# Bayesian Prediction of order statistics based on finite mixture of general class of distributions under Random Censoring

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## Abstract
This paper focuses on the Bayesian prediction of $k^{th}$ ordered future observations modelled by a two-component mixture of general class of distributions. Samples under consideration are subject to random censoring. A closed form of Bayesian predictive density is obtained under a two-sample scheme. Applications to Weibull and Burr XII components are presented and comparisons with previous results are made. A numerical example is presented for special cases of the exponential and Lomax components to obtain interval prediction of first and last order statistics.A simulation study has been conducted to assess the effect of sample size, hyper parameters, and level of censoring on prediction interval and point prediction of a future observation coming from the two-component exponential model.

**Keywords:**Two-sample Bayesian prediction; Random censoring; mixture survival models.

## 1. Introduction
Finite mixtures of distributions have been extensively used to model heterogeneous phenomena in various applications. Astronomy, Engineering, Psychiatry, Biology, Medicine, and Social Sciences are examples of fields in which mixture models are applied. Survival mixture models have been also used to describe population heterogeneity in reliability and life testing applications. For detailed explanation and more review of applications of finite mixture, see for example Everitt and Hand (1981), Titterington et al (1985), McLachlan and Peel (2000), Schlattmann(2009), and Schnatter (2006). Studying prediction of future observations is important for decision making. Prediction is applied in many fields such as medicine, Engineering, and in Business. It is also used in Economics. The problem of prediction can also be found in other areas such as safety analysis of nuclear reactors and of Aerospace systems. For more details on the history of statistical predictionanalysis and applications, see for example Aitchison and Dunsmore (1975). In many practical problems it is seldom to have a complete sample for a number of reasons such as limited budget or time. Censored samples mean that only some of the units fail before the end of the experiment and are called censored units. There are three different types of censoring; left censoring, right censoring and interval

censoring. Right censoring is most commonly used in life testing. These types of samples occur when we test n units and only r units fail during the test. Right censoring implies that if the censored data where to continue operating the failure would occur at some time after the data point observed. There are several situations where right censoring may occur. Three of them are: type I censoring, type II censoring, and random censoring.

The problem of Bayesian interval predictions of future observations based on homogeneous populations have been studied by several researchers under several types of right censoring. AbdEllah (2003) and Pradhan and Kundu (2011) studied the problem of prediction when the sample is complete. Wang (2008), Wang and Veraverbeke (2009), Dunsmore (1974) investigated the case of random censoring. Several authors undertaken the problem assuming type II censoring. Among others are Dunsmore (1974), Evans and Nigm (1980a), Evans and Nigm (1980b),Tziafetas (1987), Nigm (1988), Nigm (1989), Calabria and Pulcini (1994), Howlader and Hossaing (1995), Jaheen and Matrafi (2002), Nigmet al. (2003), Pal and Chattopadhyay (2007), Ateya (2011),AL-Hussaini and Hussain (2011), Singh et al. (2013a). The problem of prediction of homogeneous populations under double type II censoring was studied by Ferandez (2000), Raqab and Madi (2002), Ferntindez (2004), Yanlinget al. (2005), Ferozeet al. (2014). While Balakrishnan and Shafay (2012), Singh et al. (2013b), Asgharzadehet al. (2013), Sadek (2016) assumed hyprid censoring. Finally the problem of prediction under progressively type II censoring was studied by Mousa and Jaheen (2002),Wu et al. (2006), Solimanet al. (2011), Mohie El-Din and Shafay (2013), Jung and Chung (2013), Mohie El-Din et al. (2017).

Bayesian interval predictions of future observations based on heterogeneous populations have also been studied by several researchers. AL-Hussaini(1999a), Al-Hussaini et al (2001), Jaheen (2003), Al-Jarallah and AL-Hussaini (2007), Ahmad et al (2012), Feroze and Aslam (2013), Rahman and Aslam (2014), and Haq and Al-Omari (2016) assumed samples under study exposed to type I censoring. Bayesian prediction of heterogeneous populations under type II censoring was investigated by Mahmoud et al (2014) and Rahman and Aslam (2015). Prediction of order statistics when samples are subject to type II censoring based on the two-component generalized exponential model. Finally, Feroze and Aslam (2016) considered the problem under type II double censoring.

Other authors studied the Bayesian prediction of future observations modelled by a mixture of general class of distributions. AL-Hussaini(1999b) obtained the one-sample and two-sample Bayesian predictive density of future ordered statistic of general model under the homogeneous population under type II censoring. The same model proposed was then used afterwards in AL-Hussaini(2001) to obtain the Bayesian two-sample predictive density of median of future observations and Al-Hussaini (2003) to obtain the two-sample Bayesian predictive density of future ordered observation from two-mixture models under type I censoring. Abdel-Aty (2012) obtained one-sample Bayesian prediction of the number of components which will fail in a future time interval when the population density was modeled by a finite k-component general survival model assuming type I censoring.

The general model adopted in this paper resembles that used by Abd El-Baset and Al-Zaydi (2015). The finite mixture of k components has density function the takes the form,

$$f(t) = \sum_{i=1}^{k} p_i f_i(t), \qquad (0.1)$$

where$p_i$is a non-negative real number (known as the ith mixing proportion) such that $\sum_{i=1}^{k} p_i$=1 and $f_i(t)$ is known as theithmixing component, $i = 1, \ldots\ldots, k$.

Let the density of the ith component has the form,

$$f_i(t) = f_i(t;\alpha_i) = \lambda_i^{'}(t)/\alpha_i \, exp(-\lambda_i(t)/\alpha_i), t > 0, \alpha_i > 0, \qquad (0.2)$$

where $\lambda_i(t)$ is monotonically increasing, $\lambda_i(t) \to 0$ $as$ $t \to 0$ and $\lambda_i(t) \to \infty$ $as$ $t \to \infty$ and $\alpha_i \in \Omega$.

The corresponding survival function of the ith component is,

$$R_i(t) = R_i(t;\alpha_i) = exp(-\lambda_i(t)/\alpha_i), t > 0, \alpha_i > 0 \qquad (0.3)$$

Where

$$R_i(t;\alpha_i) = 1 - F_i(t;\alpha_i) \qquad (0.4)$$

Under suitable choices of $\alpha_i$ and $\lambda_i(t)$, k-component of Weibull, exponential, Rayleigh, Burr XII, Lomax, Gompertz, and power function distributions are special cases of the k-component general model proposed.

Suppose the population under study is known to have two types of failure leading to two subpopulations each with density $f_i(t)$ then the model suitable for modeling such a population is two-component mixture. The model under study is two-component mixture general model whose density and survival functions take respectively the form,

$$f(t) = pf_1(t) + qf_2(t) \qquad (0.5)$$

$$R(t) = pR_1(t) + qR_2(t), \qquad (0.6)$$

where q=1-p

The interest of this paper is to obtain a closed form for the Bayesian predictive density of the kth ordered future observation from the proposed two-component general class of distributions under random censoring.Random censoring is one in which each individual is assumed to have a lifetime T and a censoring time C, with T and C independent continuous random variables, with reliability functions R(t) and G(t),respectively. All lifetimes and censoring times are assumed to be mutually independent, and it is assumed that G(t) does not depend on any of the parameters of R(t). Random censoring occurs frequently in practice especially in clinical and medical trials.

$$t_i = min(T_i, C_i), \delta_i = \begin{cases} 1, T_i \le C \\ 0, T_i > C \end{cases}$$

The joint distribution of the two variables $t_i$ and $\delta_i$ can be written as,

$$f(t_i,\delta_i) = [G(t_i)f(t_i)]^{\delta_i}[g(t_i)R(t_i)]^{1-\delta_i}, \qquad (0.7)$$

where g(t) is the pdf of the random variable C. If we are concerned only with the parameters of failure time rather than the parameters of censoring distribution then G(t) and g(t) can be dropped from the likelihood function since they are considered as constants. (See Lawless (2003)).

The likelihood in the homogeneous population is:

$$L(\underline{t}|\theta) = \prod_{i=1}^{n} f(t_i)^{\delta_i}[R(t_i)]^{1-\delta_i}$$

$$L(\underline{t}|\theta) = [\prod_{i=1}^{r} f(t_i)][\prod_{i=1}^{n-r} R(t_i)],$$

The likelihood in the heterogeneous population is

$$L(\underline{\alpha}, p|\underline{t}) = \prod_{j=1}^{r_1} pf_1(t_{1j}) \prod_{j=1}^{r_2} qf_2(t_{2j}) \prod_{i=1}^{n-r} R(t_i), \qquad (0.8)$$

where

$\underline{t} = (t_{11}, t_{12}, ..., t_{1r_1}, t_{21}, t_{22}, ..., t_{2r_2}),$

$\underline{\alpha} = (\alpha_1, \alpha_2)$is the vector of parameters, and ri, i=1,2 denotes the number of observed items from the i$^{th}$ component, r=r$_1$+r$_2$ and n-r denotes the number of censored items (See for example Contreras-Cristán (2007)).

## 2. Bayesian Prediction under two-component mixture of general model

2.1 Bayesian predictive density function

The likelihood function under random censoring for the general distribution, takes the form,

$$L(\underline{\alpha}, p|\underline{t}) \propto \frac{p^{r_1} q^{r_2}}{{\alpha_1}^{r_1} {\alpha_2}^{r_2}} \prod_{j}^{r_1} \lambda_1'(t_{1j}) \prod_{j}^{r_2} \lambda_2'(t_{2j})\, exp[-\frac{\sum_{j=1}^{r_1} \lambda_1(t_{1j})}{\alpha_1} - \frac{\sum_{j=1}^{r_2} \lambda_2(t_{2j})}{\alpha_2} \times \prod_{i=1}^{n-r} p\, exp(\frac{-\lambda_1(t_i)}{\alpha_1})[1 + \frac{q}{p} exp(\frac{\lambda_1(t_i)}{\alpha_1} - \frac{\lambda_2(t_i)}{\alpha_2})] \quad , \tag{0.9}$$

wheret, α, ri, and n-r are as defined in (0.8)

Using mathematical induction we have proved in Appendix Athat

$$\prod_{i=1}^{n}(1 + k\, exp(x_i)) = \sum_{j_1=0}^{n} \sum_{j_2=0}^{j_1-1} \sum_{j_3=0}^{j_2-1} \cdots \sum_{j_{n-1}=0}^{j_{n-2}-1} \sum_{j_n=0}^{j_{n-1}-1} exp(\sum_{l=1}^{n} x_{j_l})(k)^{\sum_{l=1}^{n} I_{j_l}}, I_{j_l} = \begin{Bmatrix} 0, j_l = 0 \\ 1, j_l \geq 1 \end{Bmatrix}, (j_l \leq 0 \Rightarrow x_{j_l} = 0) \tag{0.10}$$

where

Let $x_i = (\frac{\lambda_1(t_i)}{\alpha_1} - \frac{\lambda_2(t_i)}{\alpha_2}), k = q/p, n = n - r$

The likelihood can be written as,

$$L(\underline{\alpha}, p|\underline{t}) \propto \prod_{j}^{r_1} \lambda_1'(t_{1j}) \prod_{j}^{r_2} \lambda_2'(t_{2j})\, p^{n-d} q^d / {\alpha_1}^{r_1} {\alpha_2}^{r_2}$$

$$\times exp(-[\sum_{j=1}^{r_1} \lambda_1(t_{1j}) + \sum_{i=1}^{n-r} \lambda_1(t_i) - \sum_{l=1}^{n-r} \lambda_1(t_{j_l})]/\alpha_1 - [\sum_{l=1}^{n-r} \lambda_2(t_{j_l}) + \sum_{j=1}^{r_2} \lambda_2(t_{2j})]/\alpha_2), \tag{0.11}$$

where

$$d = r_2 + \sum_{l=1}^{n-r} I_{j_l}, \sum = \sum_{j_1=0}^{n-r} \sum_{j_2=0}^{j_1-1} .. \sum_{j_{n-r-1}=0}^{j_{n-r-2}-1} \sum_{j_{n-r}=0}^{j_{n-r-1}-1}$$

The model parameters;$p, \alpha_1, \alpha_2$are assumed to be random variables having the following prior assumptions,

The prior distribution of $\alpha_i$ is assumed to follow an inverse gamma distribution given by,

$$\Pi_i(\alpha_i) \propto 1/\alpha_i^{b_i+1} exp(-a_i/\alpha_i), \alpha_i > 0, a_i > 0, b_i > 0$$

(0.12)

The prior distribution of the mixing proportion p is $Beta(\delta_1, \delta_2)$

Then assuming that $\alpha_1, \alpha_2, p$ are independent, the joint prior distribution of $\alpha_1, \alpha_2, p$is given by,

$$\Pi(\underline{\alpha},p) \propto \frac{p^{\delta_1-1}q^{\delta_2-1}}{\alpha_1^{b_1+1}\alpha_2^{b_2+1}} exp(-[\frac{a_1}{\alpha_1}+\frac{a_2}{\alpha_2}]) \quad (0.13)$$

Using Equations (0.11)and(0.13), the posterior density of two-component general model under the assumed prior knowledge is,

$$g(\underline{\alpha},p|\underline{t}) = 1/Q \sum \frac{p^{n-d+\delta_1-1}q^{d+\delta_2-1}}{\alpha_1^{r_1+b_1+1}\alpha_2^{r_2+b_2+1}} exp(-\frac{h_1}{\alpha_1}-\frac{h_2}{\alpha_2}), \quad (0.14)$$

where

$$h_1 = a_1 + \sum_{j=1}^{r_1} \lambda_1(t_{1j}) + \sum_{i=1}^{n-r} \lambda_1(t_i) - \sum_{l=1}^{n-r} \lambda_1(t_{jl}), h_2 = a_2 + \sum_{l=1}^{n-r} \lambda_2(t_{jl}) + \sum_{j=1}^{r_2} \lambda_2(t_{2j}) \quad (0.15)$$

and

$$Q = \sum Beta(n-d+\delta_1, d+\delta_2) \frac{\Gamma(r_1+b_1)}{(h_1)^{r_1+b_1}} \frac{\Gamma(r_2+b_2)}{(h_2)^{r_2+b_2}} \quad (0.16)$$

The two-sample predictive density of the k[th] observation is,

$$p(y_k|\underline{t}) = \int_p \int_{\alpha_1} \int_{\alpha_2} f_{Y_k}(y_k|\underline{\alpha},p) g(\underline{\alpha},p|\underline{t}) d\alpha_2 d\alpha_1 dp, k = 1,2,\ldots,m \quad (0.17)$$

where $f_{Y_k}(y_k|p,\underline{\alpha})$ is the pdf of $Y_k$ defined as

$$f_{Y_k}(y_k|\underline{\alpha},p) = k\binom{m}{k} f(y_k)(1-R(y_k))^{k-1} R(y_k)^{m-k}, y_k \geq 0, \alpha_i > 0, \quad (0.18)$$

By using binomial expansion of the term $(1-R(y_k))$ given in Equation(0.18), the predictive density of $Y_k$ can be written as,

$$p(y_k|\underline{t}) = \binom{m}{k} / Q \int_p \int_{\theta_1} \int_{\theta_2} \sum \sum_{s_1=0}^{k-1} [\binom{k-1}{s_1} (-1)^{s_1} f(y_k) R(y_k)^{m+s_1-k} g(\underline{\alpha},p|\underline{t})] d\alpha_2 d\alpha_1 dp \quad (0.19)$$

By substituting for $f(y_k)$, $R(y_k)$ and $g(\underline{\alpha},p|\underline{t})$, the predictive density $p(y_k|\underline{t})$ is obtained as,

$$p(y_k|\underline{t}) = k\binom{m}{k} / Q \int_p \int_{\alpha_1} \int_{\alpha_2} \sum \sum_{s_1=0}^{k-1} [\binom{k-1}{s_1} (-1)^{s_1} exp(-\frac{h_1}{\alpha_1}-\frac{h_2}{\alpha_2})(F_1 + F_2)] d\alpha_2 d\alpha_1 dp, \quad (0.20)$$

where

$$F_1 = \frac{\lambda_1'(y_k) p^{n-d+\delta_1} q^{d+\delta_2-1} exp(-\lambda_1(y_k)/\alpha_1)}{\alpha_1^{r_1+b_1+2}\alpha_2^{r_2+b_2+1}} [p\, exp(-\lambda_1(y_k)/\alpha_1) + q\, exp(-\lambda_2(y_k)/\alpha_2)]^{m-k+s_1}$$

$$F_2 = \frac{\lambda_2'(y_k) p^{n-d+\delta_1} q^{d+\delta_2} exp(-\lambda_2(y_k)/\alpha_2)}{\alpha_1^{r_1+b_1+1}\alpha_2^{r_2+b_2+2}} [p\, exp(-\lambda_1(y_k)/\alpha_1) + q\, exp(-\lambda_2(y_k)/\alpha_2)]^{m-k+s_1} \quad (0.21)$$

Again using binomial expansion of $(p\, exp(-\lambda_1(y_k)/\alpha_1) + q\, exp(-\lambda_2(y_k)/\alpha_2))$, the predictive density is

$$p(y_k|\underline{t}) = \frac{1}{Q} \int_0^1 \int_0^\infty \int_0^\infty \sum * \sum \zeta (F_1^* + F_2^*) exp(-\frac{h_1}{\alpha_1}-\frac{h_2}{\alpha_2}) d\alpha_2 d\alpha_1 dp, \quad (0.22)$$

where

$$F_1^* = \frac{\lambda_1'(y_k) exp(-\lambda_1(y_k)/\alpha_1)}{\alpha_1^{r_1+b_1+2}\alpha_2^{r_2+b_2+1}} p^{n-d+s_2+\delta_1} q^{d+E+\delta_2-1} exp(-(s_2+1)\lambda_1(y_k)/\alpha_1)\, exp(-E\lambda_2(y_k)/\alpha_2)$$

$$F_2^* = \frac{\lambda_2^{'}(y_k)\, exp(-\lambda_2(y_k)/\alpha_2)}{\alpha_1^{r_1+b_1+1}\alpha_2^{r_2+b_2+2}} p^{n-d+s_2+\delta_1-1} q^{d+E+\delta_2}\, exp(-s_2\lambda_1(y_k)/\alpha_1)\, exp(-(E+1)\lambda_2(y_k)/\alpha_2 \quad (0.23)$$

and

$$\zeta = k\binom{m}{k}\binom{k-1}{s_1}\binom{m-k+s_1}{s_2}(-1)^{s_1}, E = m-k+s_1-s_2, \textstyle\sum * = \sum_{s_1=0}^{k-1}\sum_{s_2=0}^{m-k+s_1} \quad (0.24)$$

Finally the predictive density of $y_k$ in (0.17)is,

$$p(y_k|\underline{t}) = \frac{1}{Q'}\textstyle\sum * \sum \zeta(F_1^{**} + F_2^{**}), \quad (0.25)$$

Where

$$F_1^{**} = \frac{\lambda_1^{'}(y_k) Beta(n-d+s_2+\delta_1+1, d+E+\delta_2)(b_1+r_1)}{[h_1+(s_2+1)\lambda_1(y_k)]^{b_1+r_1+1}[h_2+E\lambda_2(y_k)]^{b_2+r_2}},$$

$$F_2^{**} = \frac{\lambda_2^{'}(y_k) Beta(n-d+s_2+\delta_1, d+E+\delta_2+1)(b_2+r_2)}{[h_1+s_2\lambda_1(y_k)]^{b_1+r_1}[h_2+(E+1)\lambda_2(y_k)]^{b_2+r_2+1}} \quad (0.26)$$

and

$$Q' = \sum \frac{Beta(n-d+1, d+1)}{(h_1)^{r_1+b_1}(h_2)^{r_2+b_2}} \quad (0.27)$$

The predictive interval (a,b):
The lower prediction limit for the future k ordered observation $Y_k$ is given as

$$\int_0^a p(y_k|\underline{t})dy_k = \frac{1-\gamma}{2} \quad (0.28)$$

The upper prediction limit for the future k ordered observation $Y_k$ is given as

$$\int_b^\infty p(y_k|\underline{t})dy_k = \frac{1-\gamma}{2} \quad (0.29)$$

Given specified values of k, (0.28) and (0.29) can be solved numerically for a and b to get the predictive interval (a,b) for the future k ordered observation $Y_k$.

2.2 Invariance property of the predictive density of the general model

In this section, it will be shown that the predictive density$p(y_k|\underline{t})$ is invariant under one-to one transformation of the parameter$\alpha_i$. Specifically, if the pdf of the ithcomponent is indexed by a parameter$\theta_i = g(\alpha_i)$,i=1,2, where $\theta_i > 0$ and g(.) is a one-to-one transformation of $\alpha_i$, then the predictive density based on this reparametrized form $g(\alpha_i)$will also be given by Equation(0.25).

Proof:

It's known that the posterior density is,

$$g(\underline{\alpha}, p|\underline{t}) = \frac{L(\underline{\alpha}, p|\underline{t})\Pi(\underline{\alpha}, p)}{\int_{p=0}^1\int_{\alpha_1=0}^\infty\int_{\alpha_2=0}^\infty L(\underline{\alpha}, p|\underline{t})\Pi(\underline{\alpha}, p)d\alpha_2 d\alpha_1 dp}$$

(2.22)

Recall that $p, \alpha_1, \alpha_2$are independent hence$\Pi(p, \underline{\alpha}) = \Pi_1(\alpha_1)\Pi_2(\alpha_2)\Pi_3(p)$, where$\Pi_i(\alpha_i)$, i=1,2 denotes the prior distribution of $\alpha_i$and $\Pi_3(p)$denotes the prior distribution of p.

Using the transformation$\theta_i = g(\alpha_i)$, i=1,2 gives$d\theta_i = g'(\alpha_i)d\alpha_i$

Now let the prior distribution of $\theta_i$be denoted by$\Pi^*(\theta_i)$.

$\Pi^*(\theta_i)$can be obtained by the technique of transformation of variables as follows,

$$\Pi^*(\theta_i) = \Pi(\alpha_i)|d\alpha_i/d\theta_i|$$
$$= \Pi(\alpha_i)|1/g'(\alpha_i)| \quad (0.30)$$

Using the previous equation, the posterior distribution $g(\underline{\alpha}, p|\underline{t})$is,

$$g(\underline{\alpha},p|\underline{t})=\frac{L(\underline{t}|\theta_1,\theta_2,p)\Pi^*(\theta_1,\theta_2,p)\frac{d\theta_1 d\theta_2}{d\alpha_1 d\alpha_2}}{\int_{p=0}^{1}\int_{\theta_1=0}^{\infty}\int_{\theta_2=0}^{\infty}L(\underline{t}|\theta_1,\theta_2,p)\Pi^*(\theta_1,\theta_2,p)d\theta_1 d\theta_2 dp} \quad (0.31)$$

Hence $g(p,\underline{\alpha}|\underline{t})$can be written as,

$$g(\underline{\alpha},p|\underline{t})=g^*(\theta_1,\theta_2,p|\underline{t})\frac{d\theta_1}{d\alpha_1}\frac{d\theta_2}{d\alpha_2}, \quad (0.32)$$

where$g^*(p,\theta_1,\theta_2|\underline{t})$is the joint posterior density of p,$\underline{\theta}$

Substituting by (0.31) in$p(y|\underline{t})$, it follows that the predictive density is,

$$p(y_k|\underline{t})=\int_p\int_{\theta_1}\int_{\theta_2}f(y_k|\theta_1,\theta_2,p)g^*(\theta_1,\theta_2,p|\underline{t})d\theta_1 d\theta_2 dp \quad (0.33)$$

The invariance property of Equation (0.25)follows in a similar manner.

Hence, by using the one-to-one transformation between$\alpha_i$and$\theta_i$ the results deduced for the general model in the previous sections can be applied to many survival models according to the transformation $\theta_i=g(\alpha_i)$, as will be shown in the upcoming section.

