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On Modified Burr XII-Inverse Weibull Distribution: Properties and Applications

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Abstract

In this paper, a new lifetime distribution called modified Burr XII-inverse Weibull (MBXII-IW) distribution is developed from T-X family technique. The MBXII-IW distribution is very flexible and its hazard rate function accommodates various shapes. The density function of the MBXII-IW is exponential, left-skewed, right-skewed and symmetrical shaped. To show the importance of the proposed distribution, we derive mathematical properties such as ordinary moments, generating function, residual life functions, reliability measures and characterizations. We address the maximum likelihood estimation technique for the model parameters. We evaluate the performance of the maximum likelihood estimators via a simulation study. We consider an application to a real data set to clarify the potentiality and utility of the MBXII-IW model.

Key Words: Moments, Reliability, Characterizations, Maximum Likelihood Estimation.

Mathematical Subject Classification: 60E05, 62E10, 62E20

1. Introduction

Data analysis is imperious in every aspect of statistical analysis. The statistical characteristics such as skewness, kurtosis, bimodality, monotonic and non-monotonic failure rates are obtained from datasets. The selection of a suitable model for data analysis is challenging task because it depends on the nature of the dataset. However, if a wrong model is applied to analyze the dataset it leads to loss of information and invalid inferences. It is obligatory to identify the most suitable model for the given dataset. In the recent decade, many continuous distributions have been introduced in statistical literature. Some of these distributions, however, are not flexible enough for data sets from survival analysis, life testing, reliability, finance, environmental sciences, biometry, hydrology, ecology and geology. Hence, the applications of the generalized models to these fields are clear requisite. The generalization techniques such as either inserting one or more shape parameters or transforming of the parent distribution are useful to (i) increase the applicability of a parent distribution; (ii) explore skewness and tail properties and (iii) improve the goodness-of-fit of the generalized distributions.

A flexible model for the analysis of lifetime data sets is often attractive to the researchers. The inverse Weibull (IW) distribution is of interest due to its flexibility and simplicity. The IW distribution (Keller and Kanath, 1982) was developed to study the decay of mechanical components in survival and reliability analysis. During the recent years, the inverse Weibull distribution has been of great interest in literature: beta inverse Weibull (B-IW) (Khan, 2010), Kumaraswamy-Inverse Weibull (Kw-IW) (Shahbaz et al., 2012), reflected generalized beta inverse Weibull (Elbatal et al., 2016), Topp-Leone inverse Weibull (Abbas et al., 2017), Odd Frechet inverse Weibull (OF-IW) (Fayomi, 2019) and gamma-inverse Weibull(G-IW) (Abbas et al.,2020).

The idea here is to incorporate the IW distribution into a larger family through an application of the modified Burr XII (MBXII) distribution. In fact, based on the T-X transform defined by Alzaatreh et al. (2016), we construct the MBXII-IW distribution.

The study of the MBXII-IW distribution is based on the following motivations: (i) to generate distributions with symmetrical, right-skewed, left-skewed and exponential shaped as well as high kurtosis; (ii) to have monotone and non-monotone failure rate function; (iii) to derive mathematical properties such as random number generator, ordinary moments, generating function, residual life functions, reliability measures and characterizations; (iv) to estimate the precision of the maximum likelihood estimators via a simulation study; (v) to reveal the potentiality of the MBXII-IW model; (vi) to work as the preeminent substitute model to other existing models; (vii) to deliver better fits than other models and (viii) to infer empirically. The contents of the article are structured as follows. Section 2 derives the MBXII-IW model from (i) the T-X family technique and (ii) linking the exponential and gamma variables. We study basic structural properties, random number generator and sub-models for the MBXII-IW model. Section 3 presents certain mathematical properties such as ordinary moments, generating function, residual life functions, reliability measures and characterizations. In Section 4, we address the maximum likelihood estimation for the MBXII-IW parameters. We evaluate the precision of the maximum likelihood estimators via a simulation study. We consider an application to elucidate the potentiality of the MBXII-IW model. In Section 5, we offer some conclusions.

2. THE MBXII-IW DISTRIBUTION

In this section, we derive the MBXII-IW distribution from the T-X family technique. The MBXII-IW model from link concerning the exponential and gamma variables is also obtained. Basic structural properties are studied. Then, we highlight the nature of the density and failure rate functions.

2.1 T-X Family Technique

The probability density function (pdf) and cumulative distribution function (cdf) of the IW are given, respectively, by

$$g(x;\lambda,\eta)=\lambda\eta x^{-\eta-1}e^{-\lambda x^{-\eta}}, \qquad x>0,\lambda>0,\eta>0,$$

and

$$G(x;\lambda)=e^{-\lambda x^{-\eta}}, \qquad x\geq 0,\lambda>0,\eta>0.$$

The odds ratio for the IW random variable X is

$$W\left(G(x)\right)=\frac{G(x;\lambda)}{\bar{G}(x;\lambda)}=\frac{e^{-\lambda x^{-\eta}}}{1-e^{-\lambda x^{-\eta}}}=\left[\exp\left(\lambda x^{-\eta}\right)-1\right]^{-1}.$$

The cdf of the T-*X* family (Alzaatreh et al., 2016) of distributions has the form

$$F\left(x\right)=\int_{a}^{W\left[G(x;\xi)\right]}r(t)\,dt, \qquad x\in\mathbb{R}, \tag{1}$$

where $r\left(t\right)$ is the pdf of the random variable (rv) *T*, where $T\in\left[a,\ b\right]$ for $-\infty\leq a<b<\infty$ and $W\left[G\left(x;\xi\right)\right]$ is a function of the baseline cdf of a rv *X* with the vector parameter ξ, which satisfies the conditions:

i) $W\left[G\left(x;\xi\right)\right]\in\left[a,\ b\right]$,

ii) $W\left[G\left(x;\xi\right)\right]$ is differentiable and monotonically non-decreasing and

iii) $\lim_{x\to-\infty}W\left[G\left(x;\xi\right)\right]\to a$ and $\lim_{x\to\infty}W\left[G\left(x;\xi\right)\right]\to b$.

