Extended Poisson Fréchet Distribution and its Applications

Mohamed G. Khalil
Department of Statistics, Mathematics and Insurance,
Benha University, Benha, Egypt.
Email: hndaoy@gmail.com

Rezk H.
Mathematical Sciences Department, College of Science,
Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.
Department of Statistics, Al-Azhar University, Cairo, Egypt.
Email: hrmohamed@pnu.edu.sa

Abstract

A new version of the Fréchet model is investigated and studied. Some of its properties are mathematically derived. The well-known maximum likelihood classical method is used in estimating the unknown parameters. The new version is much better than other important competitive Fréchet versions in modeling reliability real data.

Keywords: Zero Truncated Poisson Distribution; Fréchet Distribution; Maximum Likelihood

1. Introduction and physical motivation

A random variable (R-V) U is said to have the Fréchet (Fr) model if its probability density function (P-D-F) and survival function (S-F) are given by (for $u \geq 0$)

$$g_{Fr}^{(\alpha,\beta)}(u) = u^{-(\beta+1)}\alpha^{\beta}\beta\, exp\left[-u^{-\beta}\alpha^{\beta}\right]$$

and

$$1 - G_{Fr}^{(\alpha,\beta)}(u) = -exp\left[-u^{-\beta}\alpha^{\beta}\right] + 1,$$

respectively, where $\alpha, \beta > 0$. On the other hand, the probability mass function (P-M-F) of T where T has a zero truncated Poisson (Z-T-P) model is given as

$$PMF_{Z.T.P.}^{(\lambda)}(T = t)|_{(t=1,2,\ldots)} = \frac{exp(-\lambda)\lambda^{t}}{t!\mathrm{C}_{(\lambda)}}, \quad (1)$$

Where

$$\mathrm{C}_{(\lambda)} = -\,exp(-\lambda) + 1.$$

Consider the Burr-X-Fr version defined by the cumulative distribution function (C-D-F) and P-D-F given by

$$\mathbf{H}_{BXFr}^{(\theta,\alpha,\beta)}(z) = \left[-\,exp\left(-\left\{\frac{exp\left[-z^{-\beta}\alpha^{\beta}\right]}{1-exp\left[-z^{-\beta}\alpha^{\beta}\right]}\right\}^{2}\right) + 1\right]^{\theta}. \quad (2)$$

and

$$\mathbf{h}_{BXFr}^{(\theta,\alpha,\beta)}(z) = 2\theta\beta\alpha^{\beta}z^{-(\beta+1)}\left\{1 - exp\left[-z^{-\beta}\alpha^{\beta}\right]\right\}^{-3}$$

$$\times exp\left[-2z^{-\beta}\alpha^{\beta}\right] exp\left(-\left\{\frac{exp\left[-z^{-\beta}\alpha^{\beta}\right]}{1-exp\left[-z^{-\beta}\alpha^{\beta}\right]}\right\}^{2}\right)$$
$$\times\left[-exp\left(-\left\{\frac{exp\left[-z^{-\beta}\alpha^{\beta}\right]}{1-exp\left[-z^{-\beta}\alpha^{\beta}\right]}\right\}^{2}\right)+1\right]^{\theta-1}, \quad (3)$$

respectively. Let

$$min\{Y_1, Y_2, \cdots, Y_T\} = Z .$$

Then, the conditional C-D-F of $X \mid T$ is

$$F(z \mid T) = 1 - Pr(Z > t \mid T) = 1 - \left[1 - \mathbf{H}_{BXFr}^{(\theta,\alpha,\beta)}(z)\right]^{T}. \quad (4)$$

Therefore, the unconditional C-D-F of the Poisson Burr-X-Fr (P-BX-Fr) version, can be expressed as

$$F_{PBXFr}^{(\lambda,\theta,\alpha,\beta)}(z)|_{(\lambda\in R)} = \frac{1-exp\left\{-\lambda\left[-exp\left(-\left\{\frac{exp\left[-z^{-\beta}\alpha^{\beta}\right]}{1-exp\left[-z^{-\beta}\alpha^{\beta}\right]}\right\}^{2}\right)+1\right]^{\theta}\right\}}{C_{(\lambda)}}, \quad (5)$$

and its corresponding P-D-F is

$$f_{PBrXFr}^{(\lambda,\theta,\alpha,\beta)}(z)|_{(\lambda\in R)} = \frac{2\theta\lambda\beta\alpha^{\beta}}{C_{(\lambda)}} z^{-(\beta+1)}\left\{1-exp\left[-z^{-\beta}\alpha^{\beta}\right]\right\}^{-3}$$
$$\times exp\left(-2z^{-\beta}\alpha^{\beta} - \left\{\frac{exp\left[-z^{-\beta}\alpha^{\beta}\right]}{1-exp\left[-z^{-\beta}\alpha^{\beta}\right]}\right\}^{2}\right)$$
$$\times\left[1-exp\left(-\left\{\frac{exp\left[-z^{-\beta}\alpha^{\beta}\right]}{1-exp\left[-(z^{-1}\alpha)^{\beta}\right]}\right\}^{2}\right)\right]^{\theta-1}$$
$$\times exp\left\{-\lambda\left[-exp\left(-\left\{\frac{exp\left[-z^{-\beta}\alpha^{\beta}\right]}{1-exp\left[-z^{-\beta}\alpha^{\beta}\right]}\right\}^{2}\right)+1\right]^{\theta}\right\}. \quad (6)$$

