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Generalized Lindley Family with application on Wind Speed Data

Selen Çakmakyapan[1*], Gamze Ozel[2]

* Corresponding Author

1. Department of Statistics Istanbul Medeniyet University, Turkey, selen.cakmakyapan@medeniyet.edu.tr
2. Department of Statistics Hacettepe University, Turkey, gamzeozl@hacettepe.edu.tr

Abstract

In this study we introduce a new extended class of continuous distributions named generalized Lindley family of distributions. Some properties of the new generator, including ordinary moments, quantile, generating and entropy functions, which hold for any baseline model, are presented. The method of maximum likelihood is used for estimating the model parameters. The flexibility of the new family of distributions is shown via an application on the wind speed data set. The results shows that the proposed family is better than well-known distributions including log-logistic, Burr, Dagum, Fréchet, Pearson, Dagum, Lindley, Weibull and exponential distributions.

Key Words: Generated Family; Lindley Distribution; Maximum Likelihood; Moment; Quantile Function; Entropy.

Mathematical Subject Classification: 62E99, 62P12

1. Introduction

Probability distributions are very useful in describing and predicting real world phenomena. . In the literature, different pdfs are used to model various data with different characteristics (Sahin et al. (2005), Dokur and Kurban (2015), Morgan et al. (2011), Tiwari et. al (2020) etc.). For the evaluation of wind energy potential, probability density functions (pdfs) are usually used to model wind speed distributions. The selection of the appropriate pdf reduces the estimation error and also allows to achieve characteristics. Over the past few years, many procedures are used to combine two or more distributions to have the most of characteristics of these distributions in the resulting distribution. Therefore, the new distribution becomes more flexible and gives a better fit to the practical data in many areas of study. Among those procedures, the definition of new generators or families of distributions by introducing additional parameter(s) to the baseline distribution is very popular. The well-known generators are as follows: the Marshall-Olkin generated family (MO-G) by Marshall and Olkin (1997), beta-G by Eugene et al. (2002), gamma-G by Zografos et al. (2009), exponentiated T-X by Alzaghal et al. (2013), Kumaraswamy-G (Kw-G) by Cordeiro & de Castro (2011), transformed-transformer (T-X) by Alzaatreh et al. (2013), additive Weibull-G family by Hassan & Hemada (2016), Lindley-G family by Cakmakyapan & Ozel,(2016), odd Lindley-G family by Gomez-Silva et al. (2017), Weibull-power Cauchy-G family by Tahir et al. (2017), Frechet Topp Leone-G family by Reyad et al. (2020) etc.

The Lindley distribution is as a mixture of exponential and gamma distributions It was proposed by Lindley (1958) and was studied in detail by Ghitany et al. (2008, 2011) and Mazucheli & Achcar (2011). They showed that several properties of the Lindley distribution are more flexible than those of the exponential or Weibull distributions. These distributions can be used for evaluating entrepreneurial opportunities, see Adel Rastkhiz et al. (2011) for more information on this.

The probability density function (pdf) of the Lindley distribution with a scale parameter is given by

$$l(x;\lambda)\frac{\lambda^2}{1+\lambda}(1+x)e^{-\lambda x}, \quad x>0,\ \lambda>0 \tag{1}$$

and the cumulative distribution function (cdf) is defined as follows:

$$L(x;\lambda)=1-\frac{1+\lambda+\lambda x}{1+\lambda}e^{-\lambda x}, \quad x>0,\ \lambda>0 \tag{2}$$

The Lindley distribution has not enough flexibility to analyze different types of lifetime data. Hence, the aim of this study is to obtain a new extended family of distributions from the Lindley distribution. The term generator shows that we have a different distribution F for each baseline distribution G. Let $F(x;\xi)$ be a baseline cdf having a (rx1) parameter vector ξ of unknown parameters. Cakmakyapan & Ozel (2016) defined Lindley family of distributions. In this study, motivated by Cakmakyapan & Ozel (2016), the cdf of the generalized Lindley-G (GL-G in short) family of distributions is obtained as

$$G(x;\lambda,a,\xi)=\int_0^{-\log[1-F^a(x;\xi)]}\frac{\lambda^2}{1+\lambda}(1+t)e^{-\lambda t}dt=1-\frac{\left\{1+\lambda+\lambda\log\left[1-F^a(x;\xi)\right]\right\}\left[1-F^a(x;\xi)\right]^{\lambda}}{1+\lambda}, \quad x>0, \tag{3}$$

where λ, a>0 are two shape parameters. The pdf corresponding to (3) is as follows:

$$g(x;\lambda,a,\xi)=\frac{\lambda^2 a}{\lambda+1}f(x;\xi)F^{a-1}(x;\xi)\left[1-F^a(x;\xi)\right]^{\lambda-1}\left[1-\log(1-F^a(x;\xi))\right], \quad x>0,\ \lambda>0,\ a>0 \tag{4}$$

