journal of Statistics and Operation Research

Exact Moments of Generalized Pareto Distribution based on Generalized Order Statistics and Characterizations

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Abstract
In this paper, we present simple explicit expressions for single and product moments of generalized order statistics from generalized Pareto distribution. These relations are deduced for the moments of order statistics and record values and tabulated the mean and variance of this distribution. Further, conditional expectation, recurrence relations for single as well as for product moments of generalized order statistics and truncated moment are used to characterize this distribution.

Key Words: Order statistics, record values, generalized order statistics, generalized Pareto distribution, single moments, product moments, recurrence relations, conditional expectation, truncated moment and characterization.

Mathematical Subject Classification: 62E10, 60E15, 62G30.

1. Introduction

In this section, we will deriveA random variable X is said to have generalized Pareto distribution (Castillo and Hadi, 1997) if its probability density function (pdf) is of the form

$$f(x)=\lambda(1-\beta\lambda x)^{(1/\beta)-1}, \quad 0\le x\le 1/\beta\lambda\,,\ \lambda>0\,,\ \beta\ne 0\,. \tag{1}$$

with the corresponding distribution function df

$$F(x)=1-(1-\beta\lambda x)^{(1/\beta)}, \quad 0\le x\le 1/\beta\lambda\,,\ \lambda>0\,,\ \beta\ne 0\,. \tag{2}$$

In view of (1) and (2), we have

$$\overline{F}(x)=\frac{1}{\lambda}(1-\beta\lambda x)\,f(x)\,. \tag{3}$$

The generalized Pareto distribution has emerged as an powerful distribution in modeling heavy tailed datasets in various applications in finance, operational risk, insurance and environmental studies.

Kamps (1995) introduced the concept of generalized order statistics (gos) as follows:

Let $n\ge 2$ be a given integer and $\tilde{m}=(m_1,m_2,\ldots,m_{n-1})\in\Re^{n-1}$, $k\ge 1$ be the parameters such that

$$\gamma_i=k+n-i+\sum_{j=i}^{n-1}m_j\ge 0, \quad \text{for } 1\le i\le n-1\,.$$

The random variables $X(1,n,\tilde{m},k), X(2,n,\tilde{m},k),\ldots, X(n,n,\tilde{m},k)$ are said to be *gos* from a continuous population with the *df* $F(x)$ and the *pdf* $f(x)$, if their joint *pdf* is of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[1-F(x_i)]^{m_i} f(x_i)\right)[1-F(x_n)]^{k-1} f(x_n) \tag{4}$$

on the cone $F^{-1}(0+) < x_1 \le \ldots \le x_n < F^{-1}(1)$ of $\Re^n$.

The model of *gos* contains special cases such as ordinary order statistics $(\gamma_i = n-i+1;\ \ i=1,2,\ldots,n$ i.e. $m_1 = m_2 = \ldots = m_{n-1} = 0,\ k=1)$, k-th record values $(\gamma_i = k$ i.e. $m_1 = m_2 = \ldots = m_{n-1} = -1,\ \ k \in N)$, sequential order statistics $(\gamma_i = (n-i+1)\alpha_i;\ \alpha_1,\alpha_2,\ldots,\alpha_n > 0)$, order statistics with non-integral sample size $(\gamma_i = \alpha - i + 1;\ \alpha > 0)$, Pfeifer's record values $(\gamma_i = \beta_i;\ \beta_1,\beta_2,\ldots,\beta_n > 0)$ and progressive type II censored order statistics $(m_i \in N,\ k \in N)$ [Kamps (1995), Kamps and Cramer (2001)].

For simplicity we assume $m_1 = m_2 = \ldots = m_{n-1} = m$.

The *pdf* of the $r-$th *gos* $X(r,n,m,k)$, is given by (Kamps, 1995)

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[\bar{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)), \tag{5}$$

and the joint *pdf* of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \le r < s \le n$, is given by (Kamps, 1995)

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[\bar{F}(x)]^m f(x) g_m^{r-1}(F(x))$$

$$\times[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s - 1} f(y), \quad x < y, \tag{6}$$

where

$$\bar{F}(x) = 1 - F(x), \quad C_{r-1} = \prod_{i=1}^{r}\gamma_i, \qquad \gamma_i = k + n - i + \sum_{j=i}^{n-1} m_j = k + (n-i)(m+1),$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1}, & m \ne -1 \\ -\ln(1-x), & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(0) = \int_0^x (1-t)^m dt, \quad x \in [0,1).$$

Characterization of probability distribution plays an important role in probability and statistics. Before a particular probability distribution model is applied to fit the real world data, it is necessary to confirm whether the given probability satisfies the essential requirements by its characterization. Recently, there has been a great interest in the characterizations of probability distributions through truncated moment. For detailed survey one may refers to Galambos and Kotz (1978), Glanzel (1987), Keseling (1999), Athar *et al.* (2003), Bieniek and Szynal (2003), Khan and Alzaid (2004), Ahsanullah and Raqab (2004), Cramer *et al.* (2004), Khan *et al.* (2006, 2009, 2011), Khan *et al.* (2010), Nofal (2011), Khan and Zia (2014), Khan *et al.* (2015), Ahsanullah *et al.* (2016), Khan and Khan (2016), Hamedani (2017), Khan and Khan (2017), Nofal and EI Gebal (2017) and Singh *et al.* (2018) among others. In this article, we mainly focus on the exact moments and characterizations of the generalized Pareto distribution through *gos* in different aspects and truncated moment.

2. Exact expressions for moments

In this section, we will derive the exact expressions for the single moments as well as for the product moments based on *gos* from the generalized Pareto distribution. Further by putting specific value of m and k, we will obtain the moments of order statistics and upper records.

Theorem 2.1. For the distribution given in (2) with fixed value of parameters β, λ, when $m \neq -1$

$$E[X(r,n,m,k)]^j = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}(\beta\lambda)^j}\sum_{u=0}^{r-1}\sum_{p=0}^{j}(-1)^{u+p}\binom{r-1}{u}\binom{j}{p}\frac{1}{(\gamma_{r-u}+\beta p)}. \tag{7}$$

$$= \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}\beta(\beta\lambda)^j}\sum_{u=0}^{r-1}(-1)^u\binom{r-1}{u}B\left(\frac{\gamma_{r-u}}{\beta}, j+1\right), \tag{8}$$

where $B(a,b)$ is the beta function.

When $m = -1$

$$E[X(r,n,-1,k)]^j = E[Y_r^{(k)}]^j = \frac{k^r}{(\beta\lambda)^j}\sum_{p=0}^{j}(-1)^p\binom{j}{p}\frac{1}{(k+\beta p)^r}, \tag{9}$$

where $Y_r^{(k)}$ denotes the $r-$th k upper record values.

Proof. From (5), we have

$$E[X(r,n,m,k)]^j = \frac{C_{r-1}}{(r-1)!}\int_0^{1/\beta\lambda} x^j [\bar{F}(x)]^{\gamma_r-1} f(x)\, g_m^{r-1}(F(x))\, dx. \tag{10}$$

On expanding binomially $g_m^{r-1}(F(x)) = \left(\frac{1}{m+1}\{1-\bar{F}(x)^{m+1}\}\right)^{r-1}$ in (10), we get

$$E[X(r,n,m,k)]^j = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{u=0}^{r-1}(-1)^u\binom{r-1}{u}\int_0^{1/\beta\lambda} x^j[\bar{F}(x)]^{\gamma_{r-u}-1} f(x)\, dx.$$

$$= \frac{C_{r-1}}{(r-1)!}\sum_{u=0}^{r-1}(-1)^u\binom{r-1}{u}J(\gamma_{r-u}-1, r-1), \tag{11}$$

where

$$J(a,b) = \frac{1}{(m+1)^b}\int_0^{1/\beta\lambda} x^j[\bar{F}(x)]^a f(x)\, dx. \tag{12}$$

On using the transformation $t = [\bar{F}(x)]^\beta$ in (12), we find that

$$J(a,b) = \frac{1}{\beta(\beta\lambda)^j(m+1)^b}\int_0^1 t^{\frac{a+1}{\beta}-1}(1-t)^j\, dt \tag{13}$$

$$= \frac{1}{\beta(\beta\lambda)^j(m+1)^b}B\left(\frac{a+1}{\beta}, j+1\right). \tag{14}$$

By using binomial expansion, we can rewrite (13) as

$$J(a,b)=\frac{1}{\beta(\beta\lambda)^{j}(m+1)^{b}}\sum_{p=0}^{j}(-1)^{p}\binom{j}{p}\int_{0}^{1}t^{\frac{a+1}{\beta}+p-1}\,dt$$

$$=\frac{1}{(\beta\lambda)^{j}(m+1)^{b}}\sum_{p=0}^{j}(-1)^{p}\binom{j}{p}\frac{1}{(a+\beta p+1)}. \tag{15}$$

Now on substituting (15) and (14) in (11), we get the relations given in (7) and (8), respectively.

When $m=-1$, from (7), we have

$$E[X(r,n,m,k)]^{j}=\frac{0}{0} \quad \text{as} \quad \sum_{u=0}^{r-1}(-1)^{u}\binom{r-1}{u}=0.$$

Since (7) is of the form $\frac{0}{0}$ at $m=-1$, therefore, we have

$$E[X(r,n,m,k)]^{j}=A\sum_{u=0}^{r-1}(-1)^{u}\binom{r-1}{u}\frac{[k+(n-r+u)(m+1)+\beta p]^{-1}}{(m+1)^{r-1}}, \tag{16}$$

where

$$A=\frac{C_{r-1}}{(r-1)!(\beta\lambda)^{j}}\sum_{p=0}^{j}(-1)^{p}\binom{j}{p} \quad \text{and} \quad \gamma_{r-u}=k+(n-r+u)(m+1).$$

Differentiating numerator and denominator of (16) $(r-1)$ times with respect to m, we get

$$E[X(r,n,m,k)]^{j}=A\sum_{u=0}^{r-1}(-1)^{u}\binom{r-1}{u}\frac{(r-n-u)^{r-1}}{[k+(n-r+u)(m+1)+\beta p]^{r}}.$$

On applying L' Hospital rule, we have

$$\lim_{m\to-1}E[X(r,n,m,k)]^{j}=A\sum_{u=0}^{r-1}(-1)^{u}\binom{r-1}{u}\frac{(r-n-u)^{r-1}}{(k+\beta p)^{r}}. \tag{17}$$

But for all integers $n\geq 0$ and for all real number x, we have Ruiz (1996)

$$\sum_{i=0}^{n}(-1)^{i}\binom{n}{i}(x-i)^{n}=n!. \tag{18}$$

Now on substituting (18) in (17) and simplifying, we get the relation given in (9).

Identity 2.1. For $\gamma_r\geq 1$, $k\geq 1$, $1\leq r\leq n$ and $m\neq -1$

$$\sum_{u=0}^{r-1}(-1)^{u}\binom{r-1}{u}\frac{1}{\gamma_{r-u}}=\frac{(r-1)!(m+1)^{r-1}}{\prod_{t=1}^{r}\gamma_{t}}. \tag{19}$$

Proof. (19) can be proved by setting $j=0$ in (7), as obtained by Khan *et al.* (2012).

Remark 2.1. Putting $m=0$ and $k=1$ in (7), the exact expression for the single moments of order statistics from generalized Pareto distribution can be obtained as

$$E(X_{r:n})^j = \frac{C_{r:n}}{(\beta\lambda)^j}\sum_{u=0}^{r-1}\sum_{p=0}^{j}(-1)^{u+p}\binom{r-1}{u}\binom{j}{p}\frac{1}{(n-r+u+\beta p+1)}, \tag{20}$$

where $C_{r:n} = \frac{n!}{(r-1)!(n-r)!}$.

Remark 2.2. Setting $k=1$ in (9), the exact expression for the single moments of upper record statistics from generalized Pareto distribution has the form

$$E[(Y_r^{(1)})^j] = E(X_{U(r)})^j = \frac{1}{(\beta\lambda)^j}\sum_{p=0}^{j}(-1)^p\binom{j}{p}\frac{1}{(1+\beta p)^r}, \tag{21}$$

where, $X_{U(r)}$ denotes the $r-$th upper record values.

The following theorem gives the exact expression for product moments of the *gos* from the generalized Pareto distribution.

Theorem 2.2. For the distribution given in (2) with fixed parameters β, λ, when $m \neq -1$

$$E[\{X(r,n,m,k)\}^i\{X(s,n,m,k)\}^j] = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}(\beta\lambda)^{i+j}}$$

$$\times\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}\sum_{p=0}^{j}\sum_{q=0}^{i}(-1)^{u+v+p+q}\binom{r-1}{u}\binom{s-r-1}{v}\binom{j}{p}\binom{i}{q}\frac{1}{(\gamma_{s-v}+\beta p)[\gamma_{r-u}+\beta(p+q)]}, \tag{22}$$

when $m=-1$

$$E[(Y_r^{(k)})^i(Y_s^{(k)})^j] = \frac{k^s}{(\beta\lambda)^{(i+j)}}\sum_{p=0}^{j}\sum_{q=0}^{i}(-1)^{p+q}\binom{j}{p}\binom{i}{q}\frac{1}{(k+\beta p)^{s-r}[k+\beta(p+q)]^r}. \tag{23}$$

Proof. From (6), we have

$$E[\{X(r,n,m,k)\}^i\{X(s,n,m,k)\}^j] = \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^{1/\beta\lambda}\int_x^{1/\beta\lambda} x^i y^j[\bar{F}(x)]^m f(x)g_m^{r-1}(F(x))$$

$$\times[h_m(F(y))-h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}f(y)\,dydx. \tag{24}$$

On expanding binomially $g_m^{r-1}(F(x))$ and $[h_m(F(y))-h_m(F(x))]^{s-r-1}$ in (24), we get

$$E[\{X(r,n,m,k)\}^i\{X(s,n,m,k)\}^j] = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}$$

$$\times\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v}\int_0^{1/\beta\lambda}\int_x^{1/\beta\lambda} x^i y^j[\bar{F}(x)]^{(s-r+u-v)(m+1)-1}f(x)$$

$$\times[\bar{F}(y)]^{\gamma_{s-v}-1}f(y)\,dy\,dx.$$

$$= \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v}$$

$$\times I_{i,j}((s-r+u-v)(m+1)-1, 0, \gamma_{s-v}-1), \tag{25}$$

where

$$I_{i,j}(a,0,c)=\int_0^{1/\beta\lambda}\int_x^{1/\beta\lambda} x^i\, y^j [\bar{F}(x)]^a f(x)[\bar{F}(y)]^c f(y)dydx$$

$$=\int_0^{1/\beta\lambda} x^i [\bar{F}(x)]^a f(x) I(x)dx \tag{26}$$

and

$$I(x)=\int_x^{1/\beta\lambda} y^j [\bar{F}(y)]^c f(y)dy\,. \tag{27}$$

By setting $z=[\bar{F}(y)]^{\beta}$ in (27), we find that

$$I(x)=\frac{1}{\beta(\beta\lambda)^j}\int_0^{[\bar{F}(x)]^{\beta}} z^{\frac{c+1}{\beta}-1}(1-z)^j dz$$

$$=\frac{1}{\beta(\beta\lambda)^j}\sum_{p=0}^{j}(-1)^p\binom{j}{p}\int_0^{[\bar{F}(x)]^{\beta}} z^{\frac{c+\beta p+1}{\beta}-1} dz$$

$$=\frac{1}{(\beta\lambda)^j(c+\beta\, p+1)}\sum_{p=0}^{j}(-1)^p\binom{j}{p}[\bar{F}(x)]^{c+\beta p+1}\,.$$

Now on substituting the above expression of $I(x)$ in (26), we get

$$I_{i,j}(a,0,c)=\frac{1}{(\beta\lambda)^j(c+\beta p+1)}\sum_{p=0}^{j}(-1)^p\binom{j}{p}\int_0^{1/\beta\lambda} x^i [\bar{F}(x)]^{a+c+\beta p+1} f(x)dx\,. \tag{28}$$

Again by substituting $w=[\bar{F}(x)]^{\beta}$ in (28) and simplifying the resulting expression, yields

$$I_{i,j}(a,0,c)=\frac{1}{(\beta\lambda)^{i+j}}\sum_{p=0}^{j}\sum_{q=0}^{i}(-1)^{p+q}\binom{j}{p}\binom{i}{q}\frac{1}{(c+\beta p+1)(a+c+\beta(p+q)+2)}\,. \tag{29}$$

Now substituting for $I_{i,j}((s-r+u-v)(m+1)-1,0,\gamma_{s-v}-1)$ from (29) in (25), and simplifying, we have the result given in (22).

When $m=-1$, we have

$$E[\{X(r,n,m,k)\}^i\{X(s,n,m,k)\}^j]=\frac{0}{0},\ \text{as}\ \sum_{u=0}^{r-1}(-1)^u\binom{r-1}{u}=0\ \text{and}\ \sum_{v=0}^{s-r-1}(-1)^v\binom{s-r-1}{v}=0\,.$$

Since (22) is of the form $\frac{0}{0}$ at $m=-1$, therefore, we have

$$E[\{X(r,n,m,k)\}^i\{X(s,n,m,k)\}^j]=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}(\beta\lambda)^{i+j}}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}\sum_{p=0}^{j}\sum_{q=0}^{i}(-1)^{u+v+p+q}$$

$$\times\binom{r-1}{u}\binom{s-r-1}{v}\binom{j}{p}\binom{i}{q}\frac{1}{(\gamma_{s-v}+\beta p)(\gamma_{r-u}+\beta(p+q))}\frac{1}{(\gamma_{s-v}+\beta p)(\gamma_{r-u}+\beta(p+q))}$$

$$= B\sum_{u=0}^{r-1}(-1)^u\binom{r-1}{u}\frac{[k+(n-r+u)(m+1)+\beta(p+q)]^{-1}}{(m+1)^{r-1}}\sum_{v=0}^{s-r-1}(-1)^v\binom{s-r-1}{v}\frac{[k+(n-s+v)(m+1)+\beta p]^{-1}}{(m+1)^{s-r-1}}, \tag{30}$$

where

$$B=\frac{C_{s-1}}{(r-1)!(s-r-1)!(\beta\lambda)^{i+j}}\sum_{p=0}^{j}\sum_{q=0}^{i}(-1)^{p+q}\binom{j}{p}\binom{i}{q}.$$

Differentiating numerator and denominator of (30) $(r-1)$ and $(s-r-1)$ times with respect to m, we get

$$E[\{X(r,n,m,k)\}^i\{X(s,n,m,k)\}^j]=B\sum_{u=0}^{r-1}(-1)^{u+r-1}\binom{r-1}{u}\frac{(n-r+u)^{r-1}}{[k+(n-r+u)(m+1)+\beta(p+q)]^r}$$

$$\times\sum_{v=0}^{s-r-1}(-1)^{v+s-r-1}\binom{s-r-1}{v}\frac{(n-s+v)^{s-r-1}}{[k+(n-s+v)(m+1)+\beta p]^{s-r}}.$$

Therefore, on applying L' Hospital rule, we find that

$$\lim_{m\to-1}E[\{X(r,n,m,k)\}^i\{X(s,n,m,k)\}^j]=\frac{B}{[k+\beta(p+q)]^r}\sum_{u=0}^{r-1}(-1)^u\binom{r-1}{u}(r-n-u)^{r-1}$$

$$\times\frac{1}{(k+\beta p)^{s-r}}\sum_{v=0}^{s-r-1}(-1)^v\binom{s-r-1}{v}(s-n-v)^{s-r-1}. \tag{31}$$

Now on substituting (18) in (31) and simplifying, we get the required result given in (23) on using

$$\lim_{x\to0}f_1(x)\,f_2(x)=\lim_{x\to0}f_1(x)\lim_{x\to0}f_2(x).$$

Identity 2.2. For γ_r, $\gamma_s\ge1$, $k\ge1$, $1\le r<s\le n$ and $m\ne-1$

$$\sum_{v=0}^{s-r-1}(-1)^v\binom{s-r-1}{v}\frac{1}{\gamma_{s-v}}=\frac{(s-r-1)!(m+1)^{s-r-1}}{\prod_{t=r+1}^{s}\gamma_t}. \tag{32}$$

Proof. At $i=j=0$ in (22), we have

$$1=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v}\frac{1}{\gamma_{s-v}\,\gamma_{r-u}}.$$

Now on using (19), we get the result given in (32), as obtained by Khan *et al*. (2012).

At $r=0$, (32) reduces to (19).

Remarks 2.3. Setting $m=0$ and $k=1$ in (22), the exact expression for the product moments of order statistics from generalized Pareto distribution can be obtained as

$$E[(X_{r:n})^i(X_{s:n})^j]=\frac{C_{r,s:n}}{(\beta\lambda)^{i+j}}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}\sum_{p=0}^{j}\sum_{q=0}^{i}(-1)^{u+v+p+q}\binom{r-1}{u}\binom{s-r-1}{v}$$

$$\times\binom{j}{p}\binom{i}{q}\frac{1}{(n-s+1+v+\beta p)[n-r+1+u+\beta(p+q)]},$$

where

$$C_{r,s:n}=\frac{n!}{(r-1)!(s-r-1)!(n-s)!}\,.$$

Remarks 2.4. Setting $k=1$ in (23), the exact expression for the product moments of upper record statistics from generalized Pareto distribution can be obtained as

$$E[(Y_r^{(1)})^i(Y_s^{(1)})^j]=E[(X_{U(r)})^i(X_{U(s)})^j]=\frac{1}{(\beta\lambda)^{(i+j)}}\sum_{p=0}^{j}\sum_{q=0}^{i}(-1)^{p+q}\binom{j}{p}\binom{i}{q}\frac{1}{(\beta\,p+1)^{s-r}[\beta\,(p+q)+1]^r}\,.$$

Remark 2.5. At $j=0$ in (22), we have

$$E[X(r,n,m,k)]^i=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}(\beta\lambda)^i}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}\sum_{q=0}^{i}(-1)^{u+v+q}$$

$$\times\binom{r-1}{u}\binom{s-r-1}{v}\binom{i}{q}\frac{1}{[\gamma_{s-v}(\gamma_{r-u}+\beta q)]}\,.$$

Making use of (32), yields

$$E[X(r,n,m,k)]^i=\frac{C_{r-1}}{(r-1)!(m+1)^{r-1}(\beta\lambda)^i}\sum_{u=0}^{r-1}\sum_{q=0}^{i}(-1)^{u+q}\binom{r-1}{u}\binom{i}{q}\frac{1}{(\gamma_{r-u}+\beta q)}\,,$$

where

$$C_{s-1}=\prod_{i=1}^{s}\gamma_i \quad \text{and} \quad \frac{C_{s-1}}{C_{r-1}}=\prod_{t=r+1}^{s}\gamma_t\,,$$

which is the exact expression for the single moment as given in (7).

3. Characterization based on the conditional expectations of generalized order statistics

Let $X(r,n,m,k)$, $r=1,2,\ldots$ be *gos* from continuous population with *df* $F(x)$ and *pdf* $f(x)$, then the conditional *pdf* of $X(s,n,m,k)$ given $X(r,n,m,k)=x$, $1\le r<s\le n$, in view of (5) and (6), is

$$f_{X(s,n,m,k)|X(r,n,m,k)}(y\mid x)=\frac{C_{s-1}}{(s-r-1)!C_{r-1}}[\bar{F}(x)]^{m-\gamma_r+1}$$

$$\times[h_m(F(y))-h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}f(y)\,,\quad x<y\,,\quad m\ne -1\,. \tag{33}$$

Theorem 3.1. Let X be a non-negative random variable having an absolutely continuous *df* $F(x)$ with $F(0)=0$ and $0\le F(x)\le 1$ for all $x\ge 0$, then

$$E[X(s,n,m,k)\mid X(l,n,m,k)=x]=\frac{1}{\beta\lambda}\left\{1-(1-\beta\lambda x)\prod_{j=1}^{s-l}\left(\frac{\gamma_{l+j}}{\gamma_{l+j}+\beta}\right)\right\},\quad l=r\,,r+1\,,\ m\ne -1 \tag{34}$$

if and only if

$$\bar{F}(x)=(1-\beta\lambda x)^{(1/\beta)}\,,\quad 0\le x\le 1/\beta\lambda\,,\ \lambda>0\,.$$

Proof. From (33), for $s>r+1$, we have

$$E[X(s,n,m,k) \mid X(r,n,m,k)=x]=\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}$$

$$\times\int_x^{1/\beta\lambda} y\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_s-1}\left\{1-\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{m+1}\right\}^{s-r-1}\frac{f(y)}{\bar{F}(x)}dy\,. \qquad (35)$$

By setting $u=\frac{\bar{F}(y)}{\bar{F}(x)}=\frac{(1-\beta\lambda y)^{(1/\beta)}}{(1-\beta\lambda x)^{(1/\beta)}}$, from (2) in (35), we obtain

$$E[X(s,n,m,k) \mid X(r,n,m,k)=x]=A\left[\int_0^1 u^{\gamma_s-1}(1-u^{m+1})^{s-r-1}du\right.$$

$$\left.-(1-\beta\lambda x)\int_0^1 u^{\gamma_s+\beta-1}(1-u^{m+1})^{s-r-1}du\right], \qquad (36)$$

where

$$A=\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}(\beta\lambda)}\,.$$

Again by setting $t=u^{m+1}$ in (36), we get

$$E[X(s,n,m,k) \mid X(r,n,m,k)=x]=\frac{A}{m+1}\left[\int_0^1 t^{\frac{\gamma_s}{m+1}-1}(1-t)^{s-r-1}dt-(1-\beta\lambda x)\int_0^1 t^{\frac{\gamma_s+\beta}{m+1}-1}(1-t)^{s-r-1}dt\right] \qquad (37)$$

$$=\frac{A}{m+1}\left[B\left(\frac{\gamma_s}{m+1},s-r\right)-(1-\beta\lambda x)\,B\left(\frac{\gamma_s+\beta}{m+1},s-r\right)\right]$$

and hence the necessary part as given in (34).

To prove the sufficient part, we have from (33) and (34)

$$\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}\int_x^{1/\beta\lambda} y[\bar{F}(x)^{m+1}-\bar{F}(y)^{m+1}]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}f(y)\,dy$$

$$=g_{s|r}(x)[\bar{F}(x)]^{\gamma_{r+1}}, \qquad (38)$$

where

$$g_{s|r}(x)=\frac{1}{\beta\lambda}\left\{1-(1-\beta\lambda x)\prod_{j=1}^{s-r}\left(\frac{\gamma_{r+j}}{\gamma_{r+j}+\beta}\right)\right\}.$$

Differentiating (38) both sides with respect to x, we get

$$-\frac{C_{s-1}[\bar{F}(x)]^m f(x)}{(s-r-2)!C_{r-1}(m+1)^{s-r-2}}\int_x^{1/\beta\lambda} y[\bar{F}(x)^{m+1}-\bar{F}(y)^{m+1}]^{s-r-2}$$

$$\times[\bar{F}(y)]^{\gamma_s-1}f(y)dy=g'_{s|r}(x)[\bar{F}(x)]^{\gamma_{r+1}}-\gamma_{r+1}g_{s|r}(x)[\bar{F}(x)]^{\gamma_{r+1}-1}f(x)$$

or

$$-\gamma_{r+1} g_{s|r+1}(x)[\bar{F}(x)]^{\gamma_{r+2}+m} f(x) = g'_{s|r}(x)[\bar{F}(x)]^{\gamma_{r+1}} - \gamma_{r+1} g_{s|r}(x)[\bar{F}(x)]^{\gamma_{r+1}-1} f(x) \,,$$

where

$$g'_{s|r}(x) = \prod_{j=1}^{s-r}\left(\frac{\gamma_{r+j}}{\gamma_{r+j}+\beta}\right) \quad \text{and} \quad g_{s|r+1}(x) = \frac{1}{\beta\lambda}\left\{1-(1-\beta\lambda x)\left(\frac{\gamma_{r+1}+\beta}{\gamma_{r+1}}\right)\prod_{j=1}^{s-r}\left(\frac{\gamma_{r+j}}{\gamma_{r+j}+\beta}\right)\right\}.$$

Therefore,

$$\frac{f(x)}{\bar{F}(x)} = -\frac{g'_{s|r}(x)}{\gamma_{r+1}[g_{s|r+1}(x) - g_{s|r}(x)]} \qquad \text{[Khan } et\ al. \text{ (2006)]}$$

$$= \frac{\lambda}{1-\beta\lambda x}\,. \tag{39}$$

Integrating (39) both sides with respect to x over $(0, y)$, the sufficient part is proved.

Remark 3.1. Letting $m \to -1$ in (34), the characterizing results of the generalized Pareto distribution based on $k-$ th upper record values can be obtained as

$$E(Y_s^{(k)} / Y_l^{(k)} = x) = \frac{1}{\beta\lambda}\left\{1-(1-\beta\lambda x)\left(\frac{k}{k+\beta}\right)^{s-l}\right\}, \quad l = r\,,\ r+1\,,\ r \ge k\,,\ m = -1\,.$$

and for record values $(k = 1)$

$$E(X_{U(s)} \mid X_{U(l)} = x) = \frac{1}{\beta\lambda}\left\{1-(1-\beta\lambda x)\left(\frac{1}{1+\beta}\right)^{s-l}\right\}.$$

Remark 3.2. Setting $m = 0$, $k = 1$ in (34), we obtain the characterization results of the generalized Pareto distribution based on order statistics in the form

$$E(X_{s:n} \mid X_{l:n} = x) = \frac{1}{\beta\lambda}\left\{1-(1-\beta\lambda x)\prod_{j=1}^{s-l}\left(\frac{n-(l+j)+1}{n-(l+j)+1+\beta}\right)\right\}, \quad l = r\,,\ r+1\,.$$

4. Characterizations based on truncated moment

Theorem 4.1. Suppose an absolutely continuous (with respect to Lebesgue measure) random variable X has the *df* $F(x)$ and *pdf* $f(x)$ for $0 < x < 1/\beta\lambda$, such that $f'(x)$ and $E(X \mid X \le x)$ exist for all x, then

$$E(X \mid X \le x) = g(x)\eta(x)\,, \tag{40}$$

where

$$\eta(x) = \frac{f(x)}{F(x)} \quad \text{and} \quad g(x) = -\frac{x(1-\beta\lambda x)}{\lambda} + \frac{(1-\beta\lambda x)^{1-(1/\beta)}}{\lambda}\int_0^x (1-\beta\lambda u)^{1/\beta}\, du$$

if and only if

$$f(x) = \lambda(1-\beta\lambda x)^{(1/\beta)-1}, \quad 0 \le x \le 1/\beta\lambda\,,\ \lambda > 0\,,\ \beta \ne 0\,.$$

Proof. We have

$$E(X \mid X \leq x) = \frac{\lambda}{F(x)} \int_0^x u\,(1-\beta\lambda u)^{(1/\beta)-1} du \,. \tag{41}$$

Integrating (41) by parts treating '$\lambda(1-\beta\lambda u)^{(1/\beta)-1}$' for integration and rest of the integrand for differentiation, we get

$$E(X \mid X \leq x) = \frac{1}{F(x)} [-x(1-\beta\lambda x)^{1/\beta} + \int_0^x (1-\beta\lambda u)^{1/\beta} du]. \tag{42}$$

After multiplying and dividing by $f(x)$ in (42), we have result given in (40).

To prove sufficient part, we have from (40)

$$\int_0^x u\, f(u)\, du = g(x) f(x)\,. \tag{43}$$

Differentiating (43) on both the sides with respect to x, we find that

$$x\, f(x) = g'(x) f(x) + g(x) f'(x)\,.$$

Therefore,

$$\frac{f'(x)}{f(x)} = \frac{x - g'(x)}{g(x)} \qquad \text{[Ahsanullah, } et\ al. \text{ (2016)]}$$

$$= \frac{\lambda(\beta-1)}{(1-\beta\lambda x)}, \tag{44}$$

where

$$g'(x) = x - g(x)\frac{\lambda(\beta-1)}{(1-\beta\lambda x)}\,.$$

Integrating both the sides in (44) with respect to x, we get

$$f(x) = c\,(1-\beta\lambda x)^{(1/\beta)-1}.$$

It is known that

$$\int_0^{1/\beta\lambda} f(x)\, dx = 1\,.$$

Thus,

$$\frac{1}{c} = \int_0^{1/\beta\lambda} (1-\beta\lambda x)^{(1/\beta)-1} dx = \frac{1}{\lambda}\,,$$

which proves that

$$f(x) = \lambda(1-\beta\lambda x)^{(1/\beta)-1}, \quad 0 \leq x \leq 1/\beta\lambda\,,\ \lambda > 0\,,\ \beta \neq 0\,.$$

5. Numerical Computation

The exact expression obtained for order statistics in (20) and record values in (21) allows us to evaluate the mean and variance for arbitrary chosen values of β, λ and $n = 1(1)5$ are listed in Table 1, 2, 3 and 4. All computations here we performed using Mathematica. Mathematica like other algebraic manipulation packages allow for arbitrary precisions, so the accuracy of the given values is not an issue.

Table 1. Mean of order statistics

n	r	$\beta=0.5$, $\lambda=1.5$	$\beta=1.5$, $\lambda=2$	$\beta=2.5$, $\lambda=3$
1	1	0.444445	0.200000	0.095238
2	1	0.444445	0.200000	0.095238
	2	0.622222	0.257143	0.116402
3	1	0.190476	0.111111	0.060606
	2	0.419048	0.206349	0.101011
	3	0.723810	0.282540	0.124098
4	1	0.148148	0.090909	0.051282
	2	0.317460	0.171717	0.088578
	3	0.520635	0.240981	0.113442
	4	0.790154	0.296392	0.127650
5	1	0.121212	0.076923	0.044444
	2	0.255892	0.146853	0.078633
	3	0.409812	0.209013	0.103497
	4	0.594517	0.262293	0.120073
	5	0.840789	0.304917	0.129545

We can note that the fact $\sum_{i=0}^{n} E(X_{i:n}^{j}) = n\,E(X)^{j}$ (David and Nagaraja (2003)) is satisfied here, using Table 1.

Table 2. Variance of order statistics

n	r	$\beta=0.5$, $\lambda=1.5$	$\beta=1.5$, $\lambda=2$	$\beta=2.5$, $\lambda=3$
1	1	0.098976	0.010000	0.001512
2	1	0.047407	0.008164	0.001568
	2	0.086914	0.005307	0.000560
3	1	0.027210	0.006173	0.001378
	2	0.052971	0.006097	0.000859
	3	0.072926	0.002976	0.000233
4	1	0.017558	0.004722	0.001169
	2	0.034668	0.005626	0.000959
	3	0.050632	0.004169	0.000451
	4	0.062009	0.001809	0.000108
5	1	0.012244	0.003698	0.000987
	2	0.024306	0.004907	0.000958
	3	0.035998	0.004383	0.005912
	4	0.046742	0.002889	0.000248
	5	0.053695	0.001176	0.000057

Table 3. Mean of upper record values

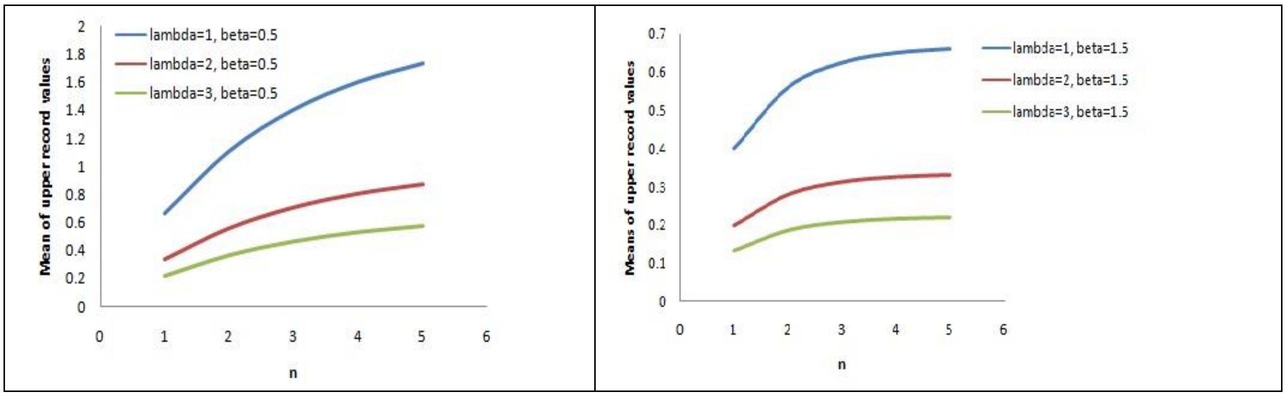
n	$\beta=0.50$			$\beta=1.5$		
	$\lambda=1$	$\lambda=2$	$\lambda=3$	$\lambda=1$	$\lambda=2$	$\lambda=3$
1	0.66667	0.333334	0.22222	0.40000	0.20000	0.133334
2	1.11111	0.55556	0.37037	0.56000	0.28000	0.186667
3	1.40742	0.703704	0.469136	0.62400	0.31200	0.20800
4	1.60494	0.802469	0.534979	0.64960	0.32480	0.216533
5	1.73663	0.868313	0.578875	0.659840	0.32990	0.219947

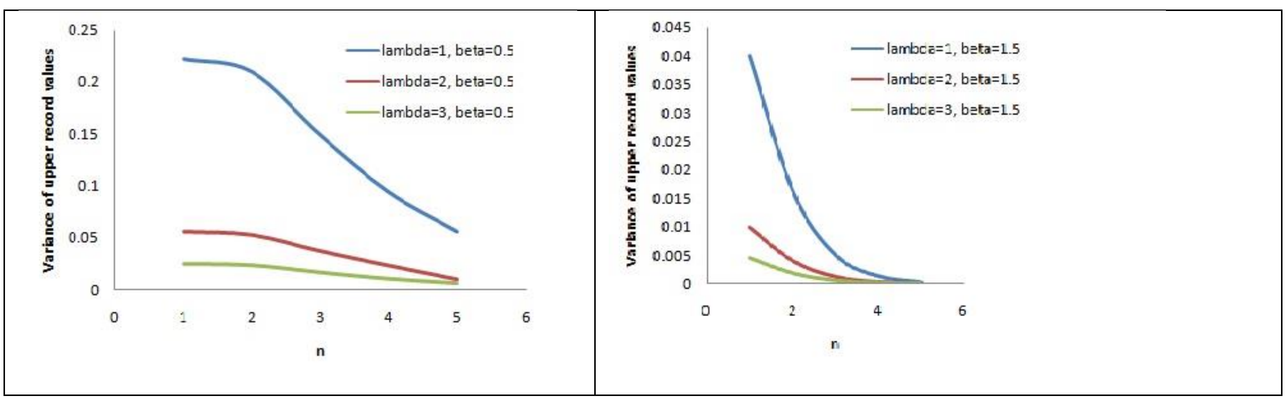
Table 4. Variance of upper record values

n	$\beta = 0.50$			$\beta = 1.5$		
	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$
1	0.222222	0.055555	0.024692	0.04000	0.01000	0.004444
2	0.209877	0.052469	0.023319	0.016400	0.004100	0.001822
3	0.148835	0.037208	0.016537	0.005124	0.001281	0.000569
4	0.093927	0.023482	0.010436	0.001444	0.000362	0.000161
5	0.055634	0.013908	0.006182	0.000387	0.000097	0.000043

Here we have presented the trend of the mean and variance of upper record values from the generalized Pareto distribution for $n = 1(1)5$. For different values of parameters, we observe that the mean of record values increases a λ increases for fixed value of β and as λ increases and Variance of record values decreases.

Behavior of the Mean of upper record values





Acknowledgements

The authors are thankful to the Editorial Board and to the learned referees for their valuable comments and suggestions that led to this improved version.

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