The hazard (failure) rate function (H-R-F) of the P-BX-Fr model can be calculated via $f_{PBrXFr}^{(\lambda,\theta,\alpha,\beta)}(z)/\left[1-F_{PBXFr}^{(\lambda,\theta,\alpha,\beta)}(z)\right]$. The P-D-F of the new Fr version can be right skewed and unimodal (see Figure 1) also it can be left skewed (see Table 1). The H-R-F of the new model can be bathtub (**U**), unimodal-bathtub (unimodal-**U**), increasing and decreasing. All Tables and Figures are listed in the Appendix. Some important Fr models are developed by Nadarajah and Kotz(2003), Barreto-Souza, et al. (2011), Krishna, et.al., (2013), Mead, et al., (2014), Yousof, et el., (2015), Mahmod and Mandoh (2013), Afify, et al., (2016-a and 2016-b), Yousof, et al., (2016), Korkmaz, et al., (2017), Yousof et al., (2018-a and 2018-b), Brito, et al., (2017), Hamedani, et al., (2017), Cordeiro, et al., (2018), Hamedani, et al., (2018), Korkmaz, et al., (2018), Chakraborty, et al., (2018), Korkmaz, et al., (2019), Hamedani, et al., (2019), among others.

2. Mathematical properties

2.1 Useful expansions

Using

$$exp(\boldsymbol{\varphi}) = \sum_{c=0}^{\infty} \varphi^{c}\frac{1}{c!},$$

the P-D-F in (6) can be written as

$$f_{PBXFr}^{(\lambda,\theta,\alpha,\beta)}(z) = \sum_{h=0}^{\infty} 2\theta\lambda^{1+h}\beta\alpha^{\beta}\frac{(-1)^h}{h!\,\mathrm{C}_{(\lambda)}}z^{-(\beta+1)}$$
$$\times\, exp\left[-\left(\frac{exp[-z^{-\beta}\alpha^{\beta}]}{1-exp[-z^{-\beta}\alpha^{\beta}]}\right)^2\right]$$
$$\times \frac{exp[-2z^{-\beta}\alpha^{\beta}]}{\{1-exp[-z^{-\beta}\alpha^{\beta}]\}^3}\left\{1-exp\left[-\left(\frac{exp[-z^{-\beta}\alpha^{\beta}]}{1-exp[-z^{-\beta}\alpha^{\beta}]}\right)^2\right]\right\}^{\theta(h+1)-1}. \qquad (7)$$

If $|\boldsymbol{\varphi}| < 1$ and $\mathrm{p} > 0$ where p is any real-non-integer, then

$$(1-\boldsymbol{\varphi})^{\mathrm{p}} = \sum_{d=0}^{\infty}\frac{(-1)^d\Gamma(1+\mathrm{p})}{d!\,\boldsymbol{\Gamma}(1+\mathrm{p}-d)}\boldsymbol{\varphi}^d. \qquad (8)$$

Applying (8) to (7) we get

$$f_{PBXFr}^{(\lambda,\theta,\alpha,\beta)}(z) = 2\theta\beta\alpha^{\beta}\frac{z^{-(\beta+1)}\,exp[-2z^{-\beta}\alpha^{\beta}]}{\mathrm{C}_{(\lambda)}}$$
$$\times\sum_{h,s=0}^{\infty}\lambda^{1+h}\frac{\boldsymbol{\Gamma}(\theta(1+h))(-1)^{h+s}}{\boldsymbol{\Gamma}(-s+(h+1)\theta)\boldsymbol{\Gamma}(1+\mathrm{s})}$$
$$\times\frac{exp\left[-(s+1)\left(\frac{exp[-z^{-\beta}\alpha^{\beta}]}{1-exp[-z^{-\beta}\alpha^{\beta}]}\right)^2\right]}{\{1-exp[-z^{-\beta}\alpha^{\beta}]\}^3}. \qquad (9)$$

Applying the series $exp(\boldsymbol{\varphi})$ to the term

$$exp\left[-(1+s)\left(\frac{exp[-z^{-\beta}\alpha^{\beta}]}{1-exp[-z^{-\beta}\alpha^{\beta}]}\right)^2\right],$$

Equation (9) becomes

$$f_{PBXFr}^{(\lambda,\theta,\alpha,\beta)}(z) = \beta\alpha^{\beta}(1+i)^j z^{-(\beta+1)}\,exp[-x^{-\beta}\alpha^{\beta}]$$
$$\times\sum_{h,s,l=0}^{\infty}2\theta\lambda^{1+h}\frac{\boldsymbol{\Gamma}(\theta(h+1))}{\boldsymbol{\Gamma}(1+\mathrm{s})\mathbf{l}!\,(-1)^{-(s+h+l)}\mathrm{C}_{(\lambda)}\boldsymbol{\Gamma}(\theta(h+1)-s)}$$
$$\times\frac{\{exp[-z^{-\beta}\alpha^{\beta}]\}^{2l+1}}{\{1-exp[-z^{-\beta}\alpha^{\beta}]\}^{2l+3}}. \qquad (10)$$

Consider

$$(1-\boldsymbol{\varphi})^{-V}\,|_{(|\boldsymbol{\varphi}|<1\ and\ V>0)} = \sum_{d=0}^{\infty}\frac{\Gamma(V+d)}{d!\,\Gamma(V)}\boldsymbol{\varphi}^d. \qquad (11)$$

