Pak.j.stat.oper.res. Vol.17 No.3 2021 pp 683-694 DOI: http://dx.doi.org/10.18187/pjsor.v17i3.2597

Pakistan Journal of Statistics and Operation Research

Estimation and Optimal Plan for Bivariate Step-Stress Accelerated Life Test under Progressive Type-I Censoring

Mashroor Ahmad Khan[1], N. Chandra[2*]

*Corresponding author

1. Department of Statistics, Pondicherry University, India, mashroor08@gmail.com
2. Department of Statistics, Pondicherry University, India, nc.stat@gmail.com

Abstract

In this paper, a step-stress accelerated life test with two stress variables for Weibull distribution under progressive Type-I censoring is considered. The stress-life relationship is a log-linear function of stress levels, and for each combination of stress levels, a cumulative exposure model is assumed. The maximum likelihood and Bayes estimates of the model parameters are obtained. The optimum test plan is developed using the variance-optimality criterion, which consists of finding out the optimal stress change time by minimizing the asymptotic variance of the maximum likelihood estimates of the log of the scale parameter at the design stress. The proposed study is illustrated by using simulated data.

Key Words: Optimum test plan; Step-stress test; Weibull distribution; Progressive censoring; Markov Chain Monte Carlo.

Mathematical Subject Classification: 62N05, 62F10, 62F15.

1. Background

Accelerated Life Testing (ALT) has received great attention from analysts of life testing to shorten the product testing time for predicting product's life and reliability. Since, human-made products (namely computers, washing machines, refrigerators, electronic cameras, and cell mobile, etc.) are well designed, highly sophisticated, and reliable. However, it is not easy to get the failure data of such types of highly reliable products within a limited testing time under normal operating conditions. Thus, ALT is the most preferable modern technology that is used to get failure time data more quickly of highly reliable product at a higher than usual level of stress (e.g., temperature, voltage, pressure, vibration, humidity, cycling conditions, etc.). Failure data collected from ALTs are then analyzed and extrapolated to estimate the life distribution of failures under usual conditions.

Moreover, designing optimum test plans and making inferences from step-stress tests based on censoring data have a great attraction in the life-testing experiment, and the most commonly used censoring schemes are Type-I and Type-II censoring. Nevertheless, these conventional censoring schemes do not allow units to be removed from the experiment at any other point than the final termination point of the test. Therefore, in the last few years, the progressive censoring (PC) schemes have received considerable attention in the step-stress ALT that is the generalization case of the Type-I and Type-II censoring. The main advantage of PC schemes is that it is possible to remove experimental units during the experiment, even if do not fail. Some key references on PC schemes are referred to the monograph by Balakrishnan and Aggarwala(2000), Balakrishnan and Han(2009) and Balakrishnan(2007).

In step-stress ALT, the problems of planning optimum test plan and making inferences of model parameters by assuming cumulative exposure model [see Nelson(1980)] based on censored data have attracted great attention in the reliability literature. Bai et al.(1989) attempted for planning step-stress ALT based on the Type-I censoring case. Khamis and Higgins(1996) proposed optimum 3-step step-stress plans for the exponential distribution. Gouno et al.(2004) proposed optimum m-step step-stress ALT plans with equal test duration and investigated in detail the case of progressive Type-I censoring with a single stress variable, assuming exponential lifetime. Balakrishnan and Han(2009) extended the Gouno et al.(2004) model with a practical modification for small to moderate sample sizes. Some recent related works are referred to as step-stress ALT [see Hakamipour and Rezaei(2015), Chandra and Khan(2015)]. All those attempts are considered the maximum likelihood method for designing a step-stress ALT plan.

The above literature referred to designing the optimum plan and making inferences of the step-stress test by using the ML method. Bayesian inferences for simple step-stress ALT are also available in bulk for a variety of life distribution and censoring (Type-I and Type-II) schemes. Van Dorp et al.(1996) and Van Dorp and Mazzuchi(2004) developed a general Bayesian inference model for simple step-stress ALT based on the exponentially distributed failure data. Lee and Pan(2008) described the Bayesian inference model for simple step-stress ALT when failure times at each stress are exponentially distributed with Type-II censoring. Sha and Pan(2014) presented a Bayesian analysis for Weibull proportional hazard (PH) model for simple step-stress ALT. In those studies, it is observed that most of the problems aforesaid involve only a single accelerating stress variable in step-stress ALT planning. Furthermore, as today's products become extremely reliable due to technological advances, a single accelerating stress variable in step-stress ALT may not yield a significant amount of failure data within a reasonable amount of time. However, it insists to include more than one stress variable in the step-stress test. For instance, an ALT of capacitors could include two accelerating stress variables, such as temperature and voltage, and ALT of circuit boards includes more than two stress variables such as temperature, humidity, and voltage [see Minford(1982), Mogilevski and Shirn(1988), Munikoti and Dhar(1988)]. However, those studies did not focus on the step-stress ALT.

Khamis(1997) presented a generalized optimum m-step step-stress ALT design with k stress variables by assuming complete knowledge of a life-stress relationship with multiple stress variables. Li and Fard(2007) proposed a bivariate step-stress ALT plan with two stress variables for Weibull failure time under Type-I censoring. Ling et al.(2011) discussed a bivariate step-stress ALT model with two stress variables to determine optimum stress change times under Type-I hybrid censored data. Some recent work in this direction refers to Hakamipour and Rezaei(2015).

The objective of this paper is to develop a bivariate step-stress test that includes two stress variables and each has two stress levels, and the stress levels are changed at different times. The expression of an optimum test plan under progressive Type-I censoring is derived by minimizing asymptotic-variance (AV) of the MLEs of the log of the scale parameter at design stress under progressive Type-I censored data. The rest of this paper is organized as follows: In section 2, the test procedure under progressive Type-I censoring and the statistical model with assumptions are presented. Likelihood function and Fisher information matrix are presented in section 3. The optimization criterion is discussed in section 4. In section 5 presents the Bayesian estimation for the model parameters. A numerical example followed by comparative study and sensitivity analysis for simulated data is given in section 6. The conclusion is contained in section 7.

