New General Transmuted Family of Distributions with Applications

Md. Mahabubur Rahman
Department of Statistics
King Abdulaziz University, Jeddah, Kingdom of Saudi Arabia
and
Department of Statistics
Islamic University, Kushtia, Bangladesh
mmriu.stat@gmail.com

Bander Al-Zahrani
Department of Statistics
King Abdulaziz University, Jeddah, Kingdom of Saudi Arabia
bmalzahrani@kau.edu.sa

Muhammad Qaiser Shahbaz
Department of Statistics
King Abdulaziz University, Jeddah, Kingdom of Saudi Arabia
qshahbaz@gmail.com

Abstract

In this paper, we have introduced a new general family of transmuted distributions and have studied the cubic transmuted family of distributions in detail. This new class of distributions offers more distributional flexibility when bi-modality appear in the data. Some special members of the proposed cubic transmuted family of distributions have been discussed. We have investigated, in detail, the proposed cubic transmuted family of distributions for parent exponential distribution. Statistical properties along with the reliability analysis for the cubic transmuted exponential distribution have been studied. We have obtained the expressions for single and joint order statistics when a sample is available from the cubic transmuted exponential distribution. Maximum likelihood estimation of parameters for cubic transmuted exponential distribution has been discussed. We have also discussed simulation study and real data applications of the proposed distribution.

Keywords: Cubic transmuted distribution, Exponential distribution, General rank transmutation, Maximum likelihood estimation, Order Statistics, Reliability analysis.

1. Introduction

Standard probability models have been extensively used in many areas of life to modeling data. The standard probability models are being extended to model more complex data sets. Several methods of extending and generalizing probability distributions have been proposed in literature. One popular method of extending the probability distributions has been proposed by Alzaatreh et al. (2013) and is known as $T-X$ family of distributions. The $T-X$ family of distributions has been used by several authors to propose new families of distributions, for example the $Beta-G$ family of distributions by Eugene et al. (2002) and $Kum-G$ family of distributions by Cordeiro and Castro (2010) are special cases of $T-X$ family of distributions.

Shaw and Buckley (2007), have proposed a different method of extending probability distributions and is known as transmuted family of distributions. The method is based upon quadratic ranking transmutation and extends a baseline distribution $G(x)$ as

$$F(x) = (1+\lambda)G(x) - \lambda G^2(x),\ \lambda \in [-1,1]. \tag{1}$$

This transmuted family of distributions introduced by Shaw and Buckley (2007), has opened doors to modify the available probability distributions to capture the quadratic behavior of the data. The quadratic transmuted family was further highlighted and extended by Aryal and Tsokos (2009, 2011); Yousof et al. (2015); Alizadeh et al. (2016); Merovci et al. (2016) and Nofal et al. (2017). The quadratic transmuted distributions are very common in the literature. Several proposed distributions have been listed in Table 1 by Tahir and Cordeiro (2016).

Recently Granzotto et al. (2017) proposed a new family of distributions to capture the complexity of the data and to increases the flexibility of transmuted distributions. Rahman et al. (2018a) introduced a general transmuted family of distributions with emphasis on the cubic transmuted family of distributions which provides additional flexibility in modeling bi-modal data.

In this article, we have proposed a general transmuted family of distributions and have introduced the cubic transmuted family of distributions. This family captures the complexity of the data arising in engineering, finance, environmental sciences and other areas of life. We have focused on exponential distribution to illustrate the applicability of this new class of distributions.

1.1 Organization of the Paper

The article is structured as follows. In Section 2, we have proposed a new general class of transmuted distributions. Section 3 provides a new family of cubic transmuted distributions. Special cases for the cubic transmuted family of distributions have been discussed in Section 4. The cubic transmuted exponential distribution have been introduced in Section 5. Some desirable properties of the cubic transmuted exponential distribution have been explored in Section 6. In section 7, we have obtained the distribution of single order statistic and joint distribution of two order statistics when sample is available from cubic transmuted exponential distribution. The estimation of parameters and inference are described in Section 8. In Section 9, simulation results and applications to real-life data sets are presented. Finally, Section 10 provides some concluding remarks.

2. General Transmuted Family of Distributions

In this section, we have proposed a general family of transmuted distributions. For this, let X be a random variable with *cdf* $G(x)$, then a general transmuted family; called $k-$ transmuted family; is defined as

$$F(x) = G(x)\left[1 + \sum_{i=1}^{k} \lambda_i \left[1 - G(x)\right]^i\right],$$

where $\lambda_1 \in [-1, 1]$ and $\lambda_i \in [0, 1]$ for $i = 2, 3, \cdots, k$. The *cdf* can also be written as

$$F(x) = \sum_{i=0}^{k} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \lambda_i G^{j+1}(x), \tag{2}$$

with $\lambda_0 = 1$. The proposed family of distributions reduces to the base distribution for, $\lambda_i = 0$ for $i = 1, 2, \cdots, k$. The *pdf* corresponding to (2) is

$$f(x) = g(x) \sum_{i=0}^{k} \sum_{j=0}^{i} (-1)^j (1 + j) \binom{i}{j} \lambda_i G^j(x), \tag{3}$$

where $\lambda_0 = 1$, $\lambda_1 \in [-1, 1]$ and $\lambda_i \in [0, 1]$ for $i = 2, 3, \cdots, k$.

2.1 Moments

The expression for the moments of general transmuted family of distributions has been obtained in terms of probability weighted moments of base distribution $G(x)$. For this, consider the expression for probability weighted moments of a distribution $G(x)$ is given as

$$\begin{aligned} M_{r,s,t} &= E\left[X^r \{G(X)\}^s \{1 - G(X)\}^t\right] \\ &= \int_{-\infty}^{\infty} x^r \{G(x)\}^s \{1 - G(x)\}^t g(x)\, dx. \end{aligned} \tag{4}$$

The rth moment for general transmuted family of distributions is given as

$$\mu_r^{'} = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx. \tag{5}$$

Now using (3) in (5), the expression for rth moment is given as

$$\begin{aligned} \mu_r^{'} &= \int_{-\infty}^{\infty} x^r g(x)\, dx + \sum_{i=1}^{k} \lambda_i \int_{-\infty}^{\infty} x^r g(x) \{1 - G(x)\}^i dx \\ &\quad - \sum_{i=1}^{k} i\lambda_i \int_{-\infty}^{\infty} x^r g(x) G(x) \{1 - G(x)\}^{i-1} dx. \end{aligned}$$

Using (4) the rth moment for general transmuted family of distributions is

$$\mu_r^{'} = M_{r,0,0} + \sum_{i=1}^{k} \lambda_i M_{r,0,i} - \sum_{i=1}^{k} i\lambda_i M_{r,1,i-1}. \tag{6}$$

From (6) it can be easily observed that the moments for general transmuted family of distributions can be expressed as weighted sum of probability weighted moments of the base distribution $g(x)$.

3. The Cubic Transmuted Family of Distributions

The cubic transmuted (CT) family of distributions is obtained by setting $k = 2$ in (2). The *cdf* of cubic transmuted family of distributions is given as

$$F(x) = G(x) + \lambda_1 G(x)[1 - G(x)] + \lambda_2 G(x)[1 - G(x)]^2,$$

which can also be written as

$$F(x) = (1 + \lambda_1 + \lambda_2)G(x) - (\lambda_1 + 2\lambda_2)G^2(x) + \lambda_2 G^3(x), \tag{7}$$

where $\lambda_1 \in [-1, 1]$ and $\lambda_2 \in [0, 1]$.

The family given in (7) is new as compared with the families developed by Granzotto et al. (2017) and Rahman et al. (2018a).

In following theorems, we have provided link of cubic transmuted family given in (7) with distribution of order statistics and with $T - X$ family of distributions.