Applying (11) to (10) for the term $\{-exp[-z^{-\beta}\alpha^{\beta}]+1\}^{2l+3}$, Equation (10) becomes

$$f_{PBXFr}^{(\lambda,\theta,\alpha,\beta)}(x) = \sum_{l,b=0}^{\infty}\upsilon_{j,b}\mathbf{h}_{[b+2+2l]}(z\,;\alpha\,,\beta), \qquad (12)$$

where

$$\upsilon_{l,k} = 2\theta\lambda^{1+h}(-1)^l\ \frac{\boldsymbol{\Gamma}(3+2l+b)}{l!\,k!\,\mathrm{C}_{(\lambda)}\boldsymbol{\Gamma}(2l+3)[b+2+2l]}$$

$$\sum_{h,s=0}^{\infty}\frac{(-1)^{h+i}\boldsymbol{\Gamma}(\theta h+\theta)}{\boldsymbol{\Gamma}(1+s)\boldsymbol{\Gamma}(-s+\theta h+\theta)(1+s)^{-l}},$$

and the function $h_{[b+2+2j]}(z,\beta)$ is the Fr P-D-F with $[b+2+2j]^{\beta^{-1}}\alpha$ as scale parameter. The C-D-F of the P-BX-Fr can also be re-expressed as a mixture of Fr C-D-Fs given by

$$F_{PBXFr}^{(\lambda,\theta,\alpha,\beta)}(z)=\sum_{l,b=0}^{\infty}\upsilon_{j,k}\ \mathbf{H}_{[b+2+2l]}(z\,;\alpha\,,\beta), \qquad (13)$$

where function $\mathbf{H}_{[2(1+j)+b]}(x\,;\alpha\,,\beta)$ is the Fr C-D-F.

2.2 Quantile and random number generation

The Q-F of a R-V Z, where $Z\sim \mathrm{P-BX-Fr}(\lambda,\theta,\alpha,\beta)$, is obtained by inverting (5) as

$$Q(u)=\alpha\left\{-ln\left[\left(1+\left\{-ln\left[1-\left(\frac{-ln\{1-u\mathrm{C}_{(\lambda)}\}}{\lambda}\right)^{\frac{1}{\theta}}\right]\right\}^{\frac{1}{2}}\right)\right]\right\}^{-\frac{1}{\beta}}.$$

2.3 Moments

The $\mathbf{E}(Z^{\mathbf{r}})$ or $\mu'_{\mathbf{r}}$, comes from (12) as

$$\mathbf{E}(Z^{\mathbf{r}})=\sum_{j,k=0}^{\infty}\upsilon_{j,k}\ \alpha^{\mathbf{r}}[k+2+2j]^{\mathbf{r}\beta^{-1}}\boldsymbol{\Gamma}(1-\mathbf{r}\beta^{-1})=\mu'_{\mathbf{r}}|_{(\mathbf{r}<\beta)}, \qquad (14)$$

The mean of Z ($\mathbf{E}(Z)$) will be

$$\mathbf{E}(Z)=\sum_{j,k=0}^{\infty}\upsilon_{j,k}\ \alpha[k+2+2j]^{\beta^{-1}}\boldsymbol{\Gamma}(1-\beta^{-1})=\mu'_{\mathbf{1}}|_{(\mathbf{1}<\beta)},$$

where

$$\boldsymbol{\Gamma}(\boldsymbol{\varphi}+1)|_{(\boldsymbol{\varphi}\in R^{+})}=\prod_{w=0}^{\boldsymbol{\varphi}-1}(\boldsymbol{\varphi}-w),$$

and

$$\int_{0}^{\infty} z^{\boldsymbol{\varphi}-1}\,exp(-z)\,dz=\boldsymbol{\Gamma}(\boldsymbol{\varphi})$$

Skewness of the P-BX-Fr can range in $(-0.067, 1.339)$, whereas the kurtosis of the P-BX-Fr varies in $(2.916, 6.004)$ also the $\mathbf{E}(Z)$ increases as λ increases (see Table 1).

2.4 Incomplete moments (IM)

The $\mathbf{r}$ th IM of Z is defined as $\mathbf{m}_{\mathbf{r}}(y)=\int_{-\infty}^{y} z^{\mathbf{r}}f(z)dz$. We can write from (12)

$$\mathbf{m}_{\mathbf{r}}(y)|_{(\mathbf{r}<\beta)}=\sum_{l,k=0}^{\infty}\upsilon_{l,k}\ \alpha^{\mathbf{r}}[k+2+2j]^{\mathbf{r}\beta^{-1}}\ \boldsymbol{\Gamma}\left(1-\mathbf{r}\beta^{-1},\left(\frac{\alpha}{y}\right)^{\beta}\right). \qquad (15)$$

Setting $\mathbf{r}=1$ in (15) gives the 1st incomplete moment of X as

$$\mathbf{m}_{1}(y)|_{(1<\beta)}=\sum_{l,k=0}^{\infty}\upsilon_{l,k}\ \alpha[k+2+2l]^{\beta^{-1}}\ \boldsymbol{\Gamma}\left(1-\beta^{-1},\left(\frac{\alpha}{y}\right)^{\beta}\right),$$

where $\gamma(\boldsymbol{\varphi},q)$ is the incomplete gamma function, where

$$\gamma(\boldsymbol{\varphi}, q)|_{(\boldsymbol{\varphi}\neq 0,-1,-2,\ldots)} = \int_0^q \quad z^{\boldsymbol{\varphi}-1}\, exp(-z)\, dz$$

$$= q^{\boldsymbol{\varphi}}\{1\mathrm{F}_1[\boldsymbol{\varphi}\,;\boldsymbol{\varphi}\, a + 1\,; -q]\}\frac{1}{\boldsymbol{\varphi}}$$

$$= \sum_{\mathbf{k}=0}^{\infty} q^{\boldsymbol{\varphi}+\mathbf{k}} \frac{(-1)^{\mathbf{k}}}{k!\,(\varphi + \mathbf{k})},$$

the function $1\mathrm{F}_1[\cdot,\cdot,\cdot]$ is a called the confluent hypergeometric function,

$$\boldsymbol{\Gamma}(\boldsymbol{\varphi}, q)|_{(x>0)} = \int_q^\infty \quad z^{\boldsymbol{\varphi}-1}\exp{(-z)}dz,$$

and

$$\boldsymbol{\Gamma}(\boldsymbol{\varphi}, q) \; + \; \gamma(\boldsymbol{\varphi}, q) = \boldsymbol{\Gamma}(\boldsymbol{\varphi}).$$