2. Test procedure and model description

2.1. Test procedure under progressive Type-I censoring

We consider the step-stress ALT problem with two stress variables, and each has two stress levels. Let x_{lk} be the $k\text{th}$ stress level for a variable l, for $l = 1,\ 2$, and for $k = 1,\ 2$. Let, N_k denote the number of units operating and remaining on test at the start of $k\text{th}$ stress level.

The bivariate step-stress test procedure under progressive Type-I censoring stars with $N_1 \equiv n$ identical units initially placed at first step with low-stress levels $(x_{11},\ x_{21})$. Then, at prefixed stress change time τ_1, when the first stress variable x_{11} changed to x_{12} and the numbers of failed units n_1 recorded and R_1^* surviving units randomly removed from test. Now, $N_2 = n - n_1 - R_1^*$, the non-removed surviving units are put on a higher stresses $(x_{12},\ x_{21})$ and run until prefixed stress change time τ_2, when the second stress variable x_{21} changed to x_{22} and the number of failed units n_2 is recorded and R_2^* surviving units are randomly withdrawn from test, and so on. Then the test is continued with stresses (x_{12}, x_{22}) and run until a predefined censoring time T, the number of failed units n_3 is recorded, and remaining surviving items $R_3^* = n - \sum_{i=1}^{3} n_i - \sum_{i=1}^{2} R_i^*$ are withdrawn from the test, thereby terminate the life test. The procedure is shown in Figure 1. Note that, when there is no intermediate censoring (viz.,$R_1^* = R_2^* = 0$), this situation corresponds to 3-level step-stress testing under Type-I censoring as a special case.

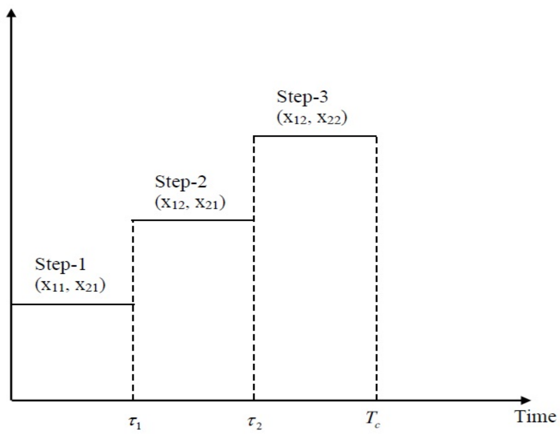

Figure 1: Graphical representation of test procedure.

2.2. Assumptions

Under the progressive Type-I censoring scheme, the optimum test plan and statistical inferences of bivariate step-stress ALT depend on the following assumptions are made:

1. The life of a test unit at each stress level follows a Weibull distribution.

$$F(t) = 1 - \exp\left(-\frac{t^\delta}{\theta_i^\delta}\right), \qquad \theta_i > 0,\ \delta > 0;\ 0 \le t < \infty, \tag{1}$$

   where δ and θ are the shape and scale parameters respectively.

2. The scale parameter θ_i at test step i,for $i = 1, 2, 3$, is assumed a log-linear function of stress levels. That is,

$$\left.\begin{array}{l} \text{Step 1}:\ \log(\theta_1) = \beta_0 + \beta_1 x_{11} + \beta_2 x_{21} \\ \text{Step 2}:\ \log(\theta_2) = \beta_0 + \beta_1 x_{12} + \beta_2 x_{21} \\ \text{Step 3}:\ \log(\theta_3) = \beta_0 + \beta_1 x_{12} + \beta_2 x_{22} \end{array}\right\}, \tag{2}$$

   where $\beta_0,\ \beta_1$ and β_2 are unknown parameters depending on the nature of the product and the test method and it is assumed that there are no interactions between the stress variables [see, Li and Fard(2007)].

3. A cumulative exposure (CE) model holds: the remaining life of a test unit depends only on its present CE.

4. For all stress levels, the shape parameter δ is common, constant, and independent of time and stress, i.e. known.

From the assumption (i) and (iii), the cumulative density function (CDF) of a test unit under step-stress test follows the K-H model [See Khamis and Higgins(1998)], can be expressed as:

$$F(t) = \begin{cases} 1 - \exp\left(-\frac{t^\delta}{\theta_1^\delta}\right), & 0 \le t < \tau_1 \\ 1 - \exp\left(-\left(\frac{t^\delta - \tau_1^\delta}{\theta_2^\delta} + \frac{\tau_1^\delta}{\theta_1^\delta}\right)\right), & \tau_1 \le t < \tau_2 \\ 1 - \exp\left(-\left(\frac{t^\delta - \tau_2^\delta}{\theta_3^\delta} + \frac{\tau_2^\delta - \tau_1^\delta}{\theta_2^\delta} + \frac{\tau_1^\delta}{\theta_1^\delta}\right)\right), & \tau_2 \le t < \infty \end{cases} \tag{3}$$

The corresponding probability density (PDF) function of a test unit is obtained as:

$$f(t)=\begin{cases}\delta\frac{t^{\delta-1}}{\theta_1^\delta}\exp\left(-\frac{t^\delta}{\theta_1^\delta}\right), & 0\le t<\tau_1\\ \delta\frac{t^{\delta-1}}{\theta_2^\delta}\exp\left(-\left(\frac{t^\delta-\tau_1^\delta}{\theta_2^\delta}+\frac{\tau_1^\delta}{\theta_1^\delta}\right)\right), & \tau_1\le t<\tau_2\\ \delta\frac{t^{\delta-1}}{\theta_3^\delta}\exp\left(-\left(\frac{t^\delta-\tau_2^\delta}{\theta_3^\delta}+\frac{\tau_2^\delta-\tau_1^\delta}{\theta_2^\delta}+\frac{\tau_1^\delta}{\theta_1^\delta}\right)\right), & \tau_2\le t<\infty\end{cases} \tag{4}$$