Here, $f(x;\xi)$ is the baseline pdf and a random variable X with the density function in (4) is denoted by X~GL-G(λ,a,ξ). Then, the hazard rate function (hrf) of X becomes

$$h(x;\lambda,a,\xi)=\frac{\lambda^2 af(x;\xi)F^{a-1}(x;\xi)\left[1-\log(1-F^a(x;\xi))\right]}{\left[1+\lambda+\lambda\log\left(1-F^a(x;\xi)\right)\right]\left[1-F^a(x;\xi)\right]}. \tag{5}$$

The rest of the paper is organized as follows: In Section 2, we introduce some special models of the new family. Some statistical properties are given in Section 3, including quantile function, moments and generated functions. The entropy and reliability functions are presented in Section 4. The maximum likelihood estimation of the family is given in Section 5. In Section 6, application to real data sets is presented to prove empirically the flexibility of the GL-G family. The paper is concluded in Section 7.

2. Special Models

Now, we discuss two special model under the GL-G family.

Generalized Lindley-Weibull (GL-W) distribution

Let $F(x;\alpha,\beta)=1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$ and $f(x;\alpha,\beta)=\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$ be the cdf and the pdf of the Weibull distribution, respectively. Then, from Eq. (4), the pdf of the GL-W distribution is given by

$$g(x;\lambda,a,\alpha,\beta)=\frac{\lambda^2 a\alpha}{(\lambda+1)\beta^{\alpha}}x^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{a-1}\left\{1-\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{a}\right\}^{\lambda-1}\left[1-\log\left(1-\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{a}\right)\right], \quad (6)$$

for $x > 0$. Here, $\alpha > 0$ is a shape parameter and $\beta > 0$ is a scale parameter with $\xi = (\alpha, \beta)^T$. A random variable with pdf (6) is denoted by $X \sim GL-W(\lambda, a, \alpha, \beta)$. For $\lambda = a = 1$, it reduces to the Weibull distribution. For $a = 1$, we obtain the Lindley-Weibull distribution. Further, if $\lambda = 1$, we have the generalized Lindley-exponential (GL-E) distribution.

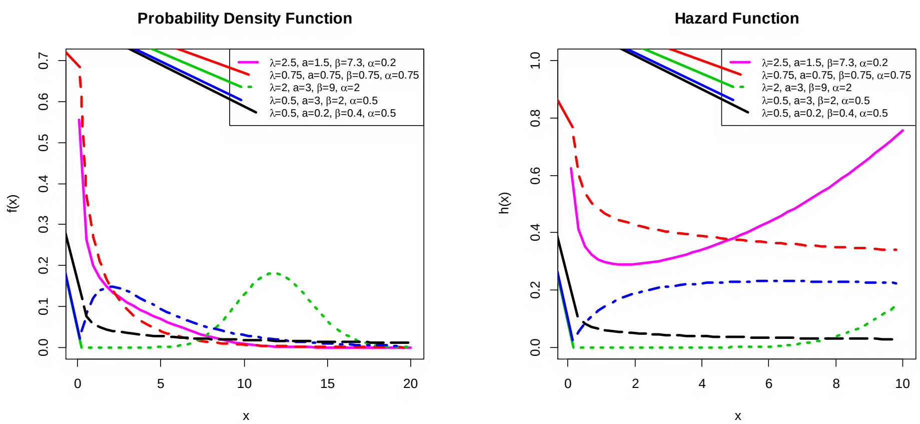

Figure 1: Plots of the GL-W density and hazard functions for some parameter values.

Figure 1 shows that the plots of the GL-W distribution with several values of parameters. As seen in Figure 1, the density function of the GL-W distribution can be right-skewed, symmetric and J-shaped. Furthermore, its hrf can be increasing, decreasing, upside-down bathtub or bathtub shaped.