Theorem 3.1. *Let X_1, X_2 and X_3 be iid random variables each with distribution function $G(x)$, then the cubic transmuted family of distributions given in (7) can be obtained as a weighted sum of three order statistics.*

Proof. Consider following order statistics

$$X_{1:3} = min(X_1, X_2, X_3),\ X_{2:3} \text{ and } X_{3:3} = max(X_1, X_2, X_3),$$

and set the random variable Y as

$$Y \stackrel{d}{=} X_{3:3}, \quad \text{with probability } p_1,$$

$$Y \stackrel{d}{=} X_{2:3}, \quad \text{with probability } p_2,$$

$$Y \stackrel{d}{=} X_{1:3}, \quad \text{with probability } p_3,$$

where $\sum_{i=1}^{3} p_i = 1 \Rightarrow p_3 = 1 - p_1 - p_2$. Hence, evidently $F_Y(x)$ is given by

$$\begin{aligned} F_Y(x) &= p_1 F_{3:3}(x) + p_2 F_{2:3}(x) + p_3 F_{1:3}(x) \\ &= (3 - 3p_1 - 3p_2)G(x) - (3 - 3p_1 - 6p_2)G^2(x) + (1 - 3p_2)G^3(x). \end{aligned} \tag{8}$$

Now if we set $1 - 3p_1 = \lambda_1$ and $1 - 3p_2 = \lambda_2$ then the distribution function given in (8) corresponds to the cubic transmuted family of distributions given in (7). □

Theorem 3.2. *Let $G(x)$ be cdf of a random variable X and $p(t)$ be pdf of a bounded random variable T with support on [0,1], then cubic transmuted family given in (7) can be obtained by using $T-X$ family of distributions, introduced by Alzaatreh et al. (2013), for suitable choice of $p(t)$. Also, the pdf $p(t)$ can be written as weighted sum of three bounded densities $p_1(t),\ p_2(t)$ and $p_3(t)$ with support on [0,1].*

Proof. A new family of distributions, introduced by Alzaatreh et al. (2013), for a general baseline *cdf* $G(x)$ is given as

$$F(x) = \int_0^{G(x)} p(t)\ \mathrm{d}t,\ x \in \mathbb{R}, \tag{9}$$

where $p(t)$ denotes a *pdf* with support on $[0,1]$. As noticed by Alizadeh et al. (2017), the *cdf* given in (1), corresponds to the *cdf* given by (9) defined with the *pdf* $p(t) = 1+\lambda-2\lambda t$. We have developed a new *pdf* $p(t) = 1+\lambda_1-2\lambda_1 t+\lambda_2-4\lambda_2 t+3\lambda_2 t^2$, that can be expressed as a mixture of three *pdf*'s with support on $[0,1]$ as follows

$$p(t) = (1-\lambda_1-\lambda_2)p_1(t)+\lambda_1 p_2(t)+\lambda_2 p_3(t), \tag{10}$$

where $p_1(t) = 1,\ p_2(t) = 2(1-t)$ and $p_3(t) = 2(1-2t)+3t^2$.

Now using (10) in (9) we have

$$F(x) = \int_0^{G(x)} \left[(1-\lambda_1-\lambda_2)p_1(t)+\lambda_1 p_2(t)+\lambda_2 p_3(t)\right]\ \mathrm{d}t. \tag{11}$$

On simplification, the *cdf* provided in (11), turned out to be the *cdf* of cubic transmuted family of distributions given in (7). □

Definition 3.1 (Cubic transmuted distribution)**.** *A random variable X is said to have a cubic transmuted distribution, with parameters $\lambda_1 \in [-1,1]$ and $\lambda_2 \in [0,1]$ if its pdf is given by*

$$f(x) = g(x)\left[1+\lambda_1+\lambda_2-2(\lambda_1+2\lambda_2)G(x)+3\lambda_2 G^2(x)\right],\ x \in \mathbb{R}, \tag{12}$$

where $g(x)$ is the pdf associated with cdf $G(x)$.

It can be seen that the *pdf* of quadratic transmuted distribution is easily obtained from the *pdf* of cubic transmuted distribution by setting $\lambda_2 = 0$. Also observe that for $\lambda_1 = \lambda_2 = 0$, we have the *pdf* of the baseline random variable. The density function of cubic transmuted family of distributions given in (12) can be obtained by using k = 2 in the density function of general transmuted family of distributions given in (3).

Lemma 3.1. *The pdf $f(x)$ given in (12) is well defined.*

Proof. The proof is simple. □

4. Special Cases for Cubic Transmuted Family of Distributions

In the following we have given some members of cubic transmuted family of distributions obtained by using various $G(x)$ in (7).

4.1 Cubic Transmuted Normal Distribution

Let X has normal distribution with mean μ, variance σ^2 and *cdf* $\Phi(X)$. Using (7), the cubic transmuted normal distribution has the *cdf*

$$F(x) = (1+\lambda_1+\lambda_2)\Phi(x) - (\lambda_1+2\lambda_2)\Phi^2(x) + \lambda_2\Phi^3(x),$$

where $\lambda_1 \in [-1,1]$ and $\lambda_2 \in [0,1]$.

4.2 Cubic Transmuted Gamma Distribution

Let X be a continuous random variable having gamma distribution with shape and scale parameters α and $1/\beta$ respectively. The *cdf* of X is given by

$$G(x) = \frac{\Gamma(\alpha) - \Gamma(\alpha, \beta x)}{\Gamma(\alpha)},\ x \in \mathbb{R}^+,$$

where $\alpha, \beta \in \mathbb{R}^+$. Here, $\Gamma(\alpha)$ and $\Gamma(\alpha, \beta x)$ are the complete and incomplete gamma function. Using (7), the *cdf* of cubic transmuted gamma distribution is

$$F(x) = \frac{[\Gamma(\alpha) - \Gamma(\alpha, x\beta)]\left[\Gamma(\alpha)^2 + \lambda_2\Gamma(\alpha, x\beta)^2 + \lambda_1\Gamma(\alpha)\Gamma(\alpha, x\beta)\right]}{\Gamma(\alpha)^3},\ x \in \mathbb{R}^+,$$

where $\lambda_1 \in [-1,1]$ and $\lambda_2 \in [0,1]$.

4.3 Cubic Transmuted Log-logistic Distribution

Let X follows Log-logistic distribution with *cdf* given as

$$G(x) = \frac{x^\beta}{\alpha^\beta + x^\beta},\ x \in [0, \infty),$$

where $\alpha, \beta \in \mathbb{R}^+$ are the scale and shape parameters respectively. Using (7), the *cdf* of cubic transmuted log-logistic distribution is given by

$$F(x) = \frac{x^\beta\left[\lambda_2\alpha^{2\beta} + \lambda_1\alpha^\beta\left(\alpha^\beta + x^\beta\right) + \left(\alpha^\beta + x^\beta\right)^2\right]}{\left(\alpha^\beta + x^\beta\right)^3},\ x \in [0, \infty),$$

where $\lambda_1 \in [-1,1]$ and $\lambda_2 \in [0,1]$.

4.4 Cubic Transmuted Pareto Distribution

Suppose the random variable X has Pareto distribution with *cdf*

$$G(x) = 1 - \left(\frac{k}{x}\right)^\theta,\ x \in [k, \infty),$$

where $k, \theta \in \mathbb{R}^+$ are the scale and shape parameters. Using (7), the *cdf* of the cubic transmuted Pareto distribution is given by