3. Likelihood Function and Fisher Information Matrix

Let $t_{ij},\ i=1,\ 2,\ 3;\ j=1,\ 2,...,n_i$ be the observed values of lifetime Tobtained from a progressive Type-I censoring. From the CDF given in (3) and corresponding PDF has given in (4), the likelihood function based on the progressive Type-I censoring sample is derived as follows:

$$L(\delta,\theta_1,\theta_2,\theta_3|\,t)=\prod_{i=1}^{3}\prod_{j=1}^{n_i}f_i(t_{ij})\left[1-F_i(\tau)\right]^{R_i^*} \tag{5}$$

From substituting (3) and (4) in (5), we get

$$\begin{aligned}L&=L(t_{ij}|\delta,\ \theta_1,\ \theta_2,\ \theta_3)\\&=\prod_{j=1}^{n_1}\delta\frac{t_{1j}^{\delta-1}}{\theta_1^\delta}\exp\left(-\frac{t_{1j}^\delta}{\theta_1^\delta}\right)\left[\exp\left(-\frac{\tau_1^\delta}{\theta_1^\delta}\right)\right]^{R_1^*}\\&\times\prod_{j=1}^{n_2}\delta\frac{t_{2j}^{\delta-1}}{\theta_2^\delta}\exp\left(-\left(\frac{t_{2j}^{\delta-1}-\tau_1^\delta}{\theta_2^\delta}+\frac{\tau_1^\delta}{\theta_1^\beta}\right)\right)\left[\exp\left(-\left(\frac{\tau_2^\delta-\tau_1^\delta}{\theta_2^\delta}+\frac{\tau_1^\delta}{\theta_1^\delta}\right)\right)\right]^{R_2^*}\\&\times\prod_{j=1}^{n_3}\delta\frac{t_{3j}^{\delta-1}}{\theta_3^\delta}\exp\left(-\left(\frac{t_{3j}^{\delta-1}-\tau_2^\delta}{\theta_3^\delta}+\frac{\tau_2^\delta-\tau_1^\delta}{\theta_2^\delta}+\frac{\tau_1^\delta}{\theta_1^\delta}\right)\right)\times\left[\exp\left(-\left(\frac{\tau_3^\delta-\tau_1^\delta}{\theta_2^\delta}+\frac{\tau_2^\delta-\tau_1^\delta}{\theta_2^\delta}+\frac{\tau_1^\delta}{\theta_1^\beta}\right)\right)\right]^{R_3^*}\end{aligned} \tag{6}$$

From (2) and (6), we get

$$L=L(\delta,\beta_0,\beta_1,\beta_3|\,t)=\prod_{i=1}^{3}\prod_{j=1}^{n_i}\delta\frac{t_{ij}^{\delta-1}}{\exp(C_i)}\exp\left(-\frac{U_i}{\exp(C_i)}\right) \tag{7}$$

where, $U_i=\sum_{j=1}^{n_i}\left(t_{ij}^\delta-\tau_{i-1}^\delta\right)+\frac{R_i^*}{\pi_i^*}\left(\tau_i^\delta-\tau_{i-1}^\delta\right),\ i=1,\ 2,\ 3,$ with $\tau_0=0,\ R_i^*=\text{round}((N_i-n_i)\pi_i)$, $C_1=\delta(\beta_0+\beta_1x_{11}+\beta_2x_{21}),\ C_2=\delta(\beta_0+\beta_1x_{12}+\beta_2x_{21})$ and $C_3=\delta(\beta_0+\beta_1x_{12}+\beta_2x_{22})$. Note that U_i is the total time on test statistic for the ith stage and R_i^* is the number of units censored at each stage, as defined by Balakrishnan and Han(2009). The log-likelihood function after taking the log of equation (7), can be written as

$$\log L=\sum_{i=1}^{3}\{n_i\log(\delta)\}+(\delta-1)\sum_{i=1}^{3}\sum_{j=1}^{n_i}\log(t_{ij})-\sum_{i=1}^{3}\left(n_iC_i+\frac{U_i}{\exp(C_i)}\right) \tag{8}$$

The MLEs for the model parameters $\beta_0,\ \beta_1$ and β_2 can be obtained by equating the first-order partial derivatives with respect to $\beta_0,\ \beta_1$ and β_2 of the log-likelihood function (8) to zero, respectively

$$\frac{\partial\log L}{\partial\beta_0}=-\delta(n_1+n_2+n_3)+\sum_{i=1}^{3}\frac{U_i}{\exp(C_i)}\frac{\partial C_i}{\partial\beta_0}=0 \tag{9}$$

$$\frac{\partial\log L}{\partial\beta_1}=-\delta(x_{11}n_1+x_{12}n_2+x_{12}n_3)+\sum_{i=1}^{3}\left(\frac{U_i}{\exp(C_i)}\right)\frac{\partial C_i}{\partial\beta_1}=0 \tag{10}$$

$$\frac{\partial \log L}{\partial \beta_2} = -\delta(x_{21}n_1 + x_{21}n_2 + x_{22}n_3) + \sum_{i=1}^{3} \frac{U_i}{\exp(C_i)} \frac{\partial C_i}{\partial \beta_2} = 0 \tag{11}$$

It is observed that likelihood equations (9), (10), and (11) constitute a system of 3 nonlinear equations in 3 unknowns $\beta_0,\ \beta_1$ and β_2. Since the non-linearity of $\hat{\beta}_0,\ \hat{\beta}_1$ and $\hat{\beta}_2$, it is therefore difficult to solve analytically. Thus, statistical inference with these MLEs is based on the asymptotic distributional result that the vector $(\hat{\beta}_0,\ \hat{\beta}_1,\ \hat{\beta}_2)$ is approximately distributed as a multivariate normal with mean vector $(\beta_0,\ \beta_1,\ \beta_2)$ and variance-covariance matrix $[I_n(\beta_0,\ \beta_1,\ \beta_2)]^{-1}$, where $[I_n(\beta_0,\ \beta_1,\ \beta_2)]$ is the expected value of the matrix of second derivatives of the log-likelihood, i.e., the Fisher Information Matrix. From equation (6), we have

$$\frac{\partial^2 \log L}{\partial \beta_0^2} = -\sum_{i=1}^{3} \frac{U_i}{\exp(C_i)} \left(\frac{\partial C_i}{\partial \beta_0}\right)^2 \tag{12}$$

