Pakistan Journal of Statistics and Operation Research

Bayesian Estimation and Prediction Based on Progressively First Failure Censored Scheme from a Mixture of Weibull and Lomax Distributions

Mohamed Mahmoud[1*], Manal M.Nassar[2], Marwah A.Aefa[3]

* Corresponding Author

1. Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt, mmahmoud46@hotmail.com
2. Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt, manal.m.nassar@gmail.com
3. Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt, marwaaifa87@gmail.com

Abstract

This paper develops Bayesian estimation and prediction, for a mixture of Weibull and Lomax distributions, in the context of the new life test plan called progressive first failure censored samples. Maximum likelihood estimation and Bayes estimation, under informative and non-informative priors, are obtained using Markov Chain Monte Carlo methods, based on the symmetric square error Loss function and the asymmetric linear exponential (LINEX) and general entropy loss functions. The maximum likelihood estimates and the different Bayes estimates are compared via a Monte Carlo simulation study. Finally, Bayesian prediction intervals for future observations are obtained using a numerical example.

Key Words: Mixture model; Progressive First Failure Censored Scheme; Loss Function; Maximum Likelihood Estimation; Bayesian Estimation and Prediction; Markov Chain Monte Carlo.

1. Introduction

Mixtures models have received great attention from analysts in the recent years due to their important role in life testing and reliability. In many applications, mixture models are used to analyze random duration in possibly heterogeneous populations, statistical analysis and machine learning such as modeling, classification, and survival analysis. Attention has been paid by some authors to the finite mixtures to discuss lifetime distributions, [see Everitt and Hand (1981),Titterington et al. (1985), Mclachlan and Basford (1988), Lindsay (1995), Mclachlan and Peel (2000)]. Also, mixture distributions have been considered extensively by several researchers using both classical and Bayesian techniques, [for example Shawky and Bakoban (2009),Abu-Zinadah (2010), Erisoglu et al. (2011), Feroze and Aslam (2014), Daniyal and Rajab (2015), Elshahat and Mahmoud (2016), Mahmoud et al. (2017)].

The Weibull distribution has been widely used in modeling of lifetime event data; this is due to the variety of shapes of the probability density function (pdf) based on its parameters. The Weibull distribution has been shown to be useful for modeling and analyzing lifetime data in the applied engineering sciences [see Murthy et al. (2003)]. The Lomax distribution, sometimes called Pareto of the second kind, has a considerable importance in the field of life testing because of its uses to fit business failure data [see Lomax (1954)].
A random variable X is said to have a mixture of two component Weibull and Lomax distribution if its probability density function (pdf) is given by

$$f(x) = \sum_{j=1}^{2} p_j f_j(x), \qquad j = 1,2. \tag{1}$$

$$f_1(x) = \alpha_1\theta_1 x^{\theta_1 - 1} e^{-\alpha_1 x^{\theta_1}} \qquad f_2(x) = \alpha_2\theta_2(1+\theta_2 x)^{-(\alpha_2+1)}$$

where $x > 0, (\alpha_j > 0, \theta_j > 0), j = 1, 2.$ The mixing proportions p_j are such that $0 \le p_j \le 1$, $\sum_{j=1}^{2} p_j = 1$.

The corresponding cumulative distribution function (cdf) and reliability function, respectively are given by

$$F(x) = \sum_{j=1}^{2} p_j F_j(x), \qquad j = 1,2. \tag{2}$$

$$F_1(x) = 1 - e^{-\alpha_1 x^{\theta_1}} \qquad F_2(x) = 1 - (1+\theta_2 x)^{-\alpha_2}$$

and $R(x) = \sum_{j=1}^{2} p_j R_j(x), \qquad j = 1,2.$ (3)

$$R_1(x) = e^{-\alpha_1 x^{\theta_1}} \qquad R_2(x) = (1+\theta_2 x)^{-\alpha_2}$$

Common censoring schemes of type I and type II censoring do not allow units to be removed from the test at any other point than the final termination point. Therefore, the focus in the last few years has been on progressive censoring due to its flexibility that allows the experimenter to remove active units during the experiment. Progressive censoring has been studied by many authors. Some of the early work on progressive censoring are Cohen (1963), Mann (1971), Viveros and Balakrishnan(1994), Balasooriya et al. (2000), Balakrishnan and Aggarwala (2000) and Balakrishnan (2007) have presented an elaborate overview of various developments in progressive censoring data.