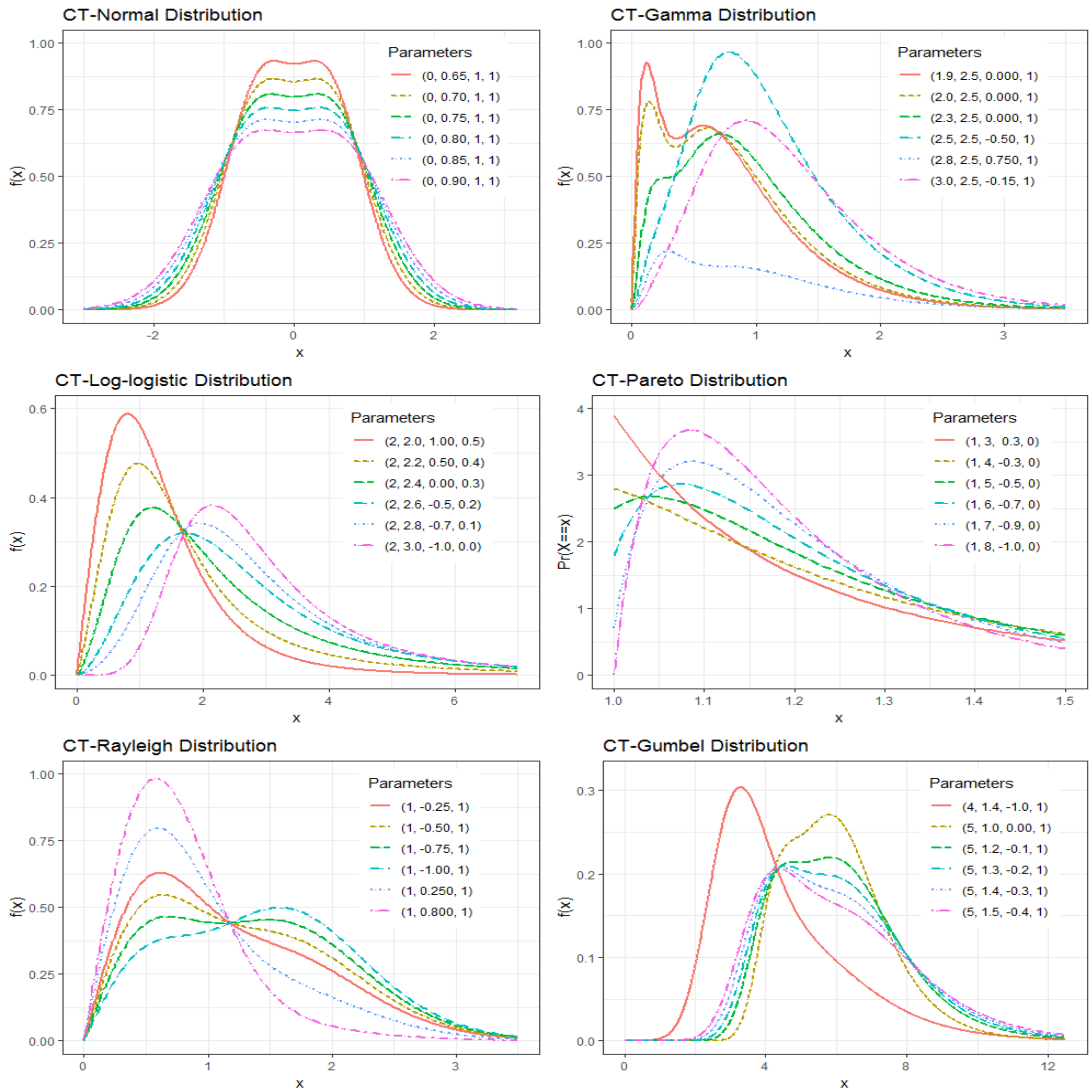

Figure 1: Plots of Density Function $f(x)$ for Selected Base Distributions.

$$F(x) = \left[1 - \left(\frac{k}{x}\right)^{\theta}\right]\left[\lambda_1\left(\frac{k}{x}\right)^{\theta} + \lambda_2\left(\frac{k}{x}\right)^{2\theta} + 1\right],\ x \in [k, \infty),$$

where $\lambda_1 \in [-1, 1]$ and $\lambda_2 \in [0, 1]$.

4.5 Cubic Transmuted Rayleigh Distribution

Let X be a Rayleigh random variable with *cdf*

$$G(x) = 1 - e^{-\frac{x^2}{2\sigma^2}},\ x \in [0, \infty),$$

where $\sigma \in \mathbb{R}^+$ is the scale parameter of the distribution. Using (7), the *cdf* of the cubic transmuted Rayleigh distribution is given by

$$F(x) = e^{-\frac{3x^2}{2\sigma^2}} \left(e^{\frac{x^2}{2\sigma^2}} - 1\right)\left(\lambda_1 e^{\frac{x^2}{2\sigma^2}} + e^{\frac{x^2}{\sigma^2}} + \lambda_2\right),\ x \in [0, \infty),$$

where $\lambda_1 \in [-1, 1]$ and $\lambda_2 \in [0, 1]$.

4.6 Cubic Transmuted Gumbel Distribution

Let X be a continuous random variable having Gumbel distribution. The *cdf* of Gumbel distribution is given as

$$G(x) = 1 - e^{-e^{-\frac{x-\mu}{\beta}}},\ x \in \mathbb{R},$$

where $\mu \in \mathbb{R}$, $\beta \in \mathbb{R}^+$ are the location and scale parameters respectively. Using (7), the *cdf* of the cubic transmuted Gumbel distribution is

$$F(x) = e^{-3e^{\frac{\mu-x}{\beta}}} \left(e^{e^{\frac{\mu-x}{\beta}}} - 1\right)\left(e^{2e^{\frac{\mu-x}{\beta}}} + \lambda_1 e^{e^{\frac{\mu-x}{\beta}}} + \lambda_2\right),\ x \in \mathbb{R},$$

where $\lambda_1 \in [-1, 1]$ and $\lambda_2 \in [0, 1]$.

The plots of density functions for above defined distributions are given in Figure 1.

5. Cubic Transmuted Exponential Distribution

The exponential distribution is a widely used lifetime distribution. The distribution is popular in many applications due to it's simplicity. The distribution has been generalized by many authors, see for example generalized exponential by Gupta and Kundu (1999, 2007), exponentiated exponential (EE) by Gupta and Kundu (2001) and beta exponential (BE) distribution by Nadarajah and Kotz (2006). These proposed generalizations are more flexible than exponential distribution when applied to real-life data sets.

In the following, we propose another generalization of the exponential distribution by using it as a baseline distribution in the *cdf* of cubic transmuted family of distributions given in (7). This proposed distribution is referred as the cubic transmuted exponential (CTE) distribution and have much wider applicability as compared with the available generalizations of the exponential distribution.

Let X be a continuous random variable having the exponential distribution with *cdf*

$$G(x) = 1 - \frac{1}{\theta} e^{-\frac{x}{\theta}},\ x \in [0, \infty), \tag{13}$$

where $\theta \in \mathbb{R}^+$ is the scale parameter. Owoloko et al. (2015) have proposed the quadratic transmuted exponential distribution with *cdf*

$$F(x) = \left[1 - e^{-\frac{x}{\theta}}\right]\left[1 + \lambda e^{-\frac{x}{\theta}}\right],\ x \in [0, \infty),$$

where $\lambda \in [-1, 1]$.

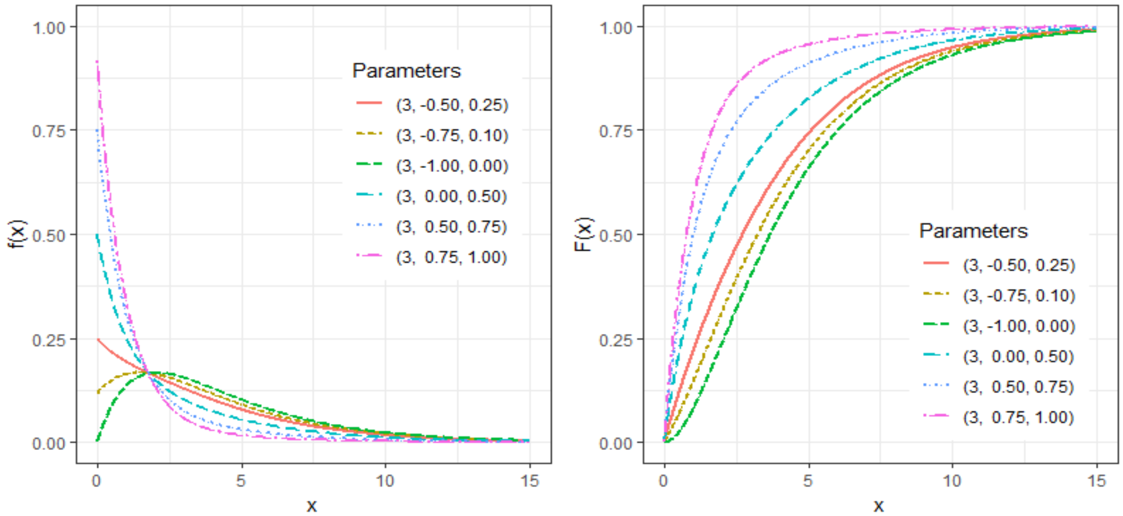

Figure 2: Density and distribution functions plots for $CTE(\theta,\lambda_1,\lambda_2)$ distribution.

The cubic transmuted exponential distribution is defined below.

Proposition 5.1. *Let X has an exponential distribution with parameter $\theta \in \mathbb{R}^+$, then the pdf of cubic transmuted exponential distribution with parameters $\theta \in \mathbb{R}^+$, $\lambda_1 \in [-1,1]$ and $\lambda_2 \in [0,1]$ is given by*

$$f(x) \;=\; \left(\frac{1}{\theta}e^{-\frac{3x}{\theta}}\right)\left[(1-\lambda_1)\,e^{\frac{2x}{\theta}}+2\,(\lambda_1-\lambda_2)\,e^{\frac{x}{\theta}}+3\lambda_2\right],\;\; x\in[0,\infty). \qquad (14)$$

Proof. Using the *cdf* (13) in (7), we can obtain the *cdf* of cubic transmuted exponential distribution and is given as

$$F(x) \;=\; e^{-\frac{3x}{\theta}}\left(e^{\frac{x}{\theta}}-1\right)\left[e^{\frac{2x}{\theta}}+\lambda_1 e^{\frac{x}{\theta}}+\lambda_2\right],\;\; x\in[0,\infty). \qquad (15)$$

The *pdf* in (14) can be easily obtained by differentiating (15) with respect to x. □

The plots of density and distribution functions of cubic transmuted exponential distribution are given in Figure 2.

6. Statistical Properties

We have discussed some important properties of cubic transmuted exponential distribution. We have obtained expression for raw moments and moment generating function of the distribution. We have also obtained quantiles by using quantile function along with the method of generating random sample from the distribution. The survival and hazard functions are also obtained.