$$\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_1} = -\sum_{i=1}^{3} \frac{U_i}{\exp(C_i)} \frac{\partial^2 C_i}{\partial \beta_0 \partial \beta_1} \tag{13}$$

$$\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_2} = -\sum_{i=1}^{3} \frac{U_i}{\exp(C_i)} \frac{\partial^2 C_i}{\partial \beta_0 \partial \beta_2} \tag{14}$$

$$\frac{\partial^2 \log L}{\partial \beta_1^2} = -\sum_{i=1}^{3} \frac{U_i}{\exp(C_i)} \left(\frac{\partial C_i}{\partial \beta_1}\right)^2 \tag{15}$$

$$\frac{\partial^2 \log L}{\partial \beta_1 \partial \beta_2} = -\sum_{i=1}^{3} \frac{U_i}{\exp(C_i)} \frac{\partial^2 C_i}{\partial \beta_1 \partial \beta_2} \tag{16}$$

$$\frac{\partial^2 \log L}{\partial \beta_2^2} = -\sum_{i=1}^{3} \frac{U_i}{\exp(C_i)} \left(\frac{\partial C_i}{\partial \beta_2}\right)^2 \tag{17}$$

To obtain the Fisher information matrix, we need the expectations of (12)-(17). To obtain their expectations, the following properties of the count and order statistics are used:
Properties:

(i) The random variable n_1 has a binomial distribution with parameters $(n, F(\tau_1))$. For $i = 2,\ 3$, given $n_1, ..., n_{i-1}$, the random variable n_i has a binomial distribution with parameters $(N_i, F_i(\tau))$, where

$$F_i(\tau) = \frac{F(\tau_i) - F(\tau_{i-1})}{1 - F(\tau_{i-1})} \tag{18}$$

is the probability that a unit fail in the interval $(\tau_{i-1}\ ,\ \tau_i]$ with $\tau_0 = 0,$ and $F(\tau_i)$ is as given in (3).

(ii) For each $i = 1,\ 2,\ 3$, the random variables $\left(t_{i,j}^{\delta} - \tau_{i-1}^{\delta}\right),\ j = 1,\ 2, ..., n_i$ constitute a random sample from a truncated Weibull distribution on $(\tau_{i-1}, \tau_i]$ where $\tau_0 = 0$, with the p.d.f

$$f_{i,\tau}(z) = \frac{f_i(z)}{F(\tau_i) - F(\tau_{i-1})} \text{ for } \tau_{i-1} \le z \le \tau_i.$$

Using property (i) and the property of conditional expectation, we have $E(n_i) = E(N_i)F_i(\tau)$. Now, let us compute the expectation of N_i and $R_i,\ i = 1,\ 2,\ 3$. Beginning with $E(N_1) = n$ and$N_{i+1} = N_i - n_i - R_i^*$, we obtain, by induction,

$$E(N_i) = n \prod_{j=1}^{i-1} S_j(\tau)(1 - \pi_j^*), \tag{19}$$

$$E(R_i^*) = E(N_i)\left[1 - F_i(\tau)\right]\pi_i^*. \tag{20}$$

Hence, the expected value of U_i is obtained as

$$E(U_i) = n\theta_i^{\delta} F_i(\tau) \prod_{j=1}^{i-1} S_j(\tau)\left(1-\pi_j^*\right),\ i=1,2,3. \tag{21}$$

Thus, the expected values of (12)-(17) are obtained as follows

$$E\left[-\frac{\partial^2 \log L}{\partial \beta_0^2}\right] = n\delta^2 \sum_{i=1}^{3} F_i(\tau) \prod_{j=1}^{2} S_j(\tau)(1-\pi_j^*) \tag{22}$$

$$E\left[-\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_1}\right] = n\delta^2 \sum_{i=1}^{3} \left(x_{11}+2x_{12}\right) F_i(\tau) \prod_{j=1}^{2} S_j(\tau)(1-\pi_j^*) \tag{23}$$

$$E\left[-\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_2}\right] = n\delta^2 \sum_{i=1}^{3} \left(2x_{21}+x_{22}\right) F_i(\tau) \prod_{j=1}^{2} S_j(\tau)(1-\pi_j^*) \tag{24}$$

$$E\left[-\frac{\partial^2 \log L}{\partial \beta_1^2}\right] = n\delta^2 \sum_{i=1}^{3} \left(x_{11}^2+2x_{12}^2\right) F_i(\tau) \prod_{j=1}^{2} S_j(\tau)(1-\pi_j^*) \tag{25}$$

$$E\left[-\frac{\partial^2 \log L}{\partial \beta_1 \partial \beta_2}\right] = n\delta^2 \sum_{i=1}^{3} \left(2x_{11}x_{21}+x_{12}x_{22}\right) F_i(\tau) \prod_{j=1}^{2} S_j(\tau)(1-\pi_j^*) \tag{26}$$

$$E\left[-\frac{\partial^2 \log L}{\partial \beta_2^2}\right] = n\delta^2 \sum_{i=1}^{3} \left(2x_{21}^2+x_{22}^2\right) F_i(\tau) \prod_{j=1}^{2} S_j(\tau)(1-\pi_j^*) \tag{27}$$

Hence, the Fisher information matrix is

$$I_n(\beta_0,\beta_1,\beta_2) = \begin{bmatrix} \sum_{i=1}^3 A_i(\tau) & \sum_{i=1}^3 A_i(\tau)B_1(x) & \sum_{i=1}^3 A_i(\tau)B_2(x) \\ \sum_{i=1}^3 A_i(\tau)B_1(x) & \sum_{i=1}^3 A_i(\tau)B_3(x) & \sum_{i=1}^3 A_i(\tau)B_4(x) \\ \sum_{i=1}^3 A_i(\tau)B_2(x) & \sum_{i=1}^3 A_i(\tau)B_4(x) & \sum_{i=1}^3 A_i(\tau)B_5(x) \end{bmatrix}, \tag{28}$$

where,

$A_i(\tau) = \sum_{i=1}^3 F_i(\tau)\prod_{j=1}^2 S_j(\tau)(1-\pi_j^*)$,
$B_1(x) = (x_{11}+2x_{12}),\ B_2(x) = (2x_{21}+x_{22}),\ B_3(x) = (x_{11}^2+2x_{12}^2)$,
$B_4(x) = (2x_{11}x_{21}+x_{12}x_{22})$ and $B_5(x) = (2x_{21}^2+x_{22}^2)$.

Since the MLEs of the model parameters are not in closed-form, it is not possible to derive the exact confidence intervals, so the asymptotic confidence intervals instead of exact confidence intervals are derived.
Then the two-sided $100(1-\alpha)\%$ confidence interval of the model parameter β_0 can be obtained from

$$\hat{\beta}_0 \pm Z_{\alpha/2}\sqrt{AVar(\hat{\beta}_0)} \tag{29}$$

where, *AVar*- stands for asymptotic-variance and $Z_{\alpha/2}$ is the $(1-\alpha/2)^{\text{th}}$ quantile of the standard normal distribution. Similarly, the two-sided $100(1-\alpha)\%$ CIs of the model parameters β_1 and β_2 can obtain.

4. Optimization Criterion

Here, we are interested in estimating the scale parameter θ_0 at usual stress conditions with maximum precision. The criterion function is then defined by

$$\begin{aligned} \text{nAVar}\left(\log\hat{\theta}_0\right) &= n\text{AVar}\left(\hat{\beta}_0+\hat{\beta}_1 x_{10}+\hat{\beta}_2 x_{20}\right) \\ &= \begin{pmatrix} 1 & x_{10} & x_{20} \end{pmatrix} I_n^{-1}\left(\hat{\beta}_0,\ \hat{\beta}_1,\ \hat{\beta}_2\right) \begin{pmatrix} 1 & x_{10} & x_{20} \end{pmatrix}' \end{aligned} \tag{30}$$

where, $I_n^{-1}\left(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2\right)$ is the asymptotic variance-covariance matrix, can be obtained from (28).

5. Bayes Estimation

In the section, Bayes estimates $(\beta_0, \beta_1, \beta_2)$ are obtained by assuming that the shape parameter δ is known. Let us consider Jeffrey's rule for choosing the independent and non-informative prior for the parameters $\beta_0,\ \beta_1$ and β_2 as suggested by Sinha(1998). The joint prior for $\beta_0,\ \beta_1$ and β_2 is the product of their independent priors. The considered priors are

$$\left.\begin{array}{l} g_1(\beta_0) \propto \frac{1}{2c_1};\ \ -c_1 \le \beta_0 \le c_1, \\ g_2(\beta_1) \propto \frac{1}{2c_2};\ \ -c_2 \le \beta_1 \le c_2, \\ g_3(\beta_2) \propto \frac{1}{2c_3};\ \ -c_3 \le \beta_2 \le c_3, \end{array}\right\} \tag{31}$$

where $c_1,\ c_2$ and c_3 are the prior hyperparameters. The expression for the posterior density by combining the likelihood function (7) with the priors (31) via the Bayes theorem up to proportionality, can be written as

$$\begin{array}{r} \pi(\beta_0, \beta_1, \beta_2|\mathbf{t}) \propto \prod_{i=1}^{3}\prod_{j=1}^{n_i} \delta \frac{t_{ij}^{\delta-1}}{\exp(C_i)} \exp\left(-\frac{U_i}{\exp(C_i)}\right) \\ \times I_{(-c_1,c_1)}\left(\beta_0\right) I_{(-c_2,c_2)}\left(\beta_1\right) I_{(-c_3,c_3)}\left(\beta_2\right) \end{array} \tag{32}$$

where, $I_{(-c,c)}\left(\beta\right) = \begin{cases} 1, \text{ if } -c < \beta < c, \\ 0, \text{ otherwise.} \end{cases}$

The posterior function given in (32) is analytically intractable to draw the desired inferences. Therefore, we consider MCMC methods to simulate samples from the posterior to obtain the Bayes estimate numerically, which is the easiest alternative way to get reliable results [see Gelman et al.(2003)]. The Gibbs sampler is an important algorithm in the MCMC technique, which provides a way for extracting samples from the posteriors distribution. To implement the Gibbs strategy, the basic steps are as follows:

Step-1: First derive the posterior distribution, up to proportionality, and specify the full conditionals of the model parameters$\beta_0,\ \beta_1$ and β_2, using (30) as

$$\pi_1(\beta_0|\beta_1, \beta_2, \mathbf{t}) \propto \exp\left(-(n_1+n_2+n_3)\delta\beta_0\right) \exp\left(-\sum_{i=1}^{3}\sum_{j=1}^{n_i} \frac{U_i}{\exp\left(C_i\right)}\right), \tag{33}$$

$$\pi_2(\beta_1|\beta_0, \beta_2, \mathbf{t}) \propto \exp\left(-\left(n_1 x_{11} + n_2 x_{12} + n_3 x_{12}\right)\delta\beta_1\right) \exp\left(-\sum_{i=1}^{3}\sum_{j=1}^{n_i} \frac{U_i}{\exp\left(C_i\right)}\right), \tag{34}$$

$$\pi_3(\beta_2|\beta_0, \beta_1, \mathbf{t}) \propto \exp\left(-\left(n_1 x_{21} + n_2 x_{21} + n_3 x_{22}\right)\delta\beta_2\right) \exp\left(-\sum_{i=1}^{3}\sum_{j=1}^{n_i} \frac{U_i}{\exp\left(C_i\right)}\right). \tag{35}$$

Step-2: Select an initial value $\underline{\theta}^{(0)} = \left(\beta_0^{(0)}, \beta_1^{(0)}, \beta_2^{(0)}\right)$to start the chain.

Step-3: Suppose at the i^{th}-step, $\underline{\theta} = (\beta_0, \beta_1, \beta_2)$ takes the value $\underline{\theta}^{(i)} = \left(\beta_0^{(i)}, \beta_1^{(i)}, \beta_2^{(i)}\right)$ then the full conditionals, generate

$$\begin{array}{l} \beta_0^{(i+1)} \text{ from } p\left(\beta_0|\, \beta_1^{(i)}, \beta_2^{(i)}, \underline{t}\right), \\ \beta_1^{(i+1)} \text{ from } p\left(\beta_1|\, \beta_0^{(i)}, \beta_2^{(i)}, \underline{t}\right) \text{ and} \\ \beta_2^{(i+1)} \text{ from } p\left(\beta_2|\, \beta_1^{(i)}, \beta_1^{(i)}, \underline{t}\right). \end{array}$$

Step-4: This completes a transition from $\underline{\theta}^{(i)}$ to $\underline{\theta}^{(i+1)}$.