The pdf of the T-*X* family can be expressed as

$$f\left(x\right)=\left\{\frac{\partial}{\partial x}W\left[G\left(x;\xi\right)\right]\right\}r\left\{W\left[G\left(x;\xi\right)\right]\right\}, \qquad x\in\mathbb{R}. \tag{2}$$

We derive the cdf of the MBXII-IW distribution from the T-X family technique by setting

$$r\left(t\right)=\alpha\beta t^{\beta-1}\left(1+\gamma t^{\beta}\right)^{-\frac{\alpha}{\gamma}-1}, \qquad t>0,\alpha>0,\gamma>0,\beta>0$$

and

$$W\left(G(x)\right)=\left[\exp\left(\lambda x^{-\eta}\right)-1\right]^{-1}.$$

Then, the cdf of MBXII-IW distribution is

$$F(x)=\int_{0}^{\left[e^{\lambda x^{-\eta}}-1\right]^{-1}}\alpha\beta t^{\beta-1}\left(1+\gamma t^{\beta}\right)^{-\frac{\alpha}{\gamma}-1}dt,$$

or

$$F(\mathrm{x})=1-\left[1+\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}}, \qquad x\geq 0, \tag{3}$$

where $\alpha,\beta,\gamma,\lambda,\eta>0,$ are parameters. The pdf corresponding to (3) is given by

$$f\left(x\right)=\alpha\beta\lambda\eta x^{-\eta-1}e^{\lambda x^{-\eta}}\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta-1}\left[1+\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}-1}, \quad x>0. \tag{4}$$

Hereafter, a rv with pdf (4) is denoted by X~MBXII-IW$\left(\alpha,\beta,\gamma,\lambda,\eta\right)$. (i) For $\gamma=1$, the MBXII-IW distribution reduces to Burr XII-inverse Weibull (BXII-IW) distribution; (ii) For $\beta=\gamma=1$, the MBXII-IW distribution reduces to the Lomax -inverse Weibull (Lomax-IW) distribution; (iii) For $\alpha=\gamma=1$, the MBXII-IW distribution reduces to the log-logistic-inverse Weibull (Log-Log–IW) distribution; (iv) For $\gamma\to 0$, the MBXII-IW distribution reduces to the generalized Weibull-IW distribution (GW-IW) distribution and (v) For $\alpha=1$ and $\gamma\to 0$, the MBXII-IW distribution reduces to the Weibull-IW distribution (W-IW) distribution.

2.2 Nexus between the Exponential and Gamma Variable

We derive the MBXII-IW distribution by linking the exponential and gamma rvs, i.e., $W_1 \sim \exp\left(1\right)$ and $W_2 \sim gamma\left(\alpha/\gamma,1\right)$.

Lemma. (i): If $W_1 \sim \exp\left(1\right)$ and $W_2 \sim gamma\left(\alpha/\gamma,1\right)$, then for $W_1=\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}W_2$, we have that X has density (4).

Proof

If $W_1 \sim \exp\left(1\right)$, i.e. $\quad f\left(w_1\right)=\mathrm{e}^{-w_1}, w_1>0$.

If $W_2 \sim gamma\left(\alpha/\gamma,1\right)$, i.e. $\quad f\left(w_2\right)=\dfrac{w_2^{\alpha/\gamma-1}\mathrm{e}^{-w_2}}{\Gamma\left(\alpha/\gamma\right)},\ w_2>0$.

Then, the joint distribution of the two rvs is $f\left(w_1,w_2\right)=\dfrac{w_2^{\alpha/\gamma-1}\mathrm{e}^{-w_2}\mathrm{e}^{-w_1}}{\Gamma\left(\alpha/\gamma\right)},\ w_1>0,\ w_2>0$.

Let $W_1=\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}W_2$.

The joint density of the rvs X and W_2 has the form

$$f\left(x,w_2\right)=\frac{w_2^{\alpha/\gamma-1}e^{-w_2}\,e^{-\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}w_2}}{\Gamma\left(\alpha/\gamma\right)}\gamma\beta\lambda\eta x^{-\eta-1}e^{\lambda x^{-\eta}}\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta-1}w_2,\ \ x>0,\ w_2>0.$$

The marginal density of X takes the form

$$f\left(x\right)=\gamma\beta\lambda\eta x^{-\eta-1}e^{\lambda x^{-\eta}}\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta-1}\frac{1}{\Gamma\left(\alpha/\gamma\right)}\int_0^{\infty}w_2^{\alpha/\gamma}\ \exp\left\{-\left[1+\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}\right]w_2\right\}dw_2\ .$$

After simplification, we obtain (for x>0)

$$f\left(x\right)=\alpha\beta\lambda\eta x^{-\eta-1}e^{\lambda x^{-\eta}}\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta-1}\left[1+\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}-1}, \quad x>0,$$

which is the MBXII-IW density.

2.3 Structural Properties

For X~MBXII-IW$(\alpha,\beta,\gamma,\lambda,\eta)$, the survival, failure rate, cumulative hazard, reverse hazard functions, Mills ratio and elasticity are given, respectively, by

$$S(x)=\left[1+\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}-1},$$

$$h(x)=\alpha\beta\lambda\eta x^{-\eta-1}e^{\lambda x^{-\eta}}\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta-1}\left[1+\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}\right]^{-1},$$

$$r(x)=\frac{d}{dx}\ln\left\{1-\left[1+\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}}\right\},$$

$$H(x)=\frac{\alpha}{\gamma}\ln\left\{1+\gamma\left[1+\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}\right]\right\},$$

$$m(x)=\frac{\left(e^{\lambda x^{-\eta}}-1\right)^{\beta+1}+\gamma\left(e^{\lambda x^{-\eta}}-1\right)}{\alpha\beta\lambda\eta x^{-\eta-1}e^{\lambda x^{-\eta}}},$$

and

$$\eta(x)=\frac{d\ln\mathrm{F}(x)}{d\ln x}=\frac{d}{d\ln x}\left(\ln\left\{1-\left[1+\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}}\right\}\right).$$

The quantile function of X (for $0<q<1$) follows from

$$x_q=\left\{\ln\left[\left(\frac{(1-q)^{-\frac{\gamma}{\alpha}}-1}{\gamma}\right)^{-\frac{1}{\beta}}+1\right]^{\frac{1}{\lambda}}\right\}^{-\frac{1}{\eta}},$$

and its random number generator with $Z\sim$ Uniform (0,1) is

$$X=\left\{\ln\left[\left(\frac{(1-Z)^{-\frac{\gamma}{\alpha}}-1}{\gamma}\right)^{-\frac{1}{\beta}}+1\right]^{\frac{1}{\lambda}}\right\}^{-\frac{1}{\eta}}.$$

2.4 Plots of the MBXII-IW Density and Failure Rate Functions

We plot the density and failure rate functions of the MBXII-IW distribution for selected values of the parameters. Figure 1 displays that the MBXII-IW density can take various shapes such as exponential, symmetrical, left-skewed and right-skewed. Figure 2 shows that the failure rate function can be increasing, decreasing, increasing-decreasing, inverted bathtub and modified bathtub shaped. Therefore, the MBXII-IW distribution is quite flexible and can be applied excellently in evaluating numerous data sets.