6.1 Moments

Moments are very useful to know the location, distribution and shape of a distribution. In the following we have obtained the expression for moments of a random variable X having cubic transmuted exponential distribution. The rth moment for cubic transmuted exponential distribution is stated by the following theorem.

Theorem 6.1. *Let X follows CTE distribution then rth moment of X is given by*

$$\mu_r^{'} = E(X^r) = \frac{\theta^r}{6^r} r! \left[6^r - 3^r(2^r - 1)\lambda_1 - (3^r - 2^r)\lambda_2\right].$$

Further the mean and variance of the distribution are

$$\mu = E(X) = \frac{\theta}{6}\left(6 - 3\lambda_1 - \lambda_2\right) \text{ and,}$$

$$\sigma^2 = V(X) = \frac{\theta^2}{36}\left(36 - 18\lambda_1 + 2\lambda_2 - 9\lambda_1^2 - 6\lambda_1\lambda_2 - \lambda_2^2\right).$$

Proof. The rth moment is obtained by

$$\mu_r^{'} = E(X^r) = \int_0^\infty x^r f(x)\mathrm{d}x.$$

Using $f(x)$ from (14) and on simplification the rth moment of cubic transmuted exponential distribution can be obtained as

$$E(X^r) = \frac{\theta^r}{6^r} r! \left[6^r - 3^r(2^r - 1)\lambda_1 - (3^r - 2^r)\lambda_2\right]. \tag{16}$$

Mean and variance can be easily obtained from (16) and are given as

$$E(X) = \frac{\theta}{6}\left(6 - 3\lambda_1 - \lambda_2\right) \text{ and,}$$

and

$$\begin{aligned} V(X) &= E(x^2) - [E(x)]^2 \\ &= \frac{\theta^2}{36}\left(36 - 18\lambda_1 + 2\lambda_2 - 9\lambda_1^2 - 6\lambda_1\lambda_2 - \lambda_2^2\right). \end{aligned}$$

The higher moments can also be obtained from (16) for $r > 2$. □

The means of CTE distribution for various combination of model parameters are given in Table 1.

6.2 Moment Generating Function

The moment generating function (MGF) is used to determine the moments of a distribution. The MGF for CTE distribution is stated by the following theorem.

Table 1: Means for the cubic transmuted exponential distribution

		$\lambda_1 = -1$	$\lambda_1 = -0.5$	$\lambda_1 = 0$	$\lambda_1 = 0.5$	$\lambda_1 = 1$
	$\lambda_2 = 0$	1.500	1.250	1.000	0.750	0.500
$\theta = 1$	$\lambda_2 = 0.5$	1.417	1.167	0.917	0.667	0.417
	$\lambda_2 = 1$	1.333	1.083	0.833	0.583	0.333
	$\lambda_2 = 0$	3.000	2.500	2.000	1.500	1.000
$\theta = 2$	$\lambda_2 = 0.5$	2.833	2.333	1.833	1.333	0.833
	$\lambda_2 = 1$	2.667	2.167	1.667	1.167	0.667
	$\lambda_2 = 0$	4.500	3.750	3.000	2.250	1.500
$\theta = 3$	$\lambda_2 = 0.5$	4.250	3.500	2.750	2.000	1.250
	$\lambda_2 = 1$	4.000	3.250	2.500	1.750	1.000
	$\lambda_2 = 0$	6.000	5.000	4.000	3.000	2.000
$\theta = 4$	$\lambda_2 = 0.5$	5.667	4.667	3.667	2.667	1.667
	$\lambda_2 = 1$	5.333	4.333	3.333	2.333	1.333
	$\lambda_2 = 0$	7.500	6.250	5.000	3.750	2.500
$\theta = 5$	$\lambda_2 = 0.5$	7.083	5.833	4.583	3.333	2.083
	$\lambda_2 = 1$	6.667	5.417	4.167	2.917	1.667

Theorem 6.2. *Let X follows the CTE distribution, then MGF, $M_X(t)$ is*

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{\theta^r}{6^r} r! \left[6^r - 3^r(2^r - 1)\lambda_1 - (3^r - 2^r)\lambda_2\right], \tag{17}$$

where $t \in \mathbb{R}$.

Proof. The moment generating function is given by

$$M_X(t) = E[e^{tx}] = \int_0^{\infty} e^{tx} f(x) \mathrm{d}x,$$

where $f(x)$ given in (14) and for exponential series representation, see Gradshteyn and Ryzhik (2007), can further expressed as

$$M_x(t) = \int_0^{\infty} \sum_{r=0}^{\infty} \frac{t^r}{r!} x^r f(x) \mathrm{d}t = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r). \tag{18}$$

Setting $E(X^r)$ from (16) in (18) turned out to be (17). $\square$

It is observed from the series expansion (18) that moments are the coefficients of $\frac{t^r}{r!}$ for different choices of r.

6.3 The Quantile Function

The qth quantile x_q of the cubic transmuted exponential distribution can be obtained as inverse function of (15) and is given as

$$x_q = \theta\ [-ln(y)], \tag{19}$$

where,

$$\left.\begin{array}{l} y = -\frac{b}{3a} - \frac{2^{1/3}\xi_1}{3a\left(\xi_2+\sqrt{4\xi_1^3+\xi_2^2}\right)^{1/3}} + \frac{\left(\xi_2+\sqrt{4\xi_1^3+\xi_2^2}\right)^{1/3}}{3(2^{1/3})a}, \\ \xi_1 = -b^2+3ac,\ \xi_2 = -2b^3+9abc-27a^2d, \\ a = -\lambda_2,\ b = \lambda_2-\lambda_1,\ c = \lambda_1-1 \text{ and } d = 1-q. \end{array}\right\} \tag{20}$$

The lower quartile, median and upper quartile can be obtained by using $q = 0.25,\ 0.50$ and 0.75 respectively in (19).

6.4 Reliability Analysis

Let T be a nonnegative random variable which represents the failure times of the components. The reliability function is defined by $R(t) = 1-F(t)$ and represents the probability of an element not failing prior to some time t. The reliability function of cubic transmuted exponential distribution is given by

$$R(t) = 1 - e^{-\frac{3t}{\theta}}\left(e^{\frac{t}{\theta}}-1\right)\left[e^{\frac{2t}{\theta}}+\lambda_1 e^{\frac{t}{\theta}}+\lambda_2\right],\ t \in \mathbb{R}^+.$$

The hazard rate function, $h(t)$, is defined by

$$h(t) = \frac{f(t)}{1-F(t)},$$

which for cubic transmuted exponential distribution is given as

$$h(t) = \frac{\left(\frac{1}{\theta}e^{-\frac{3t}{\theta}}\right)\left[(1-\lambda_1)\,e^{\frac{2t}{\theta}}+2\,(\lambda_1-\lambda_2)\,e^{\frac{t}{\theta}}+3\lambda_2\right]}{1-e^{-\frac{3t}{\theta}}\left(e^{\frac{t}{\theta}}-1\right)\left[e^{\frac{2t}{\theta}}+\lambda_1 e^{\frac{t}{\theta}}+\lambda_2\right]},\ t \in \mathbb{R}^+,$$

The hazard function specifies the instantaneous rate of death at time t, given that the individual survives up to time t.

Figure 3 provides plot of reliability and hazard functions for the selected values of model parameters λ_1 and λ_2 keeping $\theta = 3$. According to the failure time t, we observed increasing and decreasing hazard rates from the shapes.