2.5 Generating function (M-G-F)

The M-G-F, $\boldsymbol{M}(t) = \mathbf{E}(exp(tZ))$, is derived from (12) as

$$\boldsymbol{M}(t)|_{(r<\beta)} = \sum_{l,k,\mathbf{r}=0}^{\infty} \quad v_{l,k} \; \left(\frac{t^{\mathbf{r}}}{\mathbf{r}!}\right)\alpha^{\mathbf{r}}[k + 2 + 2l]^{\mathbf{r}\beta^{-1}}\, \boldsymbol{\Gamma}(1 - \mathbf{r}\beta^{-1}),$$

Let the generalized Wright hypergeometric function ($p\mathbf{W}_q$) defined as

$$p\mathbf{W}_q\begin{bmatrix}\alpha_1, A_1, & \ldots, & \alpha_p, A_p \\ \beta_1, B_1, & \ldots, & \beta_q, B_q\end{bmatrix}; \quad x \bigg] = \sum_{n=0}^{\infty} \frac{\prod_{l=1}^{p} \quad \boldsymbol{\Gamma}(\alpha_l + A_l n)}{\prod_{l=1}^{q} \quad \boldsymbol{\Gamma}(\beta_l + B_l n)}\frac{x^n}{n!}$$

Then, we can write $M(t\,;\alpha\,,\beta)$ as

$$\boldsymbol{M}(t\,;\alpha\,,\beta) = \; {}_1\mathbf{W}_0\left[\begin{matrix}(1,-\beta^{-1}) \\ -\end{matrix}; \alpha\, t\right]. \tag{16}$$

Combining (12) and (16), we obtain the M-G-F of Z , say $\boldsymbol{M}(t)$, as

$$\boldsymbol{M}(t) = \sum_{l,k=0}^{\infty} \quad v_{l,k} \quad {}_1\mathbf{W}_0\left[\begin{matrix}(1,-\beta^{-1}) \\ -\end{matrix}; \alpha\,[k + 2 + 2l]^{\beta^{-1}} t\right].$$

3. Estimation

Consider any random sample (R-S) of size m from the P-BX-Fr, then the log likelihood ($log\, L$) function can be expressed as

$$log\, L = m + m\, log\, \theta + m\, log\, \lambda + m\, log\, \beta + m\beta\, log\, \alpha$$

$$-m\, log\, \mathrm{C}_{(\lambda)} - (\beta + 1)\sum_{s=1}^{m} \quad log z_s - 3\, log(-\xi_s + 1)$$

$$+2\sum_{s=1}^{n} \quad log\xi_s - \lambda\sum_{s=1}^{m} \quad [-\, exp(-\tau_s) + 1]^{\theta} - \sum_{s=1}^{m} \quad \tau_s$$

$$+ (\theta - 1)\sum_{s=1}^{m} \quad log[-\, exp(-\tau_s) + 1]$$

where

$$\xi_s = exp\left[-(\alpha {z_s}^{-1})^{\beta}\right]$$

$$\tau_s = \left(\frac{\xi_s}{-\xi_s + 1}\right)^2.$$

The maximum likelihood method and its procedures are available with details.

4. Real data modeling

Consider the following competitive models

Competitive models	Author
Kumaraswamy-Fr (Kum Fr)	(Mead, and Abd - Eltawab, (2014))
exponentiated-Fréchet (E Fr)	(Nadarajah, and Kotz, (2003))
Weibull-Fr	(Afify, et al. (2016-b))
Mershall-Olkin-Fr (MO Fr)	(Krishna, et al., (2013))
Beta-Fr (BFr)	(Barreto-Souza, et al., (2011))
Gamma-extended-Fr (G E Fr)	(Silva, et. al., (2013))
Transmuted-Fr (T Fr)	(Mahmoud and Mandouh, (2013))

Data	n	Called	Author
$1st$	100	Stress data	Nichols & Padgett (2006)
$2nd$	63	Glass fibers data	Smith & Naylor (1987)

For comparing models, we will consider: AI_C (Akaike Information Criterion), CAI_C (Consistent AIC), BI_C (Bayesian IC), HQI_C (Hannan – Quinn-IC) and the maximized $log\, L$. The total-time-test (T-T-T) plots the two real data sets are given in Figure 3. From Figure 3, we conclude that the empirical H-R-Fs (E-H-R-Fs) of the two data can be increasing. In Tables 2 and 3, we compared the P-BX-Fr with other Fr versions. The P-BX-Fr version gives the lowest statistics among all fitted versions. So, the new one may be chosen as the best one. Figure 4 display plots E-P-D-F, P-P plot, E-C-D-F, estimated H-R-F and Kaplan-Meier survival plot (K-M-S) of the P-BX-Fr model for the $1st$ data. Figure 5 display the plots of E-P-D-F, P-P plot, E-C-D-F, E-H-R-F and K-M-S plot of the proposed P-BX-Fr model for the $2nd$ data.

5. Conclusions

In this paper, a new wider extension of the Fr model is introduced. The new Fr version was based on the well-known Z-T-P model. The P-D-F of the new Fr model can be right skewed and unimodal also it can be left skewed. The H-R-F of the new Fr model can be bathtub (**U**-shape), unimodal-bathtub (unimodal-**U**), increasing and decreasing. Some of its properties are mathematically derived and numerically studied. The method of ML is used to estimate the unknown parameters.

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Appendix

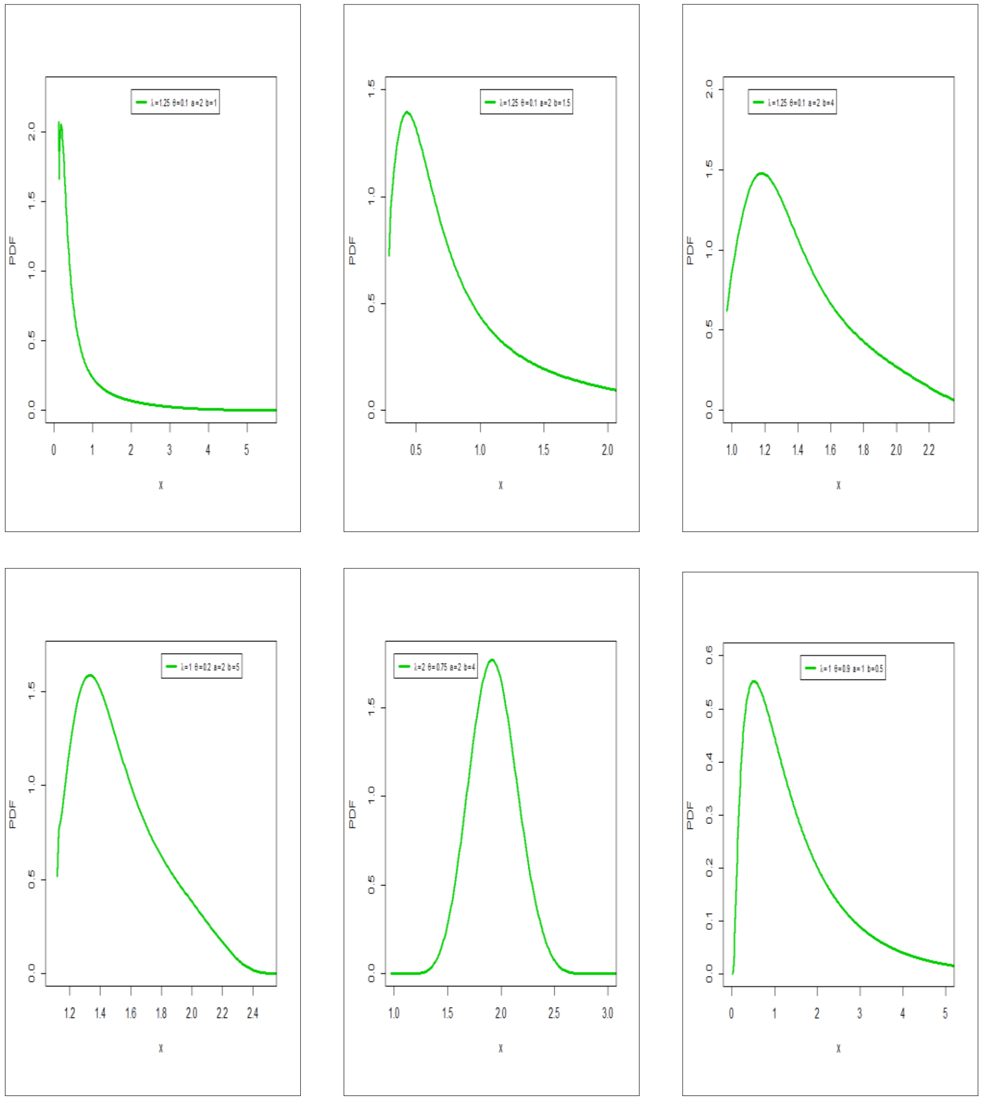


Figure 1: P.D.F.s of the new version.

