Validation of the Odd Lindley Exponentiated Exponential by a Modified Goodness of Fit test and its Applications to Censored and Complete Data

Hafida Goual
Laboratory of Probability and Statistics, University of Badji Mokhtar, Annaba, Algeria
goual.hafida@gmail.com

Haitham M. Yousof
Department of Statistics, Mathematics and Insurance, Benha University, Egypt
haitham.yousof@fcom.bu.edu.eg

M. Masoom Ali
Department of Mathematical Sciences Ball State University, Muncie, Indiana 47306 USA
mali@bsu.edu

Abstract

This paper, we firstly introduse a new extension of the exponentiated exponential distribution along with sufficient set of its mathematical properties. Secondly, we construct a modified Chi-squared goodness-of-fit test based on the Nikulin-Rao-Robson statistic in presence of censored and complete data. We describe the theory and the mechanism of the Y_n^2 statistic test which can be used in survival and reliability data analysis. We use the maximum likelihood estimators based on the initial non grouped data sets. Then, we conduct numerical simulations to reinforce the results. For showing the applicability of our model in various fields, we illustrate it and the proposed test by applications to two real data sets for complete data case and two other data sets in the presence of right censored.

Keywords: Exponential distribution; Censored data; Goodness-of-fit; Nikulin-Rao-Robson; Maximum Likelihood; Right Censoring; Simulation.

1. Introduction

The most popular continuous distributions which used for modeling lifetime data are the gamma (G), the Weibull (W), lognormal (Log-N) and exponentiated exponential (EE) distributions. However, these four models suffer from some serious drawbacks. one of them, none these four models exhibit the bathtub shapes for their hazard rate functions (H.R.F.s), the four models exhibit only monotonically decreasing, monotonically increasing or constant hazard rates and this is a major weakness point because most real-life systems exhibit bathtub shapes for their H.R.F.s. Secondly, at least three of these four models exhibit constant hazard rates, and this is a very unrealistic feature because there are hardly any real-life systems that have constant hazard rates. The aim of this work is to introduce a new three parameter alternative to the EE distribution that overcomes these mentioned drawbacks and exhibits the monotonically increasing, bathtub and monotonically decreasing shapes for its H.R.F., the main goal of the work is to introduce a new EE model using the Odd Lindley (OL-G) family whish exhibits the monotonically increasing, bathtub, constant, the monotonically decreasing hazard rates.

A random variable (R.V.) X is said to have the EE distribution (see Gupta et al. (1998)) if its probability density function (P.D.F.) given by

$$\mathbf{g}_{(\theta,\lambda)}(x) = \theta\lambda\ \left(1-\mathrm{e}^{-\lambda x}\right)^{\theta-1}\mathrm{e}^{-\lambda x}\quad \left[{}^{(x\geq 0)}_{(\lambda>0,\ \theta>0)}\right] \tag{1}$$

and cumulative distribution function (C.D.F.)

$$\mathbf{G}_{(\theta,\lambda)}(x) = \left(1-\mathrm{e}^{-\lambda x}\right)^{\theta}\quad \left[{}^{(x\geq 0)}_{(\lambda>0,\ \theta>0)}\right]$$

respectively, when $\theta = 1$, we have the standard E model. The P.D.F. and C.D.F. of the OL-G family of distribution (see Silva et al. (2017)) are given by

$$f(x;a,\varphi)|_{(a=1)} = \frac{1}{2}\exp\left[-G\left(x;\varphi\right)/\overline{G}\left(x;\varphi\right)\right] g\left(x;\varphi\right)\overline{G}\left(x;\varphi\right)^{-3}, \tag{3}$$

and

$$F(x;\theta,\varphi)|_{(a=1)} = 1-\frac{1}{2}\left[1+\overline{G}\left(x;\varphi\right)\right]\exp\left[-G\left(x;\varphi\right)/\overline{G}\left(x;\varphi\right)\right]\overline{G}\left(x;\varphi\right)^{-1}, \tag{4}$$

respectively. For more details about the OL-G family and its properties see Silva et al. (2017). To this end, we use equations (1), (2) and (3) to obtain the three-parameter OLEE density (5), a R.V. X is said to have the OLEE distribution if its P.D.F. and C.D.F. are given by

$$f(x) = \theta\lambda\ \frac{\mathrm{e}^{-\lambda x}\left(1-\mathrm{e}^{-\lambda x}\right)^{\theta-1}}{2\left[1-\left(1-\mathrm{e}^{-\lambda x}\right)^{\theta}\right]^{3}}\mathrm{e}^{\left\{-\left(1-\mathrm{e}^{-\lambda x}\right)^{\theta}/\left[1-\left(1-\mathrm{e}^{-\lambda x}\right)^{\theta}\right]\right\}}, x\geq 0, \tag{5}$$

and

$$F(x) = 1-\frac{1+\left[1-\left(1-\mathrm{e}^{-\lambda x}\right)^{\theta}\right]}{2\left[1-\left(1-\mathrm{e}^{-\lambda x}\right)^{\theta}\right]}\mathrm{e}^{\left\{-\left(1-\mathrm{e}^{-\lambda x}\right)^{\theta}/\left[1-\left(1-\mathrm{e}^{-\lambda x}\right)^{\theta}\right]\right\}}, x\geq 0, \tag{6}$$

respectively. For $\theta = 1$ the OLEE reduces to the OLiE (Silva et al. (2017)), For $\lambda = 1$ the OLEE reduces to the one parameter OLEE model. The critical points of the OLEE density function are the roots of the equation

$$\frac{\frac{d}{dx}\left\{\theta\lambda\ \left(1-\mathrm{e}^{-\lambda x}\right)^{\theta-1}\mathrm{e}^{-\lambda x}\right\}}{\theta\lambda\ \left(1-\mathrm{e}^{-\lambda x}\right)^{\theta-1}\mathrm{e}^{-\lambda x}} + 3\frac{\theta\lambda\ \left(1-\mathrm{e}^{-\lambda x}\right)^{\theta-1}\mathrm{e}^{-\lambda x}}{1-\left(1-\mathrm{e}^{-\lambda x}\right)^{\theta}} - \frac{\theta\lambda\ \left(1-\mathrm{e}^{-\lambda x}\right)^{\theta-1}\mathrm{e}^{-\lambda x}}{\left[1-\left(1-\mathrm{e}^{-\lambda x}\right)^{\theta}\right]^{2}} = 0.$$

The critical points of the of the H.R.F. of the OLEE are obtained from the following equation

$$\frac{\frac{d}{dx}\left\{\theta\lambda\ \left(1-\mathrm{e}^{-\lambda x}\right)^{\theta-1}\mathrm{e}^{-\lambda x}\right\}}{\theta\lambda\ \left(1-\mathrm{e}^{-\lambda x}\right)^{\theta-1}\mathrm{e}^{-\lambda x}}+\frac{\theta\lambda\ \left(1-\mathrm{e}^{-\lambda x}\right)^{\theta-1}\mathrm{e}^{-\lambda x}}{1+\left[1-\left(1-\mathrm{e}^{-\lambda x}\right)^{\theta}\right]}+2\frac{\theta\lambda\ \left(1-\mathrm{e}^{-\lambda x}\right)^{\theta-1}\mathrm{e}^{-\lambda x}}{\left[1-\left(1-\mathrm{e}^{-\lambda x}\right)^{\theta}\right]^{2}}=0.$$

We can examine the last two Equations to determine the local maximums and minimums and inflexion points via most computer algebra systems. The P.D.F. of X in (5) can be easily expressed as

$$f(x)=\sum_{i,k=0}^{\infty}\xi_{i,k}\ \mathbf{g}_{\{[\theta(2+i+k)],\lambda\}}(x)|\left\{\begin{smallmatrix}(x\geq 0)\\(\lambda>0,\ [\theta(2+i+k)]>0)\end{smallmatrix}\right\}, \tag{7}$$

where

$$\xi_{i,k}=\frac{(-1)^{k}\,\Gamma\left(i+k+3\right)}{2i!\left[\theta\left(2+i+k\right)\right]\Gamma\left(k+3\right)},$$

and

$$\mathbf{g}_{\{[\theta(2+i+k)],\lambda\}}(x)=\left[\theta\left(2+i+k\right)\right]\lambda\ \mathrm{e}^{-\lambda x}\left(1-\mathrm{e}^{-\lambda x}\right)^{[\theta(2+i+k)]-1}\ |\left\{\begin{smallmatrix}(x\geq 0)\\(\lambda>0,\ [\theta(2+i+k)]>0)\end{smallmatrix}\right\},$$

is P.D.F. of EE model with positive parameters $[\theta\left(2+i+k\right)]$ and λ. For more detail about the EE model and its properties see Gupta and Kundu (2001), Gupta and Kundu (2007) and Nadarajah (2011). Other useful works studied the E model such as the one-parameter odd Lindley exponential model (Korkmaz and Yousof (2017)), the two-parameter odd Lindley exponential model (see Silva et al. (2017)) and the logarithmic Burr-Hatke exponential (LogBrHE) distribution (see Abouelmagd (2018)). The C.D.F. of X can be given by integrating (7) as

$$F(x)=\sum_{i,k=0}^{\infty}\xi_{i,k}\ \mathbf{\Pi}^{(\lambda)}_{\{[\theta(2+i+k)],\lambda\}}(x)|\left\{\begin{smallmatrix}(x\geq 0)\\(\lambda>0,\ [\theta(2+i+k)]>0)\end{smallmatrix}\right\}, \tag{8}$$

where

$$\mathbf{\Pi}_{\{[\theta(2+i+k)],\lambda\}}(x)=\left(1-\mathrm{e}^{-\lambda x}\right)^{[\theta(2+i+k)]}\ |\left\{\begin{smallmatrix}(x\geq 0)\\(\lambda>0,\ [\theta(2+i+k)]>0)\end{smallmatrix}\right\},$$

is P.D.F. of EE model with positive parameters $[\theta\left(2+i+k\right)]$ and λ.

We note that the P.D.F. of the OLEE model exhibits various important unimodal shapes (SEE Figure 1), from Figure 2 we see that the H.R.F. of the OLEE distribution exhibits the monotonically increasing, constant, monotonically decreasing and bathtub hazard rates.

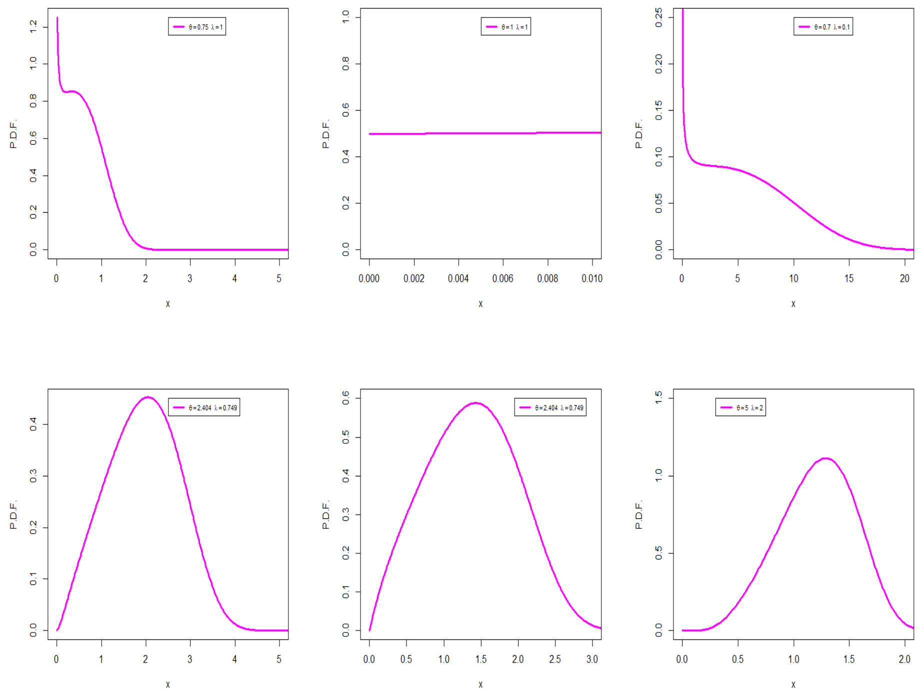


Figure 1. Plots of the OLEE P.D.F..

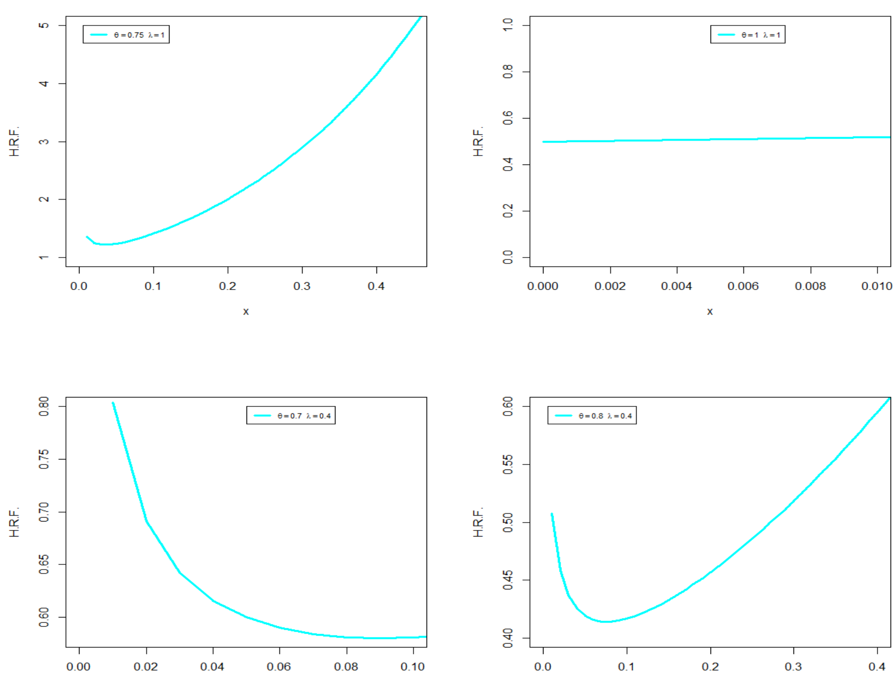


Figure 2. Plots of the OLEE H.R.F..

The major justification for the practicality of the OLEE model is based on the enormous use of the E and EE lifetime models. Also we are motivated to introduce the OLEE lifetime model since it exhibits the monotonically increasing, bathtub, constant and the monotonically decreasing hazard rates (see Figure 2). The new model can be viewed as a mixture of the EE density. It can also be considered as a convenient model for fitting the symmetric, the left skewed, the right skewed, and the unimodal data (see Figure 1). The proposed lifetime model is much better than the exponential exponential, Moment exponential, Log Butr Hatke exponential and the two parameter odd Lindley exponential models, so the new lifetime model is a good alternative to these models in modeling failure times data. Some properties of the new model are given in Appendix A. Second, we construct a modied Chi-squared goodness-of-fit test based on the Nikulin-Rao-Robson statistic in presence of censored and complete data. We describe the theory and the mechanism of the Y_n^2 statistic test which can be used in survival and reliability data analysis. We use the maximum likelihood estimators based on the initial non grouped data sets. Then, we conduct numerical simulations to reinforce the results. For showing the applicability of our model in various fields, we illustrate it and the proposed test by applications to two real data sets for complete data case and two other right censored data sets.

2. Complete data modeling

2.1 Maximum likelihood estimation

Let $x_1, \ldots, x_n$ be a R.S. from the new distribution with parameter vector $\mathbf{\Psi} = (\theta, \lambda,)^\intercal$. The log-likelihood function for $\mathbf{\Psi}$, say $\ell = \ell(\mathbf{\Psi})$, is given by

$$\begin{aligned}\ell &= \ell(\mathbf{\Psi}) = -n\log(2) + n\log\theta + n\log\lambda + (\theta-1)\sum_{i=1}^{n}\log\left(1-\mathrm{e}^{-\lambda x_i}\right) \\ &\quad -\lambda\sum_{i=1}^{n}x_i - 3\sum_{i=1}^{n}\log\left[1-\left(1-\mathrm{e}^{-\lambda x_i}\right)^{\theta}\right] - \sum_{i=1}^{n}\frac{\left(1-\mathrm{e}^{-\lambda x_i}\right)^{\theta}}{1-\left(1-\mathrm{e}^{-\lambda x_i}\right)^{\theta}}. \qquad (9)\end{aligned}$$

Equation (9) can be maximized either via the different programs like `R` (`optim` function), `SAS` (`PROC NLMIXED`) or via solving the nonlinear likelihood equations obtained by differentiating Equ. (9). The score vector elements, $\mathbf{U}(\mathbf{\Psi}) = \frac{\partial\ell}{\partial\mathbf{\Psi}} = \left(\frac{\partial\ell}{\partial\theta}, \frac{\partial\ell}{\partial\lambda}\right)^\intercal$, exist and can easily to be obtained.

2.2 real data modeling

This data set {1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2} represents the lifetime data relating to relief times (in minutes) of patients receiving an analgesic (see Gross and Clark (1975)). We shall compare thefits of the new distribution with those of other competitive models, namely: the exponential E(λ), Moment exponential MomE(λ), Log Butr Hatke exponential LogBrHE(λ) and the two parameter odd Lindley exponential OLE(a, λ) models. We consider some other goodness-of-fit measures including the Akaike information criterion (**AIC**), consistent Akaike information criterion (**CAIC**), Hannan-Quinn information criterion (**HQIC**), Bayesian information criterion (**BIC**) and $-2\widehat{\ell}$, where $\widehat{\ell}$ is the maximized log-likelihood,

$$\begin{aligned}\mathbf{AIC} &= -2\widehat{\ell}+2p,\\ \mathbf{BIC} &= -2\widehat{\ell}+p\log(n),\\ \mathbf{CAIC} &= -2\widehat{\ell}+2pn/(n-p-1),\end{aligned}$$

and

$$\mathbf{HQIC}=-2\widehat{\ell}+2p\log[\log(n)],$$

where p is the number of parameters and n is the sample size. Moreover, we consider the Cramér-Von Mises and the Anderson-Darling $\mathbf{W}^{\star}$ and $\mathbf{A}^{\star}$ statistic. The $\mathbf{W}^{\star}$and $\mathbf{A}^{\star}$ statistics are given by

$$\mathbf{W}^{\star}=(1+1/2n)\left[[1/(12n)]+\sum\nolimits_{j=1}^{n}c_j\right],$$

and

$$\mathbf{A}^{\star}=d_{(n)}\left(n+n^{-1}\sum\nolimits_{j=1}^{n}d_j\right),$$

where

$$\begin{aligned}c_j&=[z_i-(2j-1)/(2n)]^2,\\ d_{(n)}&=1+\frac{9}{4}n^{-2}+\frac{3}{4}n^{-1},\end{aligned}$$

and

$$d_j=(2j-1)\log[z_i(1-z_{n-j+1})],$$

where $z_i=F(y_j)$ and the y_j's values are the ordered observations.

Table 1: MLEs, SEs, C.I.s (in parentheses) values for the relief times data.

Models	Estimates	
E(λ)	0.526(0.117)	(0.29, 0.75)
MomE(λ)	0.950(0.150)	(0.66, 1.24)
LogBrHE(λ)	0.5263(0.118)	(0.43, 0.63)
OLE(a,λ)	0.783(0.391), 0.68(0.164)	(0, 1.56), (0.36, 1)
OLEE(θ,λ)	2.40(0.98), 0.74(0.12),	(0.6, 4.2), (0.48, 1)

Table 2: **AIC**, **BIC**, **CAIC**, **HQIC**, $\mathbf{A}^{\star}$, $\mathbf{W}^{\star}$

Models	**AIC**, **BIC**, **CAIC**, **HQIC**	$A^{\star}$, $W^{\star}$
E(λ)	67.67,68.67,67.89,67.87	4.60, 0.96
MomE(λ)	54.32,55.31,54.54,54.50	2.76, 0.53
LogBrHE(λ)	67.67,68.67,67.89,67.87	0.62, 0.105
OLiE(a,λ)	50.89,52.88,51.6,51.3	1.39, 0.24
OLEE(θ,λ)	**49.8, 51.8, 50.5, 50.2**	**1.19, 0.20**

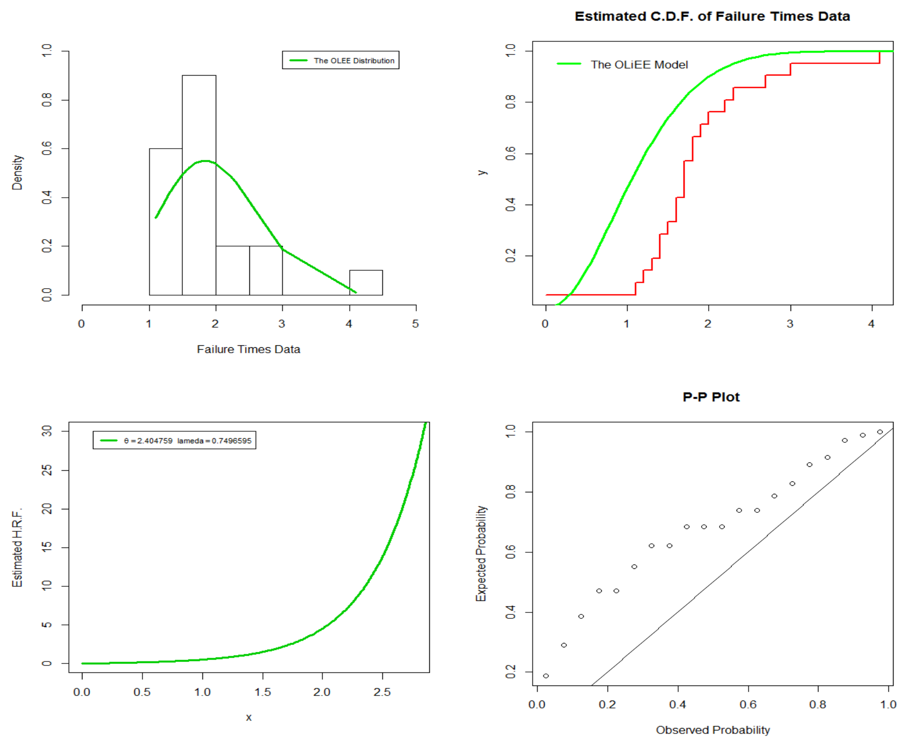


Figure 3. Estimated P.D.F., Estimated H.R.F., and P-P plot for the relief times data.

From Table 3 we conclude that the proposed lifetime model is much better than the exponential E(λ), Moment exponential MomE(λ), Log Butr Hatke exponential LogBrHE(λ) and the two parameter odd Lindley exponential OLE(a, λ) models, so the new lifetime model is a good alternative to these models in modeling relief times data.

2.3 Simulations

In this step the OLEE model is considered. The data were simulated $N = 10,000$ times; with parameter values $\theta = 0.9, \lambda = 0.5$ and sample sizes $n = 30, 100, 300, 500, 1000$. We use Barzilai-Borwein (BB) algorithm (see Ravi (2009)) in R software, for calculating the averages of the simulated values of the maximum likelihood estimators $\widehat{\theta}, \widehat{\lambda}$ parameters and their mean squared errors (MSE) presented in Table 3. From Table 3. , we can notice that the maximum likelihood estimators for the OLEE model are convergent.

Table 3: Maximum likelihood estimators $\left(\widehat{\theta}, \widehat{\lambda}\right)$ of the parameters and their MSEs.

$N = 10000$	$n = 30$	$n = 100$	$n = 300$	$n = 500$	$n = 1000$
$\widehat{\theta}$	0.86142	0.87156	0.92751	0.9214	0.9085
MSE	0.04875	0.04476	0.03924	0.02935	0.0108
$\widehat{\lambda}$	0.4821	0.4963	0.4981	0.5107	0.5078
MSE	0.0348	0.0312	0.0275	0.0201	0.0143

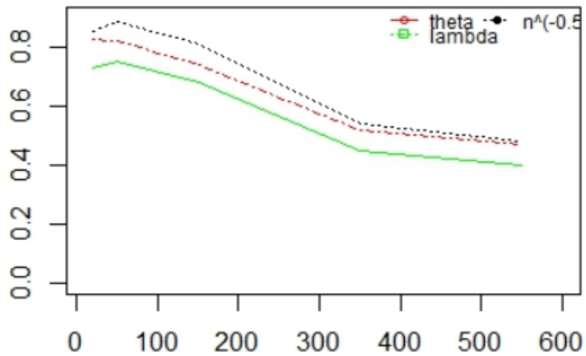


Figure 4. Simulated average absolute errors for MLEs $\hat{\theta}, \hat{\lambda}$.

From the Figure 4, we confirm that all estimates of the OLEE distribution converge faster than $n^{-0.5}$, so our maximum likelihood estimators are$\sqrt{n}$ - consistent.

3. Censored data modeling

3.1 Maximum likelihood estimation

Let T be a random variable distributed according to a OLEE distribution with $\zeta = (\widehat{\theta}, \widehat{\lambda})^T$. For i (individual); T_i is the lifetime and C_i is the censorship time, where T_i and C_i are independent random variables. The data consists of n independent observations

$$t_i = \min(T_i, C_i) \text{ for } i = 1, ..., n.$$

In case of non-informative censorship (distribution of C_i does not depend on the unknown parameters of T_i) , the likelihood function is given by:

$$\mathbf{L}(t, \zeta) = \prod_{i=1}^{n} \lambda^{\delta_i}(t_i, \zeta) S(t_i, \zeta)|_{\left(\zeta=(\theta,\lambda)^T, \delta_i = 1_{\{T_i \leq C_i\}}\right)}.$$

Let T_i be a random variable distributed with the vector of parameters $\zeta = (\theta, \lambda)^T$ of the OLEE model, so the likelihood function can be -

$$\mathbf{L}(t, \zeta) = \prod_{i=1}^{n} \left(\begin{array}{c} \frac{1+\left[1-\left(1-e^{-\lambda t_i}\right)^{\theta}\right]}{2\left[1-\left(1-e^{-\lambda t_i}\right)^{\theta}\right]} e^{\left\{-\left(1-e^{-\lambda t_i}\right)^{\theta} / \left[1-\left(1-e^{-\lambda t_i}\right)^{\theta}\right]\right\}} \\ \times \left[\frac{\theta \lambda e^{-\lambda t_i}\left(1-e^{-\lambda t_i}\right)^{\theta-1}}{2\left[1-\left(1-e^{-\lambda t_i}\right)^{\theta}\right]^2 \left\{1+\left[1-\left(1-e^{-\lambda t_i}\right)^{\theta}\right]\right\}} \right]^{\delta_i} \end{array} \right).$$

The loglikelihood function is

$$\begin{aligned} l(t,\zeta) &= \sum_{i=1}^{n} \ln\left\{1+\left[1-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right]\right\} - \ln\left\{2\left[1-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right]\right\} \\ &\quad -\sum_{i=1}^{n}\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta} / \left[1-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right] \\ &\quad \sum_{i=1}^{n}\left[\delta_i\left(\begin{array}{c} \ln\left[\theta\lambda \mathrm{e}^{-\lambda t_i}\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta-1}\right] \\ -\ln\left(2\left[1-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right]^2\left\{1+\left[1-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right]\right\}\right)\end{array}\right)\right], \end{aligned}$$

or

$$\begin{aligned} l(t,\zeta) &= \sum_{i\in F} \ln\left[\theta\lambda \mathrm{e}^{-\lambda t_i}\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta-1}\right] \\ &\quad -\sum_{i\in F}\ln\left(2\left[1-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right]^2\left\{1+\left[1-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right]\right\}\right) \\ &\quad +\sum_{i\in C}\ln\left\{\frac{1+\left[1-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right]}{2\left[1-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right]}\right\} \\ &\quad -\sum_{i\in C}\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta} / \left[1-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right], \end{aligned}$$

where F is the set of uncensored data and C is the set of censored observations. The score functions are

$$\begin{aligned} \frac{\partial l(t,\zeta)}{\partial\theta} &= \sum_{i\in F}\frac{\mathrm{e}^{-\lambda t_i}\left(1-\mathrm{e}^{-\lambda t_i}\right)^{1-\theta}\left[v\mathrm{e}^{-\lambda t_i}\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta-1}+\theta\lambda\mathrm{e}^{-\lambda t_i}\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta-1}\ln\left(1-\mathrm{e}^{-\lambda t_i}\right)\right]}{\theta\lambda} \\ &\quad +\sum_{i\in F}\frac{\left\{\begin{array}{c} 2\left[1-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right]^2\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\ln\left(1-\mathrm{e}^{-\lambda t_i}\right) \\ -4\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\left[1-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right]\ln\left(1-\mathrm{e}^{-\lambda t_i}\right)\left[2-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right]\end{array}\right\}}{2\left[1-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right]^2\left[2-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right]} \\ &\quad +\sum_{i\in C}\frac{2\left[1-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right]\left\{\frac{\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\left[2-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right]\ln\left(1-\mathrm{e}^{-\lambda t_i}\right)}{2\left[1-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right]^2}-\frac{\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\ln\left(1-\mathrm{e}^{-\lambda t_i}\right)}{2\left[1-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right]}\right\}}{2-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}} \\ &\quad -\sum_{i\in C}\frac{\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\ln\left(1-\mathrm{e}^{-\lambda t_i}\right)}{1-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}}+\frac{\left(1-\mathrm{e}^{-\lambda t_i}\right)^{2\theta}\ln\left(1-\mathrm{e}^{-\lambda t_i}\right)}{\left(1-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right)^2}, \end{aligned}$$

$$\begin{aligned}
\frac{\partial l(t,\zeta)}{\partial \lambda} &= \sum_{i\in F} \frac{\mathrm{e}^{-\lambda t_i}\left(1-\mathrm{e}^{-\lambda t_i}\right)^{1-\theta}\left[\begin{array}{c} t_i(\theta-1)\theta\lambda \mathrm{e}^{-\lambda t_i}\mathrm{e}^{-\lambda t_i}\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta-2} \\ +\theta \mathrm{e}^{-\lambda t_i}\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta-1} - \theta\lambda t_i \mathrm{e}^{-\lambda t_i}\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta-1}\end{array}\right]}{\theta\lambda} \\
&+ \sum_{i\in F} \frac{\left\{\begin{array}{c} 2t_i\theta \mathrm{e}^{-\lambda t_i}\left[1-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right]^2\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta-1} \\ -4\theta t_i \mathrm{e}^{-\lambda t_i}\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta-1}\left[1-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right]\left[2-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right]\end{array}\right\}}{2\left[1-\left(1-e^{-\lambda t_i}\right)^{\theta}\right]^2\left[2-\left(1-e^{-\lambda t_i}\right)^{\theta}\right]} \\
&+ \sum_{i\in C} \frac{2\left[1-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right]\left\{\frac{t_i\theta \mathrm{e}^{-\lambda t_i}\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta-1}\left[2-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right]}{2\left[1-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right]^2} - \frac{t_i\theta \mathrm{e}^{-\lambda t_i}\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta-1}}{2\left[1-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right]}\right\}}{2-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}} \\
&- \sum_{i\in C} \frac{t_i\theta \mathrm{e}^{-\lambda t_i}\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta-1}}{1-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}} + \frac{t_i\theta \mathrm{e}^{-\lambda t_i}\left(1-\mathrm{e}^{-\lambda t_i}\right)^{2\theta-1}}{\left[1-\left(1-\mathrm{e}^{-\lambda t_i}\right)^{\theta}\right]^2}.
\end{aligned}$$

For solving this system of score functions, we can use the Monte Carlo method, the Barzilai-Borwein (BB) and Newoton Raphson algorithms or other similar methods.

3.2 Simulations

The data from OLEE dostribution were simulated $N = 10,000$ times; with sample sizes $n = 30, n = 100, n = 300, n = 500, n = 1000$ and parameter values $\theta = 2.4, \lambda = 2$. The averages of the simulated values of the maximum likelihood estimators $\widehat{\theta}, \widehat{\lambda}$ Parameters, and their mean squared errors (MSE) are calculated and presented in Table 4. From Table 4, one can say that the the maximum likelihood estimators $\widehat{\theta}, \widehat{\lambda}$ are convergent.

Table 4: Maximum likelihood estimators $\left(\widehat{\beta}, \widehat{a}, \widehat{\lambda}\right)$ of the parameters and their MSEs (censored data)

$N = 10000$	$n = 30$	$n = 100$	$n = 300$	$n = 500$	$n = 1000$
$\widehat{\theta}$	2.3561	2.3741	2.3917	2.4142	2.4052
MSE	0.04847	0.04521	0.03841	0.2985	0.1542
$\widehat{\lambda}$	1.9576	1.9664	1.9824	2.0977	2.0276
MSE	0.03214	0.04001	0.02761	0.02223	0.0160

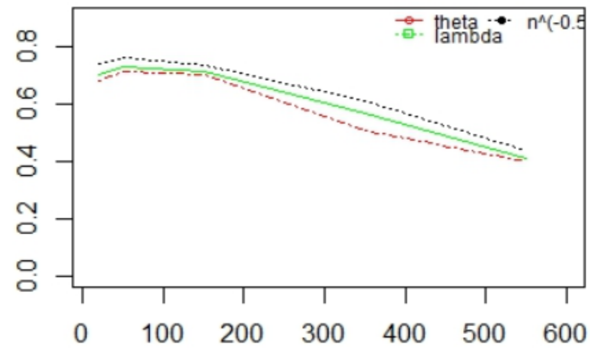


Figure 5. Simulated average absolute errors for MLEs $\hat{\theta}, \hat{\lambda}$.

From Figure 5, all estimates converge faster than $n^{-0.5}$, which affirm the fact that the maximum likelihood estimators are $\sqrt{n}$ - consistent.

4. Goodness-of-fit test

4.1 Nikulin-Rao-Robson statistic test

To verify the adequacy of mathematical models to data from observation, different techniques are used. We can apply the based test Chi-squared of Pearson. Since the middle of the last century, researchers have begun to propose modifications to take into account unknown parameters. For the complete data, Nikulin (1973) and Rao and Robson (1974) separately proposed a statistic known today as the N.R.R statistic (Nikulin-Rao-Robson). This statistical test, is a natural modification of the Pearson statistic. To test the hypothesis H_0

$$H_0 : P\{T_i \leq t\} = F(t, \zeta)|_{(t\in\mathbb{R},\ \zeta=(\zeta_1,\zeta_2,\cdots,\zeta_s)^T)},$$

where ζ represents the vector of unknown parameters, Nikulin (1973) and Rao and Robson (1974) proposed Y^2 the N.R.R statistic defined as follows:

Observations $T_1, T_2, \cdots, T_n$ are grouped in r subintervals $\mathbf{I}_1, \mathbf{I}_2, \cdots, \mathbf{I}_r$ mutually disjoint

$$\mathbf{I}_j =]a_j - 1; a_j],\ j = \overline{1; r}$$

The limits a_j of the intervals $\mathbf{I}_j$ are obtained such that

$$p_j(\zeta)|_{(\ j=1,2,\cdots,r)} = \int_{a_{j-1}}^{a_j} f(t, \zeta)dt$$

where

$$a_j|_{(j=1,\cdots,r-1)} = F^{-1}\left(\frac{j}{r}\right).$$

If

$$\nu_j = (\nu_1, \nu_2, \cdots, \nu_r)^T$$

is the vector of frequencies obtained by the grouping of data in these $\mathbf{I}_j$ intervals

$$\nu_j = \sum_{i=1}^{n} 1_{\{t_i \in \mathbf{I}_j\}} \mid_{(j=1,\dots,r)}.$$

The N.R.R statistic is given by

$$Y^2(\widehat{\zeta_n}) = X_n^2(\widehat{\zeta_n}) + n^{-1}\mathbf{L}^T(\widehat{\zeta_n})(\mathbf{I}(\widehat{\zeta_n}) - \mathbf{J}(\widehat{\zeta_n}))^{-1}\mathbf{L}(\widehat{\zeta_n})$$

where

$$X_n^2(\zeta) = \left(\frac{\nu_1 - np_1(\zeta)}{\sqrt{np_1(\zeta)}}, \frac{\nu_2 - np_2(\zeta)}{\sqrt{np_2(\zeta)}}, \cdots, \frac{\nu_r - np_r(\zeta)}{\sqrt{np_r(\zeta)}}\right)^T,$$

and $\mathbf{J}(\zeta)$ is the information matrix for the grouped data defined by

$$\mathbf{J}(\zeta) = B(\zeta)^T B(\zeta),$$

with

$$B(\zeta)|_{(i=1,2,\cdots,r \text{ and } k=1,\cdots,s)} = \left[\frac{1}{\sqrt{p_i}}\frac{\partial p_i(\zeta)}{\partial \zeta}\right]_{r\times s},$$

and

$$\mathbf{L}(\zeta) = (\mathbf{L}_1(\zeta), ..., \mathbf{L}_s(\zeta))^T, \quad \mathbf{L}_k(\zeta) = \sum_{i=1}^{r} \frac{\nu_i}{p_i}\frac{\partial}{\partial \zeta_k} p_i(\zeta),$$

where $\mathbf{I}_n(\widehat{\zeta_n})$ represents the estimated Fisher information matrix and $\widehat{\zeta_n}$ is the maximum likelihood estimator of the parameter vector. The Y^2 statistic follows a distribution of chi-square χ^2_α with $(r-1)$ degrees of freedom.

4.2 NRR statistic for the OLEE model

To verify if a sample $T = (T_1, T_2, \cdots, T_n)^T$ is distributed according to the OLEE model, $P\{T_i \le t\} = F_{OLEE}(t, \zeta)$; with unknown parameters $\zeta = (\theta, \lambda)^T$, a chi-square goodness-of-fit test is constructed by fitting the N.R.R statistic developed in the previous section. The maximum likelihood estimators $\widehat{\zeta_n}$ of the unknown parameters of the OLEE distribution are computed on the initial data. The statistic Y^2 does not depend on the parameters, we can therefore use the Fisher information matrix estimated $\mathbf{I}_n(\widehat{\zeta_n})$.

4.3 Simulation studies (N.R.R statistics Y^2)

To test the null hypothesis H_0 that a sample belongs to the OLEE model, we calculate Y^2 the NRR statistic of 10,000 simulated samples with sizes

$$n = 30, n = 50, n = 100, n = 300, n = 500, n = 1000,$$

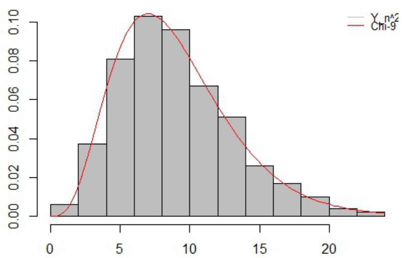
respectively. For different theoretical levels ($\epsilon = 0.02, 0.05, 0.01, 0.1$), we calculate the average of the non-rejection numbers of the null hypothesis, when $Y^2 \leq \chi^2_\epsilon\,(r-1)$ (see Table 5) then, we present the results of the corresponding empirical and theoretical levels in Table 5. As can be seen, the values of the empirical levels calculated are very close to those of their corresponding theoretical levels. Thus, we conclude that the proposed test is well adapted to the OLEE distribution.

Table 5: Empirical levels and corresponding theoretical levels ($\epsilon = 0.02, 0.05, 0.01, 0.1$)

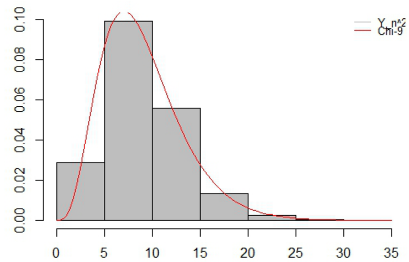
$N = 10000$	$\epsilon = 0.02$	$\epsilon = 0.05$	$\epsilon = 0.01$	$\epsilon = 0.1$
$n = 30$	0.9841	0.9535	0.9921	0.9049
$n = 50$	0.9836	0.9528	0.9919	0.9043
$n = 100$	0.9821	0.9517	0.9918	0.9023
$n = 300$	0.9811	0.9506	0.9906	0.9012
$n = 500$	0.9803	0.9502	0.9902	0.9004
$n = 1000$	0.9801	0.9501	0.99004	0.9002

4.4 Simulated distribution of Y^2 statistic for OLEE model

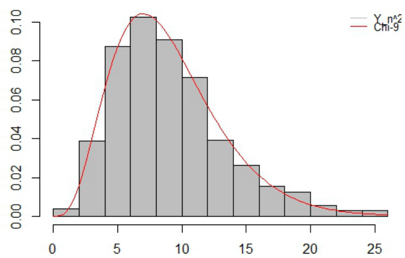
The Y^2 statistic follows in the limit; a chi-squared distribution with $k = r-1$ degrees of freedom. For demonstrating this fact, we compute $N = 10,000$ times, the simulated distribution of $Y^2(\widehat{\zeta})$ under the null hypothesis H_0 with different values of parameters, and $r = 10$ intervals, versus the chi-squared distribution with $k = r-1 = 9$ degree of freedom. Their histograms are represented in Figure 6. versus the chi-squared distribution with $k = 9$ degree of freedom.



(a) $\hat{\zeta} = (\hat{\theta} = 0.9, \hat{\lambda} = 1.8)$



(b) $\hat{\zeta} = (\hat{\theta} = 1.8, \hat{\lambda} = 1.5)$



(c) $\hat{\zeta} = (\hat{\theta} = 2.6, \hat{\lambda} = 0.8)$

Figure 6. Simulated distribution of the Y_n^2 statistic under the null hypothesis H_0, with different parameters of $\hat{\zeta}$ and the chi-squared distribution with 9 degrees of freedom, with $n = 100, N = 10000$.

From Figure 6, we can observe that the statistical distribution of Y^2 with different values of parameters and different numbers k of grouping cells -for different number of equiprobable grouping intervals and different value of parameters- ; in the limit follows a chi-squared with k degrees of freedom within the statistical errors of simulation. We can say that the limiting distribution of the generalized chi-squared Y^2 statistic for OLEE model is distribution free.

4.5 Real data modeling

4.5.1 relief times data

To test the null hypothesis H_0 that these data are adjusted by a OLEE distribution, we use the N.R.R statistic obtained previously. Using the R software and the BB algorithm (Ravi, 2009), we compute the maximum likelihood estimators (MLE) 2.4047589, 0.7496595 $\widehat{\theta} = 7.1524, \widehat{\lambda} = 0.51149$. The estimated Fisher information matrix is then

$$\mathbf{I}(\widehat{\zeta}) = \begin{pmatrix} 1.098541 & 2.04578 & 6.21866 \\ & 24.91572 & 56.84276 \\ & & 12.85441 \end{pmatrix}.$$

We then deduce the value of $Y^2 = 6.954718$. The critical value is

$$\chi^2_{0.05}(4-1) = 7.814728,$$

then, the N.R.R Y^2 statistic is less than the critical value, this allows us to say that these data correspond appropriately to the OLEE model.

5. Goodness-of-fit test for right censored data

Habib and Thomas (1986) considered the natural modifications of the N.R.R statistic. These tests are based on the differences between two probability estimators, one based on the Kaplan-Meier estimator, the other based on the maximum likelihood estimators of the unknown parameters of the cumulative distribution function of the Kaplan-Meier estimator. model tested. When to Bagdonavicius and Nikulin (2011); Bagdonavicius et al. (2013), they proposed a modification of the N.R.R statistic that takes into account random right censorship. This statistic, based on the maximum likelihood estimators on the initial data, also follows a Chi-square distribution at the limit. For more details on the construction of these statistics, we can see Voinov et al. (2013). These techniques were used to adjust observations to the generalized inverse Weibull model (Goual and Seddik-Ameur 2014), the distribution of Birbaurm Saunders (Nikuli et al. (2013)), the kumaraswamy generalized inverse Weibull distribution (Goual and Seddik-Ameur 2016) and others. In this paragraphe we develop the approach proposed by Bagdonavicius and Nikulin (2011), Bagdonavicius et al. (2013); to confirm the adequacy of OLEE model when the parameters are unknown and data are censored. Let us consider the composite hypothesis

$$H_0 : F(t) \in F_0 = F_0(t, \zeta)|_{(t \in R^1,\ \zeta \in \Psi \subset R^s)},$$

where

$$\zeta = (\zeta_1, ..., \zeta_s)^T \in \Psi \subset R^s$$

is an unknown m-dimensional parameter and F_0 is a differentiated completely specified cdf with the support $(0, \infty)$. Let us consider a finite time interval only say $[0, \tau]$, where τ is the maximum time of the study, and divide it into $k > s$ smaller intervals $\mathbf{I}_j = (a_{j-1}, a_j]$, where

$$0 =< a_0 < a_1 ... < a_{k-1} < a_k = +\infty.$$

In this case the estimated $\widehat{a}_j$ is given by

$$\widehat{a}_j = \mathbf{\Lambda}^{-1}\left((E_j - \sum_{l=1}^{i-1} \mathbf{\Lambda}(T_{(l)}, \widehat{\zeta}))/(n-i+1), \widehat{\zeta}\right), \quad \widehat{a_k} = T_{(n)},\ j = 1, ..., k,$$

where $\widehat{\zeta}$ is the Maximum likelihood estimator of the parameter ζ, $\mathbf{\Lambda}^{-1}$ is the inverse of cumulative hazard function $\mathbf{\Lambda}$, $T_{(i)}$ is the i^{th} element in the ordered statistics

$(T_{(1)},,,T_{(n)})$ and

$$E_j = (n-i+1)\mathbf{\Lambda}(\widehat{a}_j, \widehat{\zeta}) + \sum_{l=1}^{i-1} \mathbf{\Lambda}(T_{(l)}, \widehat{\zeta}),$$

and a_j are random data functions such as the k intervals chosen have equal expected numbers of failures e_j. Usually in real application we fix k. Bagdonavicius et al. (2010) and Greenwood and Nikulin (1996) give some recommendations for the choice of intervals. The test is based on the vector

$$\mathbf{Z} = (\mathbf{Z}_1, ..., \mathbf{Z}_k)^T, \quad \mathbf{Z}_j = \frac{1}{\sqrt{n}}(\mathbf{U}_j - e_j), \quad j = 1, 2, ..., k,$$

where $\mathbf{U}_j$ represent the numbers of observed failures in these intervals. The test for hypothesis H_0 can be based on the statistic

$$Y_n^2 = \mathbf{Z}^T \widehat{\mathbf{\Sigma}}^{-1} \mathbf{Z}$$

where

$$\widehat{\mathbf{\Sigma}}^{-1} = \widehat{\mathbf{A}}^{-1} + \widehat{\mathbf{C}}^{-1} \widehat{\mathbf{A}}^T \widehat{\mathbf{G}}^{-1} \widehat{\mathbf{C}} \widehat{\mathbf{A}}^{-1}$$

and

$$\widehat{\mathbf{G}} = \widehat{i} - \widehat{\mathbf{C}} \widehat{\mathbf{A}}^{-1} \widehat{\mathbf{C}}^T.$$

The test statistic can be written in the following form

$$Y_n^2 = \sum_{j=1}^{k} \frac{1}{\mathbf{U}_j}(\mathbf{U}_j - e_j)^2 + \mathbf{Q},$$

where

$$\begin{aligned}
\widehat{\mathbf{A}}_j &= n^{-1}\mathbf{U}_j, \\
\widehat{\mathbf{G}} &= [\widehat{g}_{ll'}]_{s\times s}, \\
\mathbf{U}_j &= \sum_{i:X_i \in \mathbf{I}_j} \delta_i, \\
\mathbf{Q} &= \widehat{\mathbf{W}}^T \widehat{\mathbf{G}}^{-1} \widehat{\mathbf{W}}, \\
\widehat{\mathbf{C}}_{lj} &= n^{-1} \sum_{i:X_i \in \mathbf{I}_j} \delta_i \frac{\partial}{\partial \zeta} \ln\left[\lambda_i(t_i, \widehat{\zeta})\right], \\
\widehat{\mathbf{W}}_l &= \sum_{j=1}^{k} \widehat{\mathbf{C}}_{lj} \widehat{\mathbf{A}}_j^{-1} \mathbf{Z}_j, \qquad l, l' = 1, ..., s, \\
\widehat{\mathbf{W}} &= (\widehat{\mathbf{W}}_1, \widehat{\mathbf{W}}_2, ..., \widehat{\mathbf{W}}_s)^T, \\
\widehat{i}_{ll'} &= n^{-1} \sum_{i=1}^{n} \delta_i \frac{\partial}{\partial \zeta_l} \ln\left[\lambda_i(t_i, \widehat{\zeta})\right] \frac{\partial}{\partial \zeta_{l'}} \ln\left[\lambda_i(t_i, \widehat{\zeta})\right],
\end{aligned}$$

and

$$\widehat{g}_{ll'} = \widehat{i}_{ll'} - \sum_{j=1}^{k} \widehat{\mathbf{C}}_{lj} \widehat{\mathbf{C}}_{l'j} \hat{A}_j^{-1}, \ \widehat{\mathbf{C}}_{lj} = n^{-1} \sum_{i:X_i \in \mathbf{I}_j} \delta_i \frac{\partial}{\partial \zeta} \ln \lambda_i(t_i, \widehat{\zeta}),$$

The elements of the matrices $\widehat{\mathbf{W}}$ and $\widehat{\mathbf{I}}$ are given in the Appendix B. The limit distribution of the statistic Y_n^2 is chi-square with r =rank($\mathbf{\Sigma}$) $= tr(\mathbf{\Sigma}^{-1}\mathbf{\Sigma})$ degrees of freedom. If $\mathbf{G}$ is non-degenerate then $r = k$. The hypothesis is rejected with approximate significance level ϵ if $Y_n^2 > \chi_\epsilon^2(r)$ where $\chi_\epsilon^2(r)$ is the quantile of chi-square with r degrees of freedom. For more details, see Bagdonavicius and Nikulin (2011) and Bagdonavicius et al. (2013).

5.1 Validation of OLEE model in case of censored data

In this section, we study the validity of the OLEE model, by a goodness-of-fit test based on Y_n^2, the modified N.R.R statistic presented in the previous section. Suppose H_0 is checked (T_i follows OLEE distribution), the survival function is:

$$S(t,\zeta) = 1 - F(t;\theta,\lambda) = \frac{1 + \left[1 - \left(1 - \mathrm{e}^{-\lambda x}\right)^{\theta}\right]}{2\left[1 - (1 - \mathrm{e}^{-\lambda x})^{\theta}\right]} \mathrm{e}^{\left\{-\left(1-\mathrm{e}^{-\lambda x}\right)^{\theta}/\left[1-\left(1-\mathrm{e}^{-\lambda x}\right)^{\theta}\right]\right\}},$$

when the baseline distribution is a OLEE model, the choice of $\widehat{a}_j$, is given by

$$\begin{aligned} \mathbf{\Lambda}_{OLEE}(t,\zeta) &= -\ln S(t,\zeta) = -\ln\left\{1 + \left[1 - \left(1 - \mathrm{e}^{-\lambda x}\right)^{\theta}\right]\right\} \\ &\quad + \ln\left\{2\left[1 - \left(1 - \mathrm{e}^{-\lambda x}\right)^{\theta}\right]\right\} \\ &\quad + \left\{-\left(1 - \mathrm{e}^{-\lambda x}\right)^{\theta} / \left[1 - \left(1 - \mathrm{e}^{-\lambda x}\right)^{\theta}\right]\right\}, \end{aligned}$$

$$E_j = \sum_{i:X_i > a_j} (\mathbf{\Lambda}(a_j \wedge t_i, \widehat{\zeta}) - \mathbf{\Lambda}(a_{j-1}, \widehat{\zeta}) \text{ and } E_k = \sum_{i=1}^{n} \mathbf{\Lambda}(t_i, \widehat{\zeta}),$$

with this choice of intervals, for any j we have a constant value of $e_j = E_k/k$. Intervals can be estimated by iterative method (there is no explicit form of inverse hazard function of OLEE distribution).

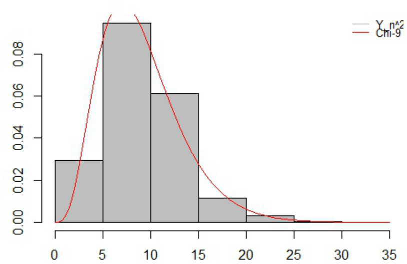
5.2 Simulation study

To test the null hypothesis H_0 that a sample comes from a OLEE model, we calculate Y_n^2 the N.R.R statistic of 10,000 simulated samples with sizes $n = 30, 100, 300, 500, 1000$, respectively. For different levels of meaning ($\epsilon = 0.02, 0.05, 0.01, 0.1$), we calculate the mean of the number of no rejections of the null hypothesis when $Y_n^2 \leq \chi_\epsilon^2(r)$, then we present the results of the empirical values and the corresponding theoretical values in Table 6.

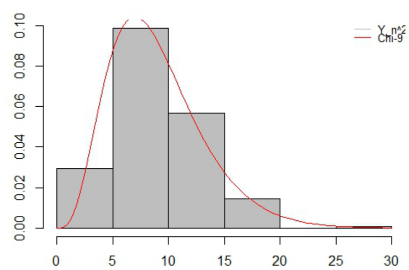
Table 6: Empirical levels and corresponding theoretical levels ($\epsilon = 0.02; 0.05; 0.01; 0.1$).

$N = 10000$	$\epsilon = 0.02$	$\epsilon = 0.05$	$\epsilon = 0.01$	$\epsilon = 0.1$
$n = 30$	0.9830	0.9528	0.9918	0.9011
$n = 100$	0.9825	0.9515	0.9917	0.9010
$n = 300$	0.9810	0.9509	0.9907	0.9008
$n = 500$	0.9805	0.9505	0.9902	0.9001
$n = 1000$	0.9802	0.9501	0.9901	0.90005

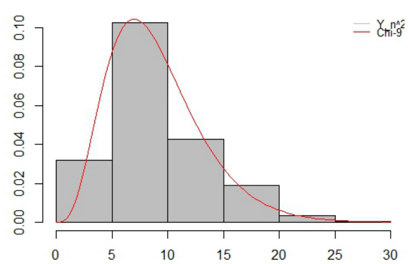
According to this results, we find that the empirical signification levels of the Y_n^2 statistic coincide with those corresponding to the theoretical levels of the chi-square distributions at r degrees of freedom. Therefore, we can say that the proposed test can properly fit censored data from the OLEE distribution.



(a) $\hat{\zeta} = (\hat{\theta} = 0.5, \hat{\lambda} = 0.5)$



(b) $\hat{\zeta} = (\hat{\theta} = 2.4, \hat{\lambda} = 1.5)$



(c) $\hat{\zeta} = (\hat{\theta} = 1.7, \hat{\lambda} = 2)$

Figure 7. Simulated distribution of the Y_n^2 statistic under the null hypothesis H_0, with different parameters of $\hat{\zeta}$ and the chi-squared distribution with 9 degrees of freedom, with $n = 100, N = 10000$.

5.3 Application to real data

5.3.1 Arm-A head and neck cancer data

The data considered below (was conducted by northern California oncology group) was used by Efron (1988) for logistic distribution. Mudholkar et al. (1996) and Nikulin and Haghighi (2006) reanalysed the same data and give the acceptable fit (chi-

square type test) to the exponentiated Weibull and generalized Weibull distribution families respectively.
The survival times in days for the patients ($n = 51$) were as below ($\delta = 42$). 7, 34, 42, 63, 64, 74*, 83, 84, 91, 108, 112, 129, 133, 133, 139, 140, 140, 146, 149, 154, 157, 160, 160, 165, 173, 176, 185*, 218, 225, 241, 248, 273, 277, 279*, 297, 319*, 405, 417, 420, 440, 523*, 523, 583, 594, 1101, 1116*, 1146, 1226*, 1349*, 1412*, 1417. * censoring
We use the data after transforming the survival times in months (1 month=30.438 days). The maximum likelihood estimator $\widehat{\theta}$ of the parameter vector θ is, if we suppose that this data are distributed according to the OLEE distribution :

$$\widehat{\zeta} = (\widehat{\theta}, \widehat{\lambda})^T = (1.8145, 2.0839)^T$$

We choose $r = 7$ as a number of classes. The elements of the test statistic Y_n^2 was presented as follow :

$\widehat{a_j}$	2.047	4.667	5.384	9.084	18.694	39.048	46.554
$\widehat{\mathbf{U}_j}$	3	14	6	9	10	5	4
e_j	1.92475	1.92475	1.92475	1.92475	1.92475	1.92475	1.92475
$\widehat{\mathbf{C}_{1j}}$	1.2895	-0.2540	3.1784	0.9574	-2.5488	3.1574	-0.7845
$\widehat{\mathbf{C}_{2j}}$	-0.1544	4.15578	-0.9854	-2.1441	5.1492	-6.2515	-0.9221
$\widehat{\mathbf{C}_{3j}}$	1.8541	2.1249	2.6588	-6.2298	1.3357	3.2549	0.0487

The Fisher's estimated matrix is

$$\widehat{\mathbf{I}} = \begin{pmatrix} 0.55796 & 1.62871 \\ 1.62871 & 2.81154 \end{pmatrix},$$

after calculate, we find $Y_n^2 = 14.00924$. The critical value

$$\chi^2_{0.05}(7) = 14.00924 > Y_n^2 = 13.67849,$$

we can say that this data can be well modelised by the our OLEE model.

5.3.2 Aluminum reduction cells data

The data of Whitmore (1983), who considered the times of failures for 20 aluminum reduction cells, and the numbers of failures in $1,000$ days units are : 0.468, 0.725, 0.838, 0.853, 0.965, 1.139, 1.142, 1.304, 1.317, 1.427, 1.554, 1.658, 1.764, 1.776, 1.990, 2.010, 2.224, 2.279*, 2.244*, 2.286*. (* censoring). Assuming that these data are distributed according to the OLEE distribution, the maximum likelihood estimator $\widehat{\zeta}$ of the parameter vector ζ is

$$\widehat{\zeta} = (\widehat{\theta}, \widehat{\lambda})^T = (1.6148, 0.92145, 0.84571)^T$$

We choose $r = 4$ a number of classes. The elements of the statistic test Y_n^2 are presented below:

$\widehat{a_j}$	1.11584	1.6671	2.1843	2.2936
$\widehat{\mathbf{U}_j}$	5	7	4	4
e_j	3.1046	3.1046	3.1046	3.1046
$\widehat{\mathbf{C}_{1j}}$	1.9658	−0.5147	3.5558	0.7423
$\widehat{\mathbf{C}_{2j}}$	1.2487	2.2293	−0.2849	−2.3691
$\widehat{\mathbf{C}_{3j}}$	2.0047	1.6497	−0.9782	1.1673

The Fisher's estimated matrix is given by

$$\widehat{\mathbf{I}} = \begin{pmatrix} 1.7538 & -4.00914 \\ -4.00914 & -3.6556 \end{pmatrix}$$

Then, we calculate the value of the statistic test $Y_n^2 = 9.49974$. The critical value is

$$\chi^2_{0.05}(4) = 9.48773 > Y_n^2 = 9.49974,$$

we reason that the data of Aluminum reduction cells is compatible with OLEE model.

6. Conclusions

This work aims to introduce a new extension of the exponentiated exponential model called the odd Lindley exponentiated exponential model. The proposed distribution can be viewed as a mixture of the exponentiated exponential density, the new density exhibits monotonically increasing, constant, monotonically decreasing and bathtub hazard rates. Some of its mathematical properties are derived. The new model can also be considered as a convenient model for fitting the right skewed, the symmetric, the left skewed and the unimodal data sets. The proposed lifetime model is much better than the exponential exponential, Moment exponential, Log Butr Hatke exponential and the two parameter odd Lindley exponential models, so the new lifetime model is a good alternative to these models in modeling failure times data. We used the well known modified goodness of fit statistics test Y_n^2 proposed by Bagdonavicius and Nikulin (2011a, b). This test statistisc is based on the NRR statistic. We have validated our model in two cases: complete and censored data. We have calculated all the theoritical elements of this two tests statistic (Y^2) for complete data and Y_n^2 for censored data). Simulation studies were performed in this work to demonstrate the appropriateness of the model in defferent elds of life time data, we analyses three real data of both complete and censored cases.

References

1. Bagdonavicius, V., Levuliene, R., J. and Nikulin, M., (2013). Chi-squared goodness-of-fit tests for parametric accelerated failure time models. Communications in Statistics-Theory and Methods, 42, 15, 2768-2785.

2. Bagdonavicius, V. and Nikulin, M. (2011). Chi-squared tests for general composite hypotheses from censored samples. Comptes Rendus de l'académie des Sciences de Paris, Mathématiques, 349, 3-4, 219-223.

3. Bagdonavicius, V., Kruopis, J. and Nikulin, M. (2010). Nonparametric tests for Censored Data. ISTE and J. Wiley.

4. Bagdonavicius, V. and Nikulin, M. (2011a). Chi-squared Goodness-of-fit Test for Right Censored Data. International Journal of Applied Mathematics and Statistics, 24, 30-50.

5. Bourguignon, M., Silva, R. B. and Cordeiro, G.M. (2014). The Weibull–G family of probability distributions, Journal of Data Science, 12, 53-68.

6. Bjerkedal, T. (1960). Acquisition of resistance in Guinea pigs infected with different doses of virulent tubercle bacilli. American Journal of Hygiene, 72, 130–148.

7. Goual, H. and Seddik Ameur, N. (2014). Chi-squared type test for the AFT-generalized inverse Weibull distribution. Communication in Statistics-Theory and Method, 43, 13, 2605-2617.

8. Goual, H. and Seddik-Ameur, N. (2016). A modified Chi-squared goodness-of-fit test for the kumaraswamy generalized inverse Weibull distribution and its applications. Journal of Statistics: Advances in Theory and Applications, 6, 2, 275-305.

9. Greenwoodand P.E. and M.S.Nikulin (1996). A guide to chi-squared testing. Wiley, New York.

10. Gross, J. and Clark, V. A. (1975). Survival Distributions: Reliability Applications in the Biometrical Sciences", John Wiley, New York, USA.

11. Gupta, R.C., Gupta, P.L. and Gupta, R.D. (1998). Modeling failure time data by Lehmann alternatives. Commun. Stat. Theory Methods, **27**, 887-904.

12. Gupta, R.D. and Kundu, D. (2001). Exponentiated exponential family: an alternative to gamma and Weibull distributions. Biom. J. 43, 117–130.

13. Gupta, R.D., Kundu, D. (2007). Generalized exponential distribution: existing results and some recent developments. J. Stat. Plann. Infer. 137, 3537–3547.

14. Habib, M.G., Thomas, D.R. (1986). Chi-squared Goodness-of-Fit Tests For Randomly Censored Data. Annals of Statistics, 14, 2, 759-765.

15. Korkmaz, M. C. and Yousof, H. M. (2017). The one-parameter odd Lindley exponential model: mathematical properties and applications. Economic Quality Control, DOI: 10.1515/eqc-2017-0008

16. Kundu, D. and Raqab, M. Z. (2009). Estimation of R = P (Y < X) for threeparameter Weibull distribution. Statistics and Probability Letters. Vol. **79**, 1839-1846.

17. Nadarajah, S. (2011). The exponentiated exponential distribution: a survey, AStA Adv Stat Anal, 95:219–251.

18. Nikulin, M.S. (1973a). Chi-square Test For Continuous Distributions with Shift and Scale Parameters. teor. Veroyatn. Primen., 18, 3, 559-568.

19. Nikulin, M.S. (1973b). Chi-squared test for continuous distributions with shift and scal parameters. Theory of Probability and its Applications, 18, 559-568.

20. Nikulin, M.S. (1973c). On a Chi-squared test for continuous distributions. Theory of Probability and its Aplications. 19, 638-639.

21. Nikulin, M.S., Gerville-Réache, L. and Tran, X. Q. (2013). Statistical Models and Methods for Reliability and Survival Analysis. chapter On Chi-Squared Goodness-of-Fit Test for Normality, Wiley Online Library, New York 213-227.

22. Nikulin, N. and Haghighi, F. (2006). A chi-squared test for the generalized power weibull family for the head-and-neck cancer censored data. Journal of Mathematical Sciences, 133(3), 1333–1341.

23. Rao, K. C., Robson, D. S. (1974). A Chi-square statistic for goodness-of-fit tests within the exponential family. Communication in Statistics, 3, 1139-1153.

24. Ravi, V., Gilbert, P. D. (2009). BB : An R package for solving a large system of nonlinear equations and for optimizing a high-dimensional nonlinear objective function. J. Statist. Software, 32, 4, 1-26.

25. Voinov, V., Nikulin, M., Balakrishnan, N. (2013). Chi-Squared Goodness of Fit Tests with Applications. Academic Press, Elsevier.

Appendix A

The r^{th} ordinary moment of X is given by $\mu'_r = \int_0^\infty x^r f(x)\,dx = \mathbf{E}(X^r)$. Using (7), we get

$$\mu'_r = \theta\lambda^{-r}\Gamma(1+r)\sum_{i,k,w=0}^{\infty}\xi_{i,k}K_w^{([\theta(2+i+k)],r)}|_{(r>-1)}, \tag{A1}$$

where

$$K_w^{([\theta(2+i+k)],r)} = \frac{\tau([\theta(2+i+k)],w)}{(1+w)^{1+r}},$$

$$\tau([\theta(2+i+k)],w) = \frac{(-1)^w\Gamma([\theta(2+i+k)])}{\Gamma([\theta(2+i+k)]-w)},$$

$$\Gamma\left(1+c\right)|_{\left(c\in\mathbb{R}^{+}\right)}=\prod_{m=0}^{c-1}\left(c-m\right)=c\left(c-1\right)\left(c-2\right)...1=c!,$$

and

$$\int_{0}^{\infty}t^{a-1}\,\mathrm{e}^{-t}dt=\Gamma\left(a\right),$$

is the complete gamma function. The r^{th} incomplete moment of X, say $I_r\left(t\right)$, is given by $I_r\left(t\right)=\int_0^t x^r\,f\left(x\right)dx$. Using (7), we obtain

$$I_r\left(t\right)=\theta\lambda^{-r}\left[\gamma\left(1+r,\frac{\lambda}{t}\right)\right]\sum_{i,k,w=0}^{\infty}\xi_{i,k}K_w^{\left(\left[\theta\left(2+i+k\right)\right],r\right)}|_{\left(r>-1\right)},$$

where $\gamma\left(\zeta,q\right)$ is the incomplete gamma function.

$$\begin{aligned}\gamma\left(c,q\right)|_{\left(c\neq0,-1,-2,\ldots\right)} &= \int_{0}^{q}t^{c-1}\exp\left(-t\right)dt\\ &= \frac{q^c}{c}\left\{{}_{(1)}\mathbf{F}_{(1)}\left[c;c+1;-q\right]\right\}\\ &= \sum_{k=0}^{\infty}\frac{\left(-1\right)^k}{k!\left(c+k\right)}q^{c+k},\end{aligned}$$

where ${}_{(1)}\mathbf{F}_{(1)}\left[\cdot,\cdot,\cdot\right]$ is a confluent hypergeometric function.

Let $X_1,\ldots,X_n$ be a random sample (R.S.) from the OLEE model of distributions and let $X_{1:1},\ldots,X_{n:n}$ be the corresponding order statistics, so the P.D.F. of the i^{th} order statistic, say $X_{i:n}$, can be expressed as

$$f_{i:n}\left(x\right)=\mathrm{B}^{-1}(i,n-i+1)\sum_{j=0}^{n-i}(-1)^j\,f(x)F(x)^{j+i-1}\binom{n-i}{j},\qquad\text{(A2)}$$

where $\mathrm{B}(\cdot,\cdot)$ is the beta function. Substituting (5) and (6) in (A2), we obtain

$$f_{i:n}\left(x\right)=\sum_{m,p=0}^{\infty}\sum_{j=0}^{k+n-i}c_{i,m,p}\;\mathbf{g}_{\{[\theta(2+j+m+p)],\lambda\}}(x)|\left\{\begin{smallmatrix}(x\geq0)\\(\lambda>0,\;[\theta(2+j+m+p)]>0)\end{smallmatrix}\right\},$$

where

$$c_{i,m,p}=\sum_{k=0}^{i-1}\frac{\left(-1\right)^{k+m}}{m!\left(2+j+m+p\right)\mathrm{B}(i,n-i+1)}\binom{j+m+p}{j+m}\binom{k+n-1}{j}\binom{i-1}{k},$$

then, the z^{th} moment of $X_{i:n}$ is given by

$$\mathbf{E}\left(X_{i:n}^{z}\right)=\theta\lambda^{-z}\Gamma\left(1+z\right)\sum_{m,p,w=0}^{\infty}\sum_{j=0}^{k+n-i}c_{i,m,p}K_{w}^{([\theta(2+j+m+p)],z)}|_{(z>-1)}\ . \qquad \text{(A3)}$$

The quantile spread (QSO) of the R.V. W ∼OLEE(θ,λ) having C.D.F. (6) is given by

$$[QSO]_{X}\left(\tau\right)|_{(\tau\in(0.5,1))}=\left[F^{-1}(\tau)\right]-\left[F^{-1}(1-\tau)\right],$$

which implies

$$\left[S^{-1}(1-\tau)\right]-\left[S^{-1}(\tau)\right]=[QSO]_{W}\left(\tau\right),$$

where

$$S(w)=1-F(w)\text{ and }F^{-1}(\tau)=S^{-1}(1-\tau)$$

is the survival function (SF). The QSO of a distribution describes how the probability mass is placed symmetrically about its median and hence can be used to formalize concepts like peakedness and tail weight traditionally associated with kurtosis. So that, it allows us to separate concepts of kurtosis and peakedness for asymmetric models. Let W_1 and W_2 be two R.V.s following the OLEE model with $[QSO]_{W_1}$ and $[QSO]_{W_2}$, respectively. Then W_1 is called smaller than W_2 in QSO, denoted as $W_1 \leq_{[QSO]} W_2$, if

$$\left([QSO]_{W_1}\left(\tau\right)\leq[QSO]_{W_2}\left(\tau\right)\right)|_{(\ \tau\in(0.5,1))}.$$

Following properties of the QSO order can be obtained:

1−The order $\left(\leq_{[QSO]}\right)$ is location-free

$$\left[W_1\leq_{[QSO]}W_2\text{ if }\left(W_1+\mathbf{c}\right)\leq_{[QSO]}W_2\right]\ |_{(\mathbf{c}\geq1)}.$$

2−The order $\left(\leq_{[QSO]}\right)$ is dilative

$$\left[W_1\leq_{[QSO]}\mathbf{c}W_1|_{(\mathbf{c}\geq1)}\text{ and }W_2\leq_{[QSO]}\mathbf{c}W_2\right]|_{(\mathbf{c}\geq1)}.$$

3− Let F_{W_1} and F_{W_2} be symmetric, then

$$\left[W_1\leq_{[QSO]}W_2\ \text{ if, and only if }F_{W_1}^{-1}\left(\tau\right)\leq F_{W_2}^{-1}\left(\tau\right)\right]|_{(\tau\in(0.5,1))}.$$

4− The order$\left(\leq_{[QSO]}\right)$ implies ordering of the mean absolute deviation around the median, say $\boldsymbol{\Upsilon}(W_i)|_{(i=1,2)}$,

$$\mathbf{E}\left[\left|-Median(W_1)+W_1\right|\right]=\boldsymbol{\Upsilon}(W_1),$$

and

$$\mathbf{E}\left[\left|-Median(W_2)+W_2\right|\right]=\boldsymbol{\Upsilon}(W_2),$$

where

$$W_1\leq_{[QSO]}W_2\Rightarrow\ \boldsymbol{\Upsilon}(W_1)\leq_{[QSO]}\boldsymbol{\Upsilon}(W_2),$$

finally

$$W_1\leq_{[QSO]}W_2\text{ if, and only if }\ -W_1\leq_{[QSO]}-W_2.$$

The n^{th} Moment of residual life (MRL) is given by

$$z_n(t) = \mathbf{E}\left\{(X-t)^n \mid \left[\begin{smallmatrix}(n=1,2,...)\\(X>t),\ (t>0)\end{smallmatrix}\right]\right\}.$$

So the n^{th} MRL of X can be given as

$$z_n(t) = \frac{\int_t^{\infty}(x-t)^n dF(x)}{1-F(t)},$$

subsequentlywe can write

$$\begin{aligned} z_n(t) &= [1-F(t)]^{-1}\sum_{i,k=0}^{\infty}\sum_{r=0}^{n}(-t)^{n-r}\ \xi_{i,k}\binom{n}{r} \\ &\times\int_t^{\infty} x^r\ \mathbf{g}_{\{[\theta(2+i+k)],\lambda\}}(x)|\left\{\begin{smallmatrix}(x\geq 0)\\(\lambda>0,\ [\theta(2+i+k)]>0)\end{smallmatrix}\right\} dx \\ &= \theta\lambda^{-n}\frac{\Gamma\left(1+n,\frac{\lambda}{t}\right)}{1-F(t)}\sum_{i,k,w=0}^{\infty}\sum_{r=0}^{n}(-t)^{n-r}\ \binom{n}{r}\xi_{i,k}K_w^{([\theta(2+i+k)],n)}\,|_{(n>-1)}, \end{aligned}$$

where

$$\Gamma\left(a,q\right)|_{(q>0)} = \int_q^{\infty} t^{a-1}\mathrm{e}^{-t}dt,$$

and

$$\Gamma\left(a,q\right)+\gamma\left(a,q\right) = \Gamma\left(a\right).$$

The n^{th} Moment of reversed residual life (MRRL) is given by

$$Z_n(t) = \mathbf{E}\left\{(t-X)^n \mid \left[\begin{smallmatrix}(n=1,2,...)\\(X\leq t),\ t>0\end{smallmatrix}\right]\right\},$$

uniquely determines $F(x)$, then we have

$$Z_n(t) = \frac{\int_0^{t}(t-x)^n dF(x)}{F(t)}.$$

Then, the n^{th} moment of the reversed residual life of X becomes

$$\begin{aligned} Z_n(t) &= F(t)^{-1}\sum_{i,k=0}^{\infty}\sum_{r=0}^{n}(-1)^r\, t^{n-r}\binom{n}{r}\xi_{i,k} \\ &\times\int_0^{t} x^r\ \mathbf{g}_{\{[\theta(2+i+k)],\lambda\}}(x)|\left\{\begin{smallmatrix}(x\geq 0)\\(\lambda>0,\ [\theta(2+i+k)]>0)\end{smallmatrix}\right\} dx \\ &= \theta\lambda^{-n}\frac{\gamma\left(1+r,\frac{\lambda}{t}\right)}{F(t)}\sum_{i,k,w=0}^{\infty}\sum_{r=0}^{n}(-1)^r\, t^{n-r}\binom{n}{r}\xi_{i,k}K_w^{([\theta(2+i+k)],n)}\,|_{(n>-1)}. \end{aligned}$$

Appendix B

The components of $\widehat{\mathbf{W}}$ and the information matrix $\widehat{\mathbf{I}}$ are required for the statistic test Y_n^2.

Calculation of the matrix $\widehat{\mathbf{W}}$

The elements of the estimated matrix $\widehat{\mathbf{W}}$ defined by

$$\widehat{\mathbf{W}}_l = \sum_{j=1}^{k} \widehat{\mathbf{C}}_{lj}\ \widehat{\mathbf{A}}_j^{-1}\mathbf{Z}_j|_{(\ l=1,2,3 \text{ and } j=1,\ldots,k)},$$

are obtained as follow

$$\widehat{\mathbf{C}_{lj}} = n^{-1} \sum_{i.t_i \in \mathbf{I}_j} \delta_i \frac{\partial}{\partial \theta} \ln \lambda(t_i, \widehat{\theta})$$

$$\begin{aligned}
\ln \lambda(t, \widehat{\theta}) &= \ln \left(\frac{\theta \lambda \mathrm{e}^{-\lambda t} \left(1 - \mathrm{e}^{-\lambda t}\right)^{\theta - 1}}{2\left[1 - (1 - \mathrm{e}^{-\lambda t})^{\theta}\right]^2 \left\{1 + \left[1 - (1 - \mathrm{e}^{-\lambda t})^{\theta}\right]\right\}} \right) \\
&= \ln \theta + \ln \lambda - \ln 2 - \lambda t + (\theta - 1) \ln \left(1 - \mathrm{e}^{-\lambda t}\right)^{\theta - 1} \\
&\quad -2 \ln \left[1 - \left(1 - \mathrm{e}^{-\lambda t}\right)^{\theta}\right] - \ln \left\{1 + \left[1 - \left(1 - \mathrm{e}^{-\lambda t}\right)^{\theta}\right]\right\}
\end{aligned}$$

The expressions of the elements of the matrix $\widehat{\mathbf{C}_{lj}}$ are given as follows

$$\widehat{\mathbf{C}_{1j}} = n^{-1} \sum_{i.t_i \in \mathbf{I}_j} \delta_i \left[\begin{array}{c} \frac{\left(1-e^{-\lambda t_i}\right)^{\theta} \ln\left(1-e^{-\lambda t_i}\right)}{2-\left(1-e^{-\lambda t_i}\right)^{\theta}} \\ +(\theta - 1) \ln \left(\ln \left(1 - e^{-\lambda t_i}\right)\right) \ln^{\theta - 1} \left(1 - e^{-\lambda t_i}\right) \\ + \ln^{\theta - 1} \left(1 - e^{-\lambda t_i}\right) + \frac{2\left(1-e^{-\lambda t_i}\right)^{\theta} \ln\left(1-e^{-\lambda t_i}\right)}{1-\left(1-e^{-\lambda t_i}\right)^{\theta}} \end{array} \right],$$

$$\widehat{\mathbf{C}_{2j}} = n^{-1} \sum_{i.t_i \in \mathbf{I}_j} \delta_i \left[\begin{array}{c} \frac{\theta t_i e^{-\lambda t_i}\left(1-e^{-\lambda t_i}\right)^{\theta-1}}{2-\left(1-e^{-\lambda t_i}\right)^{\theta}} \\ + \frac{2\theta t_i e^{-\lambda t_i}\left(1-e^{-\lambda t_i}\right)^{\theta-1}}{1-\left(1-e^{-\lambda t_i}\right)^{\theta}} + \frac{(\theta-1)^2 t_i e^{-\lambda t_i} \log^{\theta-2}\left(1-e^{-\lambda t_i}\right)}{1-e^{-\lambda t_i}} - t_i \end{array} \right].$$

Calculation of the matrix $\widehat{\mathbf{I}}$

The formulas of the elements of the Fisher's information matrix $\widehat{\mathbf{I}} = (\widehat{i}_{ll'})_{3\times 3}$ is

$$\widehat{i_{ll'}} = n^{-1} \sum_{i.t_i \in \mathbf{I}_j} \delta_i \frac{\partial \ln \lambda(t_i, \widehat{\theta})}{\partial \theta_l} \frac{\partial \ln \lambda(t_i, \widehat{\theta})}{\partial \theta_{l'}}.$$

In our case we have:

$$\widehat{i_{11}} = n^{-1} \sum_{i.t_i \in \mathbf{I}_j} \delta_i \begin{bmatrix} \frac{\left(1-e^{-\lambda t_i}\right)^{\theta} \ln\left(1-e^{-\lambda t_i}\right)}{2-\left(1-e^{-\lambda t_i}\right)^{\theta}} \\ +(\theta-1)\ln\left(\ln\left(1-e^{-\lambda t_i}\right)\right)\ln^{\theta-1}\left(1-e^{-\lambda t_i}\right) \\ +\ln^{\theta-1}\left(1-e^{-\lambda t_i}\right) + \frac{2\left(1-e^{-\lambda t_i}\right)^{\theta}\ln\left(1-e^{-\lambda t_i}\right)}{1-\left(1-e^{-\lambda t_i}\right)^{\theta}} \end{bmatrix}^2,$$

$$\widehat{i_{12}} = n^{-1} \sum_{i.t_i \in \mathbf{I}_j} \delta_{ii} \begin{pmatrix} \frac{\left(1-e^{-\lambda t_i}\right)^{\theta} \ln\left(1-e^{-\lambda t_i}\right)}{2-\left(1-e^{-\lambda t_i}\right)^{\theta}} \\ +(\theta-1)\ln\left(\ln\left(1-e^{-\lambda t_i}\right)\right)\ln^{\theta-1}\left(1-e^{-\lambda t_i}\right) \\ +\ln^{\theta-1}\left(1-e^{-\lambda t_i}\right) + \frac{2\left(1-e^{-\lambda t_i}\right)^{\theta}\ln\left(1-e^{-\lambda t_i}\right)}{1-\left(1-e^{-\lambda t_i}\right)^{\theta}} \\ \times \begin{Bmatrix} \frac{\theta t_i e^{-\lambda t_i}\left(1-e^{-\lambda t_i}\right)^{\theta-1}}{2-\left(1-e^{-\lambda t_i}\right)^{\theta}} \\ +\frac{2\theta t_i e^{-\lambda t_i}\left(1-e^{-\lambda t_i}\right)^{\theta-1}}{1-\left(1-e^{-\lambda t_i}\right)^{\theta}} + \frac{(\theta-1)^2 t_i e^{-\lambda t_i}\log^{\theta-2}\left(1-e^{-\lambda t_i}\right)}{1-e^{-\lambda t_i}} - t_i \end{Bmatrix} \end{pmatrix},$$

and

$$\widehat{i_{22}} = n^{-1} \sum_{i.t_i \in \mathbf{I}_j} \delta_i \begin{bmatrix} \frac{\theta t_i e^{-\lambda t_i}\left(1-e^{-\lambda t_i}\right)^{\theta-1}}{2-\left(1-e^{-\lambda t_i}\right)^{\theta}} \\ +\frac{2\theta t_i e^{-\lambda t_i}\left(1-e^{-\lambda t_i}\right)^{\theta-1}}{1-\left(1-e^{-\lambda t_i}\right)^{\theta}} + \frac{(\theta-1)^2 t_i e^{-\lambda t_i}\log^{\theta-2}\left(1-e^{-\lambda t_i}\right)}{1-e^{-\lambda t_i}} - t_i \end{bmatrix}^2.$$