Bayesian and Maximum Likelihood Estimation for the Weibull Generalized Exponential Distribution Parameters Using Progressive Censoring Schemes

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Abstract

In this paper we consider the estimation of the Weibull Generalized Exponential Distribution (WGED) Parameters with Progressive Censoring Schemes. In order to obtain the optimal censoring scheme for WGED, more than one method of estimation was used to reach a better scheme with the best method of estimation. The maximum likelihood method and the method of Bayesian estimation for (square error and Linex) loss function have been used. Monte carlo simulation is used for comparison between the two methods of estimation under censoring schemes. To show how the schemes work in practice; we analyze a strength data for single carbon fibers as a case of real data.

Keywords: Weibull generalized exponential distribution; Maximum likelihood estimation; Bayesian estimation; MCMC; Censoring scheme.

1. Introduction

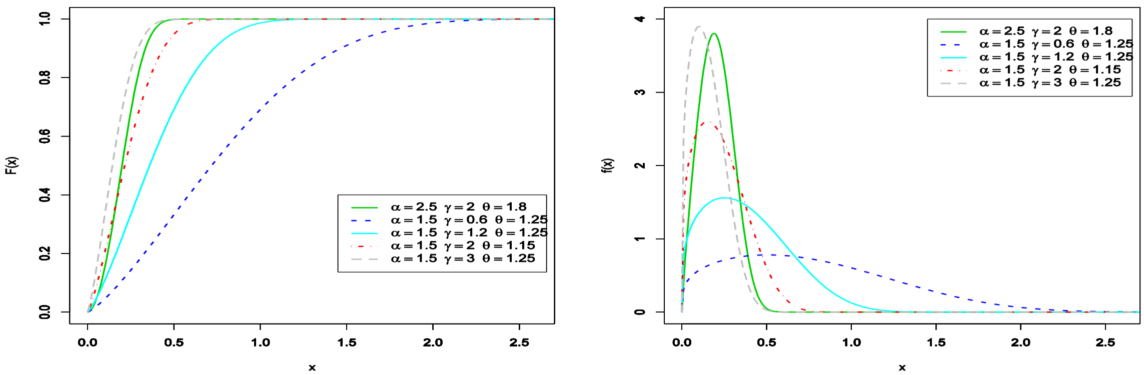
The exponential family, have more of applications including life testing experiments, reliability analysis, applied statistics and clinical studies, and the Weibull distribution is one of the most popular distributions in analyzing lifetime data. Mustafa et al. (2016) used the Weibull-G family to generating the Weibull- Generalized Exponential distribution (WGED). A random variable x has WGED with parameters $\alpha.\gamma$ and θ, say if its cumulative distribution function (cdf), probability density function (pdf) and the quantile function are given by

The cdf is

$$F(x;\alpha.\gamma.\theta) = 1 - e^{-\alpha(e^{\gamma x}-1)^{\theta}} \tag{1.1}$$

the corresponding pdf is

$$f(x;\alpha.\gamma.\theta) = \alpha\gamma\theta e^{\gamma x}(e^{\gamma x} - 1)^{\theta-1} e^{-\alpha(e^{\gamma x}-1)^{\theta}} \tag{1.2}$$



Graph 1: Plot of WGED with different parameters values

The quantile function of the WGED is:

$$x = \frac{1}{\gamma}\ln\left(1 + \left[\frac{-1}{\alpha}\ln(1-u)\right]^{1/\theta}\right) \quad . \quad 0 < u < 1 \tag{1.3}$$

Kundu and Pradhan (2009) discussed the two most common censoring schemes are termed as type-I and type-II censoring schemes. Kim and Han (2009) discussed, progressively type-II censored sampling as an important method of obtaining data in lifetime studies. Ng et al. (2004) introduced, a progressive type-II censoring scheme can be described as follows: Suppose n units are placed on a life test and the experimenter decides beforehand the quantity m, the number of failures to be observed. Now at the time of the first failure, R_1 of the remaining $n-1$ surviving units are randomly removed from the experiment. At the time of the second failure, R_2 of the remaining $n - R_1 - 1$ units are randomly removed from the experiment. Finally, at the time of the m-th failure, all the remaining surviving units $R_m = n - m - R_1 - \cdots - R_{m-1}$ are removed from the experiment. Therefore, a progressive type-II censoring scheme consists of m, and $R_1 \ldots R_m$, such that $R_1 + \cdots + R_m = n - m$. For more examples, see Dey et al. (2016), Elsherpieny et al. (2017), Mohamed et al. (2018) and Almetwaly and Almongy (2018), see for instance the book by Balakrishnan and Aggrawalla (2000), and an excellent review article by Balakrishnan et al (2007).

2. The Maximum Likelihood Estimation Method

Based on the observed $x_1 < \cdots < x_m$ from a progressive Type-II censored sample scheme, $R_1 . \ldots . R_m$ the likelihood function can be written as

$$L = A(\alpha\gamma\theta)^m e^{\gamma\sum_{i=1}^{m} x_i}\, e^{-\alpha\sum_{i=1}^{m}(e^{\gamma x_i}-1)^{\theta}} \prod_{i=1}^{m}\left[(e^{\gamma x_i}-1)^{\theta-1}\left(e^{-\alpha(e^{\gamma x_i}-1)^{\theta}}\right)^{R_i}\right] \tag{2.1}$$

Where $A = n(n - R_1 - 1) \ldots \left(n - \sum_{i=1}^{m-1} R_i - (m-1)\right)$ is a constant which doesn't depend on parameters.

The natural logarithm of the likelihood function equation can be obtained as follows

$$\ln L = \ln A + m\ln(\alpha\gamma\theta) + \gamma\sum_{i=1}^{m} x_i + (\theta-1)\sum_{i=1}^{m}\ln(e^{\gamma x_i}-1) - \alpha\sum_{i=1}^{m}(e^{\gamma x_i}-1)^{\theta} - \alpha\sum_{i=1}^{m} R_i\,(e^{\gamma x_i}-1)^{\theta} \tag{2.2}$$

respectively. To obtain the normal equations for the unknown parameters, we differentiate (2.2) partially with respect to the parameters (α, γ and θ) and equate them to zero. The estimators $\hat{\alpha}, \hat{\gamma}$ and $\hat{\theta}$ can be obtained as the solution of the following equations.

$$\frac{\partial \ln L}{\partial \alpha} = \frac{m}{\alpha} - \sum_{i=1}^{m}(e^{\gamma x_i}-1)^{\theta} - \sum_{i=1}^{m} R_i\,(e^{\gamma x_i}-1)^{\theta} \tag{2.3}$$

$$\frac{\partial \ln L}{\partial \gamma} = \frac{m}{\gamma} + \sum_{i=1}^{m} x_i + (\theta-1)\sum_{i=1}^{m}\frac{x_i e^{\gamma x_i}}{(e^{\gamma x_i}-1)} - \theta\alpha\sum_{i=1}^{m} x_i e^{\gamma x_i}(e^{\gamma x_i}-1)^{\theta-1} - \theta\alpha\sum_{i=1}^{m} R_i\, x_i e^{\gamma x_i}(e^{\gamma x_i}-1)^{\theta-1} \tag{2.4}$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{m}{\theta} + \sum_{i=1}^{m}\ln(e^{\gamma x_i}-1) - \alpha\sum_{i=1}^{m}(e^{\gamma x_i}-1)^{\theta}\ln(e^{\gamma x_i}-1) - \alpha\sum_{i=1}^{m} R_i\,(e^{\gamma x_i}-1)^{\theta}\ln(e^{\gamma x_i}-1) \tag{2.5}$$

But the three equation has to be performed numerically using a nonlinear optimization algorithm.