## 3. Applications

3. 1 Two-component Weibull model

The pdf of ith component Weibull is

$$f_i(t)=c_i t^{c_i-1}/\theta_i^{c_i}\,exp(-t^{c_i}/\theta_i^{c_i})i=1,2\theta_i>0,c_i>0,t>0$$

Thus Weibull distribution is one component of general model (0.2)when$\lambda_i(t)=t^{c_i},i=1,2$, $\underline{\alpha}=\underline{\theta^{c_i}}$As a result, the joint prior distribution has the form,

$$\Pi(\underline{\theta},p)\propto\frac{p^{\delta_1-1}q^{\delta_2-1}}{\theta_1^{c_1b_1+1}\theta_2^{c_2b_2+1}}exp-(\frac{a_1}{\theta_1^{c_1}}+\frac{a_2}{\theta_2^{c_2}}),\theta_i>0,a_i>0,b_i>0 \quad (0.34)$$

where the shape parameters c1 and c2 are assumed known.

The predictive distribution can be obtained by replacing $\lambda_i(y_k)$with $y_k^{c_i}$ in(0.25) as follows

$$p(y_k|\underline{t})=\frac{1}{Q'}\sum{*}\sum\zeta(F_1^{**}+F_2^{**}), \quad (0.35)$$

where

$$F_1^{**}=\frac{c_1y_k^{c_1-1}Beta(n-d+s_2+\delta_1+1,d+E+\delta_2)(b_1+r_1)}{(h_1+(s_2+1)y_k^{c_1})^{b_1+r_1+1}(h_2+Ey_k^{c_2})^{b_2+r_2}},$$

$$F_2^{**}=\frac{c_2y_k^{c_2-1}Beta(n-d+s_2+\delta_1,d+E+\delta_2+1)(b_2+r_2)}{(h_1+s_2y_k^{c_1})^{b_1+r_1}(h_2+(E+1)y_k^{c_2})^{b_2+r_2+1}}, \quad (0.36)$$

$$h_1=a_1+\sum_{j=1}^{r_1}t_{1j}^{c_1}+\sum_{i=1}^{n-r}t_i^{c_1}-\sum_{l=1}^{n-r}t_{j_l}^{c_1},h_2=a_2+\sum_{j=1}^{r_2}t_{2j}^{c_2}+\sum_{l=1}^{n-r}t_{j_l}^{c_2}, \quad (0.37)$$

$Q'$is as defined in equation(0.27), $\sum{*}$and $\sum$ are as defined in equation (0.24)

If $c_1=c_2=1$, the Weibull component reduces to the exponential. Using the general model, Exponential component is obtained by applying the substitution$\lambda_i(t)=t_i,i=1,2$, $\underline{\alpha}=\underline{\theta}$

3. 2 Two-component Burr type XII model

The pdf of ith component Burr XII is

$$f_i(t)=\frac{\theta_ic_i(\frac{t}{q_i})^{c_i-1}}{q_i(1+(\frac{t}{q_i})^{c_i})}exp[-\theta_i\,ln(1+(\frac{t}{q_i})^{c_i})],t\geq 0,\theta_i>0,c\quad i\ i \quad (0.38)$$

Thus Burr XII model is one component of general model (0.2)where$\lambda_i(t)=ln(1+(t/q_i)^{c_i}),i=1,2,$ and$\alpha_i=1/\theta_i$. The scale parameter ($q_1$ and $q_2$) and the first shape parameters ($c_1$ and $c_2$) are assumed to be known. Using equation(0.30), it can be shown that$\theta_i$follows the gamma prior distribution, then the joint prior distribution p, $\theta_1$ and $\theta_2$ is

$$\Pi(\underline{\theta},p) \propto p^{\delta_1-1}q^{\delta_2-1} * \theta_1^{b_1-1}\theta_2^{b_2-1}\, exp(-[\theta_1 a_1+\theta_2 a_2]) \qquad (0.39)$$

The predictive distribution can be obtained by putting replacing$\lambda_i(y_k)$ with$ln(1+(\frac{y_k}{q_i})^{c_i})$ in (0.25) as follows,

$$p(y_k|\underline{t}) = \frac{1}{Q'}\sum * \sum \zeta(F_1^{**}+F_2^{**}), \qquad (0.40)$$

where

$$F_1^{**} = \frac{c_1\left(\frac{y_k}{q_1)^{c_1-1}Beta(n-d+s_2+\delta_1+1,d+E+\delta_2)(b_1+r_1)}\right)}{q_1(1+(y_k/q_1)^{c_1})(h_1+(s_2+1)\,ln(1+(y_k/q_1)^{c_1}))^{b_1+r_1+1}(h_2+E\,ln(1+(y_k/q_2)^{c_2}))^{b_2+r_2}}$$

$$F_2^{**} = \frac{c_2(y_k/q_2)^{c_2-1}Beta(n-d+s_2+\delta_1,d+E+\delta_2+1)(b_2+r_2)}{q_2(1+(y_k/q_2)^{c_2})(h_1+s_2\,ln(1+(y_k/q_1)^{c_1}))^{b_1+r_1}(h_2+(E+1)\,ln(1+(y_k/q_2)^{c_2}))^{b_2+r_2+1}},$$

(0.41)

$$h_1 = a_1+\sum_{j=1}^{r_1} ln\left(1+\left(\frac{t_{1j}}{(q_1)^{c_1}}\right)+\sum_{i=1}^{n-r} ln\left(1+\left(\frac{t_i}{(q_1)^{c_1}}\right)-\sum_{l=1}^{n-r} ln\left(1+\left(\frac{t_{j_l}}{(q_1)^{c_1}}\right),\right.\right.\right.$$

$$h_2 = a_2+\sum_{j=1}^{r_2} ln(1+(t_{2j}/q_2)^{c_2})+\sum_{l=1}^{n-r} ln(1+(t_{j_l}/q_2)^{c_2}),$$

(0.42)

$Q'$is as defined in equation(0.27), $\sum *$and $\sum$ are as defined in equation (0.24)

If $c_1$=$c_2$=1, the Burr XII component reduces to the Lomax. Using the general model, Lomax component is obtained by applying the substitution$\lambda_i(t) = ln(1+t/q_i), i = 1,2,\alpha_i = 1/\theta_i$.

As mentioned previously, Al-Hussaini (2003) obtained the predictive density of the future kth ordered observation from a general model in type I censoring. He considered applications to the two-component Weibull and the two-component Burr XII models. It is proved that the results under random censoring reduce to those of type I censoring obtained by Al-Hussaini (2003) when each of the censoring times ti, is equal to a pre-assigned value T (proofs are available under request).

## 4. Numerical examples

In this section, two numerical examples are given to obtain the prediction bounds of first and last order statistics based on the two-component exponential and two-component Lomax when the samples are subject to random censoring.

The following steps for generating a random sample from two-component model under random censoring are common in implemented samples, I- The parameters of the two-component exponential model are generated from the prior distribution. The mixing proportion p is generated from Beta ($\delta_1$, $\delta_2$) distribution. Parameters of components are generated independently with assigned values for the hyper parameters of the assumed prior density.

II- Lifetimes $T_i$, i=1,2,..,n are obtained from the two-component model as follows, generate $U_i$, i=1,..,n from uniform (0,1). If $u_i<p$, where p has been generated in step 1 then an observation is generated from the first component, otherwise the observation is generated from the second component. The two components having same distribution with different parameters. Hence, (n) lifetimes will be obtained in this step, $n_1$ coming from the first component and $n_2$ coming from the second component ($n_1+n_2=n$).

III- Censoring time $C_i$(i=1,2,..,n) is generated from the same distribution (censoring distribution). Each lifetime ti is compared with a censoring time $c_i$. If $t_i \leq c_i$ the item is considered observed otherwise it's considered censored hence r observed items will be obtained in this step r1 coming from the first component and r2 coming from the second component ($r_1+r_2=r$) whereas n-r items will be censored.
For interval prediction$\alpha = 0.05$.

4. 1 Two-component exponential
A sample of size 25 from a two-component exponential model under random censoring is generated. Censoring distribution is exponential ($\theta = (\theta_1 + \theta_2)/2$). Both $\theta_1$and $\theta_2$ are generated from inverse gamma distribution with hyper parameters ($a_1=a_2=10$&$b_1=b_2=40$). The mixing proportion p is generated from Beta(2,4).The parameters of the mixture distribution are, p= 0.6135914, $\theta_1$=0.2292116, and $\theta_2$ =0.2466698. 20 items were observed; 15 from the first component and 5 from the other while 5 observations were censored. Using the generated sample and a future sample of size 25 the prediction interval of the first order observation is (0.0002737145, 0.04072356) and the 25thprediction interval is (0.5271723, 2.010128).

4.2 Two-component Lomax
A sample of size 20 from a two-component Lomax model under random censoring is generated.
Censoring distribution is Lomax($q = (q_1 + q_2)/2$ ,$\theta = 1/(\theta_1 + \theta_2)$). Both $\theta_1$and $\theta_2$ are generated from a gamma distribution with hyper parameters ($a_1=a_2=10$&$b_1=b_2=40$). The parameters of the mixture distribution are, p=0.7794147, the assigned scale parameters ($q_1=0.8$;$q_2=0.9$), and shape parameters are $\theta_1$=4.934642, and $\theta_2$ = 5.741062.15 items were observed; from the first component 12 items and 3 are observed from the second component while 5 are censored. Using the generated sample and a future sample of size 25 the prediction interval of the first order observation is (0.0001980174, 0.0323118) and the 25thprediction interval is (0.4909876, 1.394808).

## 5. Simulation study

For the 1000 simulated samples, the following were computed, the number of censored items on average (AC), The average number of samples where the predicted item lies in the prediction interval (Coverage), The average width (AW) of the prediction interval.

$$AC = \frac{1}{N}\sum n - r_i \, , i = 1{:}\,N$$
$$Aw = \frac{1}{N}\sum (b_i - a_i) \, , i = 1{:}\,N,$$

whereb is upper bound of the prediction interval, a is lower bound of the prediction interval.

To control the average number of items censored (AC) the parameter of censoring distribution is changed. The rate of the exponential distribution has an inverse relation with the magnitude of the generated random variables since mean=1/rate. Thus, if light censoring is desired, the rate of censoring distribution is chosen to be so small so that Ci are greater than Ti and hence n-r is small. If heavy censoring is desired, the rate of that censoring distribution is chosen to so large so that Ci are lower that Ti and hence n-r is large.

Simulation study of predicting first and last observations

1000 random censored samples of size 20 are simulated from the two-component exponential distribution to predict first and last ordered observation from a future sample of size 25. The censoring distribution is assumed to be exponential, the mixing proportion is assumed to come from beta(2,4) and scale parameters are assumed to follow inverse gamma prior (shape(b)=40, scale(a)=10). With 95% level of significance and 1.81 average number of censored items, results were as follows,

| | coverage | Average width of interval |
|---|---|---|
| y1 | 0.952 | 0.03871259 |
| y25 | 0.958 | 1.417279 |

Now, we will explore the effects of changing hyper parameters, sample size, and AC on Interval prediction through analyzing the effect on coverage, AW. The simulation process was repeated changing sample size, censoring distribution parameters, and hyper parameters in order to assess the effect of each on the prediction.

Table1: Effect of changing hyper parameters on coverage, AW for n = 10 and AC ≈2

| b1= b2, a1=a2 | Coverage<br>AW |
|---|---|
| (10,10) | 0.96<br>4.90158 |
| (10,40) | 0.96<br>19.75284 |
| (40,10) | 0.96<br>0.98929 |
| (10,70) | 0.958<br>33.67141 |
| (70,10) | 0.962,<br>0.54693 |

-As the shape parameter (b) increases while fixing the scale parameter (a) the AW of the prediction interval decreases.

-As the scale parameter increases while fixing the shape parameter the width of the prediction interval increases.
-These two results are consistent with the results obtained by Sloan and Sinha (1991) under Type I censoring.
-As the shape increase and the scale decrease the AW decreases which agrees with the first two results.
In all cases, it is observed that the coverage probability is almost stable with respect to variations in the parameters.

Table2: Effect of changing sample size (n) on coverage, AW for AC ≈ 2

| b1= b2, a1=a2 | n=10 | n=20 | n=30 |
|---|---|---|---|
| | Coverage, AW | Coverage, AW | Coverage, AW |
| 10,40 | 0.96, 19.75284 | 0.96, 18.91143 | 0.96, 18.52278 |
| 40,10 | 0.96, 0.9892897 | 0.962 0.9872877 | 0.964,0.9811134 |

As the sample size increases the AW of the prediction interval decreases and the estimated coverage probability is slightly affected.

Table3: Effect of changing AC on coverage, AW for n = 10

| b1= b2, a1=a2 | AC≈2 | AC≈5 | AC≈8 |
|---|---|---|---|
| | Coverage<br>AW | Coverage<br>AW | Coverage<br>AW |
| 10,40 | 0.96<br>19.75284 | 0.954<br>24.22422 | 0.954<br>29.55265 |
| 40,10 | 0.96<br>0.9892897 | 0.958<br>1.03671 | 0.958<br>1.092093 |

The average width of the prediction interval increases sharply as the average number of censored items increases while the estimated coverage probability is slightly affected.

## 6. Conclusions

1- A closed form of the Bayesian predictive density of the two-component mixture of general model was derived.
2- Applications to finite mixtures of two-component Weibull and two-component Burr XII have been obtained. The Bayesian predictive density $p(y_k|\underline{t})$ can be obtained for any other components.
3- Two-component exponentialand two-component Lomax were used as examples to obtain predictive bounds for first and last order statistics. Other order statistics can be obtained the same way using same Bayesian predictive density $p(y_k|\underline{t})$
4-A simulation study has been conducted to assess the effect of sample size, hyper parameters, and level of censoring on prediction interval and point prediction of a future observation coming from the two-component exponential model.
Based on this simulation study the following conclusions are valid
a- The effect of hyper parameter selection
Increasing the shape parameter and decreasing the scale parameter improves the results of both the point and interval prediction as it decreases both the width of the prediction interval and estimated risk of the point prediction.

b- The effect of sample size

Increasing the sample size decreases the average width and increases the coverage probability of the interval prediction while it decreases the estimated risk of the point predictor which is an expected result.

c- The effect of censoring level

As the number of censored items decreases, samples become closer to the complete case which improves both point and interval prediction. The prediction interval becomes narrower, the coverage probability of prediction interval increases and the estimated risk of the point predictor decreases.

## Appendix

A.1 Proof that$\sum exp(\sum_{l=1}^{n} x_{j_l})(k)^{\sum_{l=1}^{n} I_{j_l}} = \prod_{i=1}^{n}(1 + k\, exp(x_i)),$

Using mathematical induction to prove that,

$$\sum_{j_1=0}^{n}\sum_{j_2=0}^{j_1-1}\sum_{j_3=0}^{j_2-1}\ldots\ldots\sum_{j_{n-1}=0}^{j_{n-2}-1}\sum_{j_n=0}^{j_{n-1}-1} exp(\sum_{l=1}^{n} x_{j_l})(k)^{\sum_{l=1}^{n} I_{j_l}}, I_{j_l} = \begin{Bmatrix} 0, j_l = 0 \\ 1, j_l \geq 1 \end{Bmatrix}, (j_l \leq 0 \Rightarrow x_{j_l} = 0) = \prod_{i=1}^{n}(1 + k\, exp(x_i))(A.1)$$

For n=1,

$$L.H.S = \sum_{j_1=0}^{1} e^{x_{j_1}} k^{I_{j_1}} = e^{x_0}k^0 + e^{x_1}k^1 = 1 + ke^{x_1} = R.H.S \quad (A.2)$$

For n=2,

$$L.H.S = \sum_{j_1=0}^{2}\sum_{j_2=0}^{j_1-1} e^{x_{j_1}+x_{j_2}} k^{I_{j_1}+I_{j_2}} = e^{x_0}k^0 + e^{x_1}k^1 + e^{x_1+x_2}k^2 + e^{x_2}k^1$$
$$= 1 + ke^{x_1} + ke^{x_2} + k^2e^{x_1+x_2}$$
$$R.H.S = (1 + ke^{x_1})(1 + ke^{x_2}) = 1 + ke^{x_1} + ke^{x_2} + k^2e^{x_1+x_2}$$

$\therefore L.H.S = R.H.S($ .43)

Let the relation be true for n=s-1, it’s required to prove that it is true for n=s

$$n = s - 1,$$

$$\sum_{j_1=0}^{s-1}\sum_{j_2=0}^{j_1-1}\sum_{j_3=0}^{j_2-1}\ldots\ldots\sum_{j_{s-2}=0}^{j_{s-3}-1}\sum_{j_{s-1}=0}^{j_{s-2}-1} exp(\sum_{l=1}^{s-1} x_{j_l})(k)^{\sum_{l=1}^{s-1} I_{j_l}}, I_{j_l} = \begin{Bmatrix} 0, j_l = 0 \\ 1, j_l \geq 1 \end{Bmatrix}, (j_l \leq 0 \Rightarrow x_{j_l} = 0) = \prod_{i=1}^{s-1}(1 + k\; e^{\;x_i}) \quad (A.4)$$

Before moving to step (4) note that the L.H.S in equation **Error! Reference source not found.** can be expressed in another form that will ease the proof .The L.H.S in equation **Error! Reference source not found.**can be expanded as follows,

$$\sum_{j_1=0}^{n}\sum_{j_2=0}^{j_1-1}\sum_{j_3=0}^{j_2-1}\ldots\ldots\sum_{j_{n-1}=0}^{j_{n-2}-1}\sum_{j_n=0}^{j_{n-1}-1} exp(\sum_{l=1}^{n} x_{j_l})(k)^{\sum_{l=1}^{n} I_{j_l}}, I_{j_l} = \begin{Bmatrix} 0, j_l = 0 \\ 1, j_l \geq 1 \end{Bmatrix}, (j_l \leq 0 \Rightarrow x_{j_l} = 0)$$
$$= 1 + k\, exp(x_1) + k\, exp(x_2) + k^2 exp(x_2 + x_1) + k\, exp(x_3) + k^2 exp(x_3 + x_1) + k^3 exp(x_3 + x_2 + x_1) + k\, exp(x_4) + k^2 exp(x_4 + x_1)$$
$$+k^2 exp(x_4 + x_2) + k^2 exp(x_4 + x_3) + k^3 exp(x_4 + x_3 + x_2) + k^3 exp(x_4 + x_3 + x_1) + k^3 exp(x_4 + x_2 + x_1) + \ldots\ldots\ldots\ldots + k\, exp(x_n)$$

$+k^2\,exp(x_n+x_1)+k^2\,exp(x_n+x_2)+k^2\,exp(x_n+x_3)+\ldots\ldots+k^2\,exp(x_n+x_{n-1})+k^3\,exp(x_n+x_{n-1}+x_{n-2})+\ldots.+k^3\,exp(x_n+x_2+x_1)+k^4\,exp(x_n+x_{n-1}+x_{n-2}+x_{n-3})+\cdots.+k^4\,exp(x_n+x_3+x_2+x_1)+\ldots\ldots\ldots+k^n\,exp(x_n+x_{n-1}+x_{n-2}+\ldots.+x_2+x_1)$( .44)

Rearranging the terms in the above expression gives,

$$\sum_{j_1=0}^{n}\sum_{j_2=0}^{j_1-1}\sum_{j_3=0}^{j_2-1}\ldots\ldots\sum_{j_{n-1}=0}^{j_{n-2}-1}\sum_{j_n=0}^{j_{n-1}-1} exp(\sum_{l=1}^{n} x_{j_l})(k)^{\sum_{l=1}^{n} I_{j_l}}, I_{j_l}=\begin{Bmatrix}0, j_l=0\\1, j_l\geq 1\end{Bmatrix}, (j\leq 0\Rightarrow x_j=0)$$

$$=1+k[exp(x_1)+k\,exp(x_2)+k\,exp(x_3)+k\,exp(x_4)+\ldots.k\,exp(x_n)]+k^2[exp(x_1+x_2)+exp(x_1+x_3)+exp(x_1+x_4)+\ldots+exp(x_1+x_n)+exp(x_2+x_3)+exp(x_2+x_4)+\ldots+exp(x_2+x_n)+exp(x_3+x_4)+\ldots\ldots exp(x_3+x_n)+\ldots.+exp(x_n+x_{n-1})]+k^3[exp(x_1+x_2+x_3)+exp(x_1+x_2+x_4)+\ldots.+exp(x_1+x_2+x_n)+\ldots\ldots+exp(x_1+x_3+x_4)+\ldots.+exp(x_1+x_3+x_{n-1})+\ldots.+(x_n+x_{n-1}+x_{n-2})]+\ldots\ldots\ldots+k^n\,exp(x_n+x_{n-1}+x_{n-2}+\ldots.+x_2+x_1)$$ ( .45)

That can be summarized as follows,

$$\sum_{j_1=0}^{n}\sum_{j_2=0}^{j_1-1}\sum_{j_3=0}^{j_2-1}\ldots\ldots\sum_{j_{n-1}=0}^{j_{n-2}-1}\sum_{j_n=0}^{j_{n-1}-1} exp(\sum_{l=1}^{n} x_{j_l})(k)^{\sum_{l=1}^{n} I_{j_l}}, I_{j_l}=\begin{Bmatrix}0, j_l=0\\1, j_l\geq 1\end{Bmatrix}, (j\leq 0\Rightarrow x_j=0)$$

$$=1+k\sum_{j_1=1}^{n} exp(x_{j_1})+k^2\sum_{j_1=2}^{n}\sum_{j_2=1}^{j_1-1} exp(x_{j_1}+x_{j_2})+k^3\sum_{j_1=3}^{n}\sum_{j_2=2}^{j_1-1}\sum_{j_3=1}^{j_2-1} exp(x_{j_1}+x_{j_2}+x_{j_3})$$

$+..+k^{n-1}\sum_{j_1=n-1}^{n}\sum_{j_2=n-2}^{j_1-1}\sum_{j_3=n-3}^{j_2-1}..\sum_{j_{n-1}=1}^{j_{n-2}-1} exp(\sum_{l=1}^{n-1} x_{j_l})+k^n\sum_{j_1=n}^{n}\sum_{j_2=n-1}^{n-1}\sum_{j_3=n-2}^{n-2}\ldots.\sum_{j_{n-1}=2}^{2}\sum_{j_n=1}^{1} exp(\sum_{l=1}^{n} x_{j_l})$( .46)

4- Given that the relation is true for n=s-1 (step 3) proof that the relation is true for n=s

$$R.H.S=\prod_{i=1}^{s}(1+k\,e^{x_i})=\prod_{i=1}^{s-1}(1+k\,e^{x_i})(1+ke^{x_s})$$

$=(1+ke^{s})[\sum_{j_1=0}^{s-1}\sum_{j_2=0}^{j_1-1}\sum_{j_3=0}^{j_2-1}\ldots\ldots\sum_{j_{s-2}=0}^{j_{s-3}-1}\sum_{j_{s-1}=0}^{j_{s-2}-1} exp(\sum_{l=1}^{s-1} x_{j_l})(k)^{\sum_{l=1}^{s-1} I_{j_l}}], I_{j_l}=\begin{Bmatrix}0, j_l=0\\1, j_l\geq 1\end{Bmatrix}, (j\leq 0\Rightarrow x_j=0)$( .47)

By expanding the term

$(\sum_{j_1=0}^{s-1}\sum_{j_2=0}^{j_1-1}\sum_{j_3=0}^{j_2-1}\ldots\ldots\sum_{j_{s-2}=0}^{j_{s-3}-1}\sum_{j_{s-1}=0}^{j_{s-2}-1} exp(\sum_{l=1}^{s-1} x_{j_l})(k)^{\sum_{l=1}^{s-1} I_{j_l}})$, the R.H.S. is,

$$R.H.S. = (1 + ke^{s})[1 + k\sum_{j_1=1}^{s-1} exp(x_{j_1}) + k^2 \sum_{j_1=2}^{s-1}\sum_{j_2=1}^{j_1-1} exp(x_{j_1} + x_{j_2})$$
$$+ k^3 \sum_{j_1=3}^{s-1}\sum_{j_2=2}^{j_1-1}\sum_{j_3=1}^{j_2-1} exp(x_{j_1} + x_{j_2} + x_{j_3})$$
$+\ldots+k^{s-2}\sum_{j_1=s-2}^{s-1}\sum_{j_2=s-3}^{j_1-1}\sum_{j_3=s-4}^{j_2-1}\ldots.\sum_{j_{s-2}=1}^{j_{s-2}-1} exp(\sum_{l=1}^{s-2} x_{j_l}) +$
$k^{s-1}\sum_{j_1=s-1}^{s-1}\sum_{j_2=s-2}^{s-2}\sum_{j_3=s-3}^{s-3}\ldots.\sum_{j_{s-2}=2}^{2}\sum_{j_{s-1}=1}^{1} exp(\sum_{l=1}^{s-1} x_{j_l})]$( .48)

Then, applying the multiplication the R.H.S is,

$$R.H.S = [1 + k\sum_{j_1=1}^{s-1} exp(x_{j_1}) + k^2 \sum_{j_1=2}^{s-1}\sum_{j_2=1}^{j_1-1} exp(x_{j_1} + x_{j_2})$$
$$+ k^3 \sum_{j_1=3}^{s-1}\sum_{j_2=2}^{j_1-1}\sum_{j_3=1}^{j_2-1} exp(x_{j_1} + x_{j_2} + x_{j_3})$$
$$+\ldots+k^{s-2}\sum_{j_1=s-2}^{s-1}\sum_{j_2=s-3}^{j_1-1}\sum_{j_3=s-4}^{j_2-1}\ldots..\sum_{j_{s-2}=1}^{j_{s-2}-1} exp(\sum_{l=1}^{s-2} x_{j_l}) + k^{s-1} exp(x_1$$
$$+ x_2 + \ldots. + x_{s-1})]$$
$$+$$
$$[ke^{s} + k^2\sum_{j_1=1}^{s-1} exp(x_{j_1} + x_s) + k^3 \sum_{j_1=2}^{s-1}\sum_{j_2=1}^{j_1-1} exp(x_{j_1} + x_{j_2} + x_s) + \ldots\ldots$$
$$+ k^{s-1}\sum_{j_1=s-2}^{s-1}\sum_{j_2=s-3}^{j_1-1}\sum_{j_3=s-4}^{j_2-1}\ldots..\sum_{j_{s-2}=1}^{j_{s-2}-1} exp(\sum_{l=1}^{s-1} x_{j_l} + x_s)$$
$+k^{s} exp(x_1 + x_2 + \ldots. + x_{s-1} + x_s)]$( .49)

Rearranging the terms, the R.H.S can finally be shown as,

$$R.H.S = 1 + k\sum_{j_1=1}^{s} exp(x_{j_1}) + k^2 \sum_{j_1=2}^{s}\sum_{j_2=1}^{j_1-1} exp(x_{j_1} + x_{j_2})$$
$$+ k^3 \sum_{j_1=3}^{s}\sum_{j_2=2}^{j_1-1}\sum_{j_3=1}^{j_2-1} exp(x_{j_1} + x_{j_2} + x_{j_3})$$
$+\ldots\ldots\ldots\ldots.+k^{s-1}\sum_{j_1=s-1}^{s}\sum_{j_2=s-2}^{j_1-1}\sum_{j_3=s-3}^{j_2-1}\ldots.\sum_{j_{s-1}=1}^{j_{s-2}-1} exp(\sum_{l=1}^{s-1} x_{j_l}) +$
$k^{s} exp(x_1 + x_2 + \ldots. + x_{s-1} + x_s)$ ( .50)

From equation ( .46) thus R.H.S.=L.H.S.

Since the relation is true for n=1, n=2, and for n=s, then it's true for any n.

A.2 Computational algorithm of simulation

The statistical package "R" is used to simulate samples exposed to random censoring from each of the mentioned two-component mixture models. Some functions are built in R base and some others need a package inside R to be installed, listed below are the packages installed and use of each of them,

- A package called "pscl" is used to generate random variables from gamma or inverse gamma distributions
- A package called "nleqslv" is used to solve equations numerically for a root.
- A package called "R2Cuba" is used to solve integration numerically. A function called "cuhre" in this package is used. "cuhre" is a method of integration that uses an iterative technique to apply a subdivision strategy where regions are bisected as long as accuracy is not reached yet.

The following steps for generating a random sample from two-component model under random censoring are common in all implemented samples,

I- The parameters of the two-component exponential model are generated from the prior distribution. The mixing proportion p is generated from beta(2,4) distribution. Parameters of components are generated independently with assigned values for the hyper parameters of the assumed prior density.

II- Lifetimes Ti, i=1,2,..,n are obtained from the two-component model as follows, generate Ui, i=1,..,n from uniform (0,1). If ui<p, where p has been generated in step 1 then an observation is generated from the first component, otherwise the observation is generated from the second component. The two components having same distribution with different parameters. Hence, (n) lifetimes will be obtained in this step, n1 coming from the first component and n2 coming from the second component (n1+n2=n).

III- Censoring time Ci(i=1,2,..,n) is generated from the same distribution (censoring distribution). Each lifetime ti is compared with a censoring time ci. If ti≤ci the item is considered observed otherwise it's considered censored hence r observed items will be obtained in this step r1 coming from the first component and r2 coming from the second component (r1+r2=r) whereas n-r items will be censored.

## References


1. Abdel-Aty, Y. (2012). Bayesian Prediction of Future Number of Failures based on Finite Mixture of General Class of Distributions.*Statistics*, 46 (1), 111–122.
2. AbdEllah, A.H. (2003). Bayesian One-Sample Prediction Bounds for the Lomax Distribution. *Indian J. pure appl. Math*, 34(1), 101-109.
3. Abd El-Baset,A.A. and Al-Zaydi, A.M. ( 2015). Bayesian Prediction of Future Generalized Order Statistics from a Class of Finite Mixture Distributions. *Open Journal of statistics*, 5(6), 585-599.
4. Ahmad, K.E, Jaheen, Z.F, and Mohammed, H.S. (2012). Bayesian Prediction based on Type-I Censored data from a Mixture of Burr Type XII Distribution and its Reciprocal.*Statistics*, 46 (6), 787–797.
5. Aitchison, J. and Dunsmore, I.R. (1975). *Statistical Prediction Analysis*. Cambridge University Press, Cambridge.
6. AL-Hussaini, E.K. (1999a). Bayesian prediction under a mixture of two exponential components model based on type I censoring. *Journal of Applied Statistical Science*, 8 (2 and 3), 173-185.
7. AL-Hussaini, E.K. (1999b). Predicting Observables from a General Class of Distributions.*Journal of Statistical Planning and Inference*, 79, 79–91.

8. AL-Hussaini, E.K., Nigm A. M., Jaheen, Z.F. (2001). Bayesian Prediction based on Finite Mixtures of Lomax Components Model and Type I Censoring.*Statistics*, 35, 259 – 268.
9. AL-Hussaini, E.K. (2003). Bayesian Predictive Density of Order Statistics based on Finite Mixture Models.*Journal of Statistical Planning and Inference* 113, 15 – 24.
10. Al-Jarallah, R.A. and AL-Hussaini, E.K. (2007). Bayes Inference under a Finite Mixture of Two-Compound Gompertz Components Model. *Journal of Statistical Computation and Simulation*, 77(11), 915-927.
11. AL-Hussaini, E.K. and Hussein, M. (2011). Bayes Prediction of Future Observables from Exponentiated Populations with Fixed and Random Sample Size. *Open Journal of Statistics*, 1(01), 24-32.
12. Asgharzadeh, A., Valiollahi, R., and Kundu, D. (2013). Prediction for Future Failures in Weibull Distribution under Hybrid Censoring.*Journal of Statistical Computation and Simulation*, 85 (4), 824-838.
13. Ateya, S.F. (2011). Prediction under Generalized Exponential Distribution using MCMC Algorithm.*International Mathematical Forum*, 6 (63), 3111-3119.
14. Balakrishnan, N. and Shafay, A.R. (2012). One-and two-sample Bayesian Prediction Intervals based on Type-II hybrid Censored Data. *Communications in Statistics-Theory and Methods*, 41(9), 1511-1531.
15. Calabria, R. and Pulcini, G. (1994). Bayes 2-sample Prediction for the Inverse Weibull Distribution. *Communications in Statistics-Theory and Methods*, 23(6), 1811-1824.
16. Contreras-Cristán, A. (2007). Using the EM Algorithm for Inference in a Mixture of Distributions with Censored but Partially Identifiable Data.*Computational Statistics & Data Analysis*, 51, 2769 – 2781.
17. Dunsmore, I., R. (1974). The Bayesian Predictive density in Life Testing Models.*Technometrics*,16 (3), 455-460.
18. Evans, I.G and Nigm, A.H.M. (1980a). Bayesian Prediction for the Left Truncated Exponential Distribution.*Technometrics*, 22 (2), 201-204.
19. Evans, I.G. and Nigm, A.M. (1980b). Bayesian Prediction for Two-parameter Weibull LifetimeModels. *Communications in Statistics-Theory and Methods*, 9(6), 649-658.
20. Everitt, B.S and Hand, D.J (1981). *Finite Mixture Distributions*. Chapman and Hall, London.
21. Feroze, N. and Aslam, M. (2013). On Bayesian Estimation and Prediction for Two-Component Mixture of the Gompertz Distribution. *Journal of Modern Applied Statistical Methods*, 12 (2), 269-292.
22. Feroze, N., Aslam, M. and Saleem, A. (2014). Bayesian Estimation and Prediction of Burr Type XI Distribution under Singly and Doubly Censored Samples.*International Journal of Hybrid Information Technology*, 7 (2), 331-346.
23. Feroze, N. and Aslam, M. (2017). On selection of a suitable prior for the posterior analysis of censored mixture of Topp Leone distribution. *Communications in Statistics-Simulation and Computation*, 46(7), 5184-5211.

24. Fernandez, A.J. (2000). Bayesian Inference from Type II Doubly Censored Rayleigh Data. *Statistics & probability letters*, 48(4), 393-399.
25. Ferntindez, A.J. (2004). One- and Two-Sample Prediction Based on Doubly Censored Exponential Data and Prior Information.*Sociedad de Estadistica e InvestigacionOperativa*, 13 (2), 403-416.
26. Haq, A. and Al-Omari, A.I. (2016). Bayes Estimation and Prediction of a Three component Mixture of Rayleigh Distribution under Type-I Censoring.*RevistaInvestigaciónOperacional*, 37 (1), 22-37.
27. Howlader, H.A. and Hossain, A. (1995). On Bayesian Estimation and Prediction from Rayleigh based on Type II Censored Data.*Communications in Statistics-Theory and Methods*, 24(9), 2251-2259.
28. Jaheen, Z. F. (2003). Bayesian Prediction under a Mixture of Two- Component Gompertz Lifetime Model.*Sociedad de Estadistica e lnvestigaci6n Operativa,* 12 (2), 413-426.
29. Jaheen, Z. F. And Al-Matrafi, B. N. (2002). Bayesian Prediction Bounds from the Scaled Burr Type X Model.*Computers and Mathematics with Applications*, 44, 587-594.
30. Jung, J. and Chung, Y. (2013). Bayesian Prediction of Exponentiated Weibull Distribution based on Progressive Type II Censoring.*Communications for Statistical Applications and Methods*, 20 (6), 427–438.
31. Mahmoud, M., Saleh, E. and Helmy, S.M. (2014). Bayesian prediction under a Finite Mixture of Generalized Exponential Lifetime Model. *Pakistan Journal of Statistics and Operation Research*,10(4), 417-433.
32. Mclachlan, G.J. and Peel, D. (2000). *Finite Mixture Models*. Wiley Series, INC., New York, NY.
33. Mohie El-Din, M.M., Abdel-Aty, Y. and Abu-Moussa, M.H. (2017). Statistical inference for the Gompertz distribution based on Type-II progressively hybrid censored data. *Communications in Statistics-Simulation and Computation*, 1-19.
34. Mousa, A.M. and Jaheen, Z. F. (2002). Bayesian Prediction for Progressively Censored data from the Burr Model.*Statistical Papers* 43, 587-593.
35. Nigm, A.M., AL-Hussaini, E.K. and Jaheen, Z.F. (2003). Bayesian one-sample prediction of future observations under Pareto distribution. *Statistics*:*A Journal of Theoretical and AppliedStatistics*, 37(6), 527-536.
36. Nigm, A.M. (1988). Prediction Bounds for the Burr Model. *Communications in Statistics-Theory and Methods*, 17(1), 287-297.
37. Nigm, A.M. (1989). An Informative Bayesian prediction for the Weibull Lifetime Distribution.*Communications in Statistics-Theory and Methods*, 18(3), 897-911.
38. Pal, M. and Chattopadhyay, A.K. (2007). Bayesian Prediction for the Weibull Distribution.*International Journal of Statistical Sciences*,6,127-136.
39. Pradhan, B. and Kundu, D. (2011). Bayes Estimation and Prediction of the Two-parameter Gamma Distribution. *Journal of Statistical Computation and Simulation*, 81(9),1187-1198.
40. Rahman, J. and Aslam, M. (2015). On Bayesian Prediction of Future Quartiles in Mixture Geometric Distribution under Various Censoring Techniques. In *13th International Conference on Statistical Sciences*, 28, 197-204, Peshawar, Pakistan.

41. Raqab, M. and Madi, M. (2002). Bayesian Prediction of the Total Time on test using Doubly Censored Rayleigh Data. *Journal of Statistical Computation and Simulation*, 72(10), 781-789.
42. Sadek, A. (2016). Bayesian Prediction Based on Type-I Hybrid Censored Data from a General Class of Distributions. *American Journal of Theoretical and Applied Statistics*, 5(4), 192-201.
43. Schlattmann, P. (2009). *Medical applications of finite mixture models*. Springer, Germany.
44. Schnatter, S.F. (2006). *Finite Mixture and Markov Switching Models.* Springer, USA.
45. Singh, S.K, Singh, U., and Sharma, V.K. (2013b). Bayesian Prediction of Future Observations from Inverse Weibull Distribution based on Type-II Hybrid Censored Sample.*International Journal of Advanced Statistics and Probability*, 1 (2), 32-43.
46. Singh, S.K., Singh,U., and Sharma, V.K. (2013a). Bayesian Estimation and Prediction for Flexible Weibull Model under Type-II Censoring Scheme.*Journal of Probability and Statistics,* Article ID 146140.
47. Soliman, A.A, AbdEllah, A.H, Abou-Elheggag, N. A., and Modhesh, A.A (2011). Bayesian Inference and Prediction of Burr Type XII Distribution for Progressive First Failure Censored Sampling. *Intelligent Information Management*, 3 (5), 175-185.
48. Titterington, D.M, Smith, A.F.M, and Markov, U.E (1985). *Statistical Analysis of Finite Mixture Distributions.* Wiley Series, INC., New York, NY.
49. Tziafetas, G.N. (1987). On the Construction of Bayesian Prediction Limits for the Weibull Distribution. *Statistics: A Journal of Theoretical and Applied Statistics*, 18(4), 623-628.
50. Wang, L.(2008). Life Prediction under Random Censorship.*Journal of Statistical Computation and Simulation*, 78(11), 1033-1044.
51. Wang, L. and Veraverbeke, N. (2009). Bayes Prediction based on Right Censored data.*Stat Papers*, 50, 137–149.
52. Wu, S.J., Chen, D.H. and Chen, S.T. (2006). Bayesian Inference for Rayleigh Distribution under Progressive Censored Sample. *Applied Stochastic Models in Business and Industry*, 22(3), 269-279.
53. Yanling, L., Xuanmin, Z., Wenxian, X. (2005). Bayesian Prediction for The Two-Parameter Exponential Distribution based on Type I Doubly Censoring.*Appl. Math. J. Chinese Univ. Ser. B*, 20 (1), 75-84.