The experimental time is usually an important concern for the life test designers. Although conventional censoring scheme can shorten the duration of a life test, the experimental time is still too long that cannot be waited for when the units are highly reliable. Johnson (1964) introduced the first failure censoring by grouping the test units into several sets. The experimenter runs all test units simultaneously until the first failure in one of the sets. Some references that discussed this type of first failure are Balasooriya (1995), Wu et al. (2003) and Wu and Yu (2005).

Wu and Kus (2009) described new life test scheme, progressively first failure censoring scheme by combining the concept of first failure censoring with the progressive censoring. Many authors have discussed inference for different distributions based on this scheme introduced by Huang and Wu (2011), Soliman et al. (2012), Abou-Elheggag (2013), Javadkhani et al. (2014), kim and Han (2015), Dube et al. (2016) and Rashad et al. (2017).

The prediction problems of the future samples based on censored data is an important topic in statistics. The prediction techniques are used in medicine, engineering, business and other areas as well. Several authors studied Bayesian prediction for future observation; [see Al-Hussaini et al. (2001), Jaheen (2003)].

The objective of this work is to apply the Bayesian procedure to estimate the parameters and obtain two sample prediction bounds for future observations from the proposed model, based on progressive first failure censoring scheme. The rest of this paper is organized as follows: In Section 2, the progressive first-failure censoring scheme is described. In Section 3, we obtain maximum likelihood estimators of the parameters. Different loss functions are presented in Section 4. The Bayesian estimation is discussed in Section 5, using Markov Chain Monte Carlo technique. In Section 6, Monte Carlo simulation study is conducted to compare the performance of different estimation methods. Bayesian prediction with numerical data are presented in Section 7. Finally, we conclude the paper in Section 8.

2. A Progressive First-Failure Censoring Scheme

In a life- testing experiment, suppose sample of n independent groups with k items in each group are put in a life test. After units have failed, each item can be attributed to the appropriate subpopulation. Thus if the m are failed during the interval $(0, x_{(m)})$: r_1 from the first subpopulation and r_2 from the second subpopulation. When the first failure is observed, R_1 groups and the group with observed failure are randomly removed. At the second observed failure, R_2 groups and the group with observed failure are randomly removed. This experiment terminates at the time when the m^{th} failure is observed and the remaining R_m groups and the group with observed failure are all removed. Here $x_{1,m,n,k}^R < x_{2,m,n,k}^R < \cdots < x_{m,m,n,k}^R$, are known as progressive first failure censored orders statistics with the progressive censoring scheme $R = (R_1, R_2, \dots, R_m)$; $(m \le n)$, Let x_{ij} denote the failure of the j^{th} unit that belongs to the i^{th} subpopulation and $x_{ij} \le x_{(m)}$; $j = 1,2, \dots, m_i$; $m = r_1 + r_2$. where $x_{(m)}$ denotes the failure time of the m^{th} unit. For a two component mixture model, the likelihood function, is defined as

$$L(x_1, x_2, \dots, x_m) = Ck^m \left[\prod_{j=1}^{r_1} p_1 f_1(x_{1j}) \prod_{j=1}^{r_2} p_2 f_2(x_{2j})\right] \prod_{j=1}^{m} [1 - F(x_j)]^{(k(R_j+1)-1)}$$