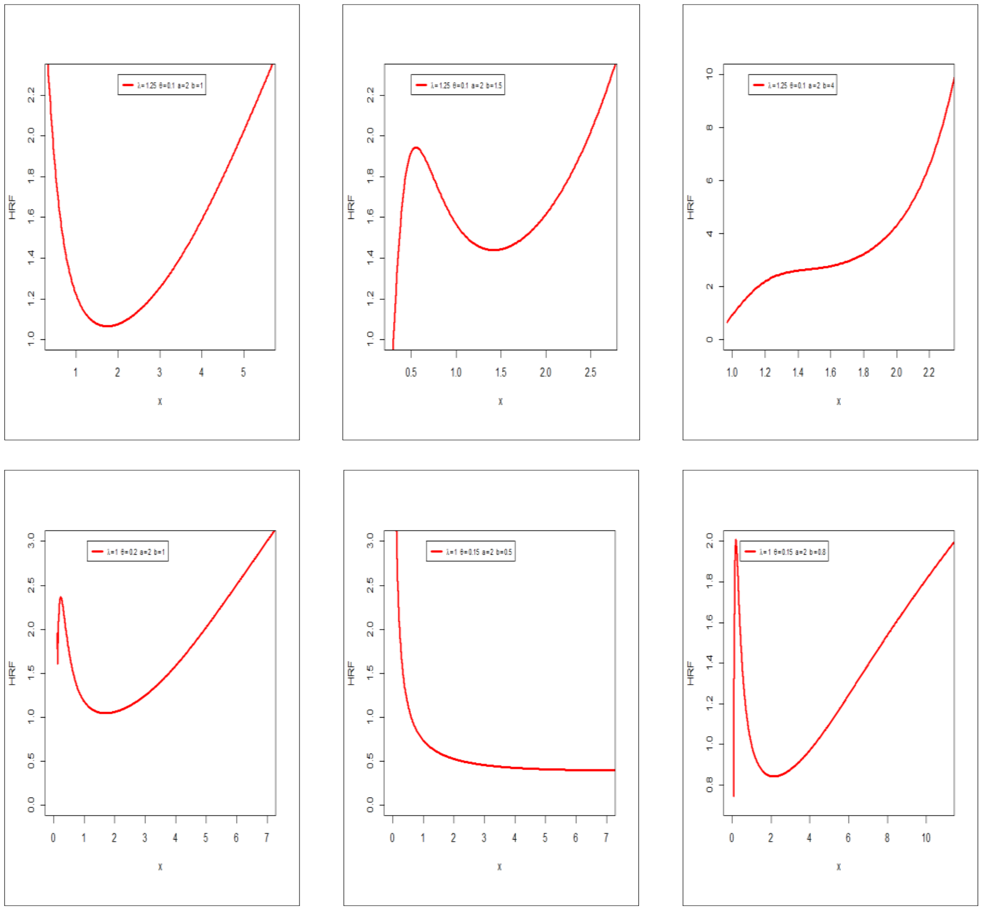
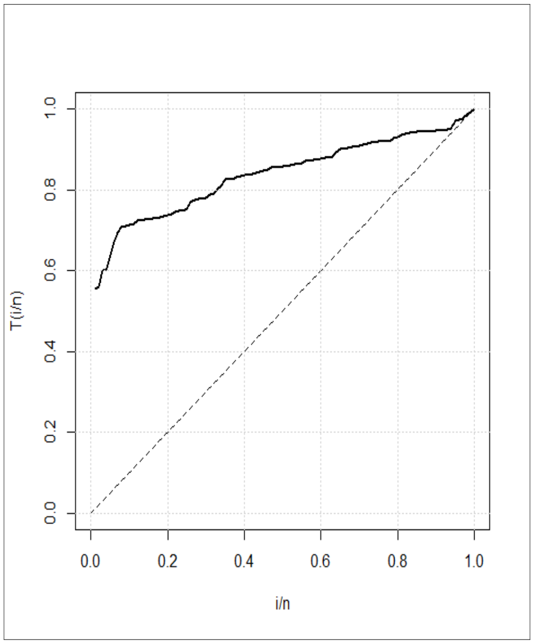


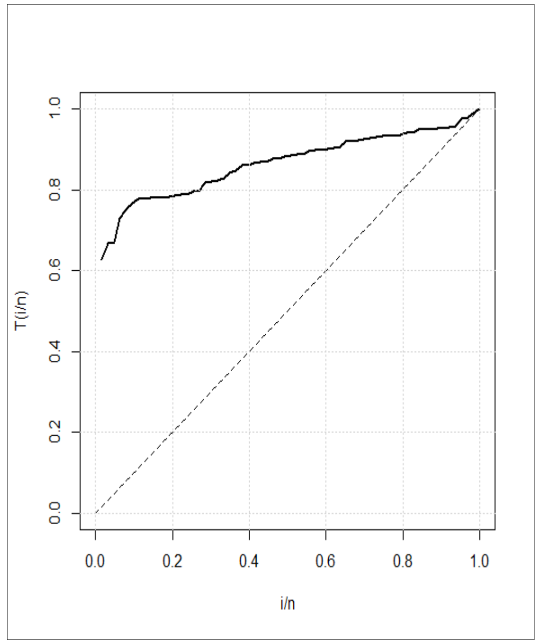
Figure 2: H.R.F.s of the new version.

Table 1: measures of the P-BX-Fr
version with α=1.25, β=0.75, different values of λ and θ.

λ	θ	Mean	Variance	Skewness	Kurtosis
50	10	2.081183	0.0437062	-0.268461	3.125746
20		2.270348	0.0684343	-0.067069	3.115156
10		2.455186	0.1095668	0.2804898	3.777957
5		2.705083	0.2181157	0.9647563	5.976804
3		2.918478	0.3435836	1.079125	5.600215
1		3.252673	0.5195756	0.8665094	4.342047
-1		3.672990	0.6303753	0.5660974	3.632238
-3		4.059918	0.6113683	0.4306407	3.640462
-5		4.345674	0.5467593	0.4745266	3.824413
-10		4.750441	0.4510728	0.6734854	4.035540
-20		5.121649	0.3945953	0.7893729	4.202697
-50		5.57532	0.3485676	0.8619313	4.362489
50	1.5	0.656871	0.0276661	0.2000253	2.916146
20		0.826048	0.0563583	0.3857478	3.196063
10		1.009971	0.1104200	0.7382564	4.535035
5		1.280310	0.2634135	1.3386010	7.028746
3		1.521799	0.4446594	1.3046860	6.003620
1		1.908314	0.7034265	0.9623336	4.29767
-1		2.401288	0.8719591	0.5815450	3.429711
-3		2.858543	0.8488771	0.3894757	3.398211
-5		3.197011	0.7534608	0.3951807	3.582307
-10		3.674373	0.6044304	0.5725431	3.790513
-20		4.106018	0.5118386	0.6976685	3.958058
-50		4.623793	0.4359177	0.7865258	4.142977



Data set I



Data set II

Figure 3: gives the TTT plots.