3. Bayesians Estimation

For Bayesian estimation, the unknown parameters α, γ and θ of WGED are assumed to be independent and have their own conjugate prior distributions. Assumed that $\alpha \sim Gamma(a.b)$, $\gamma \sim Gamma(c.d)$ and $\theta \sim Gamma(e.f)$ then, the joint prior density of $\alpha.\gamma$ and θ can be written as

$$g(\alpha.\gamma.\theta) \propto \alpha^{a-1}e^{-\alpha b}\gamma^{c-1}e^{-\gamma d}\theta^{e-1}e^{-\theta f} \tag{3.1}$$

Based on the likelihood function (2.1) and the joint prior density (3.1), the joint posterior density of α, γ and θ and the data is

$$g(\alpha,\gamma,\theta,x) \propto \alpha^{m+a-1}e^{-\alpha b}\gamma^{m+c-1}e^{-\gamma d}\theta^{m+e-1}e^{-\theta f}e^{\gamma\sum_{i=1}^{m}x_i}e^{-\alpha\sum_{i=1}^{m}(e^{\gamma x_i}-1)^{\theta}} \prod_{i=1}^{m}\left[(e^{\gamma x_i}-1)^{\theta-1}\left(e^{-\alpha(e^{\gamma x_i}-1)^{\theta}}\right)^{R_i}\right] \tag{3.2}$$

For the WGED under Progressive Type-II censoring, the full conditional posterior distributions of the parameters are given by

$$\pi(\alpha;\ \gamma,\theta,x) \propto \alpha^{m+a-1}e^{-\alpha\left(b+\sum_{i=1}^{m}(e^{\gamma x_i}-1)^{\theta}\right)} \tag{3.3}$$

$$\pi(\gamma;\ \alpha,\theta,x) \propto \gamma^{m+c-1} e^{-\gamma d} e^{\gamma \sum_{i=1}^{m} x_i} \prod_{i=1}^{m} \left[(e^{\gamma x_i} - 1)^{\theta-1} \left(e^{-\alpha (e^{\gamma x_i}-1)^{\theta}} \right)^{R_i} \right] e^{-\alpha \sum_{i=1}^{m} (e^{\gamma x_i}-1)^{\theta}} \tag{3.4}$$

$$\pi(\ \theta;\ \alpha,\gamma,x) \propto \theta^{m+e-1} e^{-\theta f} \prod_{i=1}^{m} \left[(e^{\gamma x_i} - 1)^{\theta-1} \left(e^{-\alpha (e^{\gamma x_i}-1)^{\theta}} \right)^{R_i} \right] e^{-\alpha \sum_{i=1}^{m} (e^{\gamma x_i}-1)^{\theta}} \tag{3.5}$$

Since the conditional posterior distributions do not have simple forms in perspective of sampling, we use the Metropolis-Hastings algorithm. An important sub-class of MCMC methods are Gibbs sampling and more general Metropolis within Gibbs samplers. The advantage of using the MCMC method over the MLE method is that we can always obtain a reasonable interval estimate of the parameters by constructing the probability intervals based on empirical posterior distribution. To generate samples from the conditional posterior density distributions, we use Markov chain Monte Carlo (MCMC). In more information about the Metropolis-Hastings algorithm see Metropolis et al (1953) and to more example see Amin (2017) and Nassar et al (2018).

3.1. Square Error Loss Function

Almetwaly and Almongy (2018) discussed square error (SE) loss function as a very well-known symmetric loss function which define as $L(\hat{\delta}_{SE}.\delta_{SE}) = (\hat{\delta}-\delta)^2$, after generating of parameters

$$\tilde{\alpha} = \sum_{i=1}^{M} \frac{\alpha^{(i)}}{M}, \qquad \tilde{\gamma} = \sum_{i=1}^{M} \frac{\gamma^{(i)}}{M}, \qquad \tilde{\theta} = \sum_{i=1}^{M} \frac{\theta^{(i)}}{M} \tag{3.6}$$

where M is number of periods in the MCMC process.

3.2. Linex Loss Function

One of the most commonly-used asymmetric loss functions is the Linex loss

$$L(\hat{\delta}_L, \delta_L) = e^{h(\hat{\delta}-\delta)} - h(\hat{\delta}-\delta) - 1, \qquad h \neq 0 \tag{3.7}$$

$$\tilde{\tilde{\alpha}} = \frac{-1}{v} \ln\left(\sum_{i=1}^{M} \frac{e^{-v\alpha^{(i)}}}{M} \right), \qquad \tilde{\tilde{\gamma}} = \frac{-1}{v} \ln\left(\sum_{i=1}^{M} \frac{e^{-v\theta^{(i)}}}{M} \right), \qquad \tilde{\tilde{\theta}} = \frac{-1}{v} \ln\left(\sum_{i=1}^{M} \frac{e^{-v\alpha^{(i)}}}{M} \right) \tag{3.8}$$

where v is reflects the direction and degree of asymmetry, For $v > 0$, the overestimation is more serious than underestimation, for $v < 0$. the underestimation is more serious than the overestimation, and for v closed to zero, the Linex loss is approximately squared error loss and therefore almost symmetric.

4. Asymptotic Confidence Intervals

In this section, we propose the asymptotic confidence intervals using methods of estimations. Keeping this in mind, we may propose the asymptotic confidence intervals using MLE and Bayesian estimation of SE and Linex loss function methods can be used to construct the confidence intervals for the parameters. $I(\hat{\alpha}.\hat{\gamma}.\hat{\theta})$ is the observed inverse Fishers information matrix and is define as:

$$I(\hat{\alpha}.\hat{\gamma}.\hat{\theta}) = \begin{bmatrix} -L''_{\alpha\alpha} & -L''_{\alpha\gamma} & -L''_{\alpha\theta} \\ -L''_{\gamma\alpha} & -L''_{\gamma\gamma} & -L''_{\gamma\theta} \\ -L''_{\theta\alpha} & -L''_{\theta\gamma} & -L''_{\theta\theta} \end{bmatrix}^{-1} = \begin{bmatrix} I_{\hat{\alpha}\hat{\alpha}} & I_{\hat{\alpha}\hat{\gamma}} & I_{\hat{\alpha}\hat{\theta}} \\ I_{\hat{\gamma}\hat{\alpha}} & I_{\hat{\gamma}\hat{\gamma}} & I_{\hat{\gamma}\hat{\theta}} \\ I_{\hat{\theta}\hat{\alpha}} & I_{\hat{\theta}\hat{\gamma}} & I_{\hat{\theta}\hat{\theta}} \end{bmatrix} \tag{4.1}$$

And then compensation for the value of each parameter estimated for each methods and get the matrix mathematically through simulation package. An approximate 95% two side confidence intervals for ($\alpha.\gamma$ and θ) are respectively

$$\hat{\alpha} \pm Z_{0.025}\sqrt{I_{\hat{\alpha}\hat{\alpha}}}, \qquad \hat{\gamma} \pm Z_{0.025}\sqrt{I_{\hat{\gamma}\hat{\gamma}}}, \qquad \hat{\theta} \pm Z_{0.025}\sqrt{I_{\hat{\theta}\hat{\theta}}} \tag{4.2}$$