Generalized Lindley-Gamma (GL-Γ) Distribution

Let $F(x;k,\theta) = \dfrac{\gamma(k, x/\theta)}{\Gamma(k)}$ be the cdf and $f(x;k,\theta) = x^{k-1}\dfrac{e^{-\frac{x}{\theta}}}{\Gamma(k)\theta^k}$ be the pdf of the gamma distribution where $k > 0$ is a shape parameter and $\theta > 0$ is a scale parameter. Here, $\Gamma(k) = \int_0^{\infty} t^{k-1}e^{-t}dt$ is the gamma function, $\gamma(k,z) = \int_0^{z} t^{k-1}e^{-t}dt$ is the incomplete gamma function and $\xi=(k,\theta)^T$. Then, the GL- Γ density function $x > 0$ is obtained from Eq. (4) as follows:

$$g(x;\lambda,a,k,\theta) = \frac{\lambda^2 a}{\lambda+1}\frac{x^{k-1}e^{-\frac{x}{\theta}}}{\Gamma(k)\theta^k}\left[\frac{\gamma(k,x/\theta)}{\Gamma(k)}\right]^{a-1}\left(1-\left[\frac{\gamma(k,x/\theta)}{\Gamma(k)}\right]^{a}\right)^{\lambda-1}\left[1-\log\left(1-\left[\frac{\gamma(k,x/\theta)}{\Gamma(k)}\right]^{a}\right)\right], \quad (7)$$

A random variable X with pdf (7) is denoted by $X \sim GL-\Gamma(\lambda, a, k, \theta)$. For $\lambda = a = 1$, it gives the gamma distribution. For $a = 1$, we obtain the Lindley-gamma distribution. Further, if k = 1, we have the GL-E distribution and we obtain the Lindley-exponential distribution for a = 1 and k = 1. Figure 2 shows that the hrf of the GL-Γ distribution can be increasing, decreasing or bathtub-shaped. Note that more different shapes can be obtained for several values of the parameters.

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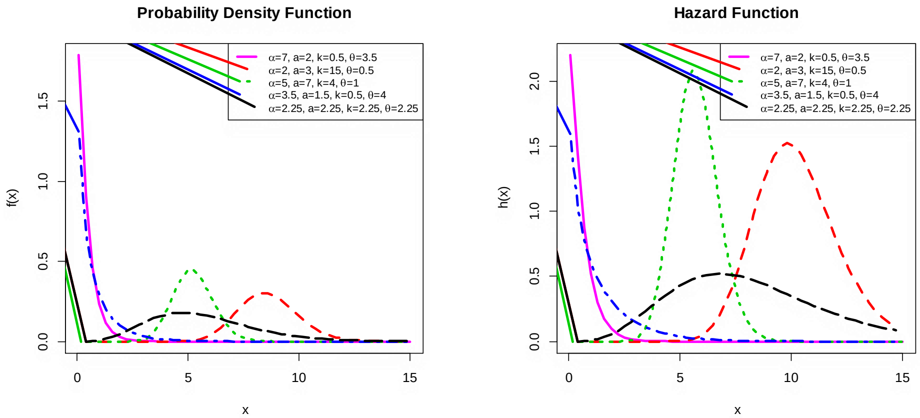

Figure 2: Plots of the GL- Γ density and hazard functions for some parameter values.

3. Main Properties

Quantile Function

Let X denote a GL-G random variable. The quantile function, $G^{-1}(u)$, $0 < u < 1$, for the T-X family of distributions is computed by using the formula of Alzaatreh et al. (2013) as

$$Q(u) = F^{-1}\left[1 - e^{L^{-1}(u)}\right]^{1/a}, \tag{8}$$

where $L^{-1}(u)$ is the inverse of the Lindley distribution function and given as

$$L^{-1}(u) = -\frac{1+\lambda+W\left[(u-1)(l+1)e^{-(\lambda+1)}\right]}{\lambda} \tag{9}$$

where W is Lambert function in equation (9) can be used to simulate the GL-G random variable. Let U be a uniform variable. Using the inverse transformation method, we can also obtain the random variable X, easily.

Useful Expansions

The expansion of Gradshteyn and Ryzhik (2007) is used for a power series raised to any positive integer, n, as

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)^n = \sum_{k=0}^{\infty} c_k x^k \tag{10}$$

where $m \geq 1$ and $c_0 = a_0^n$ are easily obtained from the recurrence equation $c_m = \frac{1}{ma_0}\sum_{k=1}^{m}(kn - m + k)a_k c_{m-k}$. In this study, we also consider the following expansions:

$$(1-z)^q = \sum_{i=0}^{\infty}(-1)^i \binom{q}{i} z^i, \quad |z| > 1, \tag{11}$$

$$\log(1-z) = -z\sum_{i=0}^{\infty}\frac{z^i}{i+1}, \quad |z| > 1, \tag{12}$$

$$\left[-\log(1-z)\right]^{a-1} = -(a-1)\sum_{k=0}^{\infty}\binom{k+1-a}{k}\sum_{j=0}^{\infty}\frac{(-1)^{j+k}\binom{k}{j}P_{j,k}}{a-1-j}z^{a+k-1}, \ a>0. \tag{13}$$

where $P_{j,k} = k^{-1}\sum_{m=1}^{k}[k-m(j+1)]P_{j,k-m}$ for $k=1,2,..$ and $P_{j,0}=1$ .Cordeiro et al. (2013).