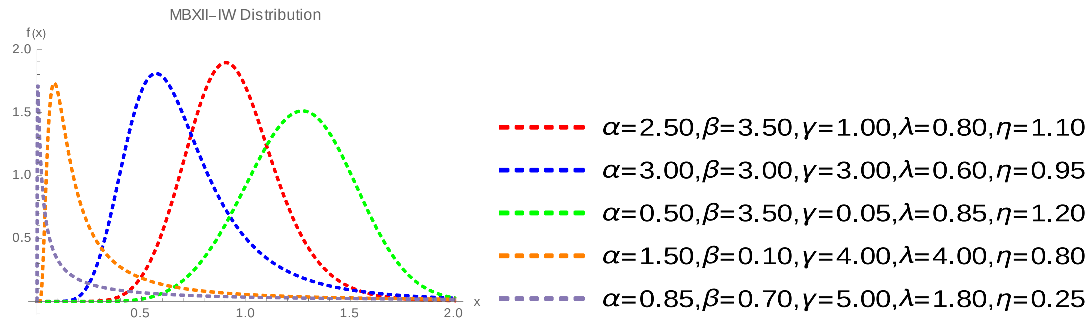

Fig. 1: Plots of pdf of the MBXII-IW density

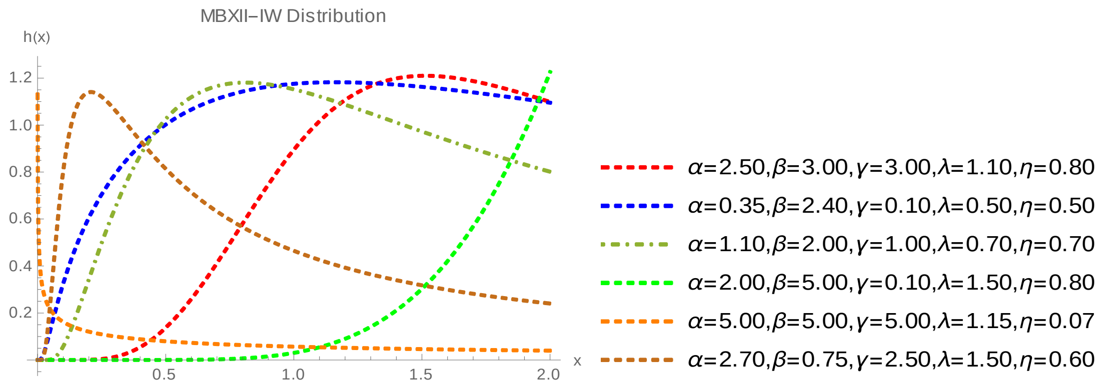

Fig. 2: Plots of the MBXII-IW hazard rate

3. Mathematical Properties

Here, we present certain mathematical and statistical properties such as the ordinary moments, generating function, residual life functions, reliability measures and characterizations.

3.1 Linear representation

In this section, we provide a linear representation for the density of X to derive some mathematical quantities for the MBXII-IW model. The cdf (3) of X can be expressed as

$$F(x) = 1 - \underbrace{\left\{1 + \gamma\left[\frac{\exp\left(-\lambda x^{-\eta}\right)}{1-\exp\left(-\lambda x^{-\eta}\right)}\right]^{\beta}\right\}}_{A}^{-\frac{\alpha}{\gamma}}. \tag{5}$$

First, we will consider the following two power series

$$\left(1+s\right)^{-c} = \sum_{k=0}^{\infty} 2^{-c-k}\left(s-1\right)^{k}\binom{-c}{k}, \tag{6}$$

and

$$\left(1-s\right)^{-c} = \sum_{j=0}^{\infty} \frac{\Gamma(c+j)}{j!\,\Gamma(c)} s^{j} \,|_{\left(|s|<1,\,c>0\right)}. \tag{7}$$

Applying (6) for A in (5) gives

$$F(x) = 1 - \sum_{k=0}^{\infty} 2^{-(\alpha/\gamma)-k}\binom{-\alpha/\gamma}{k}\left\{\gamma\left[\frac{\exp\left(-\lambda x^{-\eta}\right)}{1-\exp\left(-\lambda x^{-\eta}\right)}\right]^{\beta} - 1\right\}^{k}.$$

Second, using the binomial expansion, the last equation can be expressed as

$$F(x)=1-\sum_{k=0}^{\infty}\sum_{i=0}^{k}\frac{(-1)^{i}\gamma^{k-i}}{2^{\frac{\alpha}{\gamma}+k}}\binom{k}{i}\binom{-\alpha/\gamma}{k}\left[\exp\left(-\lambda x^{-\eta}\right)\right]^{(k-i)\beta}\underbrace{\left[1-\exp\left(-\lambda x^{-\eta}\right)\right]^{-(k-i)\beta}}_{B}.$$

Third, applying (7) for B in the last equation gives

$$F(x)=1-\sum_{j,k=0}^{\infty}\sum_{i=0}^{k}d_{i,j,k}\left[\exp\left(-\lambda x^{-\eta}\right)\right]^{(k-i)\beta+j},$$

$$F(x)=1-\sum_{j,k=0}^{\infty}\sum_{i=0}^{k}d_{i,j,k}H_{(k-i)\beta+j}(x), \tag{8}$$

where $H_c(x)$ is the cdf of the IW model with scale parameter c.

$$d_{i,j,k}=\frac{(-1)^{i}\gamma^{k-i}\Gamma\left([k-i]\beta+j\right)}{2^{\frac{\alpha}{\gamma}+k}\,j!\,\Gamma\left([k-i]\beta\right)}\binom{k}{i}\binom{-\alpha/\gamma}{k}.$$

By differentiating (8), we obtain

$$f(x)=\sum_{(j,k=0|j+k\geq 1)}^{\infty}\sum_{i=0}^{k}\tau_{i,j,k}h_{(k-i)\beta+j}(x), \tag{9}$$

where $h_c(x)$ denotes the pdf of the IW model with scale parameter c and $\tau_{i,j,k}=-d_{i,j,k}$.

Equation (9) is the main result of this section. It reveals that the MBXII-IW density is a linear combination of IW density. So, some of its mathematical properties can be easily determined from those of IW model.

3.2 Moments

The rth ordinary moment of X say $\mu_r'=E\left(X^r\right)$, is determined from (9) as

$$\mu_r'=E\left(X^r\right)=\sum_{(j,k=0|j+k\geq 1)}^{\infty}\sum_{i=0}^{k}\tau_{i,j,k}\left\{\lambda\left[(k-i)\beta+j\right]\right\}^{r}\Gamma\left(1-\frac{r}{\eta}\right)\Big|_{(r<\eta)}.$$

The rth incomplete moment of X, say $\varphi_r(t)$, can be determined from (9) as

$$\varphi_r(t)=\int_{-\infty}^{t}x^{r}f(x)dx=\sum_{(j,k=0|j+k\geq 1)}^{\infty}\sum_{i=0}^{k}b_{i,j,k}\left\{\lambda\left[(k-i)\beta+j\right]\right\}^{r}\left[\gamma\left(1-\frac{r}{\eta},\left(\frac{\lambda}{t}\right)\right)\right]\Big|_{(r<\eta)}.$$

where $\gamma(\varsigma,q)$ is the incomplete gamma function.

$$\gamma(a,q)\Big|_{(a\neq 0,-1,-2,\ldots)}=\int_{0}^{q}t^{a-1}\exp(-t)dt=\frac{q^{a}}{a}\left\{{}_1F_1\left[a;a+1;-q\right]\right\}=\sum_{k=0}^{\infty}\frac{(-1)^{k}}{k!(a+k)}q^{a+k},$$

where ${}_1F_1[.,.,.]$ is a confluent hypergeometric function.