$$0 < x_{1,m,n,k}^R < x_{2,m,n,k}^R < \cdots < x_{m,m,n,k}^R < \infty \tag{4}$$

where $C = n(n - R_1 - 1)(n - R_1 - R_2 - 2) \dots (n - \sum_{j=1}^{m-1} R_j - m + 1)$, $p_1 = p$, $p_2 = 1 - p$. Note that if $k = 1$ then, sampling scheme reduces to the progressively type II censoring, a first failure censored scheme when $R = (0, \dots, 0)$, a usual type II censored scheme when $k = 1$ and $R = (0, \dots, 0)$, and complete sample case if $k = 1$ and $R = (0, \dots, 0)$, with $n = m$. It should be noted that $x_{1,m,n,k}^R$, $x_{2,m,n,k}^R, \dots, x_{m,m,n,k}^R$ can be viewed as a progressive type II censored sample from a population with distribution function $1 - (1 - F(x))^k$. For this reason, results extended to progressive first- failure censored scheme easily. The progressive first-failure censored plan has advantages in terms of reducing the test time, in which more items are used but only m of $n \times k$ items are failures.

3. Maximum Likelihood Estimation (MLE)

Let $x_{i,m,n,k}^R$, $i = 1,2, \dots, m$, with $x_{1,m,n,k}^R \le x_{2,m,n,k}^R, \dots \le x_{m,m,n,k}^R$ denote the progressively first failure censored from the mixture of Weibull and Lomax distributions, with censored scheme $R = (R_1, R_2, \dots, R_m)$. Substituting (1) and (2) into (4), the likelihood function based on a progressive first failure censored sample is given by

$$\begin{aligned} L(p, \alpha_1, \alpha_2, \theta_1, \theta_2 | \underline{x}) \\ &= Ck^m \left[\prod_{j=1}^{r_1} p_1 \alpha_1 \theta_1 x_{1j}^{\theta_1 - 1} e^{-\alpha_1 x_{1j}^{\theta_1}} \prod_{j=1}^{r_2} p_2 \alpha_2 \theta_2 (1 + \theta_2 x_{2j})^{-(\alpha_2+1)}\right] \\ &\times \prod_{j=1}^{m} [p_1 R_1(x_j) + p_2 R_2(x_j)]^{(k(R_j+1)-1)} \end{aligned} \tag{5}$$

Assuming that the parameters θ_1 and θ_2 are known, the likelihood function (5) reduces to

$$\begin{aligned} L(p, \alpha_1, \alpha_2 | \underline{x}) \propto Ck^m \prod_{i=1}^{2} (p_i \alpha_i)^{r_i} \prod_{j=1}^{r_1} x_{1j}^{\theta_1 - 1} \times e^{-\alpha_1 \sum_{j=1}^{r_1} x_{1j}^{\theta_1}} \prod_{j=1}^{r_2} (1 + \theta_2 x_{2j})^{-(\alpha_2+1)} \\ \times \prod_{j=1}^{m} [p_1 R_1(x_j) + p_2 R_2(x_j)]^{(k(R_j+1)-1)} \end{aligned} \tag{6}$$

Thus, the log-likelihood function of the parameters α_1, α_2 and p are given by

$$\ln L\left(p,\alpha_1,\alpha_2|\underline{x}\right) \propto \ln C + m\ln k + \sum_{i=1}^{2}\{r_i \ln p_i + r_i \ln \alpha_i\} - \alpha_1 \sum_{j=1}^{r_1} x_{1j}^{\theta_1} - (\alpha_2+1)\sum_{j=1}^{r_2}\ln\left(1+\theta_2 x_{2j}\right)$$
$$+\sum_{j=1}^{m}\left(k\left(R_j+1\right)-1\right)\ln\left[p_1 e^{-\alpha_1 x_j^{\theta_1}} + p_2\left(1+\theta_2 x_j\right)^{-\alpha_2}\right] \quad (7)$$

