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Goodness of Fit Tests for Marshal-Olkin Extended Rayleigh distribution

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Abstract

A class of goodness of fit tests for Marshal-Olkin Extended Rayleigh distribution with estimated parameters is proposed. The tests are based on the empirical distribution function. For determination of asymptotic percentage points, Kolomogorov-Sminrov, Cramer-von-Mises, Anderson-Darling, Watson, and Liao-Shimokawa test statistics are used. This article uses Monte Carlo simulations to obtain asymptotic percentage points for Marshal-Olkin extended Rayleigh distribution. Moreover, power of the goodness of fit test statistics is investigated for this lifetime model against several alternatives.

Key Words: Marshal-Olkin Extended Rayleigh distribution, Goodness of fit, Monte-Carlo simulation, Asymptotic percentage points, Power test.

1. Introduction

The Rayleigh distribution is very popular among lifetime distributions. Some of the areas where it is used are the study of vibrations and waves, theory of communication to explain instantaneous peak power and hourly median of signals received at a radio, to model wind speed under certain circumstances at wind turbine sites in a year and for modeling the lifetimes of devices. There are variety of methods to add parameters to some existing probability distribution to get some new distribution, which generally provides much more flexibility to model the lifetime data. Marshall and Olkin (1997) suggested that a new lifetime distribution having survival function$\bar{F}(x;\alpha)$can be formed by adding a parameter "α" to another distribution $G(x)$. i.e.

$$\bar{F}(x,\alpha)=\frac{\alpha\,\bar{G}(x)}{1-\bar{\alpha}\,G(x)} \quad , \; x\epsilon R\,,\; \alpha>0 \tag{1}$$

where$\bar{\alpha}=1-\alpha$, and $\bar{G}(x)$ is the survival function of a continuous type. They discussed this method with the application to the Exponential and Weibull families.

Goodness of fit tests evaluate the degree of agreement between the observed sample distribution and the theoretical distribution. There are so many goodness of fit tests available in the literature, some of which endure severe limitations. Grouping of data is requirement of chi-square goodness of fit test which do not work in case of smaller sample. Tests based on correlations and tests based on moment ratios are most of the times under-estimated.

In most of practical situations, the parameters are usually unknown and are estimated from sample data, using an estimation method such as maximum likelihood. Goodness of-fit test statistics are not distribution-free when the parameters are unknown and have to be estimated from sample. The distribution of test statistics depend on sample size, the population parameters being estimated, estimation technique and on the hypothesized distribution.

Empirical distribution function (EDF) based tests are appropriate to use when the population parameters are unknown and are estimated from sample data, as these provide high power than other tests. Moreover, the empirical distribution functions based goodness of fit tests give equal weight to discrepancy between theoretical distribution and empirical distribution functions consequent to all observations.

Many researchers such as Lilliefors (1967), Lilliefors (1969), Woodruff et al. (1984), and Yen and Moore (1988) have used different test statistics to the case where parameters are unknown and to be estimated from sample. Recently, the goodness of fit for several distributions have been studied by many authors. Such as, for Generalized Pareto distribution Choulakian and Stephens (2001) used Cramer-von Mises and Anderson Darling test statistics to test goodness-of-fit and power of GP distribution against other distributions. Hassan (2005) investigated goodness of fit of Generalized Exponential distribution which was originally introduced by Gupta and Kundu (1999). Goodness of Generalized Frechet distribution was studied by Abd-Elfattah et al. (2010). They used Kolmogorov-Smirnov, Cramer-von Mises, Anderson Darling, Watson and Liao and Shimokawa statistics to test goodness of fit. Abd-Elfattah (2011) tested Generalized Rayleigh distribution for goodness of fit when the parameters were not known. Shin et al. (2011) evaluated the goodness of Generalized Extreme Value distribution and Generalized Logistic distribution with extension to shape parameter. For Gumbel distribution, goodness of fit was investigated by Zainal Abidin and Midi (2012) using six goodness of fit statistics. Al-Zahrani (2012) used Kolmogorov-Smirnov, Cramer-von Mises, Anderson-Darling, Watson and Liao and Shimokawa statistic to test the goodness of fit of Topp-Leone distribution when parameter were not known.

This article is focused on the goodness of fit testing of Marshall-Olkin Extended Rayleigh distribution $MOR(\alpha,\beta)$ which is obtained from Cordeiro and Lemonte (2013) for $\lambda = \beta^2$ *and* $\gamma = 2$. MOR distribution is right skew, symmetrical and left skew depending upon values of parameters. This new distribution has wide applications due to its characteristic of modeling the lifetime data with increasing and modified upside-down-bathtub failure rate. We have assessed the performance of this new lifetime model using five important EDF tests. These are Kolmogorov Sminrov, Cramer-Von Mises, Anderson Darling, Watson and Liao and Shimokawa test statistics. We have computed asymptotic percentage points also known as critical values of these goodness of fit test statistics using Monte-Carlo simulations. We used Newton-Raphson iterative method to obtain maximum likelihood estimates of unknown parameters of Marshall-Olkin Extended Rayleigh distribution. We have also calculated power of these test statistics for the proposed distribution against six competitive probability distributions. All simulations are performed in R.

This article is organized as follows. In section 2, we present estimation of the parameters by maximum likelihood for $MOR(\alpha,\beta)$ distribution. Simulations and goodness of fit tests are discussed in section 3 and 4. Section 5 is devoted to the calculations of critical values for the test statistics, while power comparison is given in section 6. Real data application is presented in section 7. Finally, we conclude the paper in section 8.