6.5 Random Numbers Generation

We can easily generate random numbers from the cubic transmuted exponential distribution using the method of inversion; see, for example, Aryal and Tsokos (2009). A random observation from the distribution is, therefore

$$e^{-\frac{3x}{\theta}}\left(e^{\frac{x}{\theta}}-1\right)\left[e^{\frac{2x}{\theta}}+\lambda_1 e^{\frac{x}{\theta}}+\lambda_2\right] = u,$$

where $u \sim U(0,1)$. On simplification, this can be written as

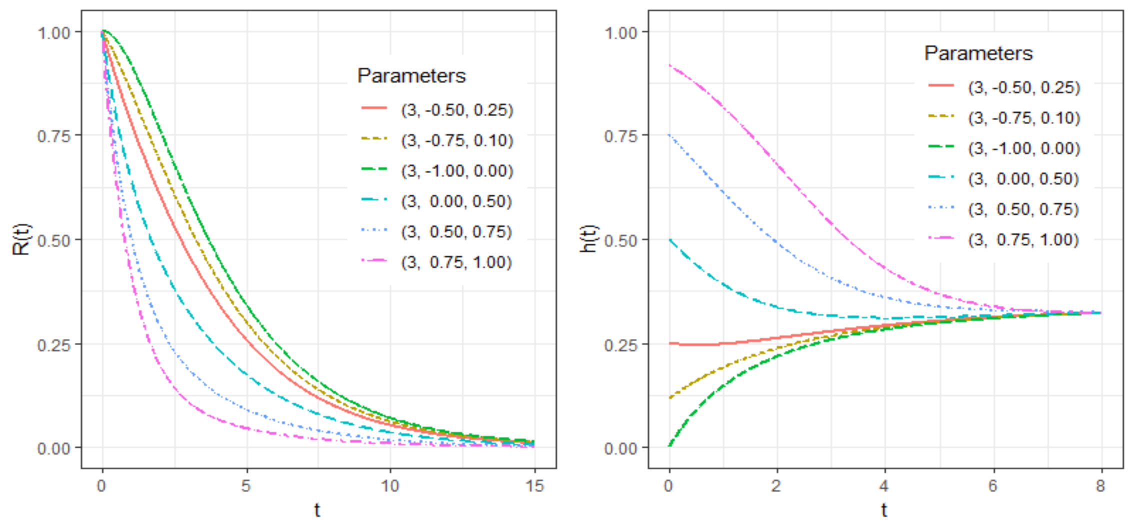

Figure 3: Reliability and hazard functions plots for $CTE(\theta,\lambda_1,\lambda_2)$ distribution.

$$X = \theta\ [-ln(y)], \tag{21}$$

where y is given in (20) with $d = 1 - u$. Now, one can generate random numbers using (21), when the parameters θ, λ_1 and λ_2 are known.

7. Order Statistics

Order statistics play an important role in statistical theory. In this section, we have obtained the *pdf* of single order statistic and joint *pdf* of two order statistics.

7.1 Distribution of a Single Order Statistic

The *pdf* of the rth order statistics for cubic transmuted exponential distribution is given below

$$\begin{aligned} f_{X_{r:n}}(x) &= \frac{n!}{(r-1)!(n-r)!}\left[e^{-\frac{3x}{\theta}}\left(e^{x/\theta}-1\right)\left(\lambda_2+\lambda_1 e^{x/\theta}+e^{\frac{2x}{\theta}}\right)\right]^{r-1} \\ &\quad \times\left[1-e^{-\frac{3x}{\theta}}\left(e^{x/\theta}-1\right)\left(\lambda_2+\lambda_1 e^{x/\theta}+e^{\frac{2x}{\theta}}\right)\right]^{n-r} \\ &\quad \times\left(\frac{1}{\theta}e^{-\frac{3x}{\theta}}\right)\left[\left(1-\lambda_1\right)e^{\frac{2x}{\theta}}+2\left(\lambda_1-\lambda_2\right)e^{\frac{x}{\theta}}+3\lambda_2\right], \end{aligned} \tag{22}$$

where $r = 1, 2, \cdots, n$. Therefor, for $r = 1$ we have the *pdf* of the smallest order statistic, $X_{1:n}$, is given by

$$\begin{aligned} f_{X_{1:n}}(x) &= \left[1-e^{-\frac{3x}{\theta}}\left(e^{x/\theta}-1\right)\left(\lambda_2+\lambda_1 e^{x/\theta}+e^{\frac{2x}{\theta}}\right)\right]^{n-1} \\ &\quad \times\left(\frac{n}{\theta}e^{-\frac{3x}{\theta}}\right)\left[\left(1-\lambda_1\right)e^{\frac{2x}{\theta}}+2\left(\lambda_1-\lambda_2\right)e^{\frac{x}{\theta}}+3\lambda_2\right]. \end{aligned}$$

Also, for $r = n$ the *pdf* of the largest order statistic, $X_{n:n}$, is given by

$$\begin{aligned} f_{X_{n:n}}(x) &= \left[e^{-\frac{3x}{\theta}}\left(e^{x/\theta}-1\right)\left(\lambda_2+\lambda_1 e^{x/\theta}+e^{\frac{2x}{\theta}}\right)\right]^{n-1} \\ &\quad \times\left(\frac{n}{\theta}e^{-\frac{3x}{\theta}}\right)\left[\left(1-\lambda_1\right)e^{\frac{2x}{\theta}}+2\left(\lambda_1-\lambda_2\right)e^{\frac{x}{\theta}}+3\lambda_2\right]. \end{aligned}$$

Note that $\lambda_1 = \lambda_2 = 0$, we have the *pdf* of the rth order statistic for exponential distribution, as follows

$$g_{X_{r:n}}(x) = \frac{n!}{(r-1)!(n-r)!}\frac{1}{\theta}e^{-\frac{x}{\theta}}\left[1-e^{-\frac{x}{\theta}}\right]^{r-1}\left[e^{-\frac{x}{\theta}}\right]^{n-r}, r = 1, 2, \cdots, n.$$

The expression of kth order moment of $X_{r:n}$ for cubic transmuted exponential distribution is given by

$$E(X_{r:n}^k) = \int_0^\infty x_r^k \cdot f_{X_{r:n}}(x)\cdot \mathrm{d}x,$$

where $f_{X_{r:n}}(x)$ is presented in (22).

7.2 Joint Distribution of Two Order Statistics

The joint *pdf* of $X_{r:n}$ and $X_{s:n}$ $(1 \le r < s \le n)$ for cubic transmuted exponential distribution is given by

$$\begin{aligned} f_{X_{r,s:n}}(x,y) = &\frac{1}{\theta^2(r-1)!(s-r-1)!(n-s)!}\Big[n!e^{-\frac{3(x+y)}{\theta}} \\ &\times\left\{e^{-\frac{3x}{\theta}}\left(e^{x/\theta}-1\right)\left(\lambda_2+\lambda_1 e^{x/\theta}+e^{\frac{2x}{\theta}}\right)\right\}^{r-1} \\ &\times\left\{\lambda_1\left(-e^{x/\theta}\right)\left(e^{x/\theta}-2\right)+\lambda_2\left(3-2e^{x/\theta}\right)+e^{\frac{2x}{\theta}}\right\} \\ &\times\Big\{e^{-\frac{x}{\theta}}+\lambda_1\left(e^{-\frac{2x}{\theta}}-e^{-\frac{x}{\theta}}-e^{-\frac{2y}{\theta}}+e^{-\frac{y}{\theta}}\right) \\ &+\lambda_2\left(e^{-\frac{3x}{\theta}}-e^{-\frac{2x}{\theta}}-e^{-\frac{3y}{\theta}}+e^{-\frac{2y}{\theta}}\right)-e^{-\frac{y}{\theta}}\Big\}^{s-r-1} \\ &\times\left\{\lambda_1\left(-e^{y/\theta}\right)\left(e^{y/\theta}-2\right)+\lambda_2\left(3-2e^{y/\theta}\right)+e^{\frac{2y}{\theta}}\right\} \\ &\times e^{-\frac{3y}{\theta}}\left\{\lambda_1\left(-e^{y/\theta}\right)\left(e^{y/\theta}-1\right)-\lambda_2\left(e^{y/\theta}-1\right)+e^{\frac{2y}{\theta}}\right\}^{n-s}\Big]. \end{aligned}$$

The joint *pdf* of smallest and largest order statistics for the cubic transmuted exponential distribution can be easily obtained from above equation by using $r = 1$ and

$s = n$ and is given as

$$
\begin{aligned}
f_{X_{1,n:n}}(x,y) = \frac{n! e^{-\frac{3(x+y)}{\theta}}}{\theta^2 (n-2)!} & \left\{ \lambda_1 \left(-e^{x/\theta}\right)\left(e^{x/\theta}-2\right) + \lambda_2 \left(3 - 2e^{x/\theta}\right) + e^{\frac{2x}{\theta}} \right\} \\
& \times \left\{ e^{-\frac{x}{\theta}} + \lambda_1 \left(e^{-\frac{2x}{\theta}} - e^{-\frac{x}{\theta}} - e^{-\frac{2y}{\theta}} + e^{-\frac{y}{\theta}} \right) \right. \\
& \left. + \lambda_2 \left(e^{-\frac{3x}{\theta}} - e^{-\frac{2x}{\theta}} - e^{-\frac{3y}{\theta}} + e^{-\frac{2y}{\theta}} \right) - e^{-\frac{y}{\theta}} \right\}^{n-2} \\
& \times \left\{ \lambda_1 \left(-e^{y/\theta}\right)\left(e^{y/\theta}-2\right) + \lambda_2 \left(3 - 2e^{y/\theta}\right) + e^{\frac{2y}{\theta}} \right\}.
\end{aligned}
$$

8. Parameter Estimation and Inference

Maximum likelihood estimation (MLE) is most commonly employed technique for the estimation of the model parameters. In the following we have given the maximum likelihood estimation for the cubic transmuted exponential distribution.

Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from the cubic transmuted exponential distribution. The likelihood function is

$$
L = \frac{1}{\theta^n} e^{-\frac{\sum_{i=1}^{n} 3x_i}{\theta}} \prod_{i=1}^{n} \left[(1-\lambda_1)\, e^{\frac{2x_i}{\theta}} + 2(\lambda_1 - \lambda_2)\, e^{\frac{x_i}{\theta}} + 3\lambda_2 \right].
$$

The log-likelihood function $l = ln(L)$ is

$$
l = \sum_{i=1}^{n} \log \left\{ (1-\lambda_1)\, e^{\frac{2x_i}{\theta}} + 2(\lambda_1 - \lambda_2)\, e^{\frac{x_i}{\theta}} + 3\lambda_2 \right\} - \sum_{i=1}^{n} \frac{3x_i}{\theta} - n \log(\theta). \tag{23}
$$

Therefore likelihood equations are obtained by differentiating (23) wrt θ, λ_1 and λ_2 and equating the resulting derivatives to zero and are

$$
\begin{aligned}
\sum_{i=1}^{n} \frac{ -\frac{2(1-\lambda_1) x_i e^{\frac{2x_i}{\theta}}}{\theta^2} - \frac{2(\lambda_1-\lambda_2) x_i e^{\frac{x_i}{\theta}}}{\theta^2} }{ (1-\lambda_1)\, e^{\frac{2x_i}{\theta}} + 2(\lambda_1 - \lambda_2)\, e^{\frac{x_i}{\theta}} + 3\lambda_2 } + \sum_{i=1}^{n} \frac{3x_i}{\theta^2} - \frac{n}{\theta} &= 0, \\
\sum_{i=1}^{n} \frac{ 2e^{\frac{x_i}{\theta}} - e^{\frac{2x_i}{\theta}} }{ (1-\lambda_1)\, e^{\frac{2x_i}{\theta}} + 2(\lambda_1 - \lambda_2)\, e^{\frac{x_i}{\theta}} + 3\lambda_2 } &= 0, \text{ and} \\
\sum_{i=1}^{n} \frac{ 3 - 2e^{\frac{x_i}{\theta}} }{ (1-\lambda_1)\, e^{\frac{2x_i}{\theta}} + 2(\lambda_1 - \lambda_2)\, e^{\frac{x_i}{\theta}} + 3\lambda_2 } &= 0.
\end{aligned}
$$

Solving above nonlinear system of equations numerically, we obtain the maximum likelihood estimators

$$
\hat{\Theta} = \left(\hat{\theta}, \hat{\lambda_1}, \hat{\lambda_2}\right)' \text{ of } \Theta = (\theta, \lambda_1, \lambda_2)'.
$$

It is well known fact that as $n \to \infty$, the asymptotic distribution of the MLE

$\left(\hat{\theta}, \hat{\lambda_1}, \hat{\lambda_2}\right)$ is given by, see for example, modified Weibull distribution by Zaindin and Sarhan (2009) and cubic transmuted Pareto distribution by Rahman et al. (2018b),

$$\begin{pmatrix} \hat{\theta} \\ \hat{\lambda_1} \\ \hat{\lambda_2} \end{pmatrix} \sim N \left[\begin{pmatrix} \theta \\ \lambda_1 \\ \lambda_2 \end{pmatrix}, \begin{pmatrix} \hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13} \\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23} \\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33} \end{pmatrix} \right],$$

where

$$V^{-1} = -E \begin{bmatrix} \frac{\delta^2 l}{\delta\theta^2} & \frac{\delta^2 l}{\delta\theta \cdot \delta\lambda_1} & \frac{\delta^2 l}{\delta\theta \cdot \delta\lambda_2} \\ \frac{\delta^2 l}{\delta\theta \cdot \delta\lambda_1} & \frac{\delta^2 l}{\delta\lambda_1^2} & \frac{\delta^2 l}{\delta\lambda_1 \cdot \delta\lambda_2} \\ \frac{\delta^2 l}{\delta\theta \cdot \delta\lambda_2} & \frac{\delta^2 l}{\delta\lambda_1 \cdot \delta\lambda_2} & \frac{\delta^2 l}{\delta\lambda_2^2} \end{bmatrix}. \tag{24}$$

The inverse of the observed information matrix given in (24) gives the asymptotic variance-covariance matrix of the maximum likelihood estimators $\hat{\theta}$, $\hat{\lambda_1}$ and $\hat{\lambda_2}$, See Appendix. An approximate $100(1-\alpha)\%$ two sided confidence intervals for θ, λ_1 and λ_2 are, respectively, given by:

$$\hat{\theta} \pm Z_{\alpha/2}\sqrt{\hat{V}_{11}}, \quad \hat{\lambda}_1 \pm Z_{\alpha/2}\sqrt{\hat{V}_{22}} \quad \text{and} \quad \hat{\lambda}_2 \pm Z_{\alpha/2}\sqrt{\hat{V}_{33}},$$

where Z_α is the αth percentile of the standard normal distribution.

9. Numerical Studies

In this section, we assess the performance of estimation procedure by simulation. In addition to this, we have applied the cubic transmuted exponential distribution on two real-life data sets to show its applicability.

9.1 Simulation Study

For simulation study, we have considered five sample sizes 50, 100, 200, 500 and 1000 from cubic transmuted exponential distribution respectively. In each setup 10000 random samples are generated for fixed parameters $\theta = 3$, $\lambda_1 = 0.3$ and $\lambda_2 = 0.7$. The maximum likelihood estimates of the parameters are obtained and the average value of these estimates along with the mean square errors ($MSE's$) are computed and are given in Table 2. It has been observed that the estimated values of parameters are very close to the fixed values of parameters used in the simulation. The $MSE's$ of the estimates decreases with increase in the sample size which shows that the estimates are consistent.

In the following we have applied the cubic transmuted exponential distribution on two real data sets.

9.2 The Wheaton River Data

The data set provides 72 exceedances of flood peaks in m^3/s of the Wheaton River for the year $1958-1984$. The data has been previously used by Choulakian and

Table 2: Average estimates of parameters and $MSE's$ for CTE distribution

Sample Size	Estimate			MSE		
	θ	λ_1	λ_2	θ	λ_1	λ_2
50	2.989	0.349	0.565	3.197	0.725	1.063
100	3.001	0.316	0.637	1.679	0.491	0.428
200	2.999	0.291	0.696	0.457	0.115	0.128
500	2.992	0.286	0.714	0.137	0.034	0.042
1000	2.999	0.289	0.716	0.062	0.016	0.020

Stephens (2001) and Akinsete et al. (2008). The summary statistics of the data are given in Table 3.

Table 3: Summary statistics for selected data sets

	Min.	Q_1	Median	Mean	Q_3	Max.
The Wheaton River data	0.10	2.13	9.50	12.20	20.13	64.00
The Floyd River data	318	1590	3570	6771	6725	71500

Table 4: $MLE's$ of parameters and respective $SE's$ for selected models

Distribution	Parameter	Estimate	SE
	θ	10.657	1.767
CTE	λ_1	-0.582	0.557
	λ_2	0.951	0.748
TE	θ	13.249	2.766
	λ	0.167	0.317
	θ	0.194	0.332
BE	λ_1	0.801	0.147
	λ_2	0.385	0.637
EE	θ	0.096	0.014
	λ	0.975	0.144

We have fitted transmuted exponential, cubic transmuted exponential, beta exponential and exponentiated exponential distributions on this data. Estimated *pdf* and *cdf* for the Wheaton River data set are presented in the upper panels of Figure 4. The $MLE's$ with their corresponding $SE's$ are given in Table 4. The computed log-likelihood, Akaike's information criterion (AIC), corrected Akaike's information criterion (AICc), Bayesian information criterion (BIC), Kolmogorov-Smirnov statistic (D_n), Anderson-Darling statistic (A^2) and Cramér-von Mises statistic (W^2) are given in Table 5. It is observed, on the basis of those criteria, that the cubic transmuted exponential distribution is the most appropriate model for fitting of this data.

Table 5: Selection criteria estimated for selected models

Distribution	logLike	AIC	AICc	BIC	D_n	A^2	W^2
CTE	-248.994	503.987	504.340	510.817	0.090	1.411	0.069
TE	-251.965	507.930	508.104	512.484	0.131	2.587	0.115
BE	-251.225	508.449	508.802	515.279	0.102	1.660	0.081
EE	-253.057	510.114	510.288	514.667	0.108	2.383	0.114

9.3 The Floyd River Data

We have considered the Floyd River data set which provides the consecutive annual flood discharge rates for the year $1935 - 1973$. The data set has been previously used by Akinsete et al. (2008). For the source and details of the data see Mudholkar and Huston (1996). The summary statistics of the data are given in Table 3.

Table 6: $MLE's$ of the parameters and respective $SE's$ for selected models

Distribution	Parameter	Estimate	SE
CTE	θ	14.69×10^{3}	8.89×10^{-5}
	λ_1	8.42×10^{-1}	1.60×10^{-1}
	λ_2	8.92×10^{-1}	5.08×10^{-1}
TE	θ	10.51×10^{3}	8.86×10^{-6}
	λ	8.07×10^{-1}	1.65×10^{-1}
BE	θ	9.71×10^{-6}	1.08×10^{-13}
	λ_1	9.21×10^{-1}	1.91×10^{-19}
	λ_2	1.40×10^{1}	1.74×10^{-6}
EE	θ	1.44×10^{-4}	2.34×10^{-5}
	λ	9.69×10^{-1}	2.42×10^{-9}

Table 7: Selection criteria estimated for selected models

Distribution	logLik	AIC	AICc	BIC	D_n	A^2	W^2
CTE	-379.399	764.799	765.484	769.789	0.090	0.044	0.003
TE	-380.662	765.323	765.657	768.650	0.109	0.048	0.003
BE	-382.906	771.812	772.497	776.802	0.147	0.059	0.004
EE	-382.986	769.972	770.305	773.299	0.152	0.063	0.005

We have considered transmuted exponential, beta exponential and exponentiated exponential distributions as an alternative of the cubic transmuted exponential distribution for the comparison purposes. Estimated *pdf* and *cdf* for the Floyd River data set are presented in the lower panels of Figure 4. Table 6 provides the $MLE's$ with their corresponding $SE's$ for selected models.

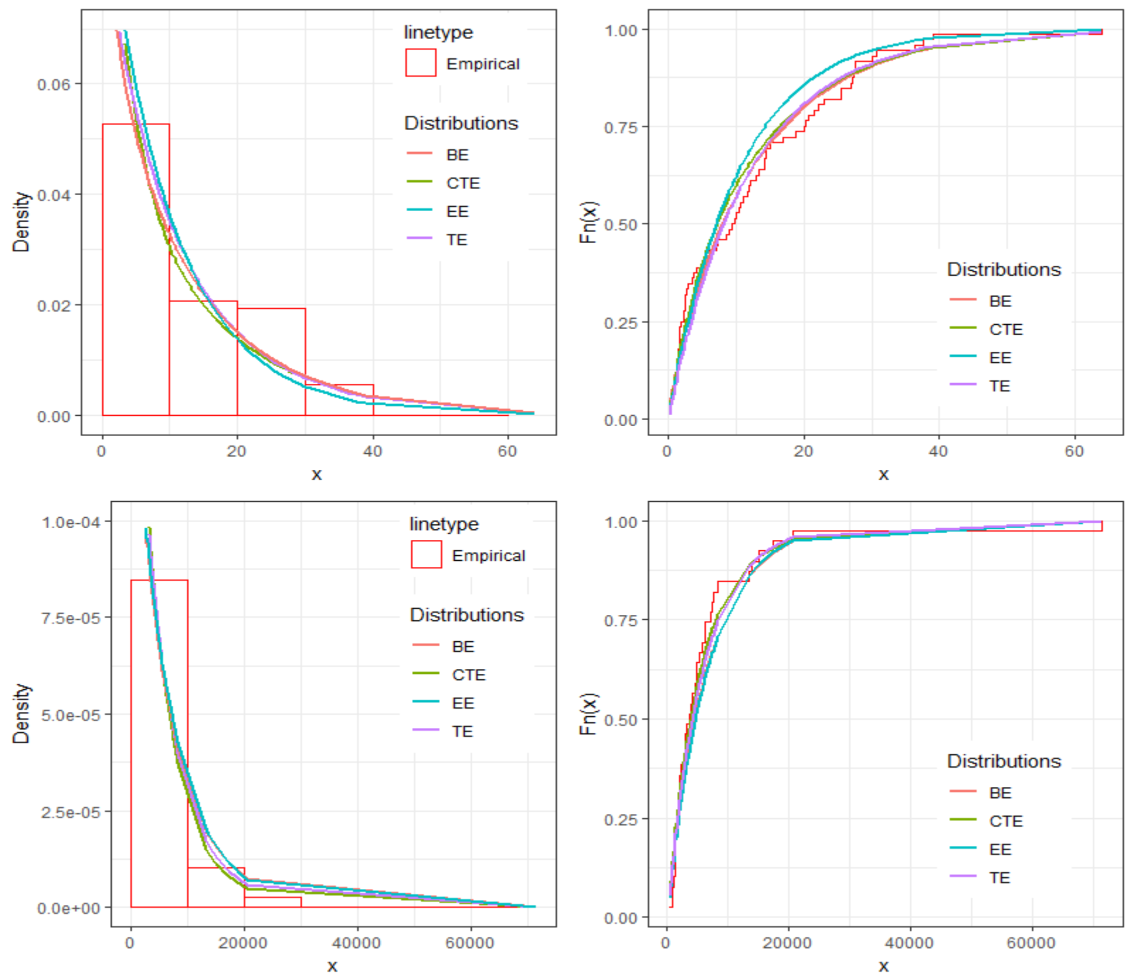

Figure 4: Estimated *pdf* and *cdf* for the Wheaton River and Floyd River data sets.

The performances of fitted models have been compared by computing values of the selection criterion's like log-likelihood, AIC, AICc, BIC, D_n, A^2 and W^2 are computed and are given in the Table 7. The above mentioned criteria provides confirmation in favor of the cubic transmuted exponential distribution.

10. Concluding Remarks

In this paper, we have developed a new general transmuted family of distributions and have introduced a cubic transmuted family of distributions as well. These cubic transmuted distributions are found to be flexible enough when bi-modality appear in the data sets. We have considered exponential distribution to clarify the applicability of this new class of distributions. We have investigated some of its statistical properties including moments, moment generating function, expressions for reliability and hazard rate functions and have discussed the random number generation. Distributions of order statistics have also been discussed. The proposed cubic transmuted exponential distribution have been applied on two real data sets and we have found that the cubic transmuted exponential distribution provides a better fit to these data as compared with the other models used. We have also conducted extensive simulation study to see the performance of estimation procedure and have found that the estimated parameters are close to the actual parameters used in the simulation study and hence the estimation is adequate.

References

1. Akinsete, A., Famoye, F., and Lee, C. (2008). The beta-Pareto distribution. *Statistics*, 6:547–563, doi:10.1080/02331880801983876.

2. Alizadeh, M., Bagheri, S. F., Alizadeh, M., and Nadarajah, S. (2016). A new four-parameter lifetime distribution. *Journal of Applied Statistics*, 44:767–797, doi:10.1080/02664763.2016.1182137.

3. Alizadeh, M., Merovci, F., and Hamedani, G. G. (2017). Generalized transmuted family of distributions: Properties and applications. *Hacettepe Journal of Mathematics and Statistics*, 46:645–667, doi:10.15672/HJMS.201610915478.

4. Alzaatreh, A., Lee, C., and Famoye, F. (2013). A new method for generating families of continuous distributions. *Metron*, 71:63–79, doi:10.1007/s40300–013–0007–y.

5. Aryal, G. R. and Tsokos, C. P. (2009). On the transmuted extreme value distribution with application. *Nonlinear Analysis: Theory, Methods and Applications*, 71:1401–1407, doi:10.1016/j.na.2009.01.168.

6. Aryal, G. R. and Tsokos, C. P. (2011). Transmuted Weibull distribution: A generalization of the Weibull probability distribution. *European Journal of Pure and Applied Mathematics*, 4:89–102.

7. Choulakian, V. and Stephens, M. A. (2001). Goodness-of-fit for the generalized Pareto distribution. *Technometrics*, 43:478–484, doi:10.1198/00401700152672573.

8. Cordeiro, G. and Castro, M. (2010). A new family of generalized distributions. *Journal of Statistical Computation and Simulation*, 81:883–898.

9. Eugene, N., Lee, C., and F., F. (2002). Beta-normal distribution and its applications. *Communications in Statistics-Theory and Methods*, 31:497–512.

10. Gradshteyn, I. S. and Ryzhik, I. M. (2007). *Table of Integrals, Series, and Products*. Elsevier, USA.

11. Granzotto, D. C. T., Louzada, F., and Balakrishnan, N. (2017). Cubic rank transmuted distributions: inferential issues and applications. *Journal of Statistical Computation and Simulation*, 87:2760–2778, doi.10.1080/00949655.2017.1344239.

12. Gupta, R. D. and Kundu, D. (1999). Generalized exponential distributions. *Australian and New Zealand Journal of Statistics*, 41:173–188, doi:10.1111/1467–842X.00072.

13. Gupta, R. D. and Kundu, D. (2001). Exponentiated exponential family: an alternative to Gamma and Weibull distributions. *Biometrical Journal*, 43:117–130, doi: 10.1002/1521–4036(200102)43:1<117::AID–BIMJ117>3.0.CO;2–R.

14. Gupta, R. D. and Kundu, D. (2007). Generalized exponential distributions: Existing results and some recent developments. *Journal of Statistical Planning and Inference*, 137:3537–3547, doi:10.1.1.468.8139.

15. Merovci, F., Alizadeh, M., and Hamedani, G. G. (2016). Another generalized transmuted family of distributions: Properties and applications. *Austrian Journal of Statistics*, 45:71–93, doi:10.17713/ajs.v45i3.109.

16. Mudholkar, G. S. and Huston, A. D. (1996). The exponentiated Weibull family: Some properties and a flood data application. *Commun. Statist. Theory Meth.*, 23:1149–1171, doi:10.1080/03610929608831886.

17. Nadarajah, S. and Kotz, S. (2006). The Beta exponential distribution. *Reliability Engineering and System Safety*, 91:689–697, doi.org/10.1016/j.ress.2005.05.008.

18. Nofal, Z. M., Afify, A. Z., Yousof, H. M., and Cordeiro, G. M. (2017). The generalized transmuted-G family of distributions. *Communications in Statistics-Theory and Methods*, 46:4119–4136, doi:10.1080/03610926.2015.1078478.

19. Owoloko, E. A., Oguntunde, P. E., and Adejumo, A. O. (2015). Performance rating of the transmuted exponential distribution: An analytical approach. *Springer Plus*, 4:8–18, doi:10.1186/s40064–015–1590–6.

20. Rahman, M. M., Al-Zahrani, B., and Shahbaz, M. Q. (2018a). A General Transmuted Family of Distributions. *Pak J Stat Oper Res*, 14:451–469, doi:10.18187/pjsor.v14i2.2334.

21. Rahman, M. M., Al-Zahrani, B., and Shahbaz, M. Q. (2018b). Cubic Transmuted Pareto Distribution. *Annals of Data Science*, doi:10.1007/s40745–018–0178–8.

22. Shaw, W. T. and Buckley, I. R. C. (2007). The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. UCL discovery repository, http://discovery.ucl.ac.uk/id/eprint/643923.

23. Tahir, M. H. and Cordeiro, G. M. (2016). Compounding of distributions: a survey and new generalized classes. *Journal of Statistical Distributions and Applications*, 3:1–35, doi:10.1186/s40488–016–0052–1.

24. Yousof, H. M., Afify, A. Z., Alizadeh, M., Butt, N. S., and Hamedani, G. G. (2015). The transmuted exponentiated generalized-G family of distributions. *Pakistan Journal of Statistics and Operation Research*, 4:441–464, doi:10.18187/pjsor.v11i4.1164.

25. Zaindin, M. and Sarhan, A. M. (2009). Parameters estimation of the modified Weibull distribution. *Applied Mathematical Sciences*, 3:541–550.

Appendix: The Hessian Matrix for the $CT-$ Exponential Distribution

The Hessian matrix is given as

$$H=\begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix} = \begin{pmatrix} -\frac{\delta^2 l}{\delta\theta^2} & -\frac{\delta^2 l}{\delta\theta\cdot\delta\lambda_1} & -\frac{\delta^2 l}{\delta\theta\cdot\delta\lambda_2} \\ -\frac{\delta^2 l}{\delta\theta\cdot\delta\lambda_1} & -\frac{\delta^2 l}{\delta\lambda_1^2} & -\frac{\delta^2 l}{\delta\lambda_1\cdot\delta\lambda_2} \\ -\frac{\delta^2 l}{\delta\theta\cdot\delta\lambda_2} & -\frac{\delta^2 l}{\delta\lambda_1\cdot\delta\lambda_2} & -\frac{\delta^2 l}{\delta\lambda_2^2} \end{pmatrix},$$

where the variance-covariance matrix V is obtained by

$$V=\begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix}^{-1},$$

with the elements

$$H_{11}=-\sum_{i=1}^{n}\left[\frac{\frac{4(1-\lambda_1)x_i^2e^{\frac{2x_i}{\theta}}}{\theta^4}+\frac{2(\lambda_1-\lambda_2)x_i^2e^{\frac{x_i}{\theta}}}{\theta^4}+\frac{4(1-\lambda_1)x_ie^{\frac{2x_i}{\theta}}}{\theta^3}+\frac{4(\lambda_1-\lambda_2)x_ie^{\frac{x_i}{\theta}}}{\theta^3}}{(1-\lambda_1)e^{\frac{2x_i}{\theta}}+2(\lambda_1-\lambda_2)e^{\frac{x_i}{\theta}}+3\lambda_2} - \frac{\left\{-\frac{2(1-\lambda_1)x_ie^{\frac{2x_i}{\theta}}}{\theta^2}-\frac{2(\lambda_1-\lambda_2)x_ie^{\frac{x_i}{\theta}}}{\theta^2}\right\}^2}{\left\{(1-\lambda_1)e^{\frac{2x_i}{\theta}}+2(\lambda_1-\lambda_2)e^{\frac{x_i}{\theta}}+3\lambda_2\right\}^2}\right]-\frac{n}{\theta^2}+\sum_{i=1}^{n}\frac{6x_i}{\theta^3},$$

$$H_{12}=-\sum_{i=1}^{n}\left[\frac{\frac{2x_ie^{\frac{2x_i}{\theta}}}{\theta^2}-\frac{2x_ie^{\frac{x_i}{\theta}}}{\theta^2}}{(1-\lambda_1)e^{\frac{2x_i}{\theta}}+2(\lambda_1-\lambda_2)e^{\frac{x_i}{\theta}}+3\lambda_2} - \frac{\left(2e^{\frac{x_i}{\theta}}-e^{\frac{2x_i}{\theta}}\right)\left\{-\frac{2(1-\lambda_1)x_ie^{\frac{2x_i}{\theta}}}{\theta^2}-\frac{2(\lambda_1-\lambda_2)x_ie^{\frac{x_i}{\theta}}}{\theta^2}\right\}}{\left\{(1-\lambda_1)e^{\frac{2x_i}{\theta}}+2(\lambda_1-\lambda_2)e^{\frac{x_i}{\theta}}+3\lambda_2\right\}^2}\right],$$

$$H_{13} = -\sum_{i=1}^{n}\left[\frac{2x_i e^{\frac{x_i}{\theta}}}{\theta^2\left\{(1-\lambda_1)\,e^{\frac{2x_i}{\theta}}+2\,(\lambda_1-\lambda_2)\,e^{\frac{x_i}{\theta}}+3\lambda_2\right\}}\right.$$
$$\left.-\frac{\left(3-2e^{\frac{x_i}{\theta}}\right)\left\{-\frac{2(1-\lambda_1)x_i e^{\frac{2x_i}{\theta}}}{\theta^2}-\frac{2(\lambda_1-\lambda_2)x_i e^{\frac{x_i}{\theta}}}{\theta^2}\right\}}{\left\{(1-\lambda_1)\,e^{\frac{2x_i}{\theta}}+2\,(\lambda_1-\lambda_2)\,e^{\frac{x_i}{\theta}}+3\lambda_2\right\}^2}\right],$$

$$H_{22} = \sum_{i=1}^{n}\frac{\left(2e^{\frac{x_i}{\theta}}-e^{\frac{2x_i}{\theta}}\right)^2}{\left\{(1-\lambda_1)\,e^{\frac{2x_i}{\theta}}+2\,(\lambda_1-\lambda_2)\,e^{\frac{x_i}{\theta}}+3\lambda_2\right\}^2},$$

$$H_{23} = \sum_{i=1}^{n}\frac{\left(3-2e^{\frac{x_i}{\theta}}\right)\left(2e^{\frac{x_i}{\theta}}-e^{\frac{2x_i}{\theta}}\right)}{\left\{(1-\lambda_1)\,e^{\frac{2x_i}{\theta}}+2\,(\lambda_1-\lambda_2)\,e^{\frac{x_i}{\theta}}+3\lambda_2\right\}^2},$$

and

$$H_{33} = \sum_{i=1}^{n}\frac{\left(3-2e^{\frac{x_i}{\theta}}\right)^2}{\left\{(1-\lambda_1)\,e^{\frac{2x_i}{\theta}}+2\,(\lambda_1-\lambda_2)\,e^{\frac{x_i}{\theta}}+3\lambda_2\right\}^2}.$$