5. Simulation Study

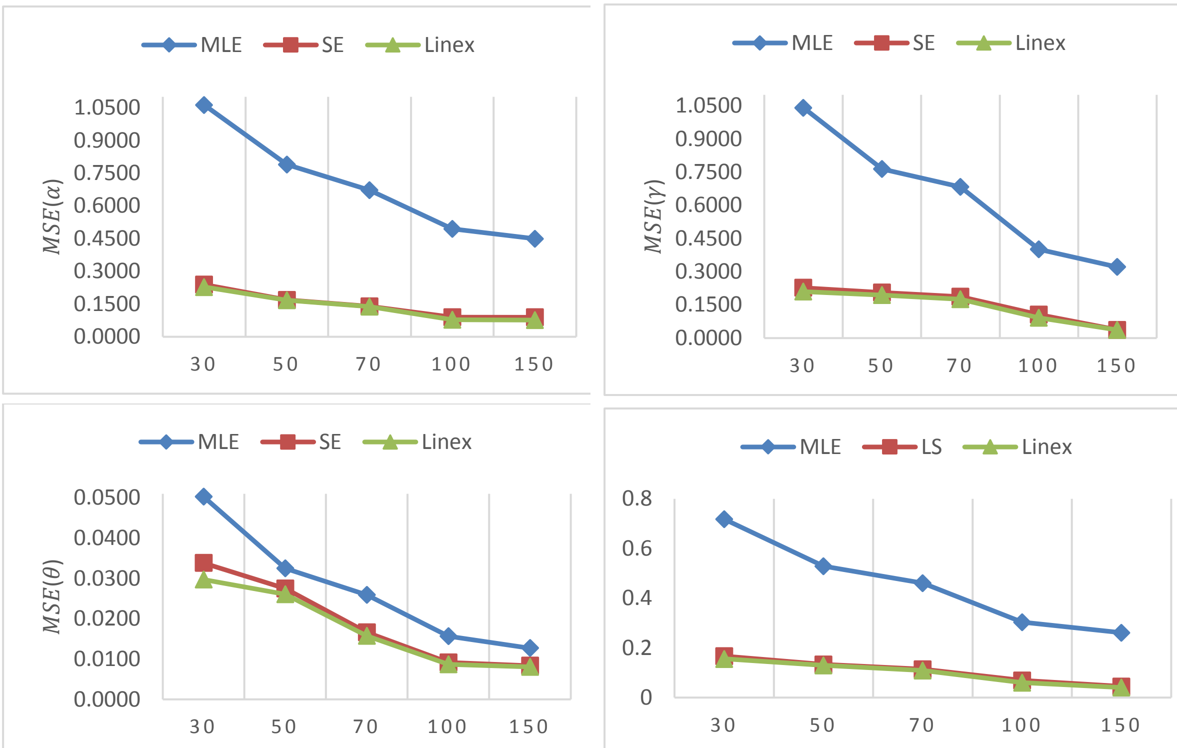
In this section; Monte Carlo simulation is done for comparison between maximum likelihood and Bayesian estimation methods under censoring scheme, for estimating parameters of WGED in life time by R language. Monte Carlo experiments were carried out based on the following data- generated from Marshall–Olkin Extended Weibull distribution by use (1.3), where x are distributed as MOEW for different shape parameters ($\alpha = 1.5, \gamma = 2$ and $\theta = 1.25$) and ($\alpha = 1.5, \gamma = 0.6$ and $\theta = 1.25$), and for different sample size $n = 30, 50, 70, 100$ *and* 150, different ratio of effective sample sizes $r = \frac{m}{n}$, and set of different samples schemes, where

- Scheme I: $R_1 = R_2 = \cdots = R_{m-1} = 0.$ *and* $R_m = n - m$. It is type-II scheme
- Scheme II: $R_1 = n - m$ *and* $R_2 = R_3 = \cdots = R_{m-1} = 0$.
- Scheme III: $R_1 = R_2 = \cdots = R_{m-1} = 1.$ *and* $R_m = n - 2m - 1$.

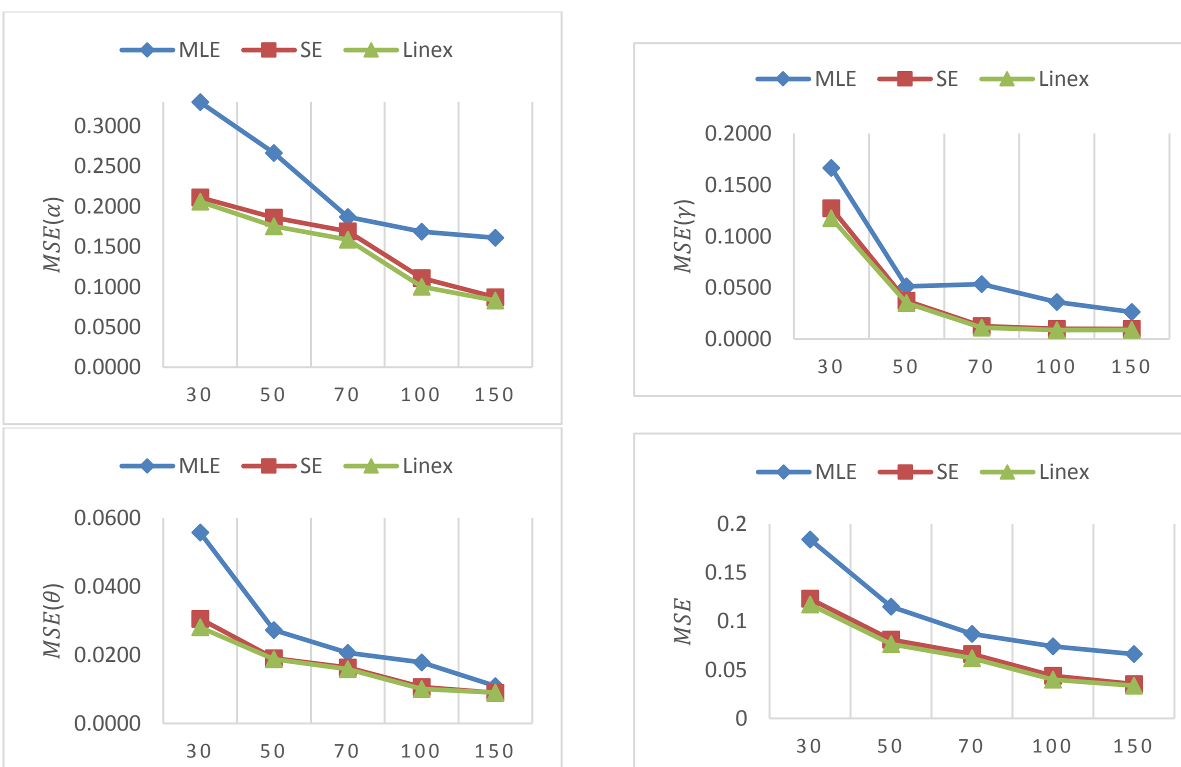
We could define the best scheme as the scheme which minimizes the mean squared error ($MSE(\theta)$), Bias of estimation and length of confidence interval (L.CI) of the estimator.