Partially differentiating equation (7) with respect to α_1, α_2 and p maximum likelihood estimates (MLEs) can obtained by solving the equations after equating it to zero

$$\frac{\partial \ln L(p,\alpha_1,\alpha_2)}{\partial \alpha_1} = \frac{r_1}{\alpha_1} - \sum_{j=1}^{r_1} x_{1j}^{\theta_1} - \sum_{j=1}^{m} \frac{\left(k\left(R_j+1\right)-1\right) p_1 x_j^{\theta_1} e^{-\alpha_1 x_j^{\theta_1}}}{p_1 e^{-\alpha_1 x_j^{\theta_1}} + p_2\left(1+\theta_2 x_j\right)^{-\alpha_2}}$$

$$\frac{\partial \ln L(p,\alpha_1,\alpha_2)}{\partial \alpha_2} = \frac{r_2}{\alpha_2} - \sum_{j=1}^{r_2} \ln\left(1+\theta_2 x_{2j}\right) - \sum_{j=1}^{m} \frac{\left(k\left(R_j+1\right)-1\right) p_2 \left(1+\theta_2 x_j\right)^{-\alpha_2} \ln\left(1+\theta_2 x_j\right)}{p_1 e^{-\alpha_1 x_j^{\theta_1}} + p_2\left(1+\theta_2 x_j\right)^{-\alpha_2}}$$

$$\frac{\partial \ln L(p,\alpha_1,\alpha_2)}{\partial p} = \frac{r_1}{p_1} - \frac{r_2}{p_2} + \sum_{j=1}^{m} \frac{\left(k\left(R_j+1\right)-1\right)\left\{e^{-\alpha_1 x_j^{\theta_1}} - \left(1+\theta_2 x_j\right)^{-\alpha_2}\right\}}{p_1 e^{-\alpha_1 x_j^{\theta_1}} + p_2\left(1+\theta_2 x_j\right)^{-\alpha_2}}$$

4. Loss Function

In decision theory, the loss criterion is specified in order to obtain the best estimator. Three loss functions are proposed, symmetric (square error) loss function and asymmetric (LINEX and general entropy) loss functions, as follows:

- Squared error loss function: A simple, and very common loss function is defined by

  $L_1(\hat{\theta},\theta) = c\left(\hat{\theta}-\theta\right)^2$; c is a constant

  which is symmetrical in nature and gives equal weight to overestimation as well as under estimation. However, in real applications, estimation of reliability and failure rate function, an overestimate is more serious than the underestimates. The use of asymmetric loss function might be inappropriate as has been recognized by Basu and Ebrahimi (1991).

- Linear exponential loss function (LINEX): One of the most commonly used asymmetric loss functions, introduced by Varian (1975) under the assumption that the minimal loss occurs at $\hat{\theta}=\theta$, it can be expressed as

  $L_2(\Delta) \propto e^{-q\Delta} - q\Delta - 1, \quad \Delta = \hat{\theta} - \theta, \quad q \neq 0$

  where q determines the shape of the loss function. If $q > 0$ means overestimation and underestimation if $q < 0$, but in a situation where $q \cong 0$, the LINEX loss is almost symmetric and approaches square error loss function.

  Under the above loss function, the Bayes estimator $\hat{\theta}_{LINEX}$ of θ can be obtained as

  $$\hat{\theta}_{LINEX} = -\frac{1}{q}\ln\left[E\left(e^{-q\theta}|\underline{x}\right)\right]$$

  provided that the expected value with respect to the posterior function of θ, $E\left(e^{-q\theta}|\underline{x}\right)$ exists and is finite.

- General entropy loss function: Another commonly asymmetric loss function is the modified LINEX loss function called a general entropy loss function proposed by Calabria and Pulcini (1996)

$$L_3(\hat{\theta},\theta) \propto \left(\frac{\hat{\theta}}{\theta}\right)^h - h\ln\left(\frac{\hat{\theta}}{\theta}\right) - 1, \qquad h \neq 0$$

which has a minimum at $\hat{\theta}=\theta$. Also, the loss function used by several authors, in the original form having the shape parameter $h=1$, for $h>0$, a positive error has a more effect than a negative error. In this case, the Bayes estimate of θ is given by

$$\hat{\theta}_{GE} = \left[E\left(\theta^{-h}|\underline{x}\right)\right]^{-\frac{1}{h}}$$

provided that the expected value with respect to the posterior function of θ, $E\left(\theta^{-h}|\underline{x}\right)$ exists and is finite.