The mathematical relation given below will be useful to obtain moments and Rényi entropy function of the GL-G family.

Moments

Now, we consider that $Y_1 \sim \text{Exp-G}\left(a(k+1)\right)$ and $Y_2 \sim \text{Exp-G}\left(a(j+1)(k+j+2)\right)$. The nth moment of X can be obtained from (14) as

$$\mu_n' = E(X^n) = \frac{\lambda^2}{(\lambda+1)\alpha}\left\{\sum_{k=0}^{\infty}w_k\left[(k+1)^{-1}E(Y_1)+\sum_{j=0}^{\infty}\left(j+1\right)^{-1}(k+j+2)^{-1}E(Y_2)\right]\right\}, \tag{14}$$

where $E(Y_1)$ and $E(Y_2)$ are moments of Y_1 and Y_2, respectively.

Nadarajah and Kotz (2006) provided the closed-form expressions for the moments of exponentiated distributions that can be used to obtain GL-G moments. The nth moment of X can also be written as

$$\mu_n' = \frac{a\lambda^2}{\lambda+1}\left[\sum_{k=0}^{\infty}w_k\left(I_{a(k+1)-1}+\sum_{j=0}^{\infty}(j+1)^{-1}I_{a(k+2+j)-1}\right)\right], \tag{15}$$

where $I_k = \int_0^1 Q^n(u)\,u^k\,du$ and $w_k = (-1)^k\binom{l-1}{k}$. Here, $Q^n(u)$ is the quantile function of the baseline distribution.

The moments can be obtained using the integral based on quantile functions. This is usually simpler to compute than if the integral is based on a pdf. By this way the moments of the several GL-G family can be obtained from Eq. (14) and (15).

Moment Generating Function

In this subsection, we obtain the moment generating function (mgf) $M(t)=E\left[\exp(tX)\right]$ of X. First, using the variables $Y_1 \sim \text{Exp-G}\left(a(k+1)\right)$ and $Y_2 \sim \text{Exp-G}\left(a(j+1)(k+j+2)\right)$, $M(t)$ is derived by the generating function of the Exp-G distribution as follows:

$$M(t) = E\left[e^{tX}\right] = \frac{\lambda^2}{(\lambda+1)a}\left\{\sum_{k=0}^{\infty}w_k\left[(k+1)^{-1}M(Y_1)+\sum_{j=0}^{\infty}\left(j+1\right)^{-1}(k+j+2)^{-1}M(Y_2)\right]\right\} \tag{16}$$

where $M(Y_1)$ and $M(Y_2)$ are mgfs of Y_1 and Y_2, respectively.

An alternative formula for $M(t)$ can be derived as

$$M(t) = \frac{a\lambda^2}{\lambda+1}\left[\sum_{k=0}^{\infty}w_k\left(S_{a(k+1)-1}+\sum_{j=0}^{\infty}(j+1)^{-1}S_{a(k+2+j)-1}\right)\right] \tag{17}$$

where $S_k = \int_0^1 Q^n(u)\,u^k\,du$ and $w_k = (-1)^k \binom{l-1}{k}$. Note that the mgfs of the GL-G family can be obtained directly from (16) and (17).

4. Other Measures

Rényi Entropy

The entropy of a random variable X is a measure of variation about the uncertainty. Well-known entropy measure is the Rényi entropy. The Rényi entropy of a random variable with pdf is defined as

$$I_R(\gamma) = \frac{1}{1-\gamma}\log\int_0^\infty f^\gamma(x)\,dx \tag{18}$$

for $\gamma > 0$ and $\gamma \neq 1$ (Rényi (1961)). In this section, we simply write $f(x;\xi) = f(x)$. Then, we obtain the Rényi entropy of the GL-G family as follows:

$$\begin{aligned} I_R(\gamma) &= \frac{1}{1-\gamma}\log\int_0^\infty f^\gamma(x)\,dx \\ &= \frac{1}{1-\gamma}\log\int_0^\infty \frac{\lambda^{2\gamma}a^\gamma}{(\lambda+1)^\gamma} f^\gamma(x)F^{a\gamma-\gamma}(x)\left[1-F^a(x)\right]^{\gamma\lambda-\gamma}\left[1-\log\left(1-F^a(x)\right)\right]^\gamma dx \\ &= \frac{1}{1-\gamma}\log\left[\frac{\lambda^{2\gamma}a^\gamma\gamma}{(\lambda+1)^\gamma}\sum_{i=0}^\infty\sum_{k=0}^\infty E_i D_k I_{i,k}\right], \end{aligned} \tag{19}$$

where $E_i = (-1)^i\binom{\gamma\lambda-\gamma}{i}$, $D_k = \sum_{j=0}^\infty\sum_{r=0}^k \binom{\gamma}{j}\binom{k-\gamma}{k}\frac{(-1)^{r+k}\binom{k}{r}P_{r,k}}{\gamma-r}$, and $I_{i,k} = \int_0^\infty F^{a(i+2\gamma+k)-\gamma}(x) f^\gamma(x)\,dx$.