Table 2

Model	Goodness of fit criteria
	BI_C, AI_C, CAI_C, HQI_C, -2ℓ
P-BX-Fr	**133.06, 123.06, 290.19, 126.86, 114.60**
W_Fr	304.93, 294.58, 294.94, 298.67, 286.52
E_Fr	303.55, 295.66, 296.01, 298.88, 289.74
Kum_Fr	307.56, 297.10, 297.52, 301.34, 289.13
B_Fr	321.60, 311.10, 311.58, 315.42, 303.19
GE_Fr	332.43, 312.00, 312.42, 316.20, 304.00
Fr	353.52, 348.30, 348.43, 350.38, 344.36
T_Fr	358.32, 350.46, 350.72, 353.62, 344.58
MO_Fr	359.10, 351.29, 351.64, 354.54, 345.34

Table 3

Model	Estimates			
P-BX-Fr(λ, θ, α, β)	**4.9**	**3.452**	**1.0310**	**0.742**
	(1.247)	**(1.024)**	**(0.193)**	**0.117**
Kum_Fr(α, β, a, b)	2.0556	0.4654	6.2815	224.18
	(0.071)	(0.00701)	(0.063)	(0.164)
W_Fr(α, β, a, b)	2.2231	0.355	6.9721	4.9179
	(11.409)	(0.411)	(113.811)	(3.756)
GE_Fr(α,β, a, b)	1.3692	0.4776	27.6452	17.4581
	(2.017)	(0.133)	(14.136)	(14.818)
B_Fr(α, β, a, b)	1.6097	0.4046	22.0143	29.7617
	(2.498)	(0.108)	(21.432)	(17.479)
T_Fr(α, β, a)	1.9315	1.7435	0.0819	
	(0.097)	(0.076)	(0.198)	
E_Fr(α, β, a)	69.1489	0.5019	145.3275	
	(57.349)	(0.08)	(122.924)	
MO_Fr(α, β, a)	2.3066	1.5796	0.5988	
	(0.498)	(0.16)	(0.3091)	
Fr(α, β)	1.8705	1.7766		
	(0.112)	(0.113)		

Table 4

Model	Goodness of fit criteria
	BI_C, AI_C, CAI_C, HQI_C, -2ℓ
P-BX-Fr	**58.15, 49.58, 50.03, 52.90, 41.40**
B_Fr	77.20, 68.62, 69.34, 72.03, 60.65
GE_Fr	78.10, 69.60, 70.32, 72.90, 61.63
Fr	102.00, 97.720, 97.920, 99.440, 93.79
T_Fr	106.50, 100.14, 100.51, 102.63, 94.19
MO_Fr	108.20, 101.74, 102.16, 104.26, 95.75

Table 5

Model	Estimates			
P-BX-Fr(λ, θ, α, β)	**4.492**	**19.999**	**0.383**	**0.506**
	(1.778)	**(9.241)**	**(0.184)**	**(0.109)**
GE_Fr(α, β, a, b)	1.6625	0.7421	32.112	13.2688
	(0.952)	(0.197)	(17.397)	(9.967)
B_Fr(α, β, a, b)	2.0518	0.6466	15.0756	36.9397
	(0.986)	(0.163)	(12.057)	(22.649)
MO_Fr(α, β, a)	1.5441	2.3876	0.4816	
	(0.226)	(0.253)	(0.252)	
T_Fr(α, β, a)	1.3068	2.7898	0.1298	
	(0.034)	(0.165)	(0.208)	
Fr(α, β)	1.264	2.888		
	(0.059)	(0.234)		