5. Bayesian Estimation

In this section, we derive Bayes estimators of the parameters α_1, α_2 and p of the considered model based first failure progressively censored sample. Assuming the following independent prior distributions for the parameters $\alpha_1 \sim Gamma(a_1,b_1)$, $\alpha_2 \sim Gamma(a_2,b_2)$, and $p \sim Beta(c,d)$ for the mixing parameter p. The joint prior distribution of α_1, α_2 and p is

$$\pi(\alpha_1,\alpha_2,p) = \pi_1(\alpha_1)\pi_2(\alpha_2)\pi_3(p)$$

where

$$\left.\begin{array}{c} \pi(\alpha_i) \propto \alpha_i^{a_i-1} e^{-b_i\alpha_i};\ \ \alpha_i>0, a_i, b_i>0;\ \ i=1,2 \\ and \qquad \pi_3(p) \propto p_1^{c-1} p_2^{d-1} \end{array}\right\} \tag{8}$$

The joint density function of α_1, α_2, p and the sample $\underline{x}$ can be written as follows

$$P(p,\alpha_1,\alpha_2,\underline{x}) \propto \left(\prod_{i=1}^{2} p_i^{r_i}\alpha_i^{r_i} \times e^{-\alpha_1\sum_{j=1}^{r_1} x_{1j}^{\theta_1}} \prod_{j=1}^{r_2}\left(1+\theta_2 x_{2j}\right)^{-(\alpha_2+1)} \prod_{j=1}^{m}\left[p_1R_1(x_j)+p_2R_2(x_j)\right]^{(k(R_j+1)-1)}\right) \times \left(p_1^{c-1}p_2^{d-1}\prod_{i=1}^{2}\alpha_i^{a_i-1}e^{-b_i\alpha_i}\right)$$

$$P(p,\alpha_1,\alpha_2,\underline{x}) \propto p_1^{r_1+c-1}p_2^{r_2+d-1}\prod_{i=1}^{2}\alpha_i^{r_i+a_i-1}e^{-\alpha_i\varphi_i}\prod_{j=1}^{m}\left[p_1R_1(x_j)+p_2R_2(x_j)\right]^{(k(R_j+1)-1)} \tag{9}$$

where $\varphi_1 = b_1 + \sum_{j=1}^{r_1} x_{1j}^{\theta_1}$ *and* $\varphi_2 = b_2 + \sum_{j=1}^{r_2}\ln\left(1+\theta_2 x_{2j}\right)$

Based on Equation (9) , the joint posterior density function of α_1, α_2 and p, is given by

$$P(p,\alpha_1,\alpha_2|\underline{x}) = \frac{P(p,\alpha_1,\alpha_2,\underline{x})}{\int_o^1\int_0^\infty\int_0^\infty P(p,\alpha_1,\alpha_2,\underline{x})\,d\alpha_1 d\alpha_2 dp} \tag{10}$$

Thus, under the squared error, LINEX and general entropy loss functions, the Bayes estimators of any function of (p, α_1, α_2), say $\phi(p, \alpha_1, \alpha_2)$, is

$$\hat{\phi}_{SE}(p, \alpha_1, \alpha_2) = \frac{\int_0^1 \int_0^\infty \int_0^\infty \phi(p, \alpha_1, \alpha_2) P(p, \alpha_1, \alpha_2, \underline{x}) d\alpha_1 d\alpha_2 dp}{\int_0^1 \int_0^\infty \int_0^\infty P(p, \alpha_1, \alpha_2, \underline{x})\, d\alpha_1 d\alpha_2 dp} \tag{11}$$