Here, we have $P_{r.k} = (k)^{-1}\sum_{m=1}^k [k-m(j+1)]P_{j,k-m}$.

Shannon Entropy

Now, we obtain the Shannon entropy for the GL-G family. It is worth mentioning that the notion of the entropy can be used for goodness of fit test for the developed distribution. See Mahdizadeh and Zamanzade (2019), Mahdizadeh and Zamanzade (2017), Zamanzade and Mahdizadeh (2017),Zamanzade (2015), Zamanzade and Arghami (2012). The Shannon entropy for a random variable X is defined by $E\left[-\log g(x)\right]$. It is the special case of Rényi entropy when $\gamma > 1$ (Shannon (1951)). Using the expansions (12) and (13), we obtain

$$\begin{aligned} -\log g(x) &= -2\log\lambda - \log a + \log(\lambda+1) - \log f(x) - (\lambda-1)\log\left(1-F^a(x)\right) \\ &\quad - a\log F(x) - \log\left[1-\log\left(1-F^a(x)\right)\right] \end{aligned} \tag{20}$$

and for the GL-G family direct calculation yields

$$\begin{aligned} E\left[-\log g(x)\right] &= -2\log\lambda - \log a + \log(\lambda+1) - E\left[\log f(x)\right] - (\lambda-1)\sum_{i=0}^\infty \frac{E\left[F^{a(i+1)}(x)\right]}{i+1} - aE\left[\log F(x)\right] \\ &\quad - E\left[\log\left[1-\log\left(1-F^a(x)\right)\right]\right]. \end{aligned} \tag{21}$$

Reliability

In reliability, the stress-strength shows a component life which has a random strength X_1 that is subjected to a random stress X_2. The component fails at the instant that the stress applied to it exceeds the strength, and the component will function satisfactorily whenever $X_1 > X_2$ Therefore, $R = P(X_1 > X_2)$ is a measure of component reliability.
Now, we define the reliability function R when $X_1 \sim GL\text{-}G(\lambda_1, a, \xi)$ and $X_2 \sim GL\text{-}G(\lambda_2, a, \xi)$ are independent random variables. Let g_i denote the pdf of X_i and G_i denote the cdf of X_i for $i = 1, 2,$ then the reliability function for the GL-G family is obtained as

$$\begin{aligned} R &= \int_0^\infty g_1(x) G_2(x)\, dx \\ &= 1 - \frac{\lambda_1 a}{\lambda_1 + 1} \sum_{i=0}^{\infty} (-1)^i \binom{\lambda_1 + \lambda_2 - 1}{i} \\ &\times \left\{ \left[a(i+1)\right]^{-1} + \sum_{k=0}^{\infty} (k+1)^{-1} \left(\frac{2\lambda_2 + 1}{\lambda_2 + 1}\right) \left[a(i+k+2)\right]^{-1} - \frac{\lambda_2}{\lambda_2 + 1} \sum_{j=0}^{\infty} c_j \left[a(i+3+j)\right]^{-1} \right\} \end{aligned} \tag{22}$$

where $c_m = m^{-1} \sum_{j=1}^{m} \frac{3j - m}{j+1} c_{m-j}$ and $m \geq 1$.

5. Maximum Likelihood Estimation

Method of moments for estimation is straightforward and produces consistent estimators these estimators are often biased. The estimates offered by the method of moments are often outside of the parameter space which is uncommon with large samples but not so uncommon with small samples; it does not make sense to rely on them in such situations. So we choose to use maximum likelihood estimation method.

Now, we obtain the MLEs of the parameters of the GL-G family. Let $x_1, x_2, ..., x_n$ be observed values from the GL-G family with parameters (λ, a, ξ). The total log-likelihood function of the parameters is obtained as

$$\begin{aligned} \log L &= n\left[2\log\lambda + \log a - \log(\lambda+1)\right] + \sum_{i=1}^{n} \log f(x_i;\xi) + (\lambda - 1)\sum_{i=1}^{n} \log\left[1 - F^a(x_i;\xi)\right] \\ &+ \sum_{i=1}^{n} \log F^a(x_i;\xi) + \sum_{i=1}^{n} \log\left[1 - \log\left(1 - F^a(x_i;\xi)\right)\right] \end{aligned} \tag{23}$$