The following are Some of the graphs of Bias and MSE in complete sample scheme which are plotted and attached for two cases:

Case-I: ($\alpha = 1.5, \gamma = 0.6$ and $\theta = 1.25$) and Case-II: ($\alpha = 1.5,\ \gamma = 2$ and $\theta = 1.25$)



Graph 2: MSE for the methods in complete sample scheme in case-I



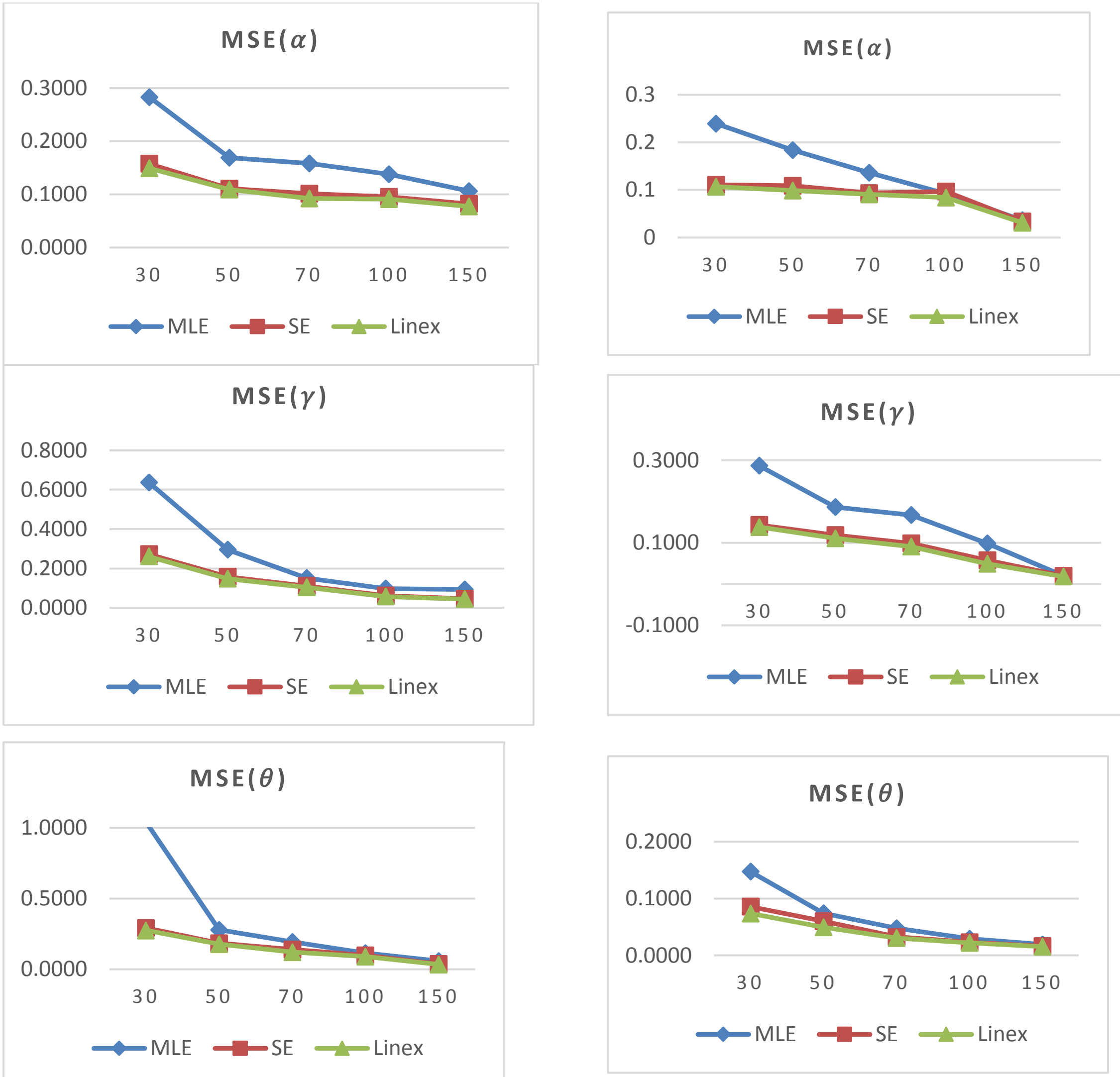
Graph 3: MSE for the methods in complete sample scheme in case-II

In the complete samples, we observe the Bayesian estimation is better and more efficient than the maximum likelihood estimation. As is evident in the previous presentation in Graph 2 and Graph 3.

Table 1: Parameter estimation for WGED under Scheme-I in case-I

($\alpha = 1.5. \gamma = 0.6.$ and $\theta = 1.25$)										
$r = 0.2$										
		MLE			SE			Linex		
n		Bias	MSE	L.C.I	Bias	MSE	L.C.I	Bias	MSE	L.C.I
30	$\hat{\alpha}$	-0.0510	0.2830	2.0769	-0.0466	0.1573	1.2051	-0.0881	0.1491	1.2431
	$\hat{\gamma}$	0.4865	0.6368	2.4810	0.1345	0.2721	0.9153	0.1055	0.2603	0.8737
	$\hat{\theta}$	0.5247	1.0402	3.4301	0.0300	0.2925	1.0544	-0.0013	0.2757	1.0096
50	$\hat{\alpha}$	-0.0518	0.1691	1.6000	0.0225	0.1109	1.2971	-0.0251	0.1093	1.2616
	$\hat{\gamma}$	0.2650	0.2956	1.8621	0.0861	0.1578	0.8574	0.0621	0.1475	0.8228
	$\hat{\theta}$	0.2278	0.2789	1.8688	0.0474	0.1846	0.9840	0.0194	0.1788	0.9519
70	$\hat{\alpha}$	-0.0213	0.1584	1.1343	0.0662	0.1013	1.2264	0.0208	0.0923	1.1940
	$\hat{\gamma}$	0.1738	0.1515	1.3656	0.0504	0.1109	0.7988	0.0307	0.1039	0.7676
	$\hat{\theta}$	0.1826	0.1929	1.5667	0.0597	0.1397	0.9334	0.0354	0.1214	0.8824
100	$\hat{\alpha}$	-0.0258	0.1381	1.4539	0.0437	0.0953	1.3268	0.0011	0.0910	1.2888
	$\hat{\gamma}$	0.1533	0.0978	1.0688	0.0766	0.0613	0.7613	0.0582	0.0577	0.7293
	$\hat{\theta}$	0.1359	0.1155	1.2218	0.0417	0.0995	0.8185	0.0217	0.0908	0.7908
150	$\hat{\alpha}$	-0.0473	0.1063	1.2649	0.0100	0.0820	1.2575	-0.0287	0.0772	1.2232
	$\hat{\gamma}$	0.1137	0.0933	1.1793	0.0854	0.0463	0.7612	0.0675	0.0439	0.7291
	$\hat{\theta}$	0.0721	0.0589	0.9090	0.0482	0.0373	0.7371	0.0315	0.0345	0.7215
$r = 0.5$										
		MLE			SE			Linex		
n		Bias	MSE	L.C.I	Bias	MSE	L.C.I	Bias	MSE	L.C.I
30	$\hat{\alpha}$	-0.1313	0.2395	1.9254	0.0098	0.1108	1.2038	-0.0329	0.1069	1.2333
	$\hat{\gamma}$	0.2206	0.2868	1.9138	0.0838	0.1431	0.748	0.0683	0.1383	0.7227
	$\hat{\theta}$	0.1226	0.1473	1.4264	0.113	0.0857	1.0637	0.0828	0.0738	1.0192
50	$\hat{\alpha}$	-0.1005	0.1837	2.256	0.0526	0.10899	1.163	0.0187	0.09856	1.1504
	$\hat{\gamma}$	0.184	0.1863	1.5316	0.0385	0.1183	0.5114	0.0281	0.1107	0.5005
	$\hat{\theta}$	0.0561	0.0739	1.043	0.0835	0.0602	0.9093	0.0582	0.0497	0.8482
70	$\hat{\alpha}$	-0.0918	0.1364	2.3377	0.0353	0.0935	1.3043	-0.0037	0.09091	1.2883
	$\hat{\gamma}$	0.1548	0.1673	1.4849	0.0367	0.0985	0.5153	0.0265	0.0907	0.4981
	$\hat{\theta}$	0.035	0.048	0.8478	0.0018	0.0324	0.7093	-0.0153	0.0302	0.6819
100	$\hat{\alpha}$	-0.0994	0.0921	1.7361	0.0152	0.0971	1.2881	-0.0271	0.0844	1.2684
	$\hat{\gamma}$	0.1182	0.0984	1.1394	0.0351	0.0575	0.5505	0.0259	0.0490	0.5328
	$\hat{\theta}$	0.0264	0.0296	0.6667	0.006	0.023	0.5974	-0.0077	0.0219	0.5826
150	$\hat{\alpha}$	-0.024	0.0365	0.7431	0.0277	0.0342	1.3268	-0.0079	0.0312	1.3132
	$\hat{\gamma}$	0.0327	0.0204	0.5446	0.0492	0.0201	0.5242	0.0417	0.0181	0.5037
	$\hat{\theta}$	0.0237	0.0192	0.536	0.0221	0.0162	0.4945	0.0126	0.0155	0.4878

$r = 0.2$ $r = 0.5$



Graph 4: MSE for the methods in censoring sample scheme-I in case-I

We can observe that as the ratio of effective sample sizes (r) increases for censoring sample, the value of the MSE decreases for the parameters of WGED.

Table 2: Parameter estimation for WGED under Scheme-II in case-I

($\alpha = 1.5. \gamma = 0.6.$ and $\theta = 1.25$)										
$r = 0.2$										
		MLE			SE			Linex		
n		Bias	MSE	L.C.I	Bias	MSE	L.C.I	Bias	MSE	L.C.I
30	$\hat{\alpha}$	-0.3243	0.9019	3.5006	-0.1355	0.1984	1.2282	-0.1272	0.1901	1.2087
	$\hat{\gamma}$	0.6573	1.2719	3.5942	0.1907	0.2414	0.7174	0.1701	0.2348	0.6811
	$\hat{\theta}$	0.0585	0.1829	1.6616	0.0257	0.1066	1.0116	-0.049	0.1014	0.9759
50	$\hat{\alpha}$	-0.2914	0.6778	3.0198	0.059	0.1262	1.2838	-0.0412	0.1185	1.2873
	$\hat{\gamma}$	0.4746	0.7994	2.9718	0.1271	0.1591	0.6992	0.1217	0.1534	0.6624
	$\hat{\theta}$	-0.0105	0.0974	1.2230	0.0445	0.0519	0.8802	0.0193	0.0458	0.8394
70	$\hat{\alpha}$	-0.2791	0.6553	2.9801	0.0329	0.1202	1.4696	-0.0111	0.1200	1.4499

	$\hat{\gamma}$	0.3955	0.5604	2.4928	0.1009	0.1223	0.5536	0.0675	0.1193	0.5271
	$\hat{\theta}$	-0.0121	0.0802	1.1095	0.0122	0.0405	0.7917	-0.0070	0.0374	0.7611
100	$\hat{\alpha}$	-0.2644	0.6079	3.1327	0.0735	0.1127	1.2905	0.0325	0.1015	1.2487
	$\hat{\gamma}$	0.3483	0.4058	2.0921	0.0934	0.0931	0.5439	0.0937	0.0907	0.5259
	$\hat{\theta}$	-0.0223	0.0570	0.9324	0.0390	0.0350	0.7206	0.0209	0.0310	0.6883
150	$\hat{\alpha}$	-0.2331	0.5022	2.6248	0.0386	0.1082	1.2940	0.0227	0.0922	1.3034
	$\hat{\gamma}$	0.2546	0.2574	1.7212	0.0617	0.0237	0.5559	0.0542	0.0213	0.5342
	$\hat{\theta}$	-0.0305	0.0406	0.7809	0.0295	0.0185	0.5228	0.0161	0.0170	0.5103
$r = 0.5$										
		MLE			SE			Linex		
n		Bias	MSE	L.C.I	Bias	MSE	L.C.I	Bias	MSE	L.C.I
30	$\hat{\alpha}$	-0.2120	0.6123	2.9016	-0.0820	0.1033	1.2660	-0.0814	0.1033	1.2556
	$\hat{\gamma}$	0.3144	0.4065	2.1752	0.0969	0.1262	0.5806	0.0921	0.1134	0.5607
	$\hat{\theta}$	0.0118	0.0897	1.1739	0.0400	0.0567	0.9251	0.0185	0.0525	0.8999
50	$\hat{\alpha}$	-0.1579	0.5989	2.9714	-0.0661	0.0908	1.1872	-0.0651	0.0913	1.1821
	$\hat{\gamma}$	0.2742	0.3282	1.9727	0.0580	0.0829	0.5508	0.0478	0.0895	0.5170
	$\hat{\theta}$	-0.0161	0.0638	0.9888	-0.0018	0.0400	0.7881	-0.0200	0.0382	0.7664
70	$\hat{\alpha}$	-0.1807	0.4007	2.3793	-0.0861	0.0881	1.3540	-0.0828	0.0826	1.0094
	$\hat{\gamma}$	0.1870	0.1695	1.4383	0.0509	0.0783	0.4940	0.0429	0.0760	0.4683
	$\hat{\theta}$	-0.0155	0.0401	0.7833	0.0393	0.0234	0.5820	0.0250	0.0215	0.5696
100	$\hat{\alpha}$	-0.1177	0.2295	1.8212	-0.0840	0.0809	1.0707	-0.0803	0.0798	1.0971
	$\hat{\gamma}$	0.1278	0.1304	1.3243	0.0740	0.0207	0.4860	0.0662	0.0180	0.4592
	$\hat{\theta}$	-0.0040	0.0299	0.6782	0.0054	0.0211	0.5725	-0.0061	0.0204	0.5622
150	$\hat{\alpha}$	-0.0071	0.2237	1.8549	-0.0533	0.0524	0.6110	-0.0504	0.0532	0.6075
	$\hat{\gamma}$	0.0520	0.0409	0.7665	0.0614	0.0188	0.4823	0.0548	0.0162	0.4533
	$\hat{\theta}$	0.0032	0.0158	0.4935	0.0057	0.0127	0.4427	-0.0022	0.0124	0.4382

Table 3: Parameter estimation for WGED under Scheme-III in case I

$(\alpha = 1.5. \gamma = 0.6. \text{and } \theta = 1.25)$										
$r = 0.2$										
		MLE			SE			Linex		
n		Bias	MSE	L.C.I	Bias	MSE	L.C.I	Bias	MSE	L.C.I
30	$\hat{\alpha}$	-0.0768	0.3210	2.2015	-0.0937	0.1303	1.4222	-0.0989	0.1299	1.4119
	$\hat{\gamma}$	0.4978	0.6690	2.5454	0.1214	0.1699	0.9250	0.0941	0.0600	0.8910
	$\hat{\theta}$	0.4905	0.9304	3.2574	0.1735	0.2801	1.0766	0.1424	0.2723	1.0459
50	$\hat{\alpha}$	-0.0759	0.1948	1.7053	-0.0834	0.1125	1.3181	-0.0822	0.1195	1.3398
	$\hat{\gamma}$	0.2736	0.3001	1.8612	0.1063	0.1157	0.9752	0.0981	0.1098	0.9105
	$\hat{\theta}$	0.2121	0.2544	1.7946	0.1209	0.1673	0.9707	0.1158	0.1581	0.9265
70	$\hat{\alpha}$	-0.0384	0.1809	1.1733	-0.0796	0.0987	1.2352	-0.0752	0.0912	1.2346
	$\hat{\gamma}$	0.1764	0.1592	1.4036	0.0864	0.0989	0.8178	0.0867	0.0930	0.7860
	$\hat{\theta}$	0.1715	0.1763	1.5029	0.1084	0.1422	0.8873	0.1046	0.1363	0.8456
100	$\hat{\alpha}$	-0.0327	0.1631	1.5787	-0.0610	0.0634	1.2031	-0.0607	0.0604	1.2024
	$\hat{\gamma}$	0.1573	0.1072	1.1263	0.0689	0.0766	0.7913	0.0616	0.0696	0.7484
	$\hat{\theta}$	0.1276	0.1063	1.1768	0.0666	0.0425	0.7916	0.0266	0.0394	0.7751
150	$\hat{\alpha}$	-0.0553	0.1229	1.3575	-0.0131	0.0232	1.0649	-0.0449	0.0756	1.0685
	$\hat{\gamma}$	0.1182	0.1103	1.2172	0.0434	0.0302	0.6629	0.0270	0.0268	0.6363