The moment generating function (mgf) $M(t)=\mathrm{E}\left(e^{tX}\right)$ of X follows from (9) as

$$M(t)=\sum_{(j,k=0|j+k\geq 1)}^{\infty}\sum_{r=0}^{\infty}\sum_{i=0}^{k}\left(t^{r}b_{i,j,k}/r!\right)\left\{\lambda\left[(k-i)\beta+j\right]\right\}^{r}\Gamma\left(1-\frac{r}{\eta}\right)\Big|_{(r<\eta)}.$$

Probability weighted moments (PWMs)

The (s,r)th PWM of X denoted by $\rho_{s,r}$ is formally defined by

$$\rho_{s,r}=E\left\{X^{s}F(X)^{r}\right\}=\int_{-\infty}^{\infty}x^{s}F(x)^{r}f(x)dx.$$

Using (3), we have

$$F(x)^r=\left(1-\left\{1+\gamma\left[\frac{\exp\left(-\lambda x^{-\eta}\right)}{1-\exp\left(-\lambda x^{-\eta}\right)}\right]^{\beta}\right\}^{-\frac{\alpha}{\gamma}}\right)^r.$$

Expanding z^{λ} in Taylor series, we can write

$$s^{\lambda}=\sum_{i=0}^{\infty}\frac{(\lambda)_i}{i!}(s-1)^i=\sum_{i=0}^{\infty}f_i(\lambda)s^i, \tag{10}$$

Where $(\lambda)_i=\lambda(\lambda-1)\dots(\lambda-i+1)$ is the descending factorial and

$$f_i(\lambda)=\sum_{i=0}^{\infty}\frac{(-1)^{h-1}(\lambda)_h}{h!}\binom{h}{i}.$$

First, applying the Taylor series in s^{λ} for $F(x)^r$, we obtain

$$F(x)^r=\sum_{i=0}^{\infty}(-1)^i f_i(r)\left\{1+\gamma\left[\frac{\exp\left(-\lambda x^{-\eta}\right)}{1-\exp\left(-\lambda x^{-\eta}\right)}\right]^{\beta}\right\}^{-\frac{\alpha i}{\gamma}}.$$

Second, using (4) and the last equation, we have

$$f(x)F(x)^r=\alpha\beta\lambda\eta x^{-\eta-1}\exp\left(-\lambda x^{-\eta}\right)\frac{\left[\exp\left(-\lambda x^{-\eta}\right)\right]^{\beta-1}}{\left[1-\exp\left(-\lambda x^{-\eta}\right)\right]^{\beta+1}}\sum_{i=0}^{\infty}(-1)^i f_i(r)\underbrace{\left\{1+\gamma\left[\frac{\exp\left(-\lambda x^{-\eta}\right)}{1-\exp\left(-\lambda x^{-\eta}\right)}\right]^{\beta}\right\}^{-(i+1)\frac{\alpha}{\gamma}-1}}_{C}.$$

Applying (6) for C in the last equation, we obtain

$$f(x)F(x)^r=\alpha\beta\lambda\eta x^{-\eta-1}\exp\left(-\lambda x^{-\eta}\right)\frac{\left[\exp\left(-\lambda x^{-\eta}\right)\right]^{\beta-1}}{\left[1-\exp\left(-\lambda x^{-\eta}\right)\right]^{\beta+1}}\sum_{i,k=0}^{\infty}(-1)^i f_i(r)2^{-(i+1)\frac{\alpha}{\gamma}-k-1}$$

$$\times\binom{-(i+1)\frac{\alpha}{\gamma}-1}{k}\underbrace{\left(-1+\gamma\left[\frac{\exp\left(-\lambda x^{-\eta}\right)}{1-\exp\left(-\lambda x^{-\eta}\right)}\right]^{\beta}\right)^k}_{D}.$$

Third, using the binomial expansion for D, the last equation be rewritten as

$$f(x)F(x)^r=\gamma^{k-j}\beta\alpha\lambda\eta x^{-\eta-1}\exp\left(-\lambda x^{-\eta}\right)\sum_{i,k=0}^{\infty}\sum_{j=0}^{k}(-1)^{i+j}\binom{-(i+1)\frac{\alpha}{\gamma}-1}{k}\binom{k}{j}$$

$$\times f_i(r)G(x)^{(k-j+1)\beta-1}2^{-(i+1)\frac{\alpha}{\gamma}-k-1}\underbrace{\left\{1-\exp\left(-\lambda x^{-\eta}\right)\right\}^{-\left[(k-j+1)\beta+1\right]}}_{E}.$$

Applying (7) for E in the last equation gives

$$f(x)F(x)^r=\sum_{i,k,m=0}^{\infty}\sum_{j=0}^{k}\gamma^{k-j}\beta\alpha\frac{(-1)^{i+j}f_i(r)\Gamma\left(\left[k-j+1\right]\beta+m+1\right)}{2^{(i+1)\left(\frac{\alpha}{\gamma}\right)+k+1}m!\Gamma\left(\left[k-j+1\right]\beta+1\right)}\frac{\left[(k-j+1)\beta+m+1\right]}{\left[(k-j+1)\beta+m+1\right]}\times$$

$$\binom{k}{j}\binom{-\left[(i+1)\frac{\alpha}{\gamma}+1\right]}{k}\lambda\eta x^{-\eta-1}\left[\exp\left(-\lambda x^{-\eta}\right)\right]^{(k-j+1)\beta+m}$$

and then

$$f(x)F(x)^r = \sum_{k,m=0}^{\infty}\sum_{j=0}^{k} a_{j,k,m}^{(r)}\, h_{(k-j+1)\beta+m}(x),$$

where $$a_{j,k,m}^{(r)} = \gamma^{k-j}\beta\alpha\,\upsilon_{j,k,m} f_i(r)\,, \qquad (11)$$

and $f_i(r)$ is defined in (10) then for ($j \le k$)

$$u_{j,k,m} = \sum_{i=0}^{\infty} \frac{(-1)^{i+j}\left[\left(\left[k-j+1\right]\beta+m+1\right)^{(m)}\right]}{2^{(i+1)\left(\frac{\alpha}{\gamma}\right)+k+1}\left[\left(k-j+1\right)\beta+m+1\right]m!}\binom{k}{j}\binom{-(i+1)\frac{\alpha}{\gamma}-k-1}{k},$$

where $a^{(n)} = \Gamma(a+n)/\Gamma(a)$ denotes the rising factorial.