The log L can be maximized by solving the nonlinear likelihood equations obtained by differentiating (23). The first derivatives of log L with respect to parameters (λ, a, ξ) are given by

$$\frac{\partial \log L}{\partial \lambda} = n\left[\frac{2}{\lambda} - \frac{1}{\lambda + 1}\right] + \sum_{i=1}^{n} \log\left[1 - F^a(x_i;\xi)\right], \tag{24}$$

$$\begin{aligned} \frac{\partial \log L}{\partial a} &= \frac{n}{a} - (\lambda - 1)\sum_{i=1}^{n} \frac{F^a(x_i;\xi)\log F(x_i;\xi)}{1 - F^a(x_i;\xi)} + \sum_{i=1}^{n} \log F(x_i;\xi) \\ &+ \sum_{i=1}^{n} \frac{\log F(x_i;\xi)}{\left[1 - \log\left(1 - F^a(x_i;\xi)\right)\right]\left[1 - F^a(x_i;\xi)\right]}, \end{aligned} \tag{25}$$

,

$$\frac{\partial \log L}{\partial \xi}=\sum_{i=1}^{n}\frac{1}{f(x_i;\xi)}\frac{\partial f(x_i;\xi)}{\partial \xi}-(\lambda-1)\sum_{i=1}^{n}\frac{aF^{a-1}(x_i;\xi)}{1-F^{a}(x_i;\xi)}\frac{\partial F(x_i;\xi)}{\partial \xi}+\sum_{i=1}^{n}\frac{aF^{a-1}(x_i;\xi)}{F(x_i;\xi)}\frac{\partial F(x_i;\xi)}{\partial \xi}$$
$$+\sum_{i=1}^{n}\frac{aF^{a-1}(x_i;\xi)}{\left[1-\log\left(1-F^{a}(x_i;\xi)\right)\right]\left[1-F^{a}(x_i;\xi)\right]}\frac{\partial F(x_i;\xi)}{\partial \xi} \quad (26)$$

The MLEs of (λ, a, ξ), say $(\hat{\lambda}, \hat{a}, \hat{\xi})$, are the simultaneous solutions of the equations $\frac{\partial \log L}{\partial \lambda}=0$, $\frac{\partial \log L}{\partial a}=0$, $\frac{\partial \log L}{\partial \xi}=0$. Maximization of the equations can be obtained from nlm or optimize in R statistical package.

6. Application

Now, the flexibility of the GL-G models is demonstrated via applications on two real data sets. The estimation of the unknown parameters are obtained by the maximum likelihood method. Then, Akaike information criterion (AIC), Corrected Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC) are provided. We also give the histograms of data sets and plot the fitted density functions to obtain a visual comparison of the adjustments of the models in Figures 3 and 4.

The data sets represent the wind speeds (m/s) from several stations in Turkey between years 2012-2015 (http://www.havaizleme.gov.tr/Default.ltr.aspx). The first data set includes 84 observations and belongs to Kırklareli (Vize) station in Turkey. The second data set includes 811 observation from Yalova station in Turkey.

We demonstrated the flexibility of the GL-E and GL-W distributions in contrast with other models including fitted log-logistic, Burr, dagum, Lindley, Weibull and exponential distributions. Table 1 presents the values of -LogL and AIC, CAIC, BIC, HQIC for the fitted models. The results in Table 1 show that the GL-E model provides the best fit to the first data. Table 2 presents the MLEs and their standard errors of the parameters from the fitted models for the first data. Figure 3 provides more information by a usual comparison of the histograms for the data with the best three fitted pdfs. It shows that the GL-E distribution yields more adequate fit than the other distributions.

Table 1: Information criterias for the first data.

Distribution	AIC	CAIC	BIC	HQIC	-Log(likelihood)
G-LE	338.8138	339.1138	346.1062	341.7453	166.4069
G-LW	340.4393	340.9456	350.1626	344.348	166.2197
Log-Logistic	341.4075	341.5556	346.2691	343.3618	168.7037
Burr	343.1571	343.4571	350.4495	346.0886	168.5785
Dagum	343.2533	343.5533	350.5458	346.1848	168.6267
Weibull	347.2001	347.3483	352.0617	349.1544	171.6001
Lindley	396.5762	396.6250	399.0070	397.5533	197.2881
Exponential	428.3656	428.4144	430.7965	429.3428	213.1828

Table 2: MLEs for the first data.

Distribution	Estimated Parameters
GL-E	0.33738, 83.45717, 0.48879
GL-W	0.23342, 30.29890, 1.28209, 0.72957
Log-Logistic	4.15890, 4.28155
Burr	3.80423, 1.32733 ,4.77772
Dagum	3.88827, 1.23275, 3.95189
Weibull	2.54382, 5.25567
Lindley	0.37148
Exponential	4.65476

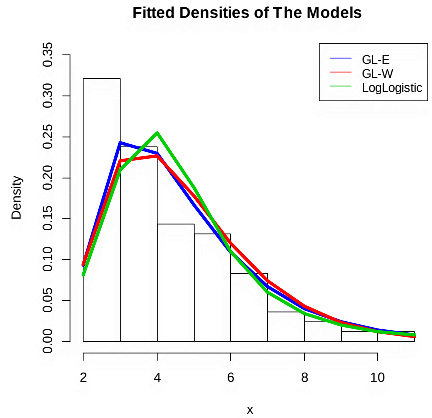

Figure 3: Estimated models for the first data set.

As the second application, we fit the Fréchet, Pearson 5, Dagum, Gamma, Lindley, Weibull, and the proposed GL-Γ distributions. The results in Table 3 show that the GL-Γ model has the lowest values for the AIC, BIC, CAIC, HQIC values among all fitted models, and it could be chosen as the best model for explaining the second data. The MLEs and their standard errors of the parameters from the fitted models for the second data are presented in Table 4. A density plot in Figure 4 also compares the fitted densities of the best three models with the empirical histogram of the second data. Figure 4 shows that the fitted density for the GL-Γ distribution is closer to the empirical histogram than the fits of the other distributions.

Table 3: Information criterias for the second data.

Distribution	AIC	CAIC	BIC	HQIC	-Log likelihood
GL-Γ	1308.517	1308.567	1327.310	1315.732	650.2587
Fréchet	1311.309	1311.338	1325.404	1316.720	652.6544
Pearson 5	1379.398	1379.427	1393.492	1384.809	686.6158
Gamma	1682.466	1682.480	1691.862	1686.073	838.7796
Weibull	1706.828	1706.842	1716.224	1710.435	851.4138
Lindley	2551.142	2551.146	2555.840	2552.945	1274.571
Dagum	1309.597	1309.647	1328.390	1316.812	650.7986

Table 4: MLEs for the second data.

Distribution	Estimated Parameters
GL-Γ	0.57676, 325.0000, 0.99726, 0.24100
Fréchlet	4.06169, 1.70736, 0.0000000001
Pearson 5	3.08617, 0.80312, 3.28057
Gamma	4.71988, 0.43108
Weibull	2.85093, 2.26683
Lindley	0.76859
Dagum	57.98648, 0.04114 -1.18864 4.78999

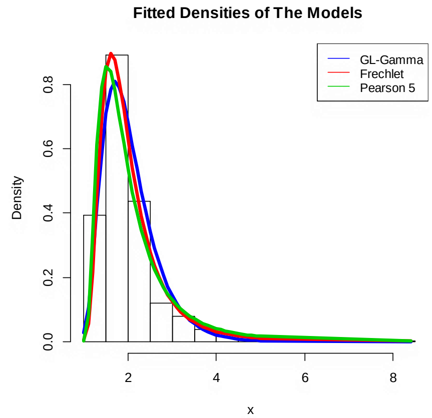

Figure 4. Estimated models for the second data set.

7. Concluding Remarks

In this paper, a new family of distributions called the "Generalized Lindley-G" which can generate all classical continuous distributions is proposed. We study some properties of the new generator including moments, generating, entropy and quantile functions, and reliability. We obtain maximum likelihood estimations of the model parameters. Two real dataset examples from the field of natural sciences were studied to demonstrate the applicability of the proposed model in real life phenomena. As a future work we will consider bivariate and multivariate extension of the proposed new distributions. Also, we will apply the models to fit various data of different areas.