$$\hat{\phi}_{LINEX}(p, \alpha_1, \alpha_2) = -\frac{1}{q} \ln\left[\frac{\int_0^1 \int_0^\infty \int_0^\infty e^{-q\phi(p,\alpha_1,\alpha_2)} P(p, \alpha_1, \alpha_2, \underline{x}) d\alpha_1 d\alpha_2 dp}{\int_0^1 \int_0^\infty \int_0^\infty P(p, \alpha_1, \alpha_2, \underline{x})\, d\alpha_1 d\alpha_2 dp}\right] \tag{12}$$

$$\hat{\phi}_{GE}(p, \alpha_1, \alpha_2) = \left[\frac{\int_0^1 \int_0^\infty \int_0^\infty \phi(p, \alpha_1, \alpha_2)^{-h} P(p, \alpha_1, \alpha_2, \underline{x}) d\alpha_1 d\alpha_2 dp}{\int_0^1 \int_0^\infty \int_0^\infty P(p, \alpha_1, \alpha_2, \underline{x})\, d\alpha_1 d\alpha_2 dp}\right]^{-\frac{1}{h}} \tag{13}$$

The ratio of the integrals in Equations (11), (12) and (13) cannot be obtained in a closed form. Therefore, the Markov Chain Monte Carlo technique will be used to approximate the integrals.

5.1. Markov Chain Monte Carlo method (MCMC)

In this subsection, we apply the importance sampling technique to obtain the approximate Bayes estimates. This technique needs no calculation of the normalizing constant. The joint posterior density function of α_1, α_2 and p given data $\underline{x}$ can be written in the form

$$P(p, \alpha_1, \alpha_2 | \underline{x}) \propto f_{Beta}(p; r_1 + c, r_2 + d) f_{GA}(\alpha_1; r_1 + a_1, \varphi_1) f_{GA}(\alpha_2; r_2 + a_2, \varphi_2) h(\alpha_1, \alpha_2, p) \tag{14}$$

where φ_1, φ_2 are defined as above, and

$$h(\alpha_1, \alpha_2, p) = \prod_{j=1}^{m} \left[p_1 R_1(x_j) + p_2 R_2(x_j)\right]^{(k(R_j+1)-1)} \tag{15}$$

According to the importance sampling technique, the approximate Bayes estimators, based on the three loss functions, can be computed by the following algorithm

Step1: $p \sim Beta(r_1 + c, r_2 + d)$

$\alpha_1 \sim Gamma(r_1 + a_1, \varphi_1)$

$\alpha_2 \sim Gamma(r_2 + a_2, \varphi_2)$

Step2: Repeat this procedure to obtain important sample $\{(p_i, \alpha_{1i}, \alpha_{2i}), i = 1, \ldots, N\}$

Step 3: Calculate Bayes estimator of $\phi(p, \alpha_1, \alpha_2)$, under squared error, LINEX and general entropy loss functions, respectively by

$$\hat{\phi}_{SE}(p, \alpha_1, \alpha_2) = \frac{\sum_{i=1}^{N} \phi(p, \alpha_1, \alpha_2) h(\alpha_{1i}, \alpha_{2i}, p_i)}{\sum_{i=1}^{N} h(\alpha_{1i}, \alpha_{2i}, p_i)}$$

$$\hat{\phi}_{LINEX}(p, \alpha_1, \alpha_2) = -\frac{1}{q} \ln\left[\frac{\sum_{i=1}^{N} e^{-q\phi(p,\alpha_1,\alpha_2)} h(\alpha_{1i}, \alpha_{2i}, p_i)}{\sum_{i=1}^{N} h(\alpha_{1i}, \alpha_{2i}, p_i)}\right]$$

$$\hat{\phi}_{GE}(p,\alpha_1,\alpha_2)=\left[\frac{\sum_{i=1}^{N}\phi(p,\alpha_1,\alpha_2)^{-h}h(\alpha_{1i},\alpha_{2i},p_i)}{\sum_{i=1}^{N}h(\alpha_{1i},\alpha_{2i},p_i)}\right]^{-\frac{1}{h}}$$

6. Comparison Study

In this section, we present the numerