A New Extended Fréchet Distribution: Properties and Estimation

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Abstract

In this work, we introduce a new extension of the Fréchet distribution. A set of the mathematical and statistical properties have been derived. The estimation of the parameters is carried out by considering the different method of estimation. The performances of the proposed estimation methods are studied by Monte Carlo simulations. The potentiality of the proposed model has been analyzed through two data sets. The weighted least square method is the best method for modelling breaking stress data, the least square method is the best method for modelling strengths data, however all other methods performed well for both data sets. On the other hand, the new model gives the best fits among all other fitted extensions of the Fréchet models to these data. So, it could be chosen as the best model for modeling breaking stress and strengths real data.

Keywords: Fréchet distribution; Maximum Likelihood; Least Square; Weighted Least Square; Jackknife; Bayesian Estimation; Simulation.

1. Introduction

The aim of this work is to firstly introduce the first generalization of the Fréchet (Fr) distribution using the Burr XII G (BXII-G) family originally introduced by Cordeiro et al. (2018) along with deriving the essential properties of the new model. Secondly, estimating the unknown parameters via considering different method of estimation, the performances of the proposed estimation methods are studied by a Monte Carlo simulation. The probability density function (PDF) and cumulative distribution function (CDF) of the Fr distribution are given by (for $x \geq 0$)

$$h(x;a,b) = ba^b x^{-(b+1)}\, exp\left[-\left(\frac{a}{x}\right)^b\right] \text{ and } H(x,a,b) = exp[-(a/x)^b], \qquad (1)$$

respectively, where $a > 0$ is a scale parameter and $b > 0$ is a shape parameter, respectively. Let Z be a random variable (RV) having the Fr distribution (1) with parameters a and b. For $r < b$, the $r^{(\text{th})}$ ordinary and incomplete moments of Z are given by

$$\mu_r' = a^r \Gamma\left(1 - \frac{r}{b}\right)$$

and

$$\phi_r(t) = a^r\, \gamma\left(1 - \frac{r}{b}, \left(\frac{a}{t}\right)^b\right),$$

respectively. The CDF of the BrXII-G family of distributions is defined by

$$F(x;\alpha,\beta,\phi) = 1 - \left\{1 + \left[\frac{H(x;\phi)}{1-H(x;\phi)}\right]^\alpha\right\}^{-\beta}. \qquad (2)$$

The PDF corresponding to (2) is given by

$$f(x;\alpha,\beta,\phi) = \alpha\beta\,\frac{h(x;\phi)H(x;\phi)^{\alpha-1}}{[1-H(x;\phi)]^{\alpha+1}}\left\{1 + \left[\frac{H(x;\phi)}{1-H(x;\phi)}\right]^\alpha\right\}^{-\beta-1}, \qquad (3)$$

where $h(x;\phi)$ is the baseline density. The hazard rate function (HRF) of X reduces to

$$\tau(x) = \alpha\beta\,\frac{h(x)H(x)^{\alpha-1}}{[1-H(x)]^{\alpha+1}}\left\{1 + \left[\frac{H(x)}{1-H(x)}\right]^\alpha\right\}^{-1}.$$

Inserting (1) in to (2) we have

$$F(x;\alpha,\beta,a,b)=\left[1-\left(1+\left\{\frac{exp[-(a/x)^b]}{1-exp[-(a/x)^b]}\right\}^{\alpha}\right)^{-\beta}\right], \tag{4}$$

Equation (4) represents the CDF of the proposed model (BrXIIFr). The PDF corresponding to (4) is given by

$$f(x;\alpha,\beta,a,b)=\alpha\beta b a^b x^{-(b+1)}\frac{exp\left[-\alpha\left(\frac{a}{x}\right)^b\right]}{\left\{1-exp\left[-\left(\frac{a}{x}\right)^b\right]\right\}^{\alpha+1}}$$
$$\times\left(1+\left\{\frac{exp[-(a/x)^b]}{1-exp[-(a/x)^b]}\right\}^{\alpha}\right)^{-\beta-1} \tag{5}$$

For $b=2$ the BrXIIFr reduces to BrXII Inverse Rayleigh model, for $b=1$ the BrXIIFr reduces to BrXII Inverse Exponential model, for $\beta=1$ the BrXIIFr reduces to Log-Logistic Fr model, for $\alpha=1$ the BrXIIFr reduces to Lomax Fr model. The hazard rate function (HRF) can be easily obtained via the well-known relationship $f(x;\alpha,\beta,a,b)/[1-F(x;\alpha,\beta,a,b)]$. From Figure 1 we conclude that the the PDF of the new model can be right skewed and unimodal, the HRF can be a unimodal and decreasing shaped.

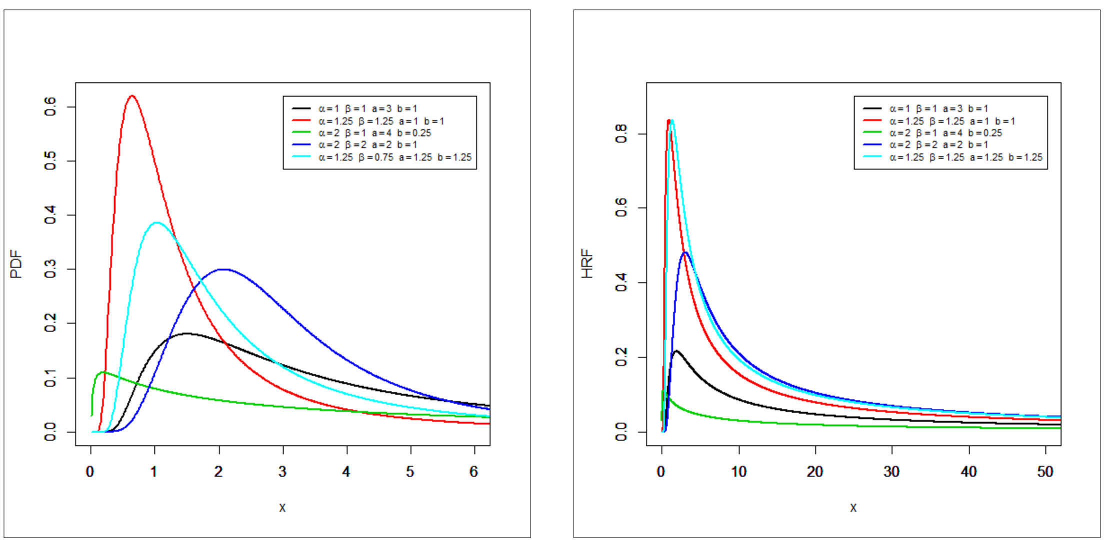

Figure 1: Plots of PDF and HRF for the new model.

The rest of this paper is outlined as follows. In Section 2, we derive some properties of the BrXIIFr distribution. In Section 3, describes four frequentist methods of estimation. In Section 4, a simulation study is carried out to compare the performance of four methods of estimation for the proposed model. In Section 5, the usefulness of the BrXIIFr distribution is illustrated by means of two real data sets. Section 6 offers some concluding remarks.

2. Properties

Linear representation

Hereafter, we denote by $X \sim$ BrXIIFr (x, α, β, a, b) a RV having density function (5). The CDF (4) of X can be expressed as

$$F(x) = 1 - \underbrace{\left(1 + \left\{\frac{exp[-(a/x)^b]}{1-exp[-(a/x)^b]}\right\}^{\alpha}\right)^{-\beta}}_{A}. \quad (6)$$

First, we consider two power series

$$(1+q)^{-c} = \sum_{h=0}^{\infty} 2^{-c-h} \binom{-c}{h} (q-1)^h \quad (7)$$

and

$$(1-q)^{-c} = \sum_{h=0}^{\infty} \frac{\Gamma(c+h)}{h!\,\Gamma(c)} q^h |_{(|q|<1,\ c>0)}. \quad (8)$$

Applying (7) for A in Equation (6) gives

$$F(x) = 1 - \sum_{k=0}^{\infty} 2^{-\beta-k} \binom{-\beta}{k} \left(\left\{\frac{exp[-(a/x)^b]}{1-exp[-(a/x)^b]}\right\}^{\alpha} - 1\right)^k.$$

Second, using the binomial expansion, the last equation can be expressed as

$$F(x) = 1 - \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{(-1)^i \binom{k}{i} \binom{-\beta}{k}}{2^{\beta+k}} \times \{exp[-(a/x)^b]\}^{(k-i)\alpha} \times \underbrace{\{1 - exp[-(a/x)^b]\}^{-(k-i)\alpha}}_{B}.$$

Third, applying (8) for B in the last equation gives

$$F(x) = 1 - \sum_{h,k=0}^{\infty} \sum_{i=0}^{k} d_{i,h,k}\ \Pi_{(k-i)\alpha+h}(x, a, b), \quad (9)$$

where

$$d_{i,h,k} = \frac{(-1)^i \Gamma([k-i]\alpha + h)}{2^{\beta+k} h!\, \Gamma([k-i]\alpha)} \binom{k}{i} \binom{-\beta}{k}.$$

and $\Pi_{(k-i)\alpha+h}(x, a, b)$ is the CDF of the Fr CDF with scale parameter $a[(k-i)\alpha + h]^{\frac{1}{b}}$ and shape parameter b. By differentiating (9), we obtain

$$f(x) = \sum_{\substack{h,k=0 \\ h+k\geq 1}}^{\infty} \sum_{i=0}^{k} b_{i,h,k}\ \pi_{(k-i)\alpha+h}(x, a, b), \quad (10)$$

where $\pi_{(k-i)\alpha+h}(x, a, b)$ is the Fr density with scale parameter $a[(k-i)\alpha + h]^{\frac{1}{b}}$ and shape parameter b and $b_{i,h,k} = -d_{i,h,k}$.

Moments and generating function

The $r^{(\text{th})}$ ordinary moment of X say $\mu'_r = E(X^r)$, is determined from (10) as

$$\mu'_r = \sum_{\substack{h,k=0 \\ h+k\geq 1}}^{\infty} \sum_{i=0}^{k} b_{i,h,k} a^r [(k-i)\alpha + h]^{\frac{r}{b}} \Gamma\left(1 - \frac{r}{b}\right) |_{(r<b)},$$

The $r^{(\text{th})}$ incomplete moment of X, say $\phi_r(t)$, can be determined from (10) as

$$\phi_r(t) = \int_{-\infty}^{t} x^r f(x)dx = \sum_{\substack{h,k=0\\h+k\geq 1}}^{\infty} \sum_{i=0}^{k} b_{i,h,k}\, a^r [(k-i)\alpha + h]^{\frac{r}{b}} \times \gamma\left(1-\frac{r}{b}, \left(\frac{a}{t}\right)^b\right)|_{(r<b)}. \quad (11)$$

The moment generating function (MGF) $M(t) = E(e^{tX})$ of X follows from (10) as

$$M(t) = \sum_{h,k,r=0}^{\infty} \sum_{i=0}^{k} b_{i,h,k}\, (t^r/r!) a^r [(k-i)\alpha + h]^{\frac{r}{b}} \Gamma\left(1-\frac{r}{b}\right)|_{(r<b),h+k\geq 1}.$$

Probability weighted moments (PWMs)

The PWMs are generally used for estimating parameters of a distribution whose inverse form cannot be expressed explicitly. The (s,r) th PWM of X denoted by $\rho_{s,r}$ is formally defined by

$$\rho_{s,r} = E\{X^s F(X)^r\} = \int_{-\infty}^{\infty} x^s\, F(x)^r f(x)dx.$$

Using (4), we have

$$F(x)^r = \left[1 - \left(1 + \left\{\frac{exp[-(a/x)^b]}{1-exp[-(a/x)^b]}\right\}^{\alpha}\right)^{-\beta}\right]^r.$$

Expanding z^λ in Taylor series, we can write

$$z^\lambda = \sum_{h=0}^{\infty} \frac{(\lambda)_h}{h!} (z-1)^h = \sum_{i=0}^{\infty} f_i(\lambda)\, z^i, \quad (12)$$

where

$$(\lambda)_h = \lambda(\lambda-1) \dots (\lambda-h+1)$$

is the descending factorial and

$$f_i(\lambda) = \sum_{h=i}^{\infty} \frac{(-1)^{h-i}(\lambda)_h}{h!} \binom{h}{i}.$$

First, applying the Taylor series in z^λ for $F(x)^r$, we obtain

$$F(x)^r = \sum_{i=0}^{\infty} (-1)^i f_i(r) \left(1 + \left\{\frac{exp[-(a/x)^b]}{1-exp[-(a/x)^b]}\right\}^{\alpha}\right)^{-i\beta}.$$

Second, using (5) and the last equation, we have

$$f(x)F(x)^r = \alpha\beta b a^b x^{-(b+1)}\, exp[-(a/x)^b] \frac{\{exp[-(a/x)^b]\}^{\alpha-1}}{\{1-exp[-(a/x)^b]\}^{\alpha+1}} \times \sum_{i=0}^{\infty} (-1)^i\, f_i(r) \underbrace{\left(1 + \left\{\frac{exp[-(a/x)^b]}{1-exp[-(a/x)^b]}\right\}^{\alpha}\right)^{-(i+1)\beta-1}}_{C}.$$

Applying (7) for C in the last equation, we obtain

$$f(x)F(x)^r = \alpha\beta \sum_{i,k=0}^{\infty} (-1)^i\, 2^{-(i+1)\beta-k-1} f_i(r) b a^b x^{-(b+1)}$$

$$\times \frac{exp[-\alpha(a/x)^b]}{\{1-exp[-(a/x)^b]\}^{\alpha+1}}$$
$$\times \underbrace{\left(-1+\left\{\frac{exp[-(a/x)^b]}{1-exp[-(a/x)^b]}\right\}^{\alpha}\right)^k}_{D} \binom{-(i+1)\beta-1}{k}.$$

Third, using the binomial expansion for D , the last equation be rewritten as

$$f(x)F(x)^r = \alpha\beta g(x) \sum_{i,k=0}^{\infty}\sum_{h=0}^{k}(-1)^{i+h}\ 2^{-(i+1)\beta-k-1}\binom{k}{h}\binom{-(i+1)\beta-1}{k}$$
$$\times f_i(r)\{exp[-(a/x)^b]\}^{(k-h+1)\alpha-1}\underbrace{\{1-exp[-(a/x)^b]\}^{-[(k-h+1)\alpha+1]}}_{E}.$$

Applying (8) for E in the last equation gives

$$f(x)F(x)^r = \sum_{k,m=0}^{\infty}\sum_{h=0}^{k} w_{h,k,m}^{(r)}\ \pi_{(k-h+1)\alpha+m}(x;a,b),$$

where

$$w_{h,k,m}^{(r)} = \alpha\beta v_{h,k,m} f_i(r),$$

and $f_i(r)$ is defined in (12) and (for $h \le k$)

$$v_{h,k,m} = \sum_{i=0}^{\infty}\frac{(-1)^{i+h}([k-h+1]\alpha+1)^{(m)}\binom{k}{h}\binom{-(i+1)\beta-1}{k}}{2^{(i+1)\beta+k+1}[(k-h+1)\alpha+m]m!},$$

where $a^{(n)} = \Gamma(a+n)/\Gamma(a)$ denotes the rising factorial. Finally, the (s,r) th PWM of X can be determined as

$$\rho_{s,r} = \sum_{k,m=0}^{\infty}\sum_{h=0}^{k} w_{h,k,m}^{(r)}\, a^s[(k-h+1)\alpha+m]^{\frac{s}{b}}\Gamma\left(1-\frac{s}{b}\right)|_{(s<b)}$$

Residual life and reversed residual life functions

The $n^{(\text{th})}$ moment of the residual life, say

$$m_n(t)|_{(X>t,n=1,2,\dots)} = E[(X-t)^n],$$

uniquely determines $F(x)$. The $n^{(\text{th})}$ moment of the residual life of X is given by

$$m_n(t) = \frac{\int_t^{\infty}(x-t)^n\, dF(x)}{1-F(t)}.$$

Therefore

$$m_n(t) = \frac{1}{1-F(t)}\sum_{h,k=0}^{\infty}\sum_{i=0}^{k} b_{i,h,k}^{*}\, a^n[(k-i)\alpha+h]^{\frac{n}{b}}\Gamma\left(1-\frac{n}{b},\left(\frac{a}{t}\right)^b\right)|_{(n<b),h+k\ge 1}$$

where

$$b_{i,h,k}^{*} = b_{i,h,k}(1-t)^n.$$

Another interesting function is the *mean residual life* (MRL) function or the life expectation at age t defined by

$$m_1(t)|_{(X>t,n=1)} = E[(X-t)],$$

which represents the expected additional life length for a unit which is alive at age t . The MRL of X can be obtained by setting $n=1$ in the last equation. The $n^{(\text{th})}$ moment of the reversed residual life, say

$$M_n(t)|_{(t>0,X\leq t,n=1,2,\dots)}=E[(t-X)^n],$$

uniquely determines $F(x)$. We obtain

$$M_n(t)=\frac{\int_0^t (t-x)^n\,dF(x)}{F(t)}.$$

Then, the $n^{(\text{th})}$ moment of the reversed residual life of X comes from

$$M_n(t)=\frac{1}{F(t)}\sum_{h,k=0}^{\infty}\sum_{i=0}^{k} b^{**}_{i,h,k}\,a^n[(k-i)\alpha+h]^{\frac{n}{b}}\gamma\left(1-\frac{n}{b},\left(\frac{a}{t}\right)^b\right)|_{(n<b),h+k\geq 1}$$

where

$$b^{**}_{i,h,k}=b_{i,h,k}\sum_{r=0}^{n}(-1)^r\binom{n}{r}t^{n-r}$$

The *mean inactivity time* (MIT), also called the mean reversed residual life function, is given by

$$M_1(t)|_{(t>0,X\leq t,n=1)}=E[(t-X)],$$

and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0,t)$. The MIT of the BrXIIFr model is obtained easily by setting $n=1$ in the above equation.

Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Let $X_1,\cdots,X_n$ be a random sample (RS) from the BrXIIFr and let $X_{1\,:\,n},\cdots,X_{n\,:\,n}$ be the corresponding order statistics. The PDF of the $i^{(\text{th})}$ order statistic, say $X_{i\,:\,n}$, is given by

$$f_{i\,:\,n}(x)=\frac{f(x)}{B(i,n-i+1)}\sum_{r=0}^{n-i}(-1)^r\binom{n-i}{r}F^{r+i-1}(x),$$

where $B(\cdot,\cdot)$ is the beta function, then we can write

$$f(x)F(x)^{r+i-1}=\sum_{k,m=0}^{\infty}\sum_{h=0}^{k} w^{(r+i-1)}_{h,k,m}\;\pi_{(k-h+1)\alpha+m}(x;a,b),$$

where $w^{(r+i-1)}_{h,k,m}$ is defined before. So, the PDF of $X_{i:n}$ follows using the last expression as

$$f_{i:n}(x)=\frac{1}{B(i,n-i+1)}\sum_{k,m=0}^{\infty}\sum_{r=0}^{n-i}\sum_{h=0}^{k}(-1)^r\binom{n-i}{r}\times w^{(r+i-1)}_{h,k,m}\;\pi_{(k-h+1)\alpha+m}(x;a,b).$$

Then, the density function of the BrXIIFr order statistics is a fourth linear combination of the Fr density. Then The $q^{(\text{th})}$ ordinary moment of $X_{i\,:\,n}$ say $E\left(X^q_{i\,:\,n}\right)$, is determined from (13) as

$$E\left(X_{i:n}^{q}\right)=\sum_{k,m=0}^{\infty}\sum_{r=0}^{n-i}\sum_{h=0}^{k}\frac{(-1)^{r}\binom{n-i}{r}w_{h,k,m}^{(r+i-1)}}{B(i,n-i+1)}$$
$$\times\ a^{q}[(k-h+1)\alpha+m]^{\frac{q}{b}}\Gamma\left(1-\frac{q}{b}\right)|_{(q<b)}$$

Numerical analysis for the $E\ (X)$, Var (X) , Ske (X) and Ku (X) measures

Numerical analysis for the $E\ (X)$, Var (X) , Ske (X) and Ku (X) is calculated in Table 1 using (10) and well-known relationships for some selected values of parameters using the R software. The same analysis is given for the Fr model in Table 2. Based on Tables 1 and we note that, the skewness of the BrXIIFr distribution can range in the interval (**−0.83**, **690.3**), whereas the skewness of the Fr distribution varies only in the interval (**1.001, 3.53**). Further, the spread for the BrXIIFr kurtosis is ranging from **− 7.092** to **476455.7**, whereas the spread for the Fr kurtosis only varies from **1.002** to **98.8** with the above parameter values.

Table 1: E(X), Var(X), Ske(X) and Ku(X) of the BrXIIFr distribution.

α	β	a	b	E(X)	Var(X)	Ske(X)	Ku(X)
1	10	1	1.5	0.537132	0.0235	1.014196	5.093263
5				0.999164	0.0107	-0.368701	3.211081
10				1.12218	0.0041	-0.661670	3.809871
20				1.194938	0.0013	-0.835348	4.340455
50				1.242866	0.00024	-0.805894	-7.091928
5	1	1.5	1.25	2.07402	0.216065	1.730015	13.64197
	5			1.622243	0.046952	-0.151036	3.153833
	20			1.396361	0.026465	-0.377502	3.147807
	50			1.27776	0.019585	-0.428751	3.185498
	150			1.157003	0.014063	-0.462287	3.227098
10	15	0.1	3	$2.4225\times e^{-7}$	$2.7961\times e^{-8}$	690.2561	476455.7
		0.5		0.524444	0.000220	-0.8080158	4.182266
		2		2.097777	0.003523	-0.8080158	4.182266
		10		10.48888	0.088085	-0.8080158	4.182266
3.5	2.5	2	1	2.353659	0.3902506	0.7464115	4.846663
			2	2.150892	0.08098219	0.2290761	3.589144
			3	2.095278	0.03429045	0.0665986	3.452645
			4	2.069519	0.0188754	-0.0140719	3.426398
			5	2.054697	0.01193425	-0.062441	3.42358

Table 2: E(X), Var(X), Ske(X) and Ku(X) of the Fr distribution.

a	b	E(X)	Var(X)	Ske(X)	Ku(X)
0.5	5	0.5821149	0.0334404	3.535071	48.0915
	10	0.5343144	0.0055656	1.910339	10.9774
	25	0.5123659	0.0007351	1.400443	6.85310
	50	0.5059737	0.0001736	1.264099	6.04447
2.5	5	2.910574	0.8360089	3.535072	48.0915

	10	2.671572	0.1391401	1.910339	10.97857
	50	2.529868	0.0043398	1.264099	6.045233
	75	2.519686	0.0018938	1.221761	5.760403
4.5	5	5.239034	2.70867	3.53507	48.0915
5	4.5	5.950756	4.60640	4.23885	98.8016
10	7.5	10.97054	4.47131	2.29491	15.5896
50	50	$1.392\times e^{-6}$	2561.83	1.00104	1.0028
60	20	61.8872	17.03792	1.473884	7.33349
60	50	60.71684	2.499703	1.2641	6.04521

3. Different methods of estimation

Maximum likelihood estimation (MLE)

Let $x_1, \ldots, x_n$ be a RS from the BrXIIFr model with parameters α, β, a and b . Let $\Theta = (\alpha, \beta, a, b)$? be the 4×1 parameter vector. For determining the MLE of Θ , we have the log-likelihood function

$$\ell = \ell(\Theta) = n\,log\,\alpha + n\,log\,\beta + n\,log\,b + nb\,log\,a - (b+1)\sum_{i=1}^{n} x_i - \alpha \sum_{i=1}^{n} (a/x_i)^b$$
$$-(\alpha+1)\sum_{i=1}^{n} log(1-s_i) - (\beta+1)\sum_{i=1}^{n} log\,z_i,$$

where

$$s_i = exp[-(a/x_i)^b] \text{ and } z_i = \left[1 + \left(\frac{s_i}{1-s_i}\right)^{\alpha}\right],$$

the score vector is given as

$$I_{(\alpha)} = \frac{n}{\alpha} - \sum_{i=1}^{n} (a/x_i)^b - \sum_{i=1}^{n} log(1-s_i) - (\beta+1)\sum_{i=1}^{n} \frac{p_i}{z_i},$$

$$I_{(\beta)} = \frac{n}{\beta} - \sum_{i=1}^{n} log\,z_i,$$

$$I_{(a)} = \frac{nb}{a} - \alpha b \sum_{i=1}^{n} \frac{1}{x_i} (a/x_i)^{b-1} - (\alpha+1) b \sum_{i=1}^{n} \frac{(a/x_i)^{b-1} \frac{1}{x_i} s_i}{1-s_i} - (\beta+1)\sum_{i=1}^{n} \frac{m_i}{z_i}$$

$$I_{(b)} = \frac{n}{b} + n\,log\,a - \sum_{i=1}^{n} x_i - \alpha \sum_{i=1}^{n} w_i - (\alpha+1)\sum_{i=1}^{n} \frac{w_i s_i}{1-s_i} - (\beta+1)\sum_{i=1}^{n} \frac{t_i}{z_i}$$

where

$$p_i = \left(\frac{s_i}{1-s_i}\right)^{\alpha} log\left(\frac{s_i}{1-s_i}\right),$$

$$m_i = -\alpha b\left(\frac{s_i}{1-s_i}\right)^{\alpha-1}\frac{\frac{s_i}{x_i}(a/x_i)^{b-1}}{[1-s_i]^2}$$

$$t_i = -\alpha\left(\frac{s_i}{1-s_i}\right)^{\alpha-1}\frac{w_i s}{[1-s_i]^2}$$

and

$$w_i = (a/x_i)^b\, log(a/x_i)$$

Setting the nonlinear system of equations $I_{(\alpha)} = 0, I_{(\beta)} = 0, I_{(a)} = 0$ and $I_{(b)} = 0$ and solving them simultaneously yields the MLE. To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize ℓ (for more details see Casella and Berger (2002)).

Method of least square and weighted least square estimation

The theory of least square estimation and weighted least square estimation was proposed by Swain et al. (1988) to estimate the parameters of Beta distributions. It is based on the minimization of the sum of the square of differences of theoretical cumulative distribution function and empirical distribution function. Suppose $F(X_{i\,:\,n}, \alpha, \beta, a, b)$ denotes the distribution function of BrXIIFr distribution and if $x_1 < x_2 < \cdots, < x_n$ be the n ordered random sample. The least square estimates are obtained by minimizing

$$Ls(\alpha,\beta,a,b) = \sum_{i=1}^{n}\left[F(x_i,\alpha,\beta,a,b) - \frac{i}{n+1}\right]^2$$

Now using (1) we get,

$$Ls(\alpha,\beta,a,b) = \sum_{i=1}^{n}\left\{\left[1-\left(1+\left\{\frac{exp[-(a/x_i)^b]}{1-exp[-(a/x_i)^b]}\right\}^{\alpha}\right)^{-\beta}\right] - \frac{i}{n+1}\right\}^2.$$

The, least square estimators (LSE) of the parameters are obtained by solving the following non-linear equations;

$$\sum_{i=1}^{n}\left\{\left[1-\left(1+\left\{\frac{exp[-(a/x_i)^b]}{1-exp[-(a/x_i)^b]}\right\}^{\alpha}\right)^{-\beta}\right] - \frac{i}{n+1}\right\}\eta_{\alpha}(x_i,\alpha,\beta,a,b) = 0,$$

$$\sum_{i=1}^{n}\left\{\left[1-\left(1+\left\{\frac{exp[-(a/x_i)^b]}{1-exp[-(a/x_i)^b]}\right\}^{\alpha}\right)^{-\beta}\right] - \frac{i}{n+1}\right\}\eta_{\beta}(x_i,\alpha,\beta,a,b) = 0,$$

$$\sum_{i=1}^{n}\left\{\left[1-\left(1+\left\{\frac{exp[-(a/x_i)^b]}{1-exp[-(a/x_i)^b]}\right\}^{\alpha}\right)^{-\beta}\right] - \frac{i}{n+1}\right\}\eta_{a}(x_i,\alpha,\beta,a,b) = 0,$$

and

$$\sum_{i=1}^{n}\left\{\left[1-\left(1+\left\{\frac{exp[-(a/x_i)^b]}{1-exp[-(a/x_i)^b]}\right\}^{\alpha}\right)^{-\beta}\right] - \frac{i}{n+1}\right\}\eta_{b}(x_i,\alpha,\beta,a,b) = 0,$$

where $\eta_\alpha(x_i,\alpha,\beta,a,b), \eta_\beta(x_i,\alpha,\beta,a,b)$, $\eta_a(x_i,\alpha,\beta,a,b)$ and $\eta_b(x_i,\alpha,\beta,a,b)$ are the values of first derivatives w. r. t. parameter of the CDF of BrXIIFr distribution. The least square estimates of the parameters α,β,a and b are obtained by solving the above simultenious equations by using any numerical approximation techniques. The weighted least squares estimate (WLSE) are obtained by minimizing the given form of equation with respect to the parameters.

$$WLs(\alpha,\beta,a,b)=\sum_{i=1}^{n}\left[F(x_i;\alpha,\beta,a,b)-\frac{i}{n+1}\right]^2 w_i.$$

The WLSE of the parameters are obtained by solving the following non-linear equations

$$\sum_{i=1}^{n} w_i\left\{\left[1-\left(1+\left\{\frac{exp[-(a/x_i)^b]}{1-exp[-(a/x_i)^b]}\right\}^{\alpha}\right)^{-\beta}\right]-\frac{i}{n+1}\right\}\eta_\alpha(x_i,\alpha,\beta,a,b)=0,$$

$$\sum_{i=1}^{n} w_i\left\{\left[1-\left(1+\left\{\frac{exp[-(a/x_i)^b]}{1-exp[-(a/x_i)^b]}\right\}^{\alpha}\right)^{-\beta}\right]-\frac{i}{n+1}\right\}\eta_\beta(x_i,\alpha,\beta,a,b)=0,$$

$$\sum_{i=1}^{n} w_i\left\{\left[1-\left(1+\left\{\frac{exp[-(a/x_i)^b]}{1-exp[-(a/x_i)^b]}\right\}^{\alpha}\right)^{-\beta}\right]-\frac{i}{n+1}\right\}\eta_a(x_i,\alpha,\beta,a,b)=0,$$

and

$$\sum_{i=1}^{n} w_i\left\{\left[1-\left(1+\left\{\frac{exp[-(a/x_i)^b]}{1-exp[-(a/x_i)^b]}\right\}^{\alpha}\right)^{-\beta}\right]-\frac{i}{n+1}\right\}\eta_b(x_i,\alpha,\beta,a,b)=0,$$

where $\eta_\alpha(x_i,\alpha,\beta,a,b), \eta_\beta(x_i,\alpha,\beta,a,b)$, $\eta_a(x_i,\alpha,\beta,a,b)$ and $\eta_b(x_i,\alpha,\beta,a,b)$ are the values of first derivatives of the CDF of BrXIIFr distribution and

$$w_i=\frac{(n+1)^2(n+2)}{i(n-i+1)}.$$

Jackknife method

The Jackknife method is similar to the bootstrap method in that it involves resampling technique, but instead of sampling with replacement, the Jackknife method samples without replacement. The delete-1 Jackknife sample**s** are selected by taking the original vector of the data and deleting one observation from the set. Thus, there are n unique Jackknife samples, and the i^{th} Jackknife sample vector can be defined as

$$X_{[i]}=\{X_1,X_2,\ldots X_{i-1},X_{i+1},\ldots,X_{n-1},X_n\}$$

This procedure is generalizable to k deletions. For more details about the Jackknife method see Quenouille (1949) and Efron (1979).

Bayesian estimation

In this subsection, the Bayesian estimation procedure has been discussed to estimate the unknown parameters of the BrXIIFr distribution. The Bayesian estimators are computed under informative gamma priors for all α,β,a and b using squared error loss function. The joint prior is given by

$$p(\alpha,\beta,a,b)\propto \alpha^{a_1-1}\beta^{a_2-1}a^{a_3-1}b^{a_4-1}\,exp(-b_1\alpha-b_2\beta-b_3a-b_4b)\quad |_{(\alpha,\beta,a,b>0)},$$

$a_1, b_1, a_2, b_2, a_3, b_3, a_a$ and b_4 are the hyperparameters assume to be known and positive. In Bayesian estimation theory, as we know that posterior distribution is needed to derive the Bayesian estimators. The posterior distribution $\pi(\alpha, \beta, a, b|\underline{x})$ using the likelihood function based on the density of BrXIIFr distribution and above joint priors has been obtained as

$$\pi(\alpha,\beta,a,b|\underline{x}) \propto \alpha^{n+a_1-1}\beta^{n+a_2-1}a^{nb+a_3-1}b^{n+a_4-1}$$
$$\times\, exp(-b_1\alpha - b_2\beta - b_3 a - b_4 b)$$
$$\times \prod_{i=1}^{n} x_i^{-(b+1)} \frac{\prod_{i=1}^{n} exp[-\alpha(a/x_i)^b]}{\prod_{i=1}^{n}\{1-exp[-(a/x_i)^b]\}^{\alpha+1}}$$
$$\times \prod_{i=1}^{n}\left(1+\left\{\frac{exp[-(a/x_i)^b]}{1-exp[-(a/x_i)^b]}\right\}^{\alpha}\right)^{-\beta-1}.$$

In Bayesian estimation theory, another most essential element is loss function. Here, we have taken the Squared Error Loss Function (SELF) which is defined as

$$L(\hat{\delta},\delta) = E(\hat{\delta}-\delta),$$

where, $\hat{\delta}$ is the estimated value of δ. Under this loss function, posterior mean is the Bayesian estimate of the respective parameter. Thus, the Bayesian estimators under SELF are easily to be obtained.

From the above equations are not solvable analytically, thus here we used Markov Chain Monte Carlo (MCMC) technique to generate the posterior sample from the full conditional posterior distribution. The full conditional posterior distributions are given by

$$\pi_1(\alpha|\beta,a,b,\underline{x}) \propto \times \left[\begin{array}{c} \alpha^{n+a_1} \\ \times\, exp(-b_1\alpha - b_2\beta - b_3 a - b_4 b) \\ \times \prod_{i=1}^{n} x_i^{-(b+1)} \dfrac{\prod_{i=1}^{n} exp\left[-\alpha\left(\frac{a}{x_i}\right)^b\right]}{\prod_{i=1}^{n}\left\{1-exp\left[-\left(\frac{a}{x_i}\right)^b\right]\right\}^{\alpha+1}} \\ \times \prod_{i=1}^{n}\left(1+\left\{\dfrac{exp[-(a/x_i)^b]}{1-exp[-(a/x_i)^b]}\right\}^{\alpha}\right)^{-\beta-1} \end{array}\right],$$

$$\pi_2(\beta|\alpha,a,b,\underline{x}) \propto \left[\begin{array}{c} \beta^{n+a_2-1} \\ \times\, exp(-b_1\alpha - b_2\beta - b_3 a - b_4 b) \\ \times \prod_{i=1}^{n} x_i^{-(b+1)} \dfrac{\prod_{i=1}^{n} exp[-\alpha(a/x_i)^b]}{\prod_{i=1}^{n}\{1-exp[-(a/x_i)^b]\}^{\alpha+1}} \\ \times \prod_{i=1}^{n}\left(1+\left\{\dfrac{exp[-(a/x_i)^b]}{1-exp[-(a/x_i)^b]}\right\}^{\alpha}\right)^{-\beta-1} \end{array}\right],$$

$$\pi_3(a|\alpha,\beta,b,\underline{x}) \propto \begin{bmatrix} a^{nb+a_3-1} \\ \times exp(-b_1\alpha - b_2\beta - b_3 a - b_4 b) \\ \times \prod_{i=1}^{n} x_i^{-(b+1)} \frac{\prod_{i=1}^{n} exp[-\alpha(a/x_i)^b]}{\prod_{i=1}^{n}\{1 - exp[-(a/x_i)^b]\}^{\alpha+1}} \\ \times \prod_{i=1}^{n} \left(1 + \left\{\frac{exp[-(a/x_i)^b]}{1 - exp[-(a/x_i)^b]}\right\}^{\alpha}\right)^{-\beta-1} \end{bmatrix}$$

and

$$\pi_4(b|\alpha,\beta,a,\underline{x}) \propto \begin{bmatrix} b^{n+a_4} \\ \times exp(-b_1\alpha - b_2\beta - b_3 a - b_4 b) \\ \times \prod_{i=1}^{n} x_i^{-(b+1)} \frac{\prod_{i=1}^{n} exp\left[-\alpha\left(\frac{a}{x_i}\right)^b\right]}{\prod_{i=1}^{n}\left\{1 - exp\left[-\left(\frac{a}{x_i}\right)^b\right]\right\}^{\alpha+1}} \\ \times \prod_{i=1}^{n} \left(1 + \left\{\frac{exp\left[-\left(\frac{a}{x_i}\right)^b\right]}{1 - exp\left[-\left(\frac{a}{x_i}\right)^b\right]}\right\}^{\alpha}\right)^{-\beta-1} \end{bmatrix}.$$

The following steps are used to extract the posterior samples from full-conditional posterior density

- Starts with $j = 1$ and set initial values of α, β, a, b say, $\alpha_0, \beta_0, a_0, b_0$;
- Generate posterior samples from full conditional distribution using Normal distribution as a proposal density;
- Repeat the above step for $j = 1,2, \cdots M$ and simulate
$$(\alpha_1, \beta_1, a_1, b_1), (\alpha_2, \beta_2, a_2, b_2), \cdots, (\alpha_M, \beta_M, a_M, b_M);$$
- Under SELF, the Bayesian estimates of α, β, a, b are given by

$$\hat{\alpha}_S = \frac{1}{M - M_0} \sum_{j=1}^{M-M_0} \alpha_j$$

$$\hat{\beta}_S = \frac{1}{M - M_0} \sum_{j=1}^{M-M_0} \beta_j$$

$$\hat{a}_S = \frac{1}{M - M_0} \sum_{j=1}^{M-M_0} a_j$$

$$\hat{b}_S = \frac{1}{M - M_0} \sum_{j=1}^{M-M_0} b_j$$

where $M = 10000$ and $M_0 = 5000$ is the burn in period.

4. Simulation study

In this section, a Monte Carlo simulation study is conducted to compare the performance of the different estimators of the unknown parameters for the BrXIIFr distribution. The performance of the different estimators proposed in the previous section is evaluated concerning their mean squared errors (MSEs). All the computations in this section are done by **Mathcad program Version 15.0**. We generate 1000 samples of the BrXIIFr distribution, where $n = (20,50,100,200,500)$ and by choosing $(\alpha, \beta, a, b) = (1.5,0.6,0.4,2)$, $(0.8,1.3,0.6,1.05)$ and $(1.2,0.9,1.5,0.9)$.

The average values of estimates and MSEs of MLEs, LSEs, WLSEs and Jackknife are obtained and reported in in Tables (3-7). The values of hyperparameters are assumed to known and chosen in such a way that prior mean is equal to the true value, and prior variance is unity.

Table 3: Average values of estimates and the corresponding MSEs (in parentheses) for $n = 20$.

Parameters	MLE	LS	WLS	Jac	Bayes
α=1.5	1.2540 (.14997)	1.4682 (3.0083)	1.8064 (2.7632)	1.4353 (0.0266)	1.3033 (0.0477)
β=0.6	0.6409 (0.0401)	0.6396 (0.0390)	0.6399 (0.0389)	0.6702 (0.0070)	0.6994 (0.0175)
a=0.4	0.4029 (0.0018)	0.4043 (0.0028)	0.4037 (0.0023)	0.3882 (0.0002)	0.3975 (0.0008)
b=2	1.6663 (0.2737)	2.3281 (2.2189)	2.4479 (2.7978)	2.1652 (0.0383)	1.7041 (0.0972)
α=0.8	0.8051 (0.0253)	0.8509 (0.7634)	1.0002 (2.0459)	0.6712 (0.0181)	0.7050 (0.0097)
β=1.3	1.3405 (0.1226)	1.3971 (0.2413)	1.3799 (0.1917)	0.8622 (0.1947)	1.0699 (0.0532)
a=0.6	0.6378 (0.0262)	0.6329 (0.0364)	0.6328 (0.0539)	0.7060 (0.0129)	0.7207 (0.0160)
b=1.05	1.0589 (0.0327)	0.8364 (0.1477)	1.1061 (0.4082)	0.8629 (0.0379)	1.0772 (0.0010)
α=1.2	1.2370 (0.0615)	1.5860 (46.5326)	1.6634 (18.7747)	1.2973 (0.0159)	1.2016 (0.0142)
β=0.9	0.9707 (0.0651)	0.9406 (0.0862)	0.9423 (0.0917)	0.9717 (0.0100)	0.8955 (0.0075)
a=1.5	1.4977 (0.1557)	1.6121 (0.2837)	1.6221 (0.3072)	1.2699 (0.0699)	1.5005 (0.0224)
b=0.9	0.9217 (0.0277)	1.0704 (0.6244)	1.0812 (0.6730)	0.7762 (0.0174)	0.8986 (0.0080)

Table 4: Average values of estimates and the corresponding MSEs (in parentheses) for $n = 50$.

Parameters	MLE	LS	WLS	Jac	Bayes
α=1.5	1.3617 (0.0695)	1.8018 (1.9513)	1.9295 (2.6831)	1.5383 (0.0036)	1.3603 (0.0277)
β=0.6	0.6069 (0.0135)	0.6135 (0.0119)	0.6094 (0.0113)	0.6196 (0.0006)	0.6403 (0.0073)
a=0.4	0.4035 (0.0007)	0.4018 (0.0009)	0.4027 (0.0008)	0.3972 (0.00002)	0.3861 (0.0007)
b=2	1.7938 (0.1259)	2.1352 (0.4952)	2.1353 (0.5087)	1.7706 (0.0536)	1.8082 (0.0458)
α=0.8	0.8071 (0.0094)	0.9281 (0.6103)	0.9624 (0.7885)	0.8569 (0.0035)	0.7771 (0.0024)
β=1.3	1.3152 (0.0382)	1.3322 (0.0656)	1.3552 (0.0667)	1.4398 (0.0202)	1.4608 (0.0275)
a=0.6	0.6206 (0.0092)	0.6119 (0.0105)	0.6027 (0.0086)	0.6256 (0.0008)	0.6349 (0.0015)
b=1.05	1.0597 (0.0114)	0.9566 (0.0657)	1.0824 (0.1859)	1.1104 (0.0039)	1.0458 (0.0004)
α=1.2	1.2083 (0.0233)	1.2926 (1.4678)	1.5831 (10.7480)	1.1564 (0.0022)	1.2019 (0.0132)
β=0.9	0.9207 (0.0229)	0.9194 (0.0254)	0.9143 (0.0234)	0.9321 (0.0016)	0.8985 (0.0074)
a=1.5	1.4969 (0.0523)	1.5263 (0.0797)	1.5352 (0.0827)	1.4614 (0.0029)	1.5046 (0.0222)
b=0.9	0.9039 (0.0107)	0.9892 (0.2223)	0.9695 (0.1839)	0.8351 (0.0045)	0.9056 (0.0066)

Table 5: Average values of estimates and the corresponding MSEs (in parentheses) for $n = 100$.

Parameters	MLE	LS	WLS	Jac	Bayes
α=1.5	1.4217 (0.0353)	1.8140 (1.6125)	1.9101 (1.8945)	1.5176 (0.0005)	1.4199 (0.0138)
β=0.6	0.6015 (0.0064)	0.6052 (0.0060)	0.6086 (0.0059)	0.5821 (0.0004)	0.6481 (0.0069)
a=0.4	0.4021 (0.0003)	0.4015 (0.0005)	0.4004 (0.0004)	0.4042 (0.00002)	0.3987 (0.0004)
b=2	1.8672 (0.0675)	2.0610 (0.1732)	2.0904 (0.2406)	1.9384 (0.0051)	1.8591 (0.0279)
α=0.8	0.8051 (0.0047)	0.8629 (0.4045)	0.8774 (0.5624)	0.8637 (0.0041)	0.7850 (0.0009)
β=1.3	1.3141 (0.0202)	1.3272 (0.0279)	1.3209 (0.0265)	1.3183 (0.0005)	1.2548 (0.0025)
a=0.6	0.6104 (0.0043)	0.5993 (0.0041)	0.6032 (0.0039)	0.5737 (0.0007)	0.5753 (0.0011)
b=1.05	1.0570 (0.0056)	0.9831 (0.0226)	1.0749 (0.0715)	1.0191 (0.0010)	1.0580 (0.0001)
α=1.2	1.2057 (0.0117)	1.2635 (0.7789)	1.3581 (2.1927)	1.2313 (0.0012)	1.2137 (0.0010)
β=0.9	0.9111 (0.0110)	0.9096 (0.0108)	0.9088 (0.0098)	0.8784 (0.0005)	0.8499 (0.0029)
a=1.5	1.5035 (0.0253)	1.5109 (0.0370)	1.5113 (0.0365)	1.5209 (0.0007)	1.5544 (0.0033)
b=0.9	0.9030 (0.0053)	0.9309 (0.0448)	0.9245 (0.0283)	0.9598 (0.0036)	0.8362 (0.0052)

Table 6: Average values of estimates and the corresponding MSEs (in parentheses) for $n = 200$.

Parameters	MLE	LS	WLS	Jac	Bayes
α=1.5	1.4425 (0.0178)	1.6136 (0.3239)	1.7455 (1.8969)	1.5002 (0.0002)	1.4886 (0.0069)
β=0.6	0.6048 (0.0035)	0.6007 (0.0028)	0.6015 (0.0028)	0.6130 (0.0002)	0.6353 (0.0049)
a=0.4	0.4001 (0.0002)	0.4012 (0.0002)	0.4010 (0.0002)	0.3970 (0.00001)	0.3947 (0.0003)
b=2	1.9168 (0.0343)	2.0189 (0.0681)	2.0272 (0.0895)	2.0545 (0.0032)	1.9577 (0.0094)
α=0.8	0.8019 (0.0023)	0.8305 (0.1741)	0.8339 (0.1789)	0.7517 (0.0023)	0.7950 (0.00004)
β=1.3	1.2993 (0.0088)	1.3117 (0.0123)	1.3142 (0.0130)	1.3037 (0.0001)	1.2987 (0.000003)
a=0.6	0.6054 (0.0018)	0.6004 (0.0020)	0.5999 (0.0019)	0.5757 (0.0006)	0.5899 (0.0001)
b=1.05	1.0526 (0.0026)	1.0215 (0.0097)	1.0490 (0.0071)	1.0301 (0.0004)	1.0632 (0.0001)
α=1.2	1.2069 (0.0055)	1.2058 (0.0746)	1.2324 (0.2346)	1.2075 (0.0001)	1.1516 (0.0036)
β=0.9	0.9081 (0.0052)	0.9024 (0.0054)	0.8997 (0.0047)	0.8884 (0.0002)	0.8920 (0.0013)
a=1.5	1.5091 (0.0125)	1.5092 (0.0180)	1.5154 (0.0187)	1.5006 (0.0001)	1.4430 (0.0051)
b=0.9	0.9041 (0.0025)	0.9078 (0.0127)	0.9081 (0.0158)	0.8593 (0.0017)	0.7986 (0.0015)

Table 7: Average values of estimates and the corresponding MSEs (in parentheses) for $n = 500$.

Parameters	MLE	LS	WLS	Jac	Bayes
α=1.5	1.4781 (0.0074)	1.5498 (0.0895)	1.5566 (0.1016)	1.5122 (0.00001)	1.4822 (0.0004)
β=0.6	0.6012 (0.0013)	0.6012 (0.0011)	0.5996 (0.0011)	0.6045 (0.00002)	0.6039 (0.0001)
a=0.4	0.4002 (0.0001)	0.4002 (0.0001)	0.4007 (0.0001)	0.3984 (0.000003)	0.3978 (0.00004)
b=2	1.9599 (0.0143)	2.0117 (0.0260)	2.0045 (0.0301)	1.9756 (0.0006)	1.9616 (0.0016)
α=0.8	0.8011 (0.0009)	0.8081 (0.0055)	0.8083 (0.0029)	0.7973 (0.000009)	0.7946 (0.00004)
β=1.3	1.3025 (0.0035)	1.3012 (0.0054)	1.2999 (0.0049)	1.3131 (0.0001)	1.3074 (0.00006)
a=0.6	0.6020 (0.0008)	0.6018 (0.0009)	0.6026 (0.0008)	0.5816 (0.0003)	0.6199 (0.00042)
b=1.05	1.0515 (0.0011)	1.0399 (0.0044)	1.0569 (0.0029)	1.0366 (0.0002)	1.0485 (0.00001)
α=1.2	1.2024 (0.0024)	1.1991 (0.0114)	1.2051 (0.0050)	1.1947 (0.00003)	1.1957 (0.00003)
β=0.9	0.9038 (0.0022)	0.8996 (0.0022)	0.9007 (0.0018)	0.9044 (0.00002)	0.9048 (0.00003)
a=1.5	1.5008 (0.0050)	1.5069 (0.0074)	1.5041 (0.0067)	1.4913 (0.00009)	1.5116 (0.0001)
b=0.9	0.9015 (0.0011)	0.9016 (0.0049)	0.9027 (0.0025)	0.8952 (0.00003)	0.8970 (0.00001)

From Tables (2-7), we observe that all the estimates show the property of consistency, i.e., the MSEs decrease as sample size increase.

5. Data analysis

Using the ML method

This section presents two applications of the BrXIIFr distribution using real data sets. We shall compare the fit of the new distribution with the WFr (Afify et al. (2017)), exponentiated Fréchet (EFr) (Nadarajah and Kotz (2003)), Kumaraswamy Fréchet (KumFr) (Mead and Abd-Eltawab (2014), beta Fréchet (BFr) (Barreto-Souza **(2004)**), transmuted Fréchet (TFr) (Mahmoud and Mandouh, (2013)), gamma extended Fréchet (GEFr) (Silva et. al. (2013)), Marshall-Olkin Fréchet (MOFr) (Krishna et al. (2013)) and Fréchet (Fr) distributions with corresponding densities (for $x > 0$):

WFr :

$$f(x) = ab\beta\alpha^{\beta}x^{-(\beta+1)}\, exp\left[-b\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1 - exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{-(b+1)}$$

$$\times\, exp\left\{-a\left[\frac{exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}{1 - exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]}\right]^{b}\right\}$$

KFr:

$$f(x) = ab\beta\alpha^{\beta}x^{-(\beta+1)}\, exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1 - exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{b-1};$$

EFr:

$$f(x) = a\beta\alpha^{\beta}x^{-(\beta+1)}\, exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1 - exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{a-1};$$

BFr:

$$f(x) = \frac{\beta\alpha^{\beta}}{B(a,b)}x^{-(\beta+1)}\, exp\left[-a\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1 - exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{b-1};$$

GEFr:

$$f(x) = \frac{a\beta\alpha^{\beta}}{\Gamma(b)}x^{-(\beta+1)}\, exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1 - exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{a-1}$$

$$\times\left\{-\,log\left\{1 - exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{a}\right\}^{b-1};$$

TFr:

$$f(x) = \beta\alpha^{\beta}x^{-(\beta+1)}\, exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{(a+1) - 2a\, exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\};$$

MOFr:

$$f(x) = a\beta\alpha^{\beta}x^{-(\beta+1)}\, exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]$$

The unknown parameters of the above PDFs are all positive real numbers except for the TFr distribution for which $|a| \leq 1$. The 1 st data set consists of 100 observations of breaking stress of carbon fbres (in Gba) given by Nichols and Padgett (2006). Figure 2: gives the total time test (TTT) plots for the 1 st data set, from Figure 2 we onclude that the empirical HRFs of the data can be increasing.

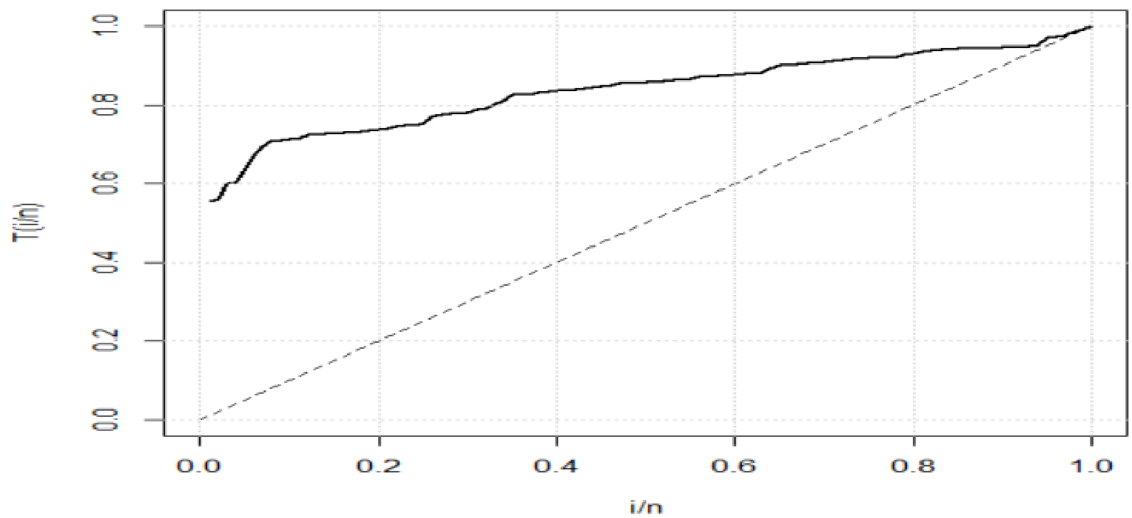

Figure 2: TTT plots for data set I.

The 2 nd data set (Smith and Naylor 1987) consists of 63 observations of the strengths of 1.5 cm glass fibers, originally obtained by workers at the UK National Physical Laboratory. Unfortunately, the units of measurement are not given in the paper. Figure 3: gives the TTT plots for the 2 nd data set, from Figure 3 we conclude that the empirical HRFs of the data can be increasing.

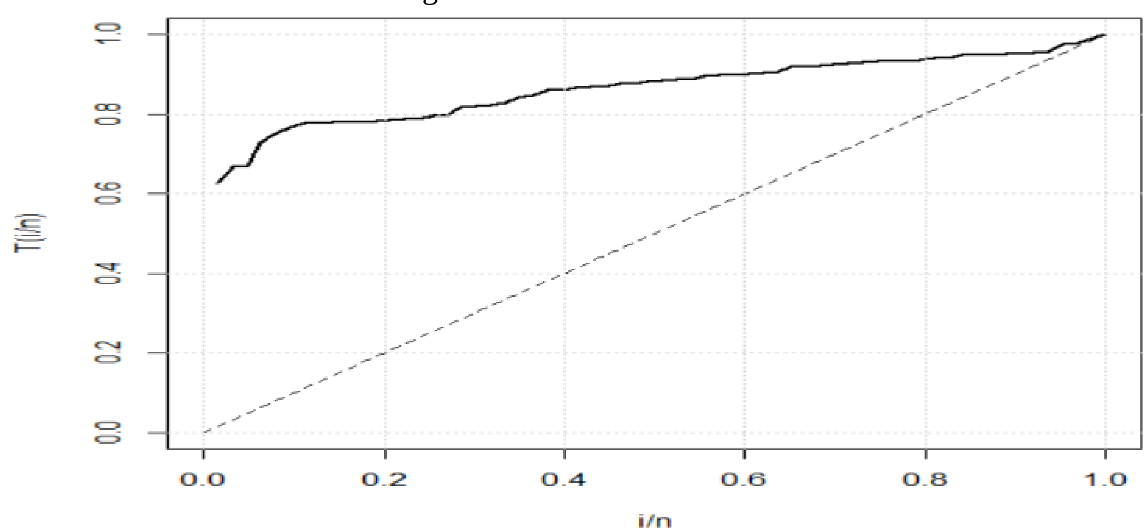

Figure 2: TTT plots for data set II.

In order to compare the distributions, we consider the following criteria: the ℓ (Maximized Log-Likelihood), AIC (Akaike Information Criterion), CAIC (Consistent Akaike Information Criterion), BIC (Bayesian Information Criterion) and HQIC (Hannan-

Quinn Information Criterion). The model with minimum values for these statistics could be chosen as the best model to fit the data. All results are obtained using the R PROGRAM.

Table 8: The statistics for carbon fiber data.

Model	Goodness of fit criteria				
	$-2\hat{\ell}$	AIC	BIC	HQIC	CAIC
BrXIIFr	**103.7**	**111.9**	**122.3**	**116.1**	**112.3**
WFr	286.5	294.5	304.9	298.7	294.9
EFr	289.7	295.7	303.5	298.9	296.0
KumFr	289.1	297.1	307.5	301.3	297.5
BFr	303.1	311.1	321.6	315.4	311.6
GEFr	304	312	332.4	316.2	312.4
Fr	344.3	348.3	353.5	350.4	348.4
TFr	344.5	350.5	358.3	353.6	350.7
MOFr	345.3	351.3	359.1	354.5	351.6

Table 9: MLEs and their standard errors (in parentheses) for breaking stress of carbon fiber data.

Model	Estimates			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
BrXIIFr	2.404	0.747	1.245	2.382
	(2.328)	(0.374)	(0.231)	(1.808)
WFr	2.2231	0.355	6.9721	4.9179
	(11.409)	(0.411)	(113.811)	(3.756)
EF	69.1489	0.5019	145.3275	
	(57.349)	(0.08)	(122.924)	
KumFr	2.0556	0.4654	6.2815	224.18
	(0.071)	(0.00701)	(0.063)	(0.164)
BFr	1.6097	0.4046	22.0143	29.7617
	(2.498)	(0.108)	(21.432)	(17.479)
GEFr	1.3692	0.4776	27.6452	17.4581
	(2.017)	(0.133)	(14.136)	(14.818)
F	1.8705	1.7766		

	(0.112)	(0.113)		
TFr	1.9315	1.7435	0.0819	
	(0.097)	(0.076)	(0.198)	
MOFr	2.3066	1.5796	0.5988	
	(0.498)	(0.16)	(0.3091)	

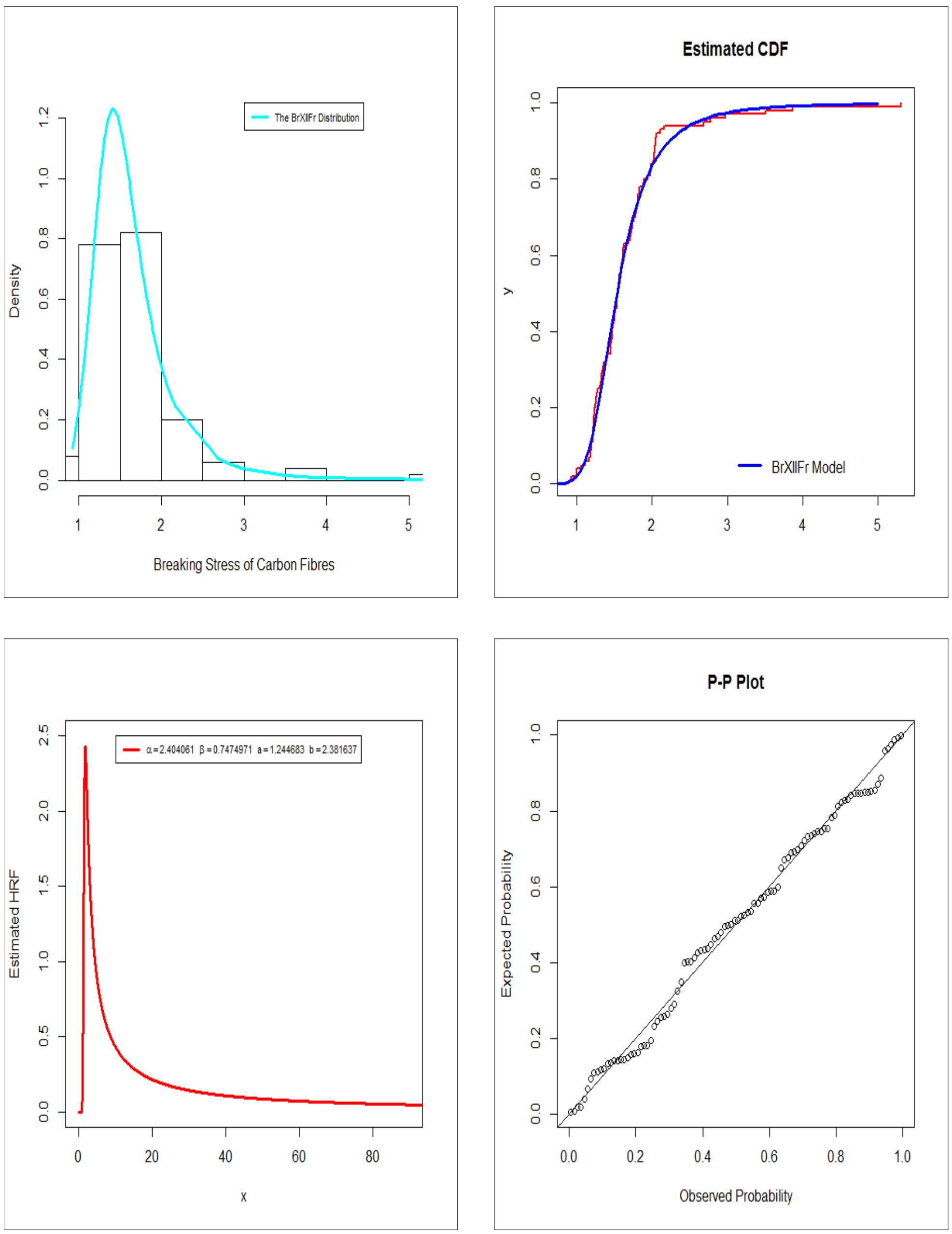

Figure 4: Estimated PDF, CDF, HRF and P-P plot for data set I.

Table 10: The statistics values for glass fiber.

Model	Goodness of fit criteria				
	$-2\hat{\ell}$	AIC	BIC	HQIC	CAIC
BrXIIFr	**38.4**	**46.5**	**55.02**	**49.8**	**47.1**
KumFr	39.6	47.6	56.2	51	48.3
EFr	44.3	50.5	56.7	52.8	50.7
BFr	60.6	68.6	77.2	72.0	69.3
GEFr	61.6	69.6	78.1	72.9	70.3
Fr	93.7	97.7	102	99.4	97.9
TFr	94.1	100.1	106.5	102.6	100.5
MOFr	95.7	101.7	108.2	104.2	102.1

Table 11: MLEs and their standard errors (in parentheses) for strengths of 1.5 cm glass fiber data.

Model	Estimates			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
BrXIIFr	3.258	0.5831	1.210	2.472
	(7.754)	(0.465)	(0.468)	(5.018)
KumFr	2.116	0.740	5.504	857.343
	(4.555)	(0.071)	(7.982)	(153.948)
EFr	7.816	0.999	132.827	
	(2.945)	(0.136)	(116.63)	
BFr	2.0518	0.6466	15.0756	36.9397
	(0.986)	(0.163)	(12.057)	(22.649)
GEFr	1.6625	0.7421	32.112	13.2688
	(0.952)	(0.197)	(17.397)	(9.967)
Fr	1.264	2.888		
	(0.059)	(0.234)		
TFr	1.3068	2.7898	0.1298	
	(0.034)	(0.165)	(0.208)	
MOFr	1.5441	2.3876	0.4816	
	(0.226)	(0.253)	(0.252)	

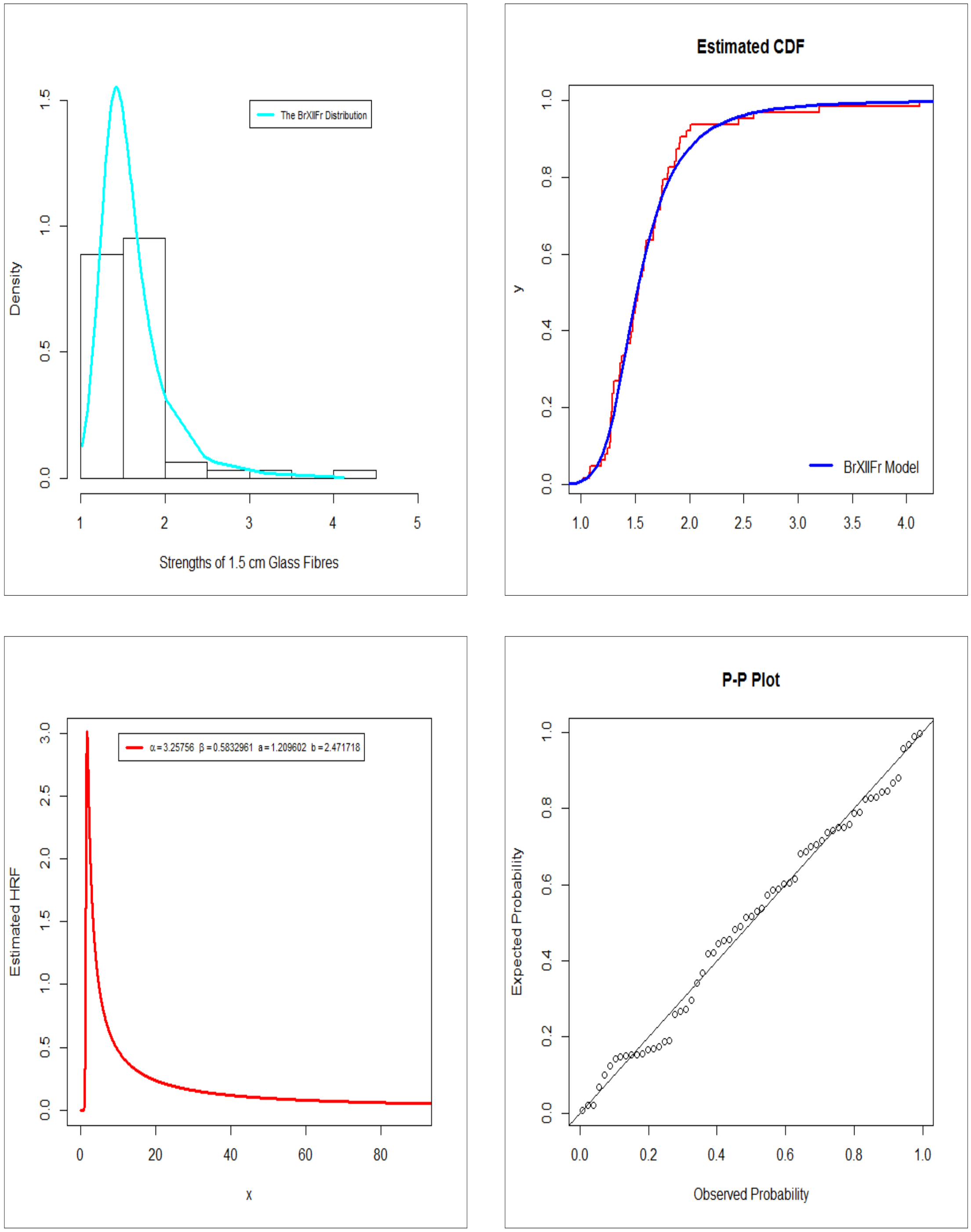

Figure 5: Estimated PDF, CDF, HRF and P-P plot for data set II.

Table 9 and 11 give the MLEs and their standard errors (in parentheses) for the two data sets. Tables 8 and 10 compare the BrXIIFr model with the WFr, the KFr, the EFr, the BFr, the GEFr, the TMOFr, the TFr, the MOFr and the Fr distributions. The BrXIIFr model gives the lowest values for the AIC, BIC, HQIC and CAIC statistics (in bold values) among all fitted models to these data and the biggest value of the $\ell 2$. So, it could be chosen as the best model. Figure 4 displays the plots of estimated PDF and estimated CDF, estimated HRF and P-P plot of the new model for the 1 st data. Figure 5 displays the plots of estimated PDF and estimated CDF, estimated HRF and P-P plot of the new model for the 2 nd data. These plots reveal that the proposed distribution yields a better ft than other nested and non-nested models for both data sets.

Using different method of estimation

In this Section, we shall use different methods of estimation considered in Section 4 to estimate the unknown parameters. The estimates using the four methods of estimation and the values of K--S (and the corresponding p-value) are displayed in Tables 12 and 13 for data sets I and II respectively.

Table 12: The values of estimators, KS and p-values for breaking stress data.

Method	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$	KS	p-value
ML	2.404	0.747	1.245	2.382	0.064	0.801
LS	5.747	0.944	1.008	0.884	0.057	0.898
WLS	2.834	0.982	1.252	1.795	0.055	0.923
Jac	2.408	0.749	1.245	2.384	0.063	0.820
Bayes	2.833	0.973	1.252	1.797	0.057	0.907

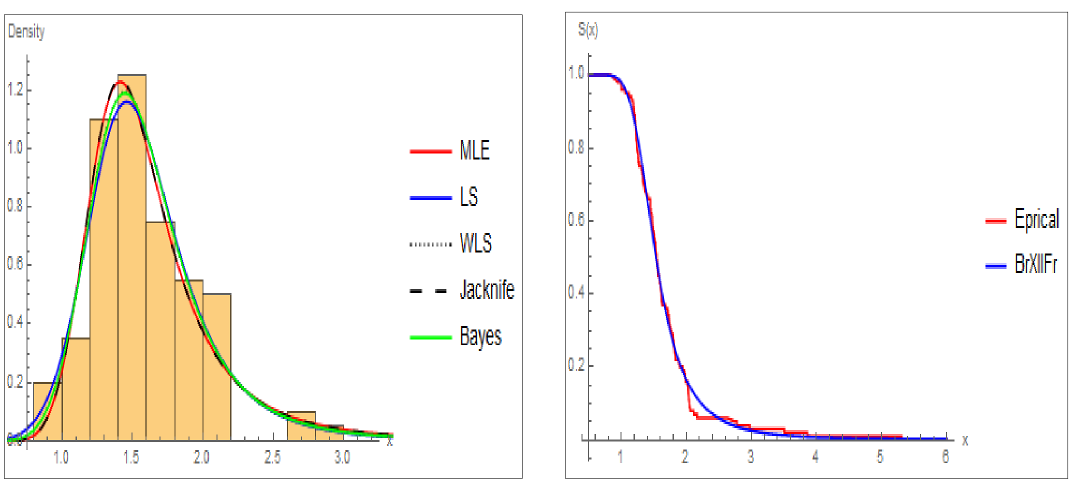

Figure 6: The relative histogram with the fitted density of the new distribution for various methods and fitted survival function for data set I.

From Table 13, the WLS method is the best method for modelling breaking stress data with KS= 0.055 and p-value= 0.923 . However, all other methods performed well.

Method	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$	KS	p-value
ML	3.258	0.5831	1.210	2.472	0.079	0.826
LS	6.022	0.957	1.061	1.028	0.068	0.930
WLS	3.299	0.857	1.239	2.012	0.074	0.877
Jac	2.531	0.637	1.257	3.069	0.077	0.854
Bayes	3.259	0.582	1.203	2.473	0.069	0.921

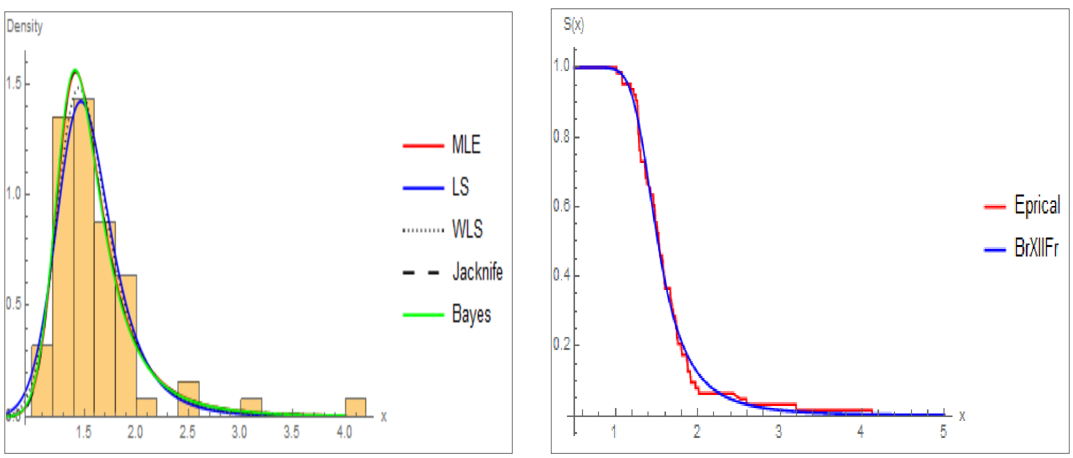

Figure 7: The relative histogram with the fitted density of the new distribution for various methods and fitted survival function for data set II.

From Table 13, the LS method is the best method for modelling strengths data with KS= 0.068 and p-value= 0.930 . However, all other methods performed well.

6. Conclusions

In this paper, we aim to introduce a new extension of the Fréchet distribution. A set of the mathematical and statistical properties have been derived. The estimation of the parameters is carried out by considering the different method of estimation namely maximum likelihood estimation, least square, weighted least square and Jackknife methods. The performances of the proposed estimation methods are studied by Monte Carlo simulations. The potentiality of the proposed model, as well as study, has been analyzed through two data sets. The WLS method is the best method for modelling breaking stress data with KS=0.055 and p-value=0.923. The least square method is the best method for modelling strengths data with KS=0.068 and p-value=0.930, however all other methods performed well for both data sets. On the other hand, The new model gives the lowest values for the AIC, BIC, HQIC and CAIC statistics among all other fitted extensions of the Fréchet models to these data. So, it could be chosen as the best model for modeling breaking stress and strengths real data.

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