References

1. Aarset, M. V. (1987). How to identify a bathtub hazard rate. IEEE Transactions on Reliability, 36(1), 106-108.
2. Adel Rastkhiz, S. E., Mobini Dehkordi, A., Yadollahi Farsi, J. & Azar, A. (2019). A new approach to evaluating entrepreneurial opportunities, Journal of Small Business and Enterprise Development, (26)(1), 67-84. https://doi.org/10.1108/JSBED-01-2018-0013
3. Afify, A. Z., Yousof, H. M., Cordeiro, G. M., Ortega, E. M. M. & Nofal, Z. M. (2016b). The Weibull Fréchet distribution and its applications. Journal of Applied Statistics, 43(14), 2608-2626.
4. Alzaatreh, A., Lee, C. & Famoye, F. (2013). A new method for generating families of distributions, Metron, (7)(1), 63-79.
5. Alzaghal, A., Famoye , F. & Lee, C. (2013). Exponentiated T-X family of distributions with some applications, Int. J. Prob. Stat., (2), 31–49.
6. Cakmakyapan, S. & Ozel, G. (2016) .The Lindley Family of Distributions: Properties and Applications
7. Cordeiro, G. M. Ortega, E. M. M. & Popovi´c B. V. (2013). The gamma-Lomax distribution, Journal of Statistical computation and Simulation , doi:10.1080/00949655.2013.822869 .
8. Cordeiro, G.M, & de Castro M. (2011). A new family of generalized distributions, J. Stat. Comput. Sim., (81), 883-893.
9. Dokur, E. & Kurban, M. (2015). Wind Speed Potential Analysis Based On Weibull Distribution. Balkan Journal of Electrical And Computer Engineering, 3(4), 231–235.
10. Eugene, N., Lee, C. & Famoye, F. (2002). Beta-normal distribution and its applications, Communication Statistical Theory,(31), 497-512.
11. Ghitany, M.E., Alqallaf, F, Al-Mutairi, D.K. & Husain, H.A. (2011). A two-parameter Lindley distribution and its applications to survival data, Math. Comput. Simulat., (81), 1190–1201.
12. Ghitany, M.E., Atieh, B. & Nadarajah, S. (2008). Lindley distribution and its application, Math. Comput. Simulat., (78) 493–506.
13. Gomes-Silva, F. Percontini, A, de Brito, E., Ramos M.W., Venancio, R. & Cordeiro, G. (2017). The Odd Lindley-G Family of Distributions, Austrian Journal of Statistics, 46, 65-87.

14. Gradshteyn, I.S. & Ryzhik, I.M. (2007). Table of Integrals, Series and Products, Academic Press, New York.
15. Hacettepe University Bulletin of Natural Sciences and Engineering Series B: Mathematics and Statistics, 46(6), 1113-1137.
16. Hassan, A. S. & Hemeda, S. E. (2016). The additive Weibull-g family of probability distributions. International Journals of Mathematics and Its Applications, (4), 151-164.
17. Lindley, D.V. (1958). Fiducial distributions and Bayes' theorem, J. Roy. Stat. Soc. B 20, 102-107.
18. Mahdizadeh, M. & Zamanzade, E. (2019). Goodness-of-t testing for the Cauchy distribution with application to nancial modeling, Journal of King Saud University - Science, (31)(4), 1167-1174.
19. Mahdizadeh, M., & Zamanzade, E. (2017). New goodness of t tests for Cauchy distribution. Journal of Applied Statistics, (44)(6), 1106-1121.
20. Marshall, A.N. & Olkin, I. (1997). A new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families, Biometrika, (84), 641-552.
21. Mazucheli, J. & Achcar, J. A. (2011). The Lindley distribution applied to competing risks lifetime data, Comput. Meth. Prog. Bio., (104), 188-92.
22. Morgan, E. C., Lackner, M., Vogel, R. M. & Baise, L. G. (2011). Probability distributions for offshore wind speeds. Energy Conversion and Management, 52(1), 15–26.
23. Nadarajah, S. & Kotz, S. (2006). The exponentiated type distributions. Acta Applicandae Mathematicae, (92), 97-111.
24. Rényi, A. (1961). On measures of entropy and information, In: Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability, 547-561. University of California Press, Berkeley.
25. Reyad, H., Korkmaz, M.C., A_fy, A.Z., Hamedani, G.G. & Othman, S.(2019). The Fr_echet Topp Leone-G family of distributions: Properties, characterizations and applications. Annals of Data Science, https://doi.org/10.1007/s40745-019-00212-9.
26. Sahin, B., Bilgili, M. & Akilli, H. (2005). The wind power potential of the eastern Mediterranean region of Turkey. Journal of Wind Engineering and Industrial Aerodynamics, 93(2), 171–183.
27. Shannon, C.E. (1951). Prediction and entropy of printed English. Bell Syst. Tech. J. (30),50-64.
28. Tiwari, A., Mishra, S. & Maurya, R. (2020). Measurement of Inequality and Study of Social Influence on Distribution of Land Holding in India. International Journal Of Agricultural Sciences. (16), 925-935.
29. Zamanzade, E. & Arghami, N.R. (2012). Testing normality based on new entropy estimators, Journal of Statistical Computation and Simulation, 82(11), 1701-1713.
30. Zamanzade, E. (2014). Testing uniformity based on new entropy estimators, Journal of Statistical Computation and Simulation, (85)(16), 3191-3205.
31. Zamanzade, E., & Mahdizadeh, M. (2017). Entropy estimation from ranked set samples with application to test of fit. Revista Colombiana de Estadstica, (40)(2), 223-241.
32. Zografos, K. & Balakrishnan, N. (2009). On families of beta- and generalized gamma-generated distributions and associated inference. Statistical Methodology, (6), 344-362.