Finally, the $(s,r)^{th}$ PWM of X can be determined as

$$\rho_{s,r} = \sum_{k,m=0}^{\infty}\sum_{j=0}^{k} a_{j,k,m}^{(r)}\left\{\lambda\left[\left(k-j+1\right)\beta+m+1\right]\right\}^{s}\Gamma(1-s)\Big|_{(s<1)}.$$

Residual life and reversed residual life functions

The n^{th} moment of the residual life, say $m_n(t) = E\left|(x-t)^n\right|_{(X>t,n=1,2,...)}$,

Uniquely determines F(x). The n^{th} moment of the residual life of X is given by

$$m_n(t) = \frac{\int_t^{\infty}(x-t)^n\, dF(x)}{1-F(t)}.$$

Therefore

$$m_n(t) = \frac{1}{1-F(t)}\sum_{(j,k=0|j+k\ge 1)}^{\infty}\sum_{i=0}^{k}\tau_{i,j,k}^{*}\left\{\lambda\left[\left(k-i\right)\beta+j\right]\right\}^{r}\Gamma\left(1-r,\left(\frac{\lambda}{t}\right)\right)\Big|_{(r<1)},$$

where

$$\Gamma(a,q)\Big|_{(q>0)} = \int_q^{\infty} t^{a-1}\exp(-t)dt,$$

$$\Gamma(a,q)+\gamma(a,q) = \Gamma(a)$$

and

$$\tau_{i,j,k}^{*} = \tau_{i,j,k}(1-t)^n.$$

The mean residual life (MRL) or the life expectation at age t defined by

$$m_1(\mathrm{t}) = \mathrm{E}\,[(\mathrm{X}-\mathrm{t})\big|_{(\mathrm{X}>\mathrm{t},\ \mathrm{n}=1)}],$$

which represents the expected additional life length for a unit which is alive at age t. The MRL of X can be obtained by setting n=1 in the last equation. The n^{th} moment of the reversed residual life, say

$$M_n(t) = E\left|(x-t)^n\right|_{(X\le t,\forall t>0,n=1,2,...)},$$

uniquely determines F(x). We obtain

$$M_n(t) = \frac{\int_0^{t}(t-x)^n\, dF(x)}{F(t)}.$$

Then, the n^{th} moment of the reversed residual life of X comes

$$M_n(t) = \sum_{(j,k=0|j+k\ge 1)}^{\infty}\sum_{i=0}^{k}\tau_{i,j,k}^{**}\left\{\lambda\left[\left(k-i\right)\beta+j\right]\right\}^{r}\left[\gamma\left(1-r,\left(\frac{\lambda}{t}\right)\right)\right]\Big|_{(r<1)},$$

where $\tau^{**}_{i,j,k} = \tau_{i,j,k} \sum_{r=0}^{n} (-1)^r \binom{n}{r} t^{n-r}$.

The mean inactivity time (MIT), also called the mean reversed residual life function, is given by

$$M_1(t) = E[(t-X)\,|_{(X \le t,\, n=1)}],$$

and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in (0,t). The MIT of the MBXII-IW model is obtained easily by setting $n=1$ in the above equation.

The r^th central moment (μ_r), coefficients of skewness (γ_1) and kurtosis (γ_2) of X are

$$\mu_r = \sum_{\ell=1}^{r} (-1)^{\ell} \binom{r}{\ell} \mu'_{\ell}\, \mu'_{r-\ell},\ \gamma_1 = \mu_3 \big/ \sqrt[3]{\mu_2} \quad \text{and}\ \beta_2 = \mu_4 \big/ (\mu_2)^2 .$$

The numerical values for the mean (μ'_1), median $(\tilde{\mu})$, standard deviation (σ), skewness (γ_1) and kurtosis (γ_2) of the MBXII-IW distribution for selected values of $\alpha,\beta,\gamma,\lambda,\eta$ are listed in Table 1.

Table 1: Quantities μ'_1, $\tilde{\mu}$, σ, γ_1 and γ_2 of the MBXII-IW Distribution

$\alpha,\beta,\gamma,\lambda,\eta$	μ'_1	$\tilde{\mu}$	σ	γ_1	γ_2
0.65,5,0.75,3.5,0.60	19.5487	16.7378	13.8215	16.2841	1152.47
3.65,3.9,4.3,0.8,0.95	1.0207	0.9238	0.4718	7.174	209.223
3.65,3.9,4.3,0.8,1.25	1.0017	0.9416	0.3233	3.8747	59.7261
3.5,5,1.3,0.8,1.25	0.951	0.95	0.1404	0.1334	3.4778
3.75,5,1.3,1.3,1.25	1.3896	1.3892	0.202	0.0869	3.4028
5,5,1.5,0.5,0.5	0.3207	0.3103	0.1107	0.7517	4.5607
5,5,1.5,1.0,0.5	1.2825	1.2411	0.4425	0.7518	4.5626
5,5,1.5,2,0.5	5.1291	4.9642	1.7686	0.7424	4.4852
5,5,1.5,0.5,3	0.8203	0.8228	0.0485	-0.2844	3.4174
5,5,1.5,1.5,0.5	2.8853	2.7923	0.9957	0.749	4.58
5,5,1.5,1.5,1	1.6735	1.6712	0.2918	0.1267	3.3422
5,5,1.5,1.5,1.5	1.4048	1.4083	0.1646	-0.0755	3.3027
5,5,1.5,1.5,2	1.2886	1.2927	0.1138	-0.181	3.3471
5,5,1.5,1.5,2.5	1.2231	1.228	0.0868	-0.2443	3.3988
5,5,1.5,1.5,3	1.1831	1.1866	0.0701	-0.2855	3.4321

3.3 Reliability Estimation of Multicomponent Stress-Strength model

Consider a system that has κ identical components out of which *s* components are functioning. The strengths of κ components are $X_i, i=1,2...\kappa$ with common cdf F while, the stress Y imposed on the components has cdf G. The strengths $X_i, i=1,2...\kappa$ and stress Y are i.i.d. The probability that the system operates properly is reliability of the system i.e.

$$R_{s,\kappa} = Pr[strengths(X_i, i=1,2\ldots\kappa) > stress(Y)],$$

$$R_{s,\kappa} = Pr[\text{at least"}s\text{"}of\,(X_i, i=1,2\ldots\kappa)\ exceed Y].$$

Then, we can write this probability (Bhattacharyya and Johnson, 1974) as follows:

$$R_{s,\kappa} = \sum_{\ell=s}^{\kappa} \binom{\kappa}{\ell} \int_{-\infty}^{\infty} [1-F(y)]^{\ell} [F(y)]^{\kappa-\ell} dG(y). \qquad (12)$$

Let X~MBXII-IW$(\alpha_1,\beta,\gamma,\lambda,\eta)$ and Y~MBXII-IW$(\alpha_2,\beta,\gamma,\lambda,\eta)$ with common parameters $\beta,\gamma,\lambda,\eta$ and unknown shape parameters α_1 and α_2. The reliability that the system operates properly in multicomponent stress- strength for the MBXII-IW distribution is

$$R_{s,\kappa}=\sum_{\ell=s}^{\kappa}\binom{\kappa}{\ell}\int_0^{\lambda}\left(\left[1+\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}\right]^{-\frac{\alpha_1}{\gamma}}\right)^{\ell}\left(1-\left[1+\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}\right]^{-\frac{\alpha_1}{\gamma}}\right)^{(\kappa-\ell)}\frac{\alpha_2\beta\lambda\eta\left[1+\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}\right]^{-\frac{\alpha_2}{\gamma}-1}}{x^{\eta+1}e^{-\lambda x^{-\eta}}\left(e^{\lambda x^{-\eta}}-1\right)^{\beta+1}}dx.$$

Letting $u=\left[1+\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}\right]^{-\frac{\alpha_2}{\gamma}}$, we obtain

$$R_{s,\kappa}=\sum_{\ell=s}^{\kappa}\binom{\kappa}{\ell}\int_0^1\left(u^{v}\right)^{\ell}\left(1-u^{v}\right)^{(\kappa-\ell)}du,\ \text{ where } v=\frac{\alpha_1}{\alpha_2}.$$

Letting $u^{v}=w$, we have

$$R_{s,\kappa}=\sum_{\ell=s}^{\kappa}\binom{\kappa}{\ell}\int_0^1 w^{\ell}\left(1-w\right)^{(\kappa-\ell)}\frac{1}{\nu}w^{\frac{1}{\nu}-1}dw,$$

$$R_{s,\kappa}=\frac{1}{\nu}\sum_{\ell=s}^{\kappa}\binom{\kappa}{\ell}B\left(\ell+\frac{1}{\nu},\kappa-\ell+1\right). \tag{13}$$

where $B(.,.)$ is the beta function. The probability in (13) is known as the reliability of multicomponent stress-strength model. For $s=\kappa=1$, the multicomponent stress-strength model reduces to the stress-strength model (Kotz et al., 2003) as

$$R_{1,1}=Pr(\mathrm{Y}<\mathrm{X})=\frac{\alpha_2}{(\alpha_1+\alpha_2)},\ \text{ where } \alpha_1+\alpha_2>0,$$

which is independent of the parameters β,γ,λ and η.

3.4 Characterizations based on Truncated Moment of a Function of the Random Variable

In this subsection, we first present a characterization of the MBXII-IW distribution in terms of a simple relationship between truncated moment of a function of X and another function. This characterization result employs a version of the theorem due to Glänzel (1987); see Theorem G of Appendix A. Note that the result holds also when the interval H is not closed. Moreover, as mentioned above, it could be also applied when the cdf F does not have a closed form. As shown in Glänzel (1990), this characterization is stable in the sense of weak convergence.

Proposition 3.4.1 Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $g(x)=\left[1+\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}\right]^{-1}, x>0.$

The rv X has pdf (4) if and only if the function $h(x)$ defined in Theorem G has the form

$$h(x)=\frac{\alpha}{\alpha+\gamma}\left[1+\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}\right]^{-1},\quad x>0\ .$$

Proof If X has pdf (4), then for (x>0),

$$\left(1-F(x)\right)E\left(g(X)\middle|X\geq x\right)=\frac{\alpha}{\alpha+\gamma}\left[1+\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}\right]^{-(\frac{\alpha}{\gamma}+1)},x>0,$$

or

$$E\left(g(X)\middle|X\geq x\right)=\frac{\alpha}{\alpha+\gamma}\left[1+\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}\right]^{-1},\ x>0,$$

and

$$h(x)-g(x)=-\frac{\gamma}{\alpha+\gamma}\left[1+\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}\right]^{-1},x>0.$$

Conversely, if $h(x)$ is given as above, then

$$h'(x)=-\frac{\alpha}{\alpha+\gamma}\frac{\beta\gamma\lambda}{x^{\eta-1}}e^{\lambda x^{-\eta}}\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta-1}\left[1+\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}\right]^{-2}<0,\ x>0,$$

and

$$s'(x)=\frac{h'(x)}{h(x)-g(x)}=\frac{\frac{\alpha}{\gamma}\frac{\beta\gamma\lambda}{x^{\eta-1}}e^{\lambda x^{-\eta}}\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta-1}}{\left[1+\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}\right]},\ x>0,$$

and hence

$$s(x)=\ln\left[1+\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}\right]^{\frac{\alpha}{\gamma}},x>0,$$

and

$$e^{-s(x)}=\left[1+\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}\right]^{-\frac{\alpha}{\gamma}},x>0.$$

In view of Theorem G, X has density (4).

Corollary 3.4.1: Let $X:\Omega\to(0,\infty)$ be a continuous random variable. The pdf of X is (4) if and only if there exist functions $h(x)$ and $g(x)$ defined in Theorem G satisfying the differential equation

$$\frac{h'(x)}{h(x)-g(x)}=\frac{\frac{\alpha}{\gamma}\frac{\beta\gamma\lambda}{x^{\eta-1}}e^{\lambda x^{-\eta}}\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta-1}}{\left[1+\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}\right]},\ x>0.$$

Remark 3.4.1: The general solution of the differential equation in Corollary 3.4.1 is

$$h(x)=\left[1+\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}\right]^{\frac{\alpha}{\gamma}}\left[-\int\frac{\frac{\alpha}{\gamma}\frac{\beta\gamma\lambda}{x^{\eta-1}}e^{\lambda x^{-\eta}}\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta-1}}{\left[1+\gamma\left(e^{\lambda x^{-\eta}}-1\right)^{-\beta}\right]^{\frac{\alpha}{\gamma}+1}}g(x)\,dx+D\right],$$

where D is a constant.

4. STATISTICAL INFERENCE

First, we adopt the maximum likelihood estimation technique for the MBXII-IW parameters. We evaluate the behavior of the maximum likelihood estimators of the MBXII-IW parameters via a simulation study. We explain the utility of the MBXII-IW model among its family and class using serum-reversal time (in days) of 143 children born from HIV-infected mothers (Silva, 2004).

4.1 Parameter Estimation

Let $\xi=(\alpha,\beta,\gamma,\lambda,\eta)^T$ be the unknown parameter vector. The log likelihood function $\ell(\xi)$ for the MBXII-IW distribution is

$$\ell(\xi)=\ln\mathrm{L}(\xi)=n\ln(\alpha)+n\ln(\beta)+n\ln(\eta)+n\ln(\lambda)-(\eta+1)\sum_{i=1}^{n}\ln x_i+\lambda\sum_{i=1}^{n}(x_i)^{-\eta}-$$
$$(\beta+1)\sum_{i=1}^{n}\ln\left(e^{\lambda x_i^{-\eta}}-1\right)-\left(\frac{\alpha}{\gamma}+1\right)\sum_{i=1}^{n}\ln\left[1+\gamma\left(e^{\lambda x_i^{-\eta}}-1\right)^{-\beta}\right]. \quad (14)$$

We can compute the maximum likelihood estimators (MLEs) of $\alpha,\beta,\gamma,\lambda$ and η by solving equations (15)-(19) simultaneously, either directly or using quasi-Newton procedure, computer packages/ softwares such as R, SAS, Ox, MATHEMATICA, MATLAB and MAPLE.

$$\frac{\partial \ell(\xi)}{\partial \alpha}=\frac{n}{\alpha}-\frac{1}{\gamma}\sum_{i=1}^{n}\ln\left\{1+\gamma\left[\exp\left(\lambda x_i^{-\eta}\right)-1\right]^{-\beta}\right\}=0, \quad (15)$$

$$\frac{\partial \ell(\xi)}{\partial \beta}=\frac{n}{\beta}-\sum_{i=1}^{n}\ln\left(e^{\frac{\lambda}{x_i}}-1\right)+(\alpha+\gamma)\sum_{i=1}^{n}\left[\left(e^{\lambda x_i^{-\eta}}-1\right)^{\beta}+\gamma\right]^{-1}\ln\gamma\left(e^{\lambda x_i^{-\eta}}-1\right)=0, \quad (16)$$

$$\frac{\partial \ell(\xi)}{\partial \gamma}=\frac{\alpha}{\gamma^2}\sum_{i=1}^{n}\ln\left[1+\gamma\left(e^{\lambda x_i^{-\eta}}-1\right)^{-\beta}\right]-\left(\frac{\alpha}{\gamma}+1\right)\sum_{i=1}^{n}\left[\left(e^{\lambda x_i^{-\eta}}-1\right)^{\beta}+\gamma\right]^{-1}=0, \quad (17)$$

$$\frac{\partial \ell(\xi)}{\partial \lambda}=\frac{n}{\lambda}+\sum_{i=1}^{n}x_i^{-\eta}-(\beta+1)\sum_{i=1}^{n}\frac{x_i^{-\eta}}{\left(1-e^{-\lambda x_i^{-\eta}}\right)}+(\alpha+\gamma)\beta\sum_{i=1}^{n}\frac{x_i^{-\eta}e^{\lambda x_i^{-\eta}}}{\left[\left(e^{\lambda x_i^{-\eta}}-1\right)^{\beta+1}+\gamma\left(e^{\lambda x_i^{-\eta}}-1\right)\right]}=0, \quad (18)$$

$$\frac{\partial \ell(\xi)}{\partial \eta}=\frac{n}{\eta}-\sum_{i=1}^{n}\ln x_i-\lambda\sum_{i=1}^{n}x_i^{-\eta}\ln x_i+(\beta+1)\lambda\sum_{i=1}^{n}\frac{x_i^{-\eta}\ln x_i}{\left(1-e^{-\lambda x_i^{-\eta}}\right)}+$$
$$(\alpha+\gamma)\beta\lambda\sum_{i=1}^{n}\frac{x_i^{-\eta}e^{\lambda x_i^{-\eta}}\ln x_i}{\left[\left(e^{\lambda x_i^{-\eta}}-1\right)^{\beta+1}+\gamma\left(e^{\lambda x_i^{-\eta}}-1\right)\right]}=0. \quad (19)$$

4.2. Simulation Study

We evaluate the behavior of the MLEs of the MBXII-IW parameters with respect to the sample size n. We generate 10000 samples of sizes *n*=50,100,200,300,500 from the inverse cdf of the MBXII-IW distribution with true parameter settings $(\alpha,\beta,\gamma,\lambda,\eta)$=(0.8, 0.9, 1.0,1.3, 0.5) and (1.5,2.0,1.5,1.5,0.75). We estimate the MLEs $(\hat{\alpha},\hat{\beta},\hat{\gamma},\hat{\lambda},\hat{\eta})$ for 10000 samples from the non-linear optimization techniques. We also compute the means, biases and mean squared errors (MSE) of the MLEs. We infer from the simulation results (Table 2) that as the sample size n increases, the means approach the true parameter value, the estimated MSE decreases, and estimated biases drop to zero. We observe that as the shape parameter increases, MSE of estimated parameters increases. Finally, we infer that the MLEs for the MBXII-IW distribution are consistent.

Table 2: Means, Bias and MSEs of MBXII-IW distribution

Sample	Statistics	*a* = 0.8	*b* = 0.9	*g* = 1.0	*l* = 1.3	*h* = 0.5
50	Means	0.8966	0.9511	0.784	1.2147	0.7145
	Bias	0.0966	0.0511	-0.216	-0.0853	0.2145
	MSE	0.908	1.296	4.5137	0.4022	0.7256
100	Means	0.913	0.944	0.8714	1.0981	0.6166
	Bias	0.113	0.044	-0.1286	-0.2019	0.1166
	MSE	0.7834	0.7403	4.2884	0.2312	0.4045
200	Means	0.9155	0.8967	0.8861	1.04	0.5549
	Bias	0.1155	-0.0033	-0.1139	-0.26	0.0549
	MSE	0.2938	0.2818	1.1419	0.1608	0.2258
300	Means	0.9171	0.8713	0.8949	1.0247	0.5306
	Bias	0.1171	-0.0287	-0.1051	-0.2753	0.0306
	MSE	0.2204	0.1328	0.2232	0.1387	0.1659
500	Means	0.9307	0.8678	0.9475	1.0032	0.5091
	Bias	0.1307	-0.0322	-0.0525	-0.2968	0.0091
	MSE	0.1846	0.075	0.1341	0.1277	0.1102