	$\hat{\theta}$	0.0669	0.0544	0.8765	0.0228	0.0275	0.6529	0.0177	0.0264	0.6359
$r = 0.5$										
		MLE			SE			Linex		
n		Bias	MSE	L.C.I	Bias	MSE	L.C.I	Bias	MSE	L.C.I
30	$\hat{\alpha}$	-0.2017	0.3195	2.5427	-0.1147	0.1004	1.1158	-0.1121	0.1008	1.1164
	$\hat{\gamma}$	0.3132	0.4383	2.2874	0.1003	0.0967	0.6118	0.0978	0.0942	0.5951
	$\hat{\theta}$	0.0663	0.1097	1.2730	0.0953	0.0866	0.7323	0.0215	0.0832	0.7123
50	$\hat{\alpha}$	-0.1799	0.3055	2.8112	-0.0751	0.0870	1.3476	-0.0710	0.0835	1.3485
	$\hat{\gamma}$	0.3059	0.3852	2.1180	0.0787	0.0781	0.5835	0.0671	0.0742	0.5535
	$\hat{\theta}$	0.0079	0.0635	0.9881	0.0812	0.0442	0.7269	-0.0766	0.0426	0.7104
70	$\hat{\alpha}$	-0.1869	0.2843	2.4642	-0.0693	0.0632	1.2634	-0.0645	0.0624	1.2582
	$\hat{\gamma}$	0.2250	0.2265	1.6446	0.0553	0.0685	0.6131	0.0553	0.0650	0.5835
	$\hat{\theta}$	-0.0003	0.0394	0.7781	0.0597	0.0220	0.5724	0.0458	0.0207	0.5632
100	$\hat{\alpha}$	-0.1468	0.2581	1.9980	-0.0162	0.0332	1.2638	-0.0507	0.0315	1.2580
	$\hat{\gamma}$	0.1595	0.1438	1.3496	0.0597	0.0483	0.4786	0.0527	0.0466	0.4628
	$\hat{\theta}$	0.0014	0.0258	0.6295	0.0470	0.0200	0.6117	0.0455	0.0195	0.5982
150	$\hat{\alpha}$	-0.0367	0.1421	1.4712	-0.0177	0.0290	1.2374	-0.0152	0.0259	1.2122
	$\hat{\gamma}$	0.0573	0.0468	0.8184	0.0524	0.0210	0.5321	0.0450	0.0187	0.5083
	$\hat{\theta}$	0.0109	0.0164	0.5001	-0.0033	0.0099	0.3921	-0.0103	0.0098	0.3880

Table 4: Parameter estimation for WGED under Scheme-I in case-II

$(\alpha = 1.5. \gamma = 2.$ and $\theta = 1.25)$										
$r = 0.2$										
		MLE			SE			Linex		
n		Bias	MSE	L.C.I	Bias	MSE	L.C.I	Bias	MSE	L.C.I
30	$\hat{\alpha}$	0.4134	1.0167	3.6071	0.0996	0.1975	1.2299	-0.0929	0.1970	1.2202
	$\hat{\gamma}$	0.8999	2.2436	4.6961	0.1264	0.2146	1.3336	0.1168	0.2160	1.3291
	$\hat{\theta}$	0.5578	1.2064	3.7108	0.0768	0.1491	0.8703	0.0773	0.1470	0.8536
50	$\hat{\alpha}$	0.1916	0.5311	2.7577	0.0425	0.1113	1.3034	0.0012	0.1051	1.2774
	$\hat{\gamma}$	0.5309	0.9718	3.2578	0.1107	0.1824	1.3110	0.1103	0.1780	1.2946
	$\hat{\theta}$	0.2374	0.3001	1.9362	0.0725	0.0941	0.8177	0.0715	0.0909	0.7963
70	$\hat{\alpha}$	0.1069	0.3840	2.3939	0.0488	0.1023	1.2455	0.0069	0.0934	1.2035
	$\hat{\gamma}$	0.4497	0.7335	2.8588	0.0627	0.1221	1.3758	-0.0664	0.1114	1.3107
	$\hat{\theta}$	0.1848	0.2031	1.6119	0.0193	0.0311	0.6906	0.0027	0.0296	0.6779
100	$\hat{\alpha}$	0.0960	0.2202	1.8013	0.0613	0.0916	1.2759	0.0601	0.0901	1.2365
	$\hat{\gamma}$	0.2966	0.4175	2.2513	-0.0526	0.0932	1.2017	-0.0502	0.0923	1.1805
	$\hat{\theta}$	0.1406	0.1203	1.2437	0.0190	0.0307	0.7190	-0.0138	0.0289	0.7130
150	$\hat{\alpha}$	0.0791	0.1443	1.4573	0.0346	0.0597	1.1723	-0.0307	0.0525	1.1523
	$\hat{\gamma}$	0.1680	0.2091	1.6679	0.0224	0.0657	1.2777	-0.0159	0.0618	1.2244
	$\hat{\theta}$	0.0787	0.0646	0.9482	-0.0134	0.0221	0.5838	-0.0148	0.0121	0.5780
r=0.5										

		MLE			SE			Linex		
n		Bias	MSE	L.C.I	Bias	MSE	L.C.I	Bias	MSE	L.C.I
30	$\hat{\alpha}$	-0.0237	0.6885	3.2530	0.1248	0.1005	1.1041	0.1100	0.0927	1.0619
	$\hat{\gamma}$	0.6043	1.4382	4.0628	0.1342	0.1928	1.1735	0.1287	0.1855	1.1463
	$\hat{\theta}$	0.1180	0.1470	1.4306	0.0422	0.0557	0.9150	0.0170	0.0505	0.8829
50	$\hat{\alpha}$	0.0561	0.4818	2.7134	0.1180	0.0944	1.1711	0.1121	0.0860	1.1437
	$\hat{\gamma}$	0.3242	0.7136	3.0592	0.1056	0.1452	1.2143	-0.1015	0.1360	1.2185
	$\hat{\theta}$	0.0698	0.0773	1.0558	0.0366	0.0387	0.7618	0.0173	0.0363	0.7472
70	$\hat{\alpha}$	-0.0934	0.2273	1.8337	0.0198	0.0799	1.1106	-0.0096	0.0756	1.0823
	$\hat{\gamma}$	0.2647	0.3784	2.1779	0.0952	0.1127	1.0579	-0.0926	0.1080	1.0569
	$\hat{\theta}$	0.0418	0.0431	0.7978	0.0053	0.0291	0.6713	-0.0093	0.0281	0.6596
100	$\hat{\alpha}$	0.0362	0.1157	1.3262	0.0187	0.0691	1.0914	0.0201	0.0657	1.0808
	$\hat{\gamma}$	0.0819	0.1190	1.3142	0.0835	0.0815	1.0522	0.0793	0.0781	1.0267
	$\hat{\theta}$	0.0426	0.0298	0.6565	0.0049	0.0222	0.5871	-0.0066	0.0215	0.5775
150	$\hat{\alpha}$	-0.0667	0.1040	2.4614	-0.0069	0.0166	1.0903	-0.0384	0.0137	1.0290
	$\hat{\gamma}$	0.3337	0.1064	2.8510	0.0612	0.0702	1.1133	0.0634	0.0691	1.0579
	$\hat{\theta}$	0.0068	0.0098	0.5509	0.0104	0.0141	0.4668	-0.0062	0.0135	0.4571

Table 5: Parameter estimation for WGED under Scheme-II in case-II

$(\alpha = 1.5. \gamma = 2. \text{and } \theta = 1.25)$										
$r = 0.2$										
		MLE			SE			Linex		
n		Bias	MSE	L.C.I	Bias	MSE	L.C.I	Bias	MSE	L.C.I
30	$\hat{\alpha}$	-0.0954	1.3120	4.4767	0.0997	0.1660	1.3046	0.0987	0.1616	1.2733
	$\hat{\gamma}$	1.1810	3.6325	5.8668	0.2502	0.2958	1.2034	0.2426	0.2812	1.1885
	$\hat{\theta}$	0.0901	0.1681	1.5685	0.0355	0.0771	1.0848	0.0029	0.0695	1.0384
50	$\hat{\alpha}$	-0.0269	1.2951	4.7944	0.0877	0.1522	1.0420	0.0824	0.1461	1.0120
	$\hat{\gamma}$	0.9265	2.6943	5.3140	0.2185	0.2791	1.1736	0.2178	0.2667	1.1579
	$\hat{\theta}$	0.0082	0.0786	1.0990	0.0644	0.0672	0.9893	0.0384	0.0597	0.9508
70	$\hat{\alpha}$	-0.1482	1.0883	4.0499	0.0814	0.1263	1.1552	0.0621	0.1119	1.1006
	$\hat{\gamma}$	0.8845	2.5794	5.5620	0.1551	0.1925	1.1965	0.1502	0.1897	1.1781
	$\hat{\theta}$	-0.0017	0.0663	1.0098	0.0542	0.0482	0.8380	0.0330	0.0435	0.8116
100	$\hat{\alpha}$	-0.0862	1.0730	4.0485	0.0535	0.0896	1.1113	0.0868	0.0804	1.1073
	$\hat{\gamma}$	0.6873	1.3635	4.2297	0.1219	0.1654	0.9541	0.1122	0.1596	0.9394
	$\hat{\theta}$	0.0011	0.0474	0.8541	0.0670	0.0435	0.7786	0.0486	0.0386	0.7499
150	$\hat{\alpha}$	-0.0590	0.8768	3.6652	0.0581	0.0397	1.0883	0.0273	0.0343	1.0688
	$\hat{\gamma}$	0.5510	1.2853	3.8859	0.0498	0.0487	0.8472	0.0241	0.0453	0.8333
	$\hat{\theta}$	-0.0156	0.0351	0.7322	0.0174	0.0222	0.5825	0.0042	0.0214	0.5755
$r = 0.5$										
		MLE			SE			Linex		
n		Bias	MSE	L.C.I	Bias	MSE	L.C.I	Bias	MSE	L.C.I
30	$\hat{\alpha}$	-0.0317	1.1912	4.1331	0.0514	0.0866	1.1418	0.0143	0.0780	1.0989

	$\hat{\gamma}$	0.6723	1.7722	4.5063	0.1189	0.0943	1.1158	0.0874	0.0840	1.0886
	$\hat{\theta}$	0.0277	0.0798	1.1026	0.0869	0.0561	0.8676	0.0613	0.0498	0.8453
50	$\hat{\alpha}$	0.1073	1.1662	4.2143	0.0973	0.0697	0.9666	0.0649	0.0627	0.9524
	$\hat{\gamma}$	0.4723	1.2740	4.0206	0.0371	0.0616	0.9672	0.0108	0.0574	0.9428
	$\hat{\theta}$	0.0105	0.0524	0.8968	0.0432	0.0385	0.7538	0.0233	0.0340	0.7205
70	$\hat{\alpha}$	0.0328	0.8682	3.6520	0.0740	0.0669	1.2547	0.0401	0.0642	1.2120
	$\hat{\gamma}$	0.3961	0.9582	3.5108	0.0303	0.0593	1.0765	0.0138	0.0583	1.0280
	$\hat{\theta}$	-0.0014	0.0362	0.7458	0.0376	0.0292	0.5249	0.0246	0.0175	0.5120
100	$\hat{\alpha}$	0.1586	0.8092	4.0515	0.0654	0.0590	1.0024	0.0360	0.0526	0.9753
	$\hat{\gamma}$	0.3063	0.8357	3.3780	0.0274	0.0541	0.9097	0.0062	0.0501	0.8819
	$\hat{\theta}$	0.0019	0.0291	0.6695	0.0262	0.0209	0.5603	0.0160	0.0200	0.5534
150	$\hat{\alpha}$	-0.0048	0.4405	2.6028	0.0402	0.0421	1.0461	0.0088	0.0393	1.0366
	$\hat{\gamma}$	0.2385	0.4769	2.5418	0.0225	0.0429	0.9329	0.0214	0.0408	0.8584
	$\hat{\theta}$	-0.0071	0.0175	0.5177	0.0108	0.0133	0.4520	0.0028	0.0128	0.4452

Table 6: Parameter estimation for WGED under Scheme-III in case-II

$(\alpha = 1.5. \gamma = 2.$ and $\theta = 1.25)$										
$r = 0.2$										
		MLE			SE			Linex		
n		Bias	MSE	L.C.I	Bias	MSE	L.C.I	Bias	MSE	L.C.I
30	$\hat{\alpha}$	0.3783	0.9428	3.5072	0.0652	0.1381	1.4412	0.0246	0.1199	1.3606
	$\hat{\gamma}$	0.8671	2.1383	4.6180	-0.0138	0.1133	1.3250	-0.0595	0.1181	1.3332
	$\hat{\theta}$	0.5236	1.0755	3.5109	0.0371	0.0597	0.9515	0.0117	0.0573	0.9423
50	$\hat{\alpha}$	0.1652	0.5327	2.7882	0.0144	0.1297	1.2293	-0.0249	0.0986	1.2331
	$\hat{\gamma}$	0.5361	0.9876	3.2818	0.0622	0.1130	1.3017	0.0223	0.1066	1.2834
	$\hat{\theta}$	0.2220	0.2733	1.8564	0.0373	0.0498	0.8666	0.0143	0.0467	0.8496
70	$\hat{\alpha}$	0.0862	0.3858	2.4124	0.0273	0.1211	1.3667	-0.0087	0.0909	1.3696
	$\hat{\gamma}$	0.4629	0.7715	2.9275	0.0265	0.1122	1.3155	-0.0129	0.1052	1.2769
	$\hat{\theta}$	0.1734	0.1862	1.5497	0.0250	0.0528	0.9000	0.0066	0.0504	0.8837
100	$\hat{\alpha}$	0.0816	0.2202	1.8122	0.0488	0.0864	1.1419	0.0155	0.0828	1.1319
	$\hat{\gamma}$	0.2912	0.3959	2.1878	0.0117	0.1049	1.4327	0.0107	0.1152	1.3365
	$\hat{\theta}$	0.1328	0.1107	1.1966	0.0493	0.0379	0.7424	0.0343	0.0353	0.7274
150	$\hat{\alpha}$	0.0738	0.1337	1.4043	0.0096	0.0675	1.2799	0.0092	0.0668	1.2255
	$\hat{\gamma}$	0.1577	0.1904	1.5955	0.0113	0.0969	1.3467	0.0083	0.0918	1.3617
	$\hat{\theta}$	0.0745	0.0600	0.9152	-0.0048	0.0343	0.7294	-0.0171	0.0326	0.7081
$r = 0.5$										
		MLE			SE			Linex		
n		Bias	MSE	L.C.I	Bias	MSE	L.C.I	Bias	MSE	L.C.I
30	$\hat{\alpha}$	-0.0461	0.8952	3.8234	0.0591	0.1208	1.3495	0.0177	0.1155	1.3371
	$\hat{\gamma}$	0.7347	1.9617	4.6768	0.0790	0.1000	1.2067	0.0477	0.0911	1.1745
	$\hat{\theta}$	0.0700	0.1029	1.2276	0.0537	0.0531	0.8825	0.0295	0.0473	0.8493
50	$\hat{\alpha}$	-0.0025	0.7047	3.2924	0.0422	0.0902	1.1714	0.0058	0.0839	1.1411
	$\hat{\gamma}$	0.5112	1.2653	3.9297	0.0705	0.0980	1.2015	0.0361	0.0880	1.1598
	$\hat{\theta}$	0.0343	0.0580	0.9353	0.0558	0.0463	0.8188	0.0365	0.0413	0.7878

70	$\hat{\alpha}$	-0.1594	0.4959	2.6900	0.0305	0.0759	1.0786	-0.0008	0.0702	1.0435
	$\hat{\gamma}$	0.5204	1.0326	3.4231	0.0594	0.0637	0.9667	0.0330	0.0598	0.9550
	$\hat{\theta}$	0.0063	0.0355	0.7388	0.0486	0.0298	0.6530	0.0338	0.0275	0.6397
100	$\hat{\alpha}$	0.0688	0.4579	2.6400	0.0299	0.0583	0.9436	0.0016	0.0553	0.9263
	$\hat{\gamma}$	0.1949	0.4451	2.5025	0.0303	0.0498	0.8715	0.0074	0.0472	0.8558
	$\hat{\theta}$	0.0227	0.0238	0.5988	0.0123	0.0138	0.4606	0.0012	0.0131	0.4516
150	$\hat{\alpha}$	0.0135	0.3511	2.8038	0.0242	0.0460	1.0777	0.0026	0.0409	1.0488
	$\hat{\gamma}$	0.2770	0.5875	2.8029	0.0209	0.0338	0.9820	0.0182	0.0201	0.9552
	$\hat{\theta}$	0.0001	0.0165	0.5042	-0.0051	0.0109	0.4114	-0.0129	0.0108	0.4070

For numerical illustration we have computed point estimation and interval estimation for the parameters of WGED, based on 1000 samples using different sample sizes 30, 50, 70, 100 and 150 generated by Monte Carlo simulation technique. A random samples was generated from the WGED for different values of parameters. In order to obtain the optimal censoring scheme for WGED by use different schemes as well as obtain the best scheme, more than one method was used to reach a better scheme with the best method of estimation. The maximum likelihood method and the method of Bayesian estimation for (SE and Linex) loss function have been used.

We note the greater the sample size, the greater the efficiency of the estimator in terms of lower values of MSE, Bias and L.C.I. If the ratio of effective sample sizes (r) increases for censoring sample, then the value of the MSE decreases for the parameters of WGED. The focus is on the output of the small ratio of effective censoring sample sizes, where the rare cases occur, and the small problems of loss. The study confirms the compatibility of the result in large Sampling sizes. The previous results confirm the compatibility of the Bayesian estimation for Linex loss function, which has the lowest of MSE values compared to other estimators, followed by SE loss function and finally the maximum likelihood estimation. scheme III is the best censoring scheme where it has the lowest MSE and the narrower L.C.I.

6. Application of Real Data

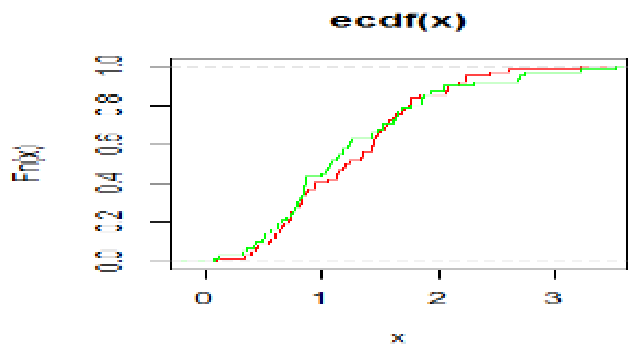
We present the numerical results of the parameter estimation of WGED under progressive censoring scheme of a cases of real data. Bader and Priest (1982) discussed the real data set of sample size 69 observed failure times. The data set is represented the strength data measured in GPA, for single carbon fibers and impregnated 1000 carbon fiber tows.

We computed the Kolmogorov-Smirnov (KS) distance between the empirical and the fitted WGED functions.

Table 7: Goodness of Fit Data

D	0.0739
p-value	0.8624

The distance (D) between the fitted and the empirical distribution functions for the data is 0.0739 and the corresponding p-value is 0.8624. Therefore, it indicates that WGED can be fitted to the data set, by use empirical cumulative distribution function (ecdf) to obtain a data graph to confirm the accuracy of the kolmogorov-smirnov test as following.



Graph 5: Kolmogorov-Smirnov Test and the Plot of Max Distance Between Two ecdf Curves

Table 8: Estimation of Coefficient and Stander Deviation for Complete Data

	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\theta}$	AIC	BIC
Coefficient	38.2650	0.1031	1.9864	117.37	123.56
Std	0.1564	0.0493	0.2121		
L.C.I	0.6131	0.1367	0.8313		

Table 9: Estimation Parameter of Model under Progressive Censoring Schemes for Real Data

		maximum likelihood estimation			Bayesian estimation		
$r = 0.2$							
Schemes		$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\theta}$
I	Coef	36.9940	0.1921	2.6025	36.9561	0.2170	2.1334
	Std	0. 5506	0.0959	0.6802	0.0723	0.0938	0.4747
II	Coef	25.8192	0.5969	3.4788	29.9150	0.4531	2.9458
	Std	0.5802	0.3549	0.65308	0.0459	0.2971	0.5852
III	Coef	36.9110	0.2451	2.9230	35.8530	0.2032	2.5322
	std	0.7509	0.0699	0.5614	0.0721	0.0805	0.5351
$r = 0.5$							
I	Coef	34.8864	0.1682	2.0337	36.9008	0.2105	1.6915
	Std	0.0974	0.1811	0.3403	0.0847	0.1142	0.3011
II	Coef	29.2420	1.1910	2.1363	29.1887	0.8916	2.2584
	Std	0.3725	0.5699	0.4892	0.3156	0.2319	0.3454
III	Coef	30.1303	1.2445	2.0700	32.2805	0.7148	2.4130
	Std	0.2429	0.4871	0.4446	0.2158	0.2049	0.4011

From the results above, we can observe that Bayesian estimation is the best especially with scheme III.

6. Conclusion

In this paper, we discussed the maximum likelihood and Bayesian estimation to estimate parameters problem of the WGED under the progressive type-II censoring schemes. We used MLE and Bayesian estimation under different loss function methods to estimate the unknown parameters for WGED. We obtained the Bayesian estimate based on square error (SE) loss function and Linex loss function under the assumption of independent gamma priors. The performance of the different estimator's optimal censoring schemes is compared based on simulation study to determine the optimal censoring schemes by using MSE, the bias and the L.C.I. It is noticeable that the Bayesian estimation is better and more efficient than the maximum likelihood estimation and that scheme III is the best censoring scheme. The application of real data set is used to show how the schemes work in practice. The application affirms that Bayesian estimation is the best especially with scheme III. The results of this paper will be useful for researchers and statisticians where the comprehensive comparison of different sample size and the ratio of effective censoring sample sizes, parameters and censoring schemes, as well as different methods of estimation the most efficient to analyses and exclusively where WGED is used.

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