Step-5: Repeat Step-3, N times.

In this case, we shall prefer to use the WinBUGS software to obtain the posterior samples. It is a powerful and flexible program for performing Bayesian analyses, developed by the BUGS project, a team of UK researchers at the MRC Biostatistics Unit at Cambridge [see Spiegelhalter et al.(2003)].

6. Numerical Illustrations

In this section, we present a simulated example to illustrate the proposed bivariate step-stress ALT model under progressive Type-I censoring. In this simulation study, we used two average censoring proportions (ACPs) $\pi_0 = 0.10$ and 0.20 with the following initial values of the model parameters $n = 40,\ x_{10} = 0.1,\ x_{20} = 0.5,\ x_{11} = 0.4,\ x_{12} = 0.7,\ x_{21} = 1.2,\ x_{22} = 2.5,\ \delta = 1.5,\ \beta_0 = 6,\ \beta_1 = -1$, and $\beta_2 = -0.5$.

6.1. Optimum Plan

To obtain the optimum values of the stress changing times τ_1 and τ_2, we minimize the asymptotic-variance of the MLEs of the log of scale parameter at usual stress conditions given in equation (30). Since the asymptotic-variance of the MLEs of the log of scale parameter at x_0 cannot be obtained in explicit forms, therefore we approach a graphical method to plot the relation between $\mathrm{AVar}(\log\hat{\theta}_0)$ and stress changing times $(\tau_1$ and $\tau_2)$ by choosing different values of τ_1 and τ_2, and then the optimum values of stress changing times are obtained from the Figure 1, i.e. $\tau_1^* = 107.5$ and $\tau_2^* = 152.0$ for $\pi_0 = 0.10$ and $\tau_1^* = 85.0$ and $\tau_2^* = 116.0$ for $\pi_0 = 0.20$.

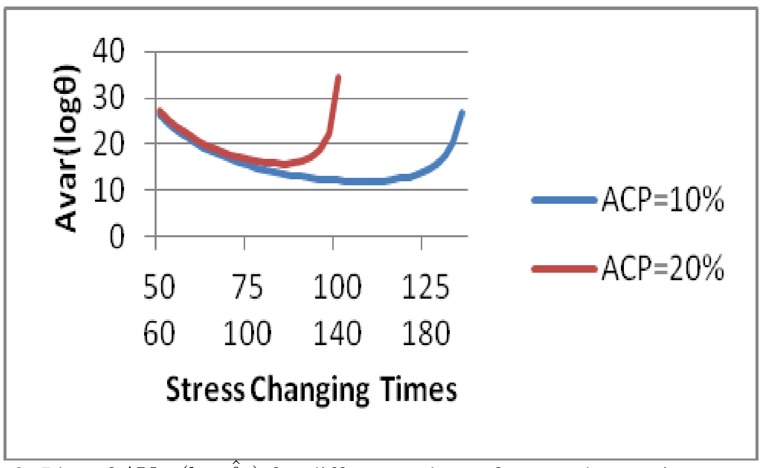

Figure 2: Plot of $\mathrm{AVar}(\log\hat{\theta}_0)$ for different values of stress change times τ_1 and τ_2.

6.2. Simulated Data

A simulation study is carried out and the failure time observations for various combinations of two stress variables are generated by using equation (3) under the progressive Type-I censoring set up for the chosen initial values of the model parameters with two average censoring proportions (ACPs) $\pi_0 = 0.10$ and 0.20. The simulated failure times data are presented in Table 1.

6.3. MLEs of the Model Parameters

The MLEs and 95% confidence intervals of the model parameters $\beta_0,\ \beta_1$ and β_2 are obtained using simulated data given in Table 1 and the results are presented in Table 2.

The MLEs of the model parameters are obtained using R software (R Development Core Team, 2017) by directly maximizing the log-likelihood function, and the confidence intervals of estimates are obtained using the Hessian matrix. From Table 2, it is observed that the model parameters $\beta_0,\ \beta_1$ and β_2 have a smaller SD for ACP $\pi_0 = 0.10$ as compared to SD for ACP $\pi_0 = 0.20$.

Table 1: Simulated failure times under progressive Type-I censoring

		$\pi_0 = 0.10$
1^{st} Stress level	2^{nd} Stress level	Failure times
$x_{11} = 0.4$	$x_{21} = 1.2$	12.68054, 16.00950, 26.41577, 27.62933, 48.50038, 53.54161, 53.65388, 59.91186, 83.68310, 83.80152, 91.96629, 92.48443, 94.23786, 94.44097, 101.73687, 107.25567
$x_{12} = 0.7$	$x_{21} = 1.2$	111.34981, 118.89825, 126.29394, 127.73697, 129.06297, 132.27548, 132.82133, 141.73262, 143.27622, 144.13275, 151.04934
$x_{12} = 0.7$	$x_{22} = 2.5$	164.07598, 164.61501, 168.88635, 174.74573
		$\pi_0 = 0.20$
$x_{11} = 0.4$	$x_{21} = 1.2$	3.177988, 14.813664, 15.850217, 18.073464, 24.723454, 32.206788, 36.952152, 39.418974, 42.696305, 46.117481, 46.378645, 54.478969, 56.877109, 58.664509, 67.072512, 68.067866, 74.555354, 81.800785, 84.186432
$x_{12} = 0.7$	$x_{21} = 1.2$	86.742937, 96.629550, 97.471344
$x_{12} = 0.7$	$x_{22} = 2.5$	116.478150, 120.527766, 129.954149, 149.598110

Table 2: The MLEs and 95% confidence intervals of the model parameters

		$\pi_0 = 0.10$		
Parameters	MLEs	Bias	SD	Confidence Interval
β_0	5.7759	-0.2241	0.7667	(4.2733, 7.2786)
β_1	-1.0716	-0.0716	1.2553	(-3.5319, 1.3887)
β_2	-0.2315	0.2685	0.2746	(-0.7698, 0.3068)
		$\pi_0 = 0.20$		
β_0	5.2879	-0.7121	0.7697	(3.7793, 6.7966)
β_1	0.5736	1.5736	1.6825	(-2.7242, 3.8714)
β_2	-0.6243	-0.1243	0.4814	(-1.5679, 0.3193)

6.4. Bayes Estimate of the Model Parameters

In this case, two MCMC chains with different initial values, for chain 1: $\beta_0 = 6,\ \beta_1 = -2,\ \beta_2 = -1$ and for chain 2: $\beta_0 = 10,\ \beta_1 = -3,\ \beta_2 = -2$, were run simultaneously in one simulation. Each chain continues for 40000 iterations. The posterior summary obtained from WinBUGS is presented in Table 3.

The results in Table 3 shows that posterior parameters β_0 and β_2 have a smaller SD for ACP $\pi_0 = 0.10$ as compared to ACP $\pi_0 = 0.20$, while in the case of parameter β_1 second ACP $\pi_0 = 0.20$ has a smaller SD in comparison to the first ACP $\pi_0 = 0.10$.

Table 3: Summary of posterior analysis

Parameters	Mean	SD	MC error	Median (50%)	Confidence Interval
			$\pi_0 = 0.10$		
β_0	9.6480	0.3235	0.0049	9.7390	(8.8140, 9.990)
β_1	-2.6890	0.8661	0.0139	-2.8110	(-3.927, -0.7291)
β_2	-0.5944	0.3692	0.0069	-0.5576	(-1.386, -0.0348)
			$\pi_0 = 0.20$		
β_0	10.08	1.027	0.03249	10.12	(7.958, 11.830)
β_1	-1.727	0.8193	0.01103	-1.815	(-2.939, -0.1424)
β_2	-1.046	0.4536	0.009132	-1.063	(-1.867, -0.147)

6.5. The Gelman-Rubin Convergence Diagnostics

The Gelman–Rubin convergence statistic, *R*, is introduced to evaluate MCMC convergence by analyzing the difference between multiple Markov chains. The convergence is assessed by comparing the estimated between-chains and within-chain variances for each model parameter. Large differences between these variances indicate non-convergence. See Gelman and Rubin(1992) for the detailed description of the method. When a WinBUGS simulation converges, *R* should be, or close to one. Figure 3 and 4 corresponding to ACPs $\pi_0 = 0.10$ and 0.20, respectively, shows the convergence pattern based on Gelman-Rubin convergence statistic of β_0, β_1 and β_2 which indicates that the simulation is believed to have converged.

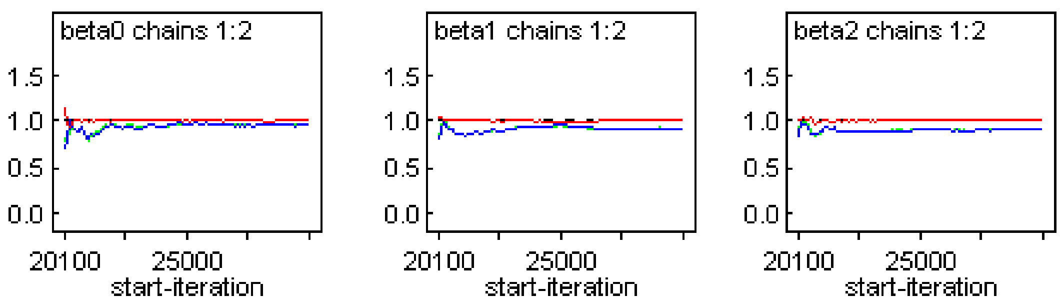

Figure 3: Gelman-Rubin statistic for β_0, β_1 and β_2 for $\pi_0 = 0.10$.

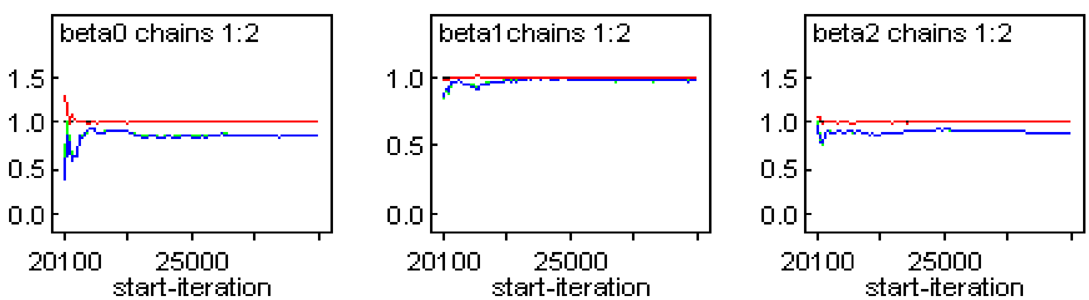

Figure 4: Gelman-Rubin statistic for β_0, β_1 and β_2 for $\pi_0 = 0.20$.

6.6. Comparative study

We have compared the proposed step-stress ALT model under progressive Type-I censoring with Type-I censoring in terms of the optimum plan, results are given in Table 4.

Table 4: Comparative study with Type-I censoring

SSALT Model	$\pi_0 = 0.10$	$\pi_0 = 0.20$
Under Progressive Type-1 censoring	$\tau_1^* = 107.5$ and $\tau_2^* = 152.0$	$\tau_1^* = 85.0$ and $\tau_2^* = 116.0$
Under Type-1 censoring	$\tau_1^* = 135.0$ and $\tau_2^* = 196.0$	

Table 4 shows that the optimal stress change times for the proposed optimum plan under modified progressive Type-I censoring are reduced as compared to Type-I censoring. Thus, the proposed plan is performing better than the plan under Type-I censoring for a given data set.

6.7. Sensitivity Analysis

The sensitivity analysis is performed to observe the effect of changes in the value of initially estimated model parameters $\beta_0,\ \beta_1$ and β_2 on the optimum value of stress change times (τ_1^* and τ_2^*), the results are displayed in Table 5.