Sample	Statistics	$a = 1.5$	$b = 2.0$	$g = 1.5$	$l = 1.5$	$h = 0.75$
50	Means	2.4159	2.0513	1.3691	1.1865	0.7906
	Bias	0.9159	0.0513	-0.1309	-0.3135	0.0406
	MSE	7.2562	4.006	8.4563	0.4718	0.3221
100	Means	2.3062	2.1084	1.5759	1.0659	0.7184
	Bias	0.8062	0.1084	0.0759	-0.4341	-0.0316
	MSE	4.4724	2.912	6.85	0.3554	0.1623
200	Means	2.2574	2.1194	1.6883	1.0032	0.6771
	Bias	0.7574	0.1194	0.1883	-0.4968	-0.0729
	MSE	3.0882	1.9918	3.8847	0.3335	0.1008
300	Means	2.234	2.1254	1.769	0.9784	0.6563
	Bias	0.734	0.1254	0.269	-0.5216	-0.0937
	MSE	2.4618	1.5797	2.7178	0.3328	0.0767
500	Means	2.2214	2.0694	1.8303	0.9671	0.633
	Bias	0.7214	0.0694	0.3303	-0.5329	-0.117
	MSE	2.0706	0.9032	1.5074	0.3234	0.0574

4.3 Data Applications for Selection and Comparison

We consider an application to serum-reversal time (in days) of 143 children born from HIV-infected mothers (Silva, 2004) for authentication of the potentiality of the MBXII-IW distribution. We compare the MBXII-IW distribution with BXII-IW, L-IW, LL-IW, OF-IW, Kw-IW, W-IW, MBXII and IW. For selection of the optimum distribution, we compute the estimate of various model selection criteria such as "likelihood ratio statistics ($-2\hat{\ell}$), Akaike information criterion (AIC), corrected Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC)" and goodness of fit statistics (GOFs) such as "Cramer-von Mises (W*), Anderson Darling (A*), and Kolmogorov- Smirnov statistics (K-S)" with p-values for all competing models. We estimate the MLEs for the parameters and their standard errors (SEs).

Table 3 reports the MLEs (SEs) and measures W*, A*, K-S (p-values). Table 4 displays the values of measures $-2\hat{\ell}$, AIC, CAIC, BIC and HQIC.

Table 3: MLEs (SEs) and W*, A*, K-S (p-values) for Serum-reversal time data

Model	α	β	γ	λ	η	W	A	K-S p-value
MBXII-IW	154.9999 (56.8548)	17.2962 (6.0595)	1.0265 (11.4959)	0.0613 (0.0219)	1.2362 (0.2308)	0.9431	5.2974	0.1588 (0.0015)
BXII-IW	58.5081 (57.9404)	11.1084 (6.6059)	1	0.0922 (0.0482)	1.5695 (0.6668)	1.0099	5.6729	0.1593 (0.0014)
L-IW	674.5684 (302.0037)	1	1	0.1850 (0.0144)	20.0969 (1.4271)	1.6853	9.3650	0.2071 (9.363e-06)
LL-IW	1	35.4684 (10.6524)	1	0.0402 (0.0119)	0.8799 (0.0619)	1.7189	9.5548	0.1407 (0.0069)
OF-IW	---	0.7290 (0.0067)	----	0.0100 (0.0018)	40.0342 (7.2587)	4.0321	21.0695	0.2944 (3.462e-11)
Kw-IW	65.0000 (18.4924)	0.3611 (0.1986)	---	0.0158 (0.0338)	1.5081 (0.8394)	4.0445	21.1287	0.2866 (1.259e-10)
W-IW	0.0034	2.3448	---	2.2303	0.5375	1.1602	6.4979	0.2062

	(0.0015)	(4.5568)		(2.5846)	(0.9676)			(1.042e-05)
MBXII	0.0111 (0.0009)	86.6803 (293.070)	5.5000 (21.2078)	1	1	3.9243	20.5617	0.4548 (< 2.2e-16)
IW	---	---	---	16.7307 (2.3641)	0.5614 (0.0289)	4.0130	20.9785	0.2924 (4.835e-11)

Table 4: $-2\hat{\ell}$, AIC, CAIC, BIC and HQIC for Serum-reversal time data

Model	$-2\hat{\ell}$	AIC	CAIC	BIC	HQIC
MBXII-IW	1982.877	**1992.877**	**1993.315**	**2007.691**	**1998.897**
BXII-IW	1986.905	1994.905	1995.195	2006.756	1999.721
L-IW	2029.848	2035.848	2036.021	2044.737	2039.46
LL-IW	2037.754	2043.755	2043.927	2052.643	2047.367
OFIW	2200.062	2206.061	2206.234	2214.95	2209.673
Kw-IW	2211.198	2219.197	2219.487	2231.049	2224.013
W-IW	1994.372	2002.372	2002.662	2014.224	2007.188
MBXII	2420.664	2426.664	2426.836	2435.552	2430.276
IW	2198.318	2202.317	2202.403	2208.243	2204.725

The MBXII-IW distribution is best fitted model than all other competing models because the values of all criteria of goodness of fit are significantly smaller for MBXII-IW distribution.

5. CONCLUDING REMARKS

We propose the MBXII-IW distribution from (i) the T-X family technique and (ii) link between the exponential and gamma random variables. The MBXII-IW density highlights various shapes as exponential, left-skewed, right-skewed and symmetrical shapes. Its hazard rate function has various shapes such as increasing, decreasing, increasing-decreasing, inverted bathtub and modified bathtub. We study some of its mathematical properties such as random number generator, ordinary moments, generating function, residual life functions, reliability measures and characterizations. We address the maximum likelihood estimation for the MBXII-IW parameters. We evaluate the precision of the maximum likelihood estimators via a simulation study. We consider an application to serum-reversal times to illustrate the potentiality of the new model. The potentiality of the MBXII-IW model clarifies that it is flexible to other existing distributions. Hence it should be included in the distribution theory to help the researchers.

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APPENDIX A

Theorem G. Let $(\Omega, \mathrm{F}, \mathrm{P})$ be a given probability space and let $\mathrm{H}=[a_1, a_2]$ be an interval with $a_1 < a_2$ ($a_1=-\infty, a_2=\infty$). Let $X:\Omega \to [a_1, a_2]$ be a continuous random variable with distribution function F and Let $g(x)$ be a real function defined on $\mathrm{H}=[a_1, a_2]$ such that $E[g(X)|\mathrm{X} \geq \mathrm{x}]=h(x)$ for $x \in H$ is defined with some real function $h(x)$ should be in simple form. Assume that $g(x) \varepsilon C([a_1, a_2])$, $h(x) \varepsilon C^2([a_1, a_2])$ and F is twofold continuously differentiable and strictly monotone function on the set $[a_1, a_2]$.We conclude, assuming that the equation $g(x)=h(x)$ has no real solution in the inside of $[a_1, a_2]$ .Then F is obtained from the functions $g(x)$ and $h(x)$ as $F(x)=\int_a^x k\left|\frac{h'(t)}{h(t)-g(t)}\right|\exp(-s(t))dt$, where $s(t)$ is the solution of equation $s'(t)=\frac{h'(t)}{h(t)-g(t)}$ and k is a constant, chosen to make $\int_{a_1}^{a_2} dF = 1$.