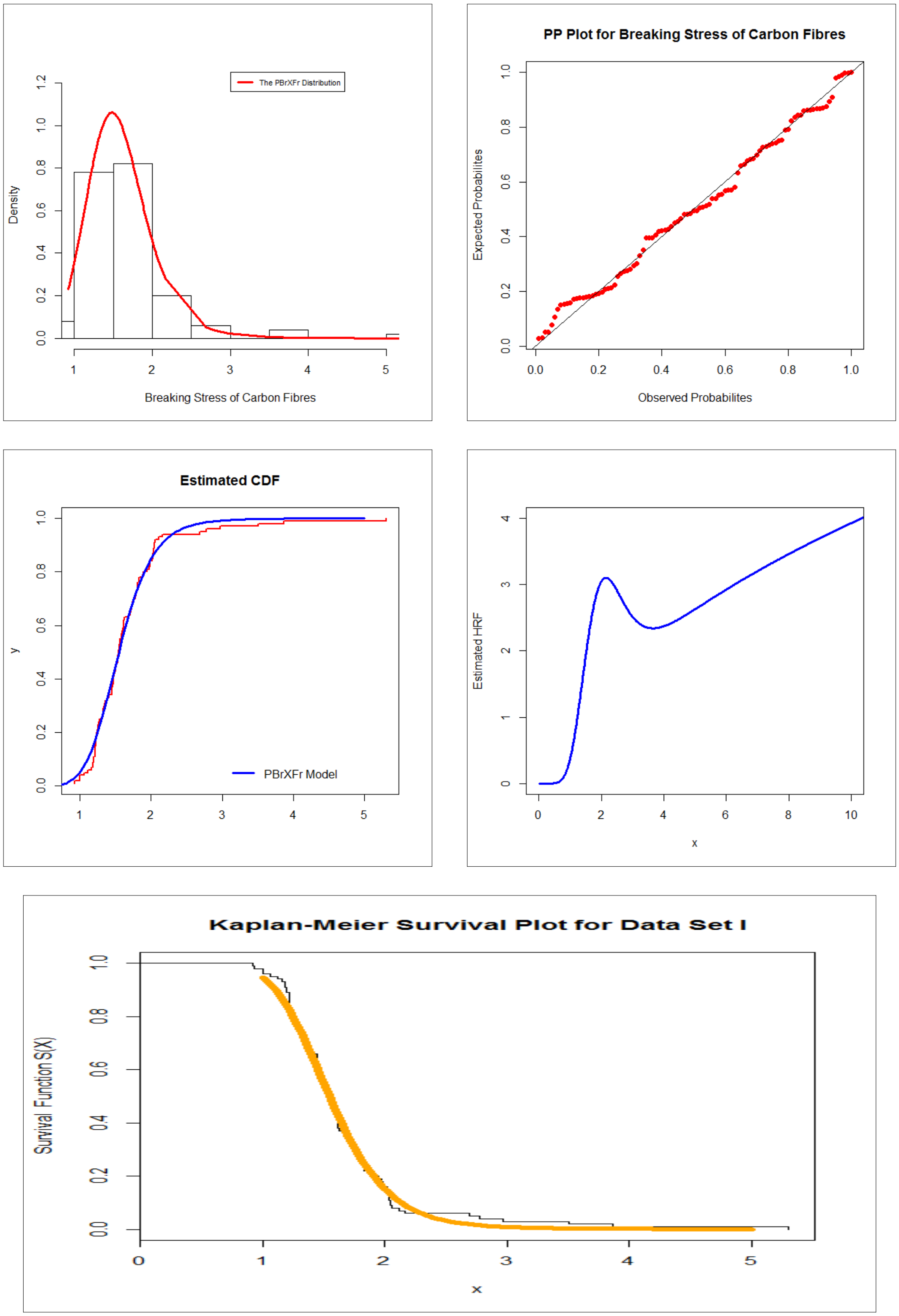

The PBrXFr Distribution
Density
Breaking Stress of Carbon Fibres
PP Plot for Breaking Stress of Carbon Fibres
Expected Probabilites
Observed Probabilites
Estimated CDF
y
PBrXFr Model
Estimated HRF
x
Kaplan-Meier Survival Plot for Data Set I
Survival Function S(X)
x

Figure 4.

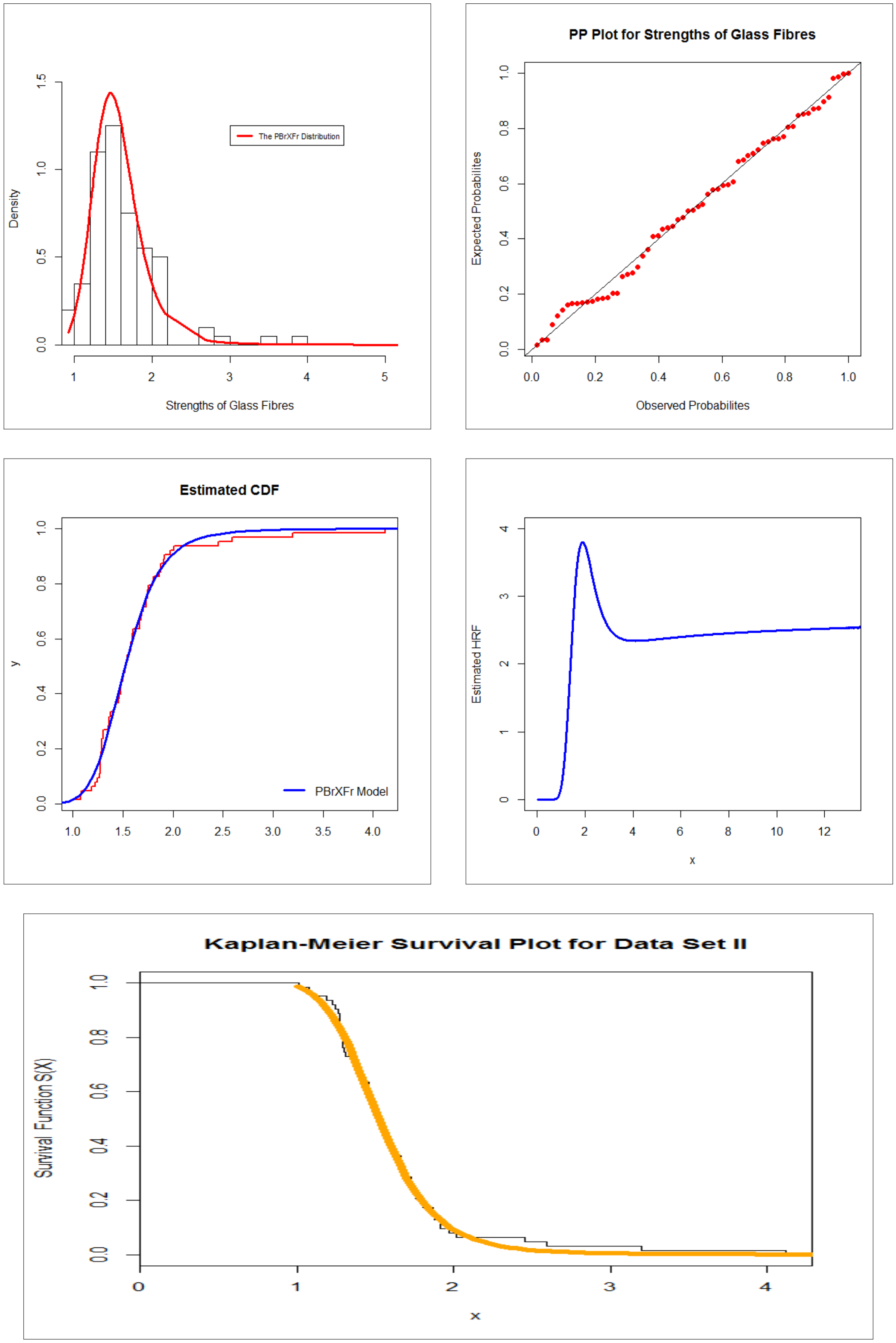

The PBrXFr Distribution
Density
Strengths of Glass Fibres
PP Plot for Strengths of Glass Fibres
Expected Probabilites
Observed Probabilites
Estimated CDF
y
PBrXFr Model
Estimated HRF
x
Kaplan-Meier Survival Plot for Data Set II
Survival Function S(X)
x

Figure 5.