Table 5: Sensitivity analysis of bivariate SSALT plan

		$\pi_0 = 0.10$		$\pi_0 = 0.10$	
Parameters	Deviation %	τ_1^*	τ_2^*	τ_1^*	τ_2^*
β_0	$+5$	105.5	148.0	82.5	112
	-5	110.0	156.0	87.0	120
β_1	$+5$	105.0	148.0	82.5	112
	-5	107.5	152.0	85.0	116
β_2	$+5$	107.5	152.0	85.0	116
	-5	107.5	152.0	85.0	116

The result in Table 5 shows that the proposed optimum test plan is robust to the deviations in true values of the model parameters. Especially, the test plan is robust to change in the model parameters β_0 and β_1 and strongly robust to the change in the parameter β_2.Therefore, the proposed optimum plan is robust.

7. Conclusion

In this paper, we have studied a step-stress ALT with two stress variables and each has two stress levels, and the stress levels are changed at different times for Weibull distribution under progressive Type-I censoring. The optimum test plan is developed using variance–optimality criteria. Based on simulated data, the model parameters $\beta_0,\ \beta_1$ and β_2 have been estimated through maximum likelihood and Bayes methods. In parameter estimation, it is observed that estimates of the model parameters through the ML method for $\pi_0 = 0.10$ performing better than $\pi_0 = 0.20$ and the corresponding 95% confidence intervals are also presented in Table 2. The Gelman–Rubin convergence diagnostics test is used to evaluate MCMC convergence of multiple Markov chains and it shows that the simulation is converged (Figure 3 & 4). Moreover, the comparative study for the optimum plan has been conducted and it shows that the proposed plan under progressive Type-I censoring performs better than the plan under traditional Type-I censoring. Sensitivity analysis results suggest that the optimum test plan is robust for small deviations in the true value of the model parameters.

References

1. Bai, D., Kim, M., and Lee, S. (1989). Optimum simple step-stress accelerated life tests with censoring. *IEEE transactions on reliability*, 38(5):528–532.
2. Balakrishnan, N. (2007). Progressive censoring methodology: an appraisal. *Test*, 16(2):211.
3. Balakrishnan, N. and Aggarwala, R. (2000). *Progressive censoring: theory, methods, and applications*. Springer Science and Business Media.
4. Balakrishnan, N. and Han, D. (2009). Optimal step-stress testing for progressively type-i censored data from exponential distribution. *Journal of statistical planning and inference*, 139(5):1782–1798.
5. Chandra, N. and Khan, M. A. (2015). Optimum generalized compound linear plan for multiple-step step-stress accelerated life tests. *International Journal of Mathematical Modelling and Computations*, 5(3):267–275.
6. Gelman, A., Carlin, J. B., Stern, H. S., and Rubin, D. B. (2003). *Bayesian Data Analysis*. Chapman and Hall/CRC, New York.
7. Gelman, A. and Rubin, D. B. (1992). Inference from iterative simulation using multiple sequences. *Statistical science*, 7(4):457–472.
8. Gouno, E., Sen, A., and Balakrishnan, N. (2004). Optimal step-stress test under progressive type i censoring. *IEEE transactions on reliability*, 53(3):383–393.
9. Hakamipour, H. and Rezaei, S. (2015). Optimal design for a bivariate simple step-stress acceleratedlife testing model with type-ii censoring and gompertz distribution. *International Journal of Information Technology and Decision Making*, 14(6):1243–1262.
10. Khamis, I. H. (1997). Optimum m-step step-stress design with k stress variables. *Communications in Statistics - Simulation and Computation*, 26(4):1301–1313.
11. Khamis, I. H. and Higgins, J. J. (1996). Optimum 3-step step-stress tests. *IEEE transactions on reliability*, 45(2):341–345.
12. Khamis, I. H. and Higgins, J. J. (1998). A new model for step-stress testing. *IEEE transactions on reliability*, 47(2):131–134.
13. Lee, J. and Pan, R. (2008). Bayesian inference model for step-stress accelerated life testing with type-ii censoring. In *2008 Annual Reliability and Maintainability Symposium*, pages 91–96.
14. Li, C. and Fard, N. (2007). Optimum bivariate step-stress accelerated life test for censored data. *IEEE Transactions on Reliability*, 56(1):77–84.
15. Ling, L., Xu, W., and Li, M. (2011). Optimal bivariate step-stress accelerated life test for type-i hybrid censored data. *Journal of Statistical Computation and Simulation*, 81(9):1175–1186.
16. Minford, W. (1982). Accelerated life testing and reliability of high k multilayer ceramic capacitors. *IEEE Transactions on Components, Hybrids, and Manufacturing Technology*, 5(3):297–300.
17. Mogilevski, B. M. and Shirn, G. A. (1988). Accelerated life tests of ceramic capacitors. In *Electronics Components Conference*, pages 362–370.
18. Munikoti, R. and Dhar, P. (1988). Highly accelerated life testing (halt) for multilayer ceramic capacitor qualification. *IEEE Transactions on Components, Hybrids, and Manufacturing Technology*, 11(4):342–345.
19. Nelson, W. (1980). Accelerated life testing step-stress models and data analysis. *IEEE Transactions on Reliability*, R-29(2):103–108.
20. Sha, N. and Pan, R. (2014). Bayesian analysis for step-stress accelerated life testing using weibull proportional hazard model. *Statistical Papers*, 55(3):715–726.
21. Sinha, S. K. (1998). *Bayesian estimation*.
22. Spiegelhalter, D., Thomas, A., Best, N., and Lunn, D. (2003). *WinBUGS user manual*.
23. Van Dorp, J. R. and Mazzuchi, T. A. (2004). A general bayes exponential inference model for accelerated life testing. *Journal of statistical planning and inference*, 119(1):55–74.
24. Van Dorp, J. R., Mazzuchi, T. A., Fornell, G. E., and Pollock, L. R. (1996). A bayes approach to step-stress accelerated life testing. *IEEE Transactions on Reliability*, 45(3):491–498.