2. Estimating the parameters of Marshal-Olkin extended Rayleigh distribution

A two parameter distribution, named as Marshal-Olkin Extended Rayleigh (MOR) distribution is obtained by using equation (1) as a generalization of standard Rayleigh distribution. Let 'X' be a continuous random variable follows the Marshal-Olkin Extended Rayleigh (MOR) distribution with shape parameter a and scale parameter ß, the cumulative distribution function for MOR distribution is

$$F(x) = \frac{1-e^{-(\beta x)^2}}{1-\bar{\alpha}e^{-(\beta x)^2}}\ ,\ \ x>0,\ \ \alpha,\beta>0\ \ where\ \bar{\alpha} = 1-\alpha. \tag{2}$$

The density function of Marshal-Olkin Extended Rayleigh (MOR) distribution is

$$f(x) = \frac{2\alpha\beta^2 x e^{-(\beta x)^2}}{\left(1-\bar{\alpha}e^{-(\beta x)^2}\right)^2}\ \ ,\ \ \ x>0,\ \ \alpha,\beta>0. \tag{3}$$

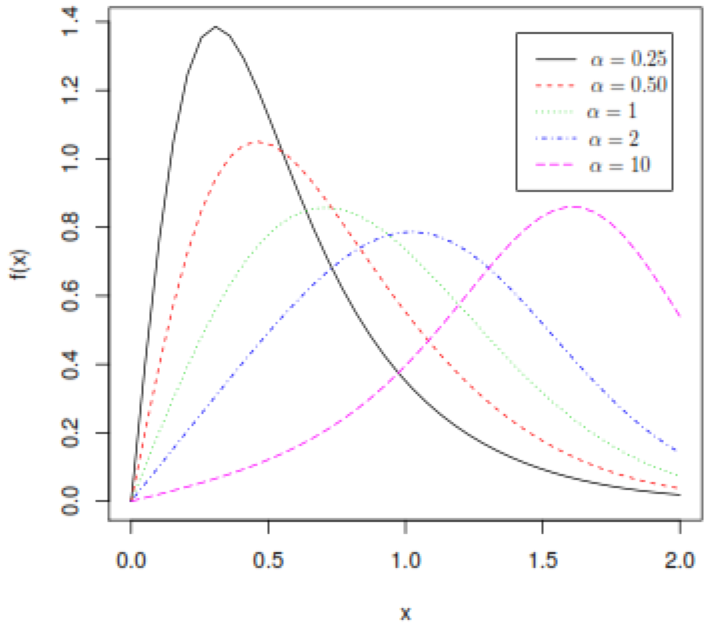

Figure 1. Probability density function of Marshal-Olkin Extended Rayleigh distribution for $\beta = 1$ and different values of shape parameter α.

Let $X_1, X_2, \dots, X_n$ be a random sample from Marshal-Olkin Extended Rayleigh (MOR) distribution, then the log-likelihood function is

$$L = n\ln 2 + n\ln\alpha + 2n\ln\beta + \sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n} (\beta x_i)^2 - 2\sum_{i=1}^{n} \ln\left(1 - \bar{\alpha} e^{-(\beta x_i)^2}\right) \quad (4)$$

The normal equations are

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} - 2\sum_{i=1}^{n}\left[\frac{e^{-(\beta x)^2}}{1-(1-\alpha)e^{-(\beta x)^2}}\right] = 0 \quad (5)$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{2n}{\beta} - 2\beta\sum_{i=1}^{n} x_i^2 - 4\sum_{i=1}^{n}\left[\frac{(1-\alpha)\beta x_i^2 e^{-(\beta x)^2}}{1-(1-\alpha)e^{-(\beta x)^2}}\right] = 0 \quad (6)$$

The MLE's of α and β can be determined from the solution of non-linear equations (5) and (6). Since the MLE's are not in explicit form so we use Newton-Raphson iterative method to obtain the numerical estimates of the parameters of MOR distribution.

2.1. Random Number Generator. The random number generator of Marshal-Olkin Extended Rayleigh (MOR) distribution is given by

$$x = \frac{1}{\beta}\sqrt{\ln\left(\frac{R\bar{\alpha}-1}{R-1}\right)} \quad (7)$$

where R is the random number from uniform distribution u(0, 1).

3. Simulations and Power Study for Marshal-Olkin extended Rayleigh distribution

We have assessed the performance of the proposed lifetime model using five important goodness of fit tests. For this purpose, we have computed critical values of these goodness of fit test statistics using Monte-Carlo simulations. We used Newton-Raphson iterative method to obtain maximum likelihood estimates of unknown parameters of Marshal-Olkin Extended Rayleigh distribution. We have also calculated power of these test statistics for MOR distribution against six competitive probability distributions.

4. Goodness of Fit Tests

Goodness of fit tests are used to assess whether a particular data set is consistent with a hypothesized null distribution or not. We use five important EDF tests. These are Kolmogorov Sminrov, Cramer-Von Mises, Anderson Darling, Watson and Liao and Shimokawa test statistics. To test the null hypothesis,
H_0: The random sample $X_1, X_2, \dots, X_n$ come from Marshal-Olkin Extended Rayleigh distribution (3) with unknown parameters, we used the following formulas of the test statistics for computational purpose :

1. The Kolomogorov Smirnov (KS) test statistic D_n is

$$D_n = max\left[\frac{i}{n} - F_0(x_i, \hat{\alpha}, \hat{\beta}), F_0(x_i, \hat{\alpha}, \hat{\beta}) - \frac{i-1}{n}\right] \quad (8)$$

Where $F_0\left(x_i, \hat{\alpha}, \hat{\beta}\right)$ is the cumulative distribution function of $MOR(\alpha, \beta)$ distribution, n is the sample size and $\hat{\alpha}, \hat{\beta}$ are the estimated parameters using maximum likelihood estimators of α and β from (5) and (6), respectively.

2. The Cramer-von-Mises (CVM) test statistic W_n^2 is

$$W_n^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left[F_0(x_i, \hat{\alpha}, \hat{\beta}) - \frac{2i-1}{2n}\right]^2 \quad (9)$$

3. The Anderson-Darling (AD) test statistic A_n^2 is

$$A_n^2 = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[ln\, F_0\,(x_i, \hat{\alpha}, \hat{\beta}) + ln\left(1 - F_0(x_{n+1-i}, \hat{\alpha}, \hat{\beta})\right)\right] \quad (10)$$

4. The Watson U_n^2 test statistic is

$$U_n^2 = W_n^2 - n\left[\frac{\sum_{i=1}^{n} F_0(x_i, \hat{\alpha}, \hat{\beta})}{n} - \frac{1}{2}\right]^2 \quad (11)$$

5. The Liao and Shimokawa (LS) test statistic is

$$L_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{max\left[\frac{i}{n} - F_0(x_i, \hat{\alpha}, \hat{\beta}), F_0(x_i, \hat{\alpha}, \hat{\beta}) - \frac{i-1}{n}\right]}{\sqrt{F_0(x_i, \hat{\alpha}, \hat{\beta})\left[1 - F_0(x_i, \hat{\alpha}, \hat{\beta})\right]}} \quad (12)$$

5. Calculation of critical values for Marshal-Olkin extended Rayleigh distribution

For calculating critical values, we proceeded as follows:

1. We generated a random sample $X_1, X_2, \dots, X_n$ from Marshal-Olkin Extended Rayleigh distribution with probability density function (3).
2. For this, we first generated a random sample of n ordered statistics i.e. $U_{(1)}, U_{(2)}, U_{(3)}, \dots\dots, U_{(n)}$ from uniform distribution $u(0,1)$.
3. Then using (7), the random number generator of MOR (α, β) with β =1 and $\alpha = 0.50$, we obtained an ordered sample of size n from MOR distribution.
4. We used this random sample to estimate the unknown parameters of MOR distribution by method of maximum likelihood. Because the normal equations (5) and (6) to obtain MLE's are non-linear, so we used Newton-Raphson iterative method to solve these non-linear equations.
5. We performed this iterative procedure using the 'rootSolve' package in R that was suggested by Soetaert (2009).
6. Then we determined the cumulative distribution function of our hypothesized Marshal-Olkin Extended Rayleigh distribution using these maximum likelihood estimates of the unknown parameters.
7. We selected the sample of size as $n = 5\,(5), 50\,(10), 100$ and calculate all five test statistics D_n, W_n^2, A_n^2, U_n^2 and L_n for these sample sizes.
8. We repeated this procedure to generate 10,000 Monte-Carlo runs, from which we obtained 10,000 independent values of each test statistic.
9. We then ranked these 10,000 values for each statistic.
10. We selected seven levels of significance (γ) as 0.01, 0.025, 0.05, 0.10, 0.15, 0.20 and 0.25 at which we compute the critical values for each test statistic against each sample size.

Following table represents the critical points of five goodness of fit test statistics against different significance level and sample sizes using Monte Carlo method.

Table 1: Critical Points of Test Statistics for Marshal-Olkin extended Rayleigh Distribution

Sample Size	Test Statistic	Significance Level γ						
		0.01	0.025	0.05	0.10	0.15	0.20	0.25
5	D_n	0.3498	0.3305	0.3143	0.2935	0.2795	0.2697	0.2611
	W_n^2	0.1386	0.1176	0.1029	0.0862	0.0766	0.0698	0.0643

	A_n^2	1.1269	0.8877	0.7380	0.6036	0.5340	0.4856	0.4480
	U_n^2	0.1386	0.1176	0.1029	0.0862	0.0766	0.0697	0.0643
	L_n	2.9346	2.1388	1.7308	1.4279	1.2850	1.2088	1.1561
10	D_n	0.2704	0.2531	0.2388	0.2222	0.2119	0.2023	0.1946
	W_n^2	0.1511	0.1266	0.1078	0.0893	0.0786	0.0706	0.0649
	A_n^2	1.0794	0.8879	0.7494	0.6115	0.5443	0.4951	0.4534
	U_n^2	0.1511	0.1266	0.1078	0.0893	0.0786	0.0706	0.0649
	L_n	1.9710	1.5744	1.3632	1.1889	1.1049	1.0524	1.0104
15	D_n	0.2316	0.2156	0.2024	0.1877	0.1773	0.1702	0.1641
	W_n^2	0.1593	0.1314	0.1123	0.0928	0.0815	0.0728	0.0668
	A_n^2	1.1057	0.9021	0.7732	0.6316	0.5594	0.5079	0.4665
	U_n^2	0.1594	0.1314	0.1123	0.0928	0.0815	0.0728	0.0668
	L_n	1.6417	1.4050	1.2447	1.0988	1.0275	0.9796	0.9412
20	D_n	0.2026	0.1888	0.1770	0.1650	0.1563	0.1494	0.1443
	W_n^2	0.1630	0.1354	0.1140	0.0933	0.0818	0.0738	0.0675
	A_n^2	1.1284	0.9041	0.7697	0.6431	0.5645	0.5083	0.4688
	U_n^2	0.1630	0.1354	0.1139	0.0932	0.0817	0.0738	0.0674
	L_n	1.4749	1.2857	1.1532	1.0435	0.9741	0.9289	0.8943
25	D_n	0.1833	0.1720	0.1614	0.1491	0.1414	0.1354	0.1303
	W_n^2	0.1652	0.1369	0.1144	0.0945	0.0832	0.0741	0.0676
	A_n^2	1.1338	0.9283	0.7876	0.6441	0.5644	0.5127	0.4729
	U_n^2	0.1652	0.1369	0.1144	0.0945	0.0832	0.0741	0.0676
	L_n	1.4167	1.2331	1.1173	1.0056	0.9402	0.8981	0.8666
30	D_n	0.1701	0.1580	0.1482	0.1369	0.1294	0.1240	0.1195
	W_n^2	0.1692	0.1381	0.1175	0.0954	0.0832	0.0745	0.0680
	A_n^2	1.1078	0.9422	0.7906	0.6519	0.5763	0.5186	0.4774
	U_n^2	0.1692	0.1381	0.1175	0.0954	0.0832	0.0745	0.0680
	L_n	1.3432	1.1842	1.0791	0.9755	0.9190	0.8772	0.8456
35	D_n	0.1570	0.1463	0.1374	0.1280	0.1207	0.1157	0.1112
	W_n^2	0.1668	0.1367	0.1164	0.0948	0.0833	0.0750	0.0683
	A_n^2	1.1120	0.9257	0.7890	0.6436	0.5731	0.5178	0.4777
	U_n^2	0.1668	0.1367	0.1164	0.0948	0.0833	0.0750	0.0683
	L_n	1.3092	1.1446	1.0411	0.9478	0.8899	0.8551	0.8240
40	D_n	0.1483	0.1386	0.1286	0.1189	0.1134	0.1087	0.1044
	W_n^2	0.1672	0.1365	0.1155	0.0945	0.0831	0.0754	0.0685
	A_n^2	1.1114	0.9276	0.7849	0.6476	0.5754	0.5220	0.4794
	U_n^2	0.1672	0.1365	0.1155	0.0945	0.0831	0.0754	0.0685
	L_n	1.2417	1.1062	1.0241	0.9311	0.8787	0.8388	0.8087
45	D_n	0.1410	0.1308	0.1223	0.1131	0.1069	0.1023	0.0985
	W_n^2	0.1718	0.1418	0.1173	0.0959	0.0834	0.0754	0.0686
	A_n^2	1.1210	0.9217	0.7816	0.6497	0.5730	0.5209	0.4786
	U_n^2	0.1718	0.1418	0.1173	0.0959	0.0834	0.0754	0.0686
	L_n	1.1922	1.0786	0.9949	0.9112	0.8629	0.8217	0.7932
50	D_n	0.1359	0.1249	0.1165	0.1071	0.1015	0.0972	0.0935
	W_n^2	0.1701	0.1393	0.1172	0.0952	0.0833	0.0752	0.0681
	A_n^2	1.1329	0.9226	0.7805	0.6483	0.5745	0.5165	0.4768
	U_n^2	0.1701	0.1393	0.1172	0.0952	0.0833	0.0752	0.0681
	L_n	1.1976	1.0708	0.9813	0.9026	0.8490	0.8125	0.7836
60	D_n	0.1224	0.1136	0.1059	0.0982	0.0930	0.0893	0.0862
	W_n^2	0.1713	0.1390	0.1165	0.0957	0.0833	0.0752	0.0685
	A_n^2	1.1345	0.9413	0.7858	0.6541	0.5786	0.5212	0.4807
	U_n^2	0.1713	0.1390	0.1165	0.0957	0.0833	0.0752	0.0685

	L_n	1.1658	1.0482	0.9652	0.8798	0.8316	0.7940	0.7641
70	D_n	0.1141	0.1056	0.0989	0.0916	0.0869	0.0832	0.0801
	W_n^2	0.1657	0.1378	0.1177	0.0969	0.0839	0.0757	0.0688
	A_n^2	1.0862	0.9124	0.7889	0.6593	0.5838	0.5291	0.4830
	U_n^2	0.1657	0.1378	0.1177	0.0969	0.0839	0.0757	0.0688
	L_n	1.1224	1.0239	0.9438	0.8674	0.8178	0.7806	0.7533
80	D_n	0.1060	0.0990	0.0922	0.0856	0.0813	0.0778	0.0750
	W_n^2	0.1630	0.1393	0.1174	0.0966	0.0844	0.0756	0.0697
	A_n^2	1.0993	0.9148	0.7907	0.6549	0.5796	0.5264	0.4830
	U_n^2	0.1630	0.1393	0.1174	0.0966	0.0844	0.0756	0.0697
	L_n	1.0954	0.9992	0.9331	0.8530	0.8045	0.7685	0.7395
90	D_n	0.1016	0.0943	0.0878	0.0816	0.0772	0.0737	0.0708
	W_n^2	0.1703	0.1411	0.1176	0.0979	0.0858	0.0768	0.0693
	A_n^2	1.1124	0.9441	0.7974	0.6580	0.5812	0.5287	0.4892
	U_n^2	0.1703	0.1411	0.1176	0.0979	0.0858	0.0768	0.0693
	L_n	1.0961	1.0028	0.9263	0.8426	0.7938	0.7613	0.7333
100	D_n	0.0967	0.0896	0.0837	0.0774	0.0733	0.0700	0.0675
	W_n^2	0.1708	0.1389	0.1187	0.0971	0.0850	0.0761	0.0692
	A_n^2	1.1280	0.9241	0.7936	0.6629	0.5827	0.5279	0.4832
	U_n^2	0.1708	0.1389	0.1187	0.0971	0.0850	0.0761	0.0692
	L_n	1.0822	0.9851	0.9127	0.8377	0.7865	0.7506	0.7211

From table 1, we can observe that critical values for all the five statistics i.e. D_n, W_n^2, A_n^2, U_n^2 and L_n decreases as the significance level γ increases. For Kolmogorov-Smirnov D_n statistic, critical values decreases as sample size become larger. We can see that Waston test which is the modified form of Cramer-von Mises test behave exactly same as Cramer-von Mises test for MOR distribution. Thus we cannot distinguish between Cramer-von Mises W_n^2 statistic and Watson U_n^2 statistic for Marshal-Olkin extended Rayleigh distribution.
There is no general behavior of Anderson-Darling A_n^2 statistic with regard to sample size. Critical values for Liao and Shimokawa L_n statistic also decreases as sample sizes increases for all significance levels.

6. Power study for Marshal-Olkin extended Rayleigh distribution

In testing of hypothesis, power is a useful tool to evaluate the goodness of a particular test or to compare two competing tests. Power of goodness of a test is denoted by $1-\beta$ and is defined as a probability that a statistic leads to reject a null hypothesis H_0, when infact it is not true. Here β is the probability of making type-II error.

For power comparison of Marshal-Olkin Extended Rayleigh (MOR) distribution, we choose eight competitive probability distributions as the alternative to our hypothesized distribution. These distributions are:
(i). Two-parameter Weibull distribution $W(5,1)$.
(ii). Standard Cauchy distribution $C(0,1)$.
(iii). Gamma distribution $G(2,1)$.
(iv). Logistic distribution $LOG(0,1)$.
(v). Exponential distribution E (0.67).
(vi). Generalized Exponential distribution GE (3,2).
(vii). Rayleigh distribution $R(1)$.
(viii). Generalized Rayleigh distribution $GR(0.5,1)$.

We calculated the power of Marshal-Olkin Extended Rayleigh distribution by simulating the data from the alternative distribution (H_1) and fitting the MOR distribution (H_0) to this data. We repeated this process 10,000 time for sample size $n = 5(5), 30$and significance level $\gamma = 0.01, 0.25, 0.05, 0.10, 0.15, 0.20, 0.25$. We consider $n = 5$ as small, $n = 15$as moderate and $n = 30$as large sample sizes. We observed the number of times each test statistics exceeded respective critical values at each level of significance to obtain power of tests. The results of power of these tests are presented in table 2 to 7:

Table 2: Power function when the alternative distribution is
Two parameter Weibull distribution $W(5,1)$

Power of the Test								
Sample Size	Test Statistic	Significance Level γ						
		0.01	0.025	0.05	0.10	0.15	0.20	0.25
5	D_n	0.9502	0.9663	0.9757	0.9843	0.9882	0.9919	0.9940
	W_n^2	0.9543	0.9701	0.9781	0.9868	0.9907	0.9935	0.9950
	A_n^2	0.8381	0.9159	0.9534	0.9741	0.9831	0.9883	0.9915
	U_n^2	0.7295	0.8263	0.8837	0.9319	0.9544	0.9671	0.9745
	L_n	0.0006	0.0820	0.4066	0.7531	0.8740	0.9208	0.9460
10	D_n	0.9958	0.9975	0.9980	0.9990	0.9993	0.9994	0.9995
	W_n^2	0.9972	0.9985	0.9994	0.9996	0.9997	0.9998	0.9998
	A_n^2	0.9933	0.9972	0.9986	0.9993	0.9995	0.9998	0.9998
	U_n^2	0.9599	0.9823	0.9913	0.9966	0.9979	0.9987	0.9991
	L_n	0.4535	0.8706	0.9656	0.9920	0.9968	0.9981	0.9988
15	D_n	0.9995	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999
	W_n^2	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
	A_n^2	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
	U_n^2	0.9960	0.9986	0.9994	0.9998	0.9999	0.9999	0.9999
	L_n	0.9332	0.9946	0.9992	0.9999	0.9999	0.9999	0.9999
20	D_n	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
	W_n^2	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
	A_n^2	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
	U_n^2	0.9996	0.9997	0.9998	0.9999	0.9999	0.9999	0.9999
	L_n	0.9989	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999
25	D_n	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
	W_n^2	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
	A_n^2	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
	U_n^2	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
	L_n	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
30	D_n	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
	W_n^2	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
	A_n^2	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
	U_n^2	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
	L_n	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999

Table 2 shows that power of all goodness of fit tests for MOR distribution increases monotonically as level of significance increases. For small to moderate sample size W_n^2 is more powerful than D_n, A_n^2, U_n^2, L_n. For large sample size all test statistics depict good power.

We computed the power of the test for Cauchy, Gamma and Logistic distribution using $D_n, W_n^2, A_n^2, U_n^2 \ and \ L_n$ statistics. The results of power of the tests are very close to one even for small sample sizes for different choices of probability of type-I error. Thus, those alternative distributions are not much informative

Table 3: Power function when the alternative distribution is Exponential distribution

Power of the Test								
Sample Size	Test Statistic	Significance Level γ						
		0.01	0.025	0.05	0.10	0.15	0.20	0.25
5	D_n	0.8206	0.8551	0.8823	0.9131	0.9291	0.9403	0.9479
	W_n^2	0.7817	0.8386	0.8767	0.9114	0.9331	0.9441	0.9528
	A_n^2	0.9005	0.9278	0.9441	0.9620	0.9700	0.9761	0.9799
	U_n^2	0.4668	0.5476	0.6282	0.7259	0.7819	0.8185	0.8447
	L_n	0.8312	0.8860	0.9252	0.9493	0.9623	0.9702	0.9763
10	D_n	0.8893	0.9172	0.9364	0.9526	0.9627	0.9697	0.9807
	W_n^2	0.8956	0.9244	0.9453	0.9648	0.9740	0.9807	0.9855
	A_n^2	0.9733	0.9835	0.9892	0.9934	0.9951	0.9959	0.9966
	U_n^2	0.6848	0.7703	0.8276	0.8824	0.9132	0.9284	0.9440
	L_n	0.9716	0.9851	0.9905	0.9936	0.9958	0.9972	0.9975
15	D_n	0.9381	0.9602	0.9703	0.9852	0.9896	0.9919	0.9935
	W_n^2	0.9520	0.9712	0.9810	0.9885	0.9925	0.9941	0.9956
	A_n^2	0.9945	0.9960	0.9974	0.9985	0.9988	0.9992	0.9995
	U_n^2	0.8278	0.8892	0.9226	0.9513	0.9664	0.9746	0.9797
	L_n	0.9950	0.9977	0.9983	0.9993	0.9996	0.9997	0.9998
20	D_n	0.9714	0.9842	0.9902	0.9946	0.9966	0.9981	0.9986
	W_n^2	0.9804	0.9892	0.9939	0.9973	0.9985	0.9991	0.9994
	A_n^2	0.9989	0.9997	0.9998	0.9998	0.9998	0.9998	0.9998
	U_n^2	0.9165	0.9506	0.9689	0.9833	0.9891	0.9932	0.9952
	L_n	0.9994	0.9997	0.9998	0.9998	0.9998	0.9998	0.9999
25	D_n	0.9904	0.9941	0.9964	0.9985	0.9988	0.9991	0.9992
	W_n^2	0.9934	0.9971	0.9981	0.9994	0.9995	0.9997	0.9997
	A_n^2	0.9995	0.9996	0.9997	0.9998	0.9998	0.9998	0.9998
	U_n^2	0.9642	0.9827	0.9903	0.9951	0.9969	0.9979	0.9986
	L_n	0.9997	0.9997	0.9997	0.9998	0.9998	0.9999	0.9999
30	D_n	0.9960	0.9975	0.9987	0.9992	0.9997	0.9997	0.9999
	W_n^2	0.9976	0.9985	0.9990	0.9997	0.9998	0.9999	0.9999
	A_n^2	0.9997	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999
	U_n^2	0.9849	0.9924	0.9954	0.9978	0.9985	0.9990	0.9994
	L_n	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999

From table 3 we have observed that with increase in significance level, power of all goodness of fit tests for MOR distribution increases. For all sample sizes, $L_n$is more powerful than $D_n, W_n^2, A_n^2\ U_n^2$. All test statistics present high power for moderate and large sample sizes. U_n^2 represented comparatively low power than other tests for all sample sizes.

Table 4: Power function when the alternative distribution is Generalized Exponential distribution

Power of the Test								
Sample Size	Test Statistic	Significance + Level γ						
		0.01	0.025	0.05	0.10	0.15	0.20	0.25
5	D_n	0.5656	0.6390	0.6985	0.7645	0.8055	0.8330	0.8557
	W_n^2	0.5352	0.6186	0.6931	0.7608	0.8050	0.8347	0.8585
	A_n^2	0.4871	0.5856	0.6684	0.7546	0.8017	0.8355	0.8594
	U_n^2	0.1390	0.2163	0.3057	0.4887	0.4887	0.5528	0.6034
	L_n	0.2352	0.3608	0.4826	0.5992	0.6777	0.7340	0.7734
10	D_n	0.5868	0.6526	0.7106	0.7706	0.8127	0.8420	0.8653
	W_n^2	0.6085	0.6700	0.7263	0.7899	0.8293	0.8598	0.8793
	A_n^2	0.5938	0.6755	0.7392	0.8063	0.8483	0.8771	0.8949

	U_n^2	0.1422	0.2179	0.3035	0.4182	0.5024	0.5661	0.6226
	L_n	0.3997	0.5459	0.6424	0.7423	0.7936	0.8318	0.8613
15	D_n	0.6075	0.6735	0.7355	0.8013	0.8386	0.8668	0.8883
	W_n^2	0.6558	0.7157	0.7671	0.8282	0.8590	0.8825	0.9022
	A_n^2	0.6824	0.7464	0.7951	0.8534	0.8840	0.9064	0.9224
	U_n^2	0.1554	0.2324	0.3071	0.4267	0.5096	0.5781	0.6311
	L_n	0.5735	0.6755	0.7520	0.8237	0.8604	0.8851	0.9040
20	D_n	0.6589	0.7303	0.7816	0.8338	0.8669	0.8892	0.9082
	W_n^2	0.7150	0.7649	0.8116	0.8581	0.8859	0.9063	0.9220
	A_n^2	0.7379	0.8003	0.8424	0.8892	0.9131	0.9292	0.9424
	U_n^2	0.1659	0.2407	0.3390	0.4574	0.5424	0.6065	0.6607
	L_n	0.6843	0.7754	0.8291	0.8749	0.9024	0.9180	0.9322
25	D_n	0.7206	0.7800	0.8254	0.8702	0.8974	0.9154	0.9292
	W_n^2	0.7632	0.8160	0.8543	0.8936	0.9140	0.9308	0.9432
	A_n^2	0.7931	0.8545	0.8879	0.9193	0.9386	0.9516	0.9615
	U_n^2	0.1835	0.2810	0.3753	0.4926	0.5717	0.6299	0.6845
	L_n	0.7788	0.8389	0.8789	0.9142	0.9334	0.9472	0.9564
30	D_n	0.7527	0.8047	0.8465	0.8879	0.9147	0.9296	0.9427
	W_n^2	0.8049	0.8471	0.8750	0.9090	0.9298	0.9438	0.9545
	A_n^2	0.8426	0.8831	0.9105	0.9362	0.9517	0.9628	0.9696
	U_n^2	0.2168	0.3040	0.3855	0.5041	0.5839	0.6457	0.6973
	L_n	0.8281	0.8759	0.9091	0.9354	0.9488	0.9599	0.9673

Table 4 presented that for MOR distribution, power of tests improves as significance level increases. We observed that for small sample sizes D_n *and* W_n^2 show high power as compared to other tests while Anderson-Darling statistic A_n^2 appears to be more powerful than D_n, W_n^2, U_n^2, L_n for moderate and large sample sizes.

Table 5: Power function when the alternative distribution is Rayleigh distribution

Power of the Test								
Sample Size	Test Statistic	Significance Level γ						
		0.01	0.025	0.05	0.10	0.15	0.20	0.25
5	D_n	0.8863	0.9098	0.9270	0.9472	0.9565	0.9634	0.9692
	W_n^2	0.8807	0.9085	0.9258	0.9453	0.9567	0.9654	0.9709
	A_n^2	0.8692	0.9078	0.9298	0.9510	0.9629	0.9696	0.9747
	U_n^2	0.4153	0.5151	0.5902	0.6896	0.7476	0.7907	0.8222
	L_n	0.6107	0.7729	0.8593	0.9185	0.9417	0.9547	0.9632
10	D_n	0.9662	0.9740	0.9812	0.9870	0.9901	0.9923	0.9945
	W_n^2	0.9723	0.9798	0.9846	0.9896	0.9918	0.9941	0.9956
	A_n^2	0.9740	0.9840	0.9890	0.9932	0.9947	0.9960	0.9966
	U_n^2	0.6222	0.7184	0.7910	0.8557	0.8957	0.9189	0.9364
	L_n	0.9232	0.9665	0.9814	0.9906	0.9936	0.9946	0.9956
15	D_n	0.9893	0.9925	0.9958	0.9968	0.9980	0.9982	0.9984
	W_n^2	0.9928	0.9956	0.9969	0.9981	0.9984	0.9987	0.9987
	A_n^2	0.9939	0.9968	0.9978	0.9988	0.9989	0.9990	0.9991
	U_n^2	0.7689	0.8475	0.8942	0.9363	0.9542	0.9666	0.9749
	L_n	0.9848	0.9940	0.9970	0.9985	0.9990	0.9991	0.9994
20	D_n	0.9975	0.9984	0.9986	0.9992	0.9995	0.9996	0.9998
	W_n^2	0.9985	0.9989	0.9992	0.9995	0.9996	0.9996	0.9996
	A_n^2	0.9988	0.9992	0.9992	0.9995	0.9996	0.9997	0.9998
	U_n^2	0.8689	0.9195	0.9505	0.9753	0.9838	0.9894	0.9928
	L_n	0.9982	0.9991	0.9993	0.9996	0.9997	0.9997	0.9998
25	D_n	0.9995	0.9996	0.9997	0.9998	0.9998	0.9999	0.9999

	W_n^2	0.9997	0.9998	0.9998	0.9998	0.9998	0.9999	0.9999
	A_n^2	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
	U_n^2	0.9281	0.9622	0.9783	0.9893	0.9936	0.9962	0.9975
	L_n	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
30	D_n	0.9998	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999
	W_n^2	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999
	A_n^2	0.9997	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
	U_n^2	0.9655	0.9827	0.9901	0.9962	0.9981	0.9990	0.9993
	L_n	0.9997	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999

Table 5 showed that for sample size as n = 5 and significance level γ = 0.01 and γ = 0.025, Kolmogorov-Smirnov D_n statistic appears to be more powerful than other tests but for the same sample size and significance level 0.05 to 0.25, Anderson-Darling A_n^2 depicts high power than D_n, W_n^2, U_n^2, L_n . For all the significance levels and sample sizes from 10 till 30, A_n^2 is the most powerful among all test statistics except for n =30, γ = 0.01.

Table 6: Power function when the alternative distribution is Generalized Rayleigh distribution

Power of the Test								
Sample Size	Test Statistic	Significance Level γ						
		0.01	0.025	0.05	0.10	0.15	0.20	0.25
5	D_n	0.5696	0.6359	0.6900	0.7651	0.8087	0.8360	0.8586
	W_n^2	0.5384	0.6141	0.6762	0.7539	0.8030	0.8375	0.8622
	A_n^2	0.5131	0.6176	0.6960	0.7743	0.8206	0.8536	0.8768
	U_n^2	0.1688	0.2421	0.3169	0.4311	0.5059	0.5654	0.6197
	L_n	0.2638	0.4240	0.5530	0.6788	0.7524	0.7912	0.8237
10	D_n	0.5657	0.6383	0.6925	0.7613	0.7995	0.8350	0.8631
	W_n^2	0.5779	0.6507	0.7162	0.7854	0.8279	0.8589	0.8789
	A_n^2	0.6299	0.7062	0.7697	0.8351	0.8699	0.8941	0.9129
	U_n^2	0.1955	0.2820	0.3726	0.4822	0.5612	0.6209	0.6645
	L_n	0.4916	0.6390	0.7319	0.8144	0.8575	0.8813	0.9011
15	D_n	0.5717	0.6510	0.7216	0.7850	0.8270	0.8573	0.8846
	W_n^2	0.6076	0.6823	0.7457	0.8113	0.8496	0.8790	0.8992
	A_n^2	0.6900	0.7683	0.8160	0.8731	0.9004	0.9219	0.9371
	U_n^2	0.2151	0.3121	0.4064	0.5154	0.5931	0.6498	0.6949
	L_n	0.6160	0.7461	0.8183	0.8785	0.9067	0.9232	0.9381
20	D_n	0.6223	0.6905	0.7526	0.8131	0.8511	0.8804	0.8999
	W_n^2	0.6517	0.7219	0.7812	0.8418	0.8773	0.8988	0.9164
	A_n^2	0.7399	0.8145	0.8577	0.8964	0.9195	0.9382	0.9489
	U_n^2	0.2509	0.3468	0.4441	0.5584	0.6301	0.6838	0.7290
	L_n	0.7381	0.8175	0.8692	0.9067	0.9298	0.9437	0.9544
25	D_n	0.6594	0.7216	0.7803	0.8394	0.8745	0.8991	0.9157
	W_n^2	0.6941	0.7615	0.8192	0.8696	0.8975	0.9203	0.9342
	A_n^2	0.7782	0.8377	0.8791	0.9174	0.9390	0.9517	0.9618
	U_n^2	0.2934	0.3945	0.4917	0.5996	0.6666	0.7250	0.7661
	L_n	0.7777	0.8488	0.8919	0.9281	0.9470	0.9573	0.9657
30	D_n	0.6898	0.7547	0.8083	0.8645	0.8965	0.9170	0.9321
	W_n^2	0.7245	0.7932	0.8373	0.8876	0.9151	0.9343	0.9471
	A_n^2	0.8253	0.8688	0.9060	0.9386	0.9532	0.9652	0.9736
	U_n^2	0.3169	0.4358	0.5306	0.6452	0.7138	0.7624	0.7988
	L_n	0.8296	0.8883	0.9218	0.9484	0.9626	0.9710	0.9773

From Table 6 we can see that for n = 5 and γ = 0.01, 0.025, Kolmogorov-Smirnov D_n statistic is more powerful than other tests but for the same sample size with γ =0.05, 0.10, 0.15, 0.20, 0.25, Anderson-Darling A_n^2 shows high power than D_n, W_n^2, U_n^2, L_n .While for all the significance levels and sample size n = 10, A_n^2 is the most powerful among all test statistics. We also observe that for sample size 15, A_n^2 represents highest power for γ = 0.01, 0.025 while for γ = 0.05 till γ = 0.25, L_n is more powerful. For sample size 20, 25 L_n appears to be most powerful except γ = 0.01 and for n = 30 and γ = 0.01, L_n is most powerful among all statistics.

7. Real life application

In this section, we used a real life application using well known data set to show the wider applicability of our proposed model over other competing models in survival and reliability. We considered Exponential, Rayleigh, Generalized Exponential (GE) (Gupta and Kundu, 2001) , Generalized Rayleigh(GR)(Kundu and Raqab, 2005) distributions.

The data set is about the strengths of 1.5 cm glass fibers, measured at the National Physical Laboratory, England. The data is taken from Smith and Naylor (1987) comprises of 63 observations: 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2,0.74, 1.04, 1.27, 1.39,1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 1.62, 1.66,1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.7, 1.77, 1.84, 0.84,1.24,1.3, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.7, 1.78, 1.89.

For each distribution, using maximum likelihood estimates we calculated AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), KS (Kolmogorov-Smirnov) statistic with its respective p-value. Recently, the data was reanalyzed in Barreto-Souza et al. (2010) to fit the beta-generalized exponential (BGE) distribution.

Table 7. Maximum likelihood estimates along with their standard errors, log-likelihood, AIC, BIC, Kolmogorov-Smirnov statistics for the strengths of fibers

Distribution	ML Estimates (Standard Errors)	Log likelihood	AIC	BIC
Weibull	5.78, 1.62 (0.03 ,(0.57)	-15.20	34.41	38.69
Cauchy	1.59,0.02 (0.13),(0.02)	-17.63	39.27	49.54
Gamma(α,β)	17.4458, 11.5745 (3.04686), (2.072)	-23.95	51.90	56.19
Logistic	1.54,0.17 (0.04),(0.02)	-16.02	36.04	40.33
Exponential(λ)	0.6636 (0.08361)	-88.83	179.7	181.8
Rayleigh(β)	0.6490 (0.04089)	-49.79	101.6	103.7
GE(α, λ)	31.349, 2.612 (9.519), (0.2380)	-31.38	66.8	71.1
GR(α, β)	5.4860, 0.9869 (1.2196), (0.0549)	-23.93	51.9	56.1
MR(α, β)	120.1841, 1.4115 (4.9197), (0.0291)	-13.00	30	34.3

Table 8. Kolmogorov-Smirnov Cramer Von Misses and Anderson-Darling statistics for and p-values for the strengths of fibers.

Distribution	Dn	p- value	W_n^2	p- value	A_n^2	p- value
Weibull	0.1522	0.1079	0.2150	0.2404	1.2410	0.2525
Cauchy	0.1029	0.5167	0.1905	0.2871	1.4001	0.2021
Gamma(α,β)	0.2163	0.005508	0.56536	0.0269	3.0856	0.02498
Logistic	0.1253	0.276	0.1724	0.3284	1.2846	0.2373
Exponential(λ)	0.418	<0.0001	3.8622	<0.0001	18.425	<0.00001
Rayleigh(β)	0.3339	<0.0001	2.32209	<0.0001	11.4249	<0.0001

GE(α, λ)	0.2293	0.0027	0.79862	0.0072	4.3375	0.0060
GR(α, β)	0.2151	0.0059	0.5833	0.02428	3.1297	0.02367
MR(α, β)	0.1065	0.4728	0.1068	0.5539	0.7367	0.5283

The results of MLEs (along with their standard errors), log-likelihood, AIC and BIC are presented in table 7. Kolmogorov-Smirnov, Cramer Von Misses and Anderson-Darling along withp-values are presented in table 8. It is evident from table 7 and 8 that Marshal-Olkin Extended Rayleigh distribution provides a better fit than the other competing models.

8. Conclusion

We obtained critical values for Marshal-Olkin Extended Rayleigh distribution using Monte Carlo simulations for different sample sizes n and significance level γ. It is visible from table of critical values that Marshal-Olkin extended Rayleigh distribution cannot distinguish between Cramer-von Mises W_n^2 test and Watson U_n^2 test. As critical values of W_n^2 and $U_n^2$are same for all sample sizes at each significance level. It can clearly be observed from tables of power study that Marshal-Olkin extended Rayleigh distribution shows high power than Weibull, Cauchy, Gamma, Logistic, Exponential, Generalized Exponential, Rayleigh and Generalized Rayleigh distribution. From the real life application, it can be seen that Marshal-Olkin Extended Rayleigh distribution outperforms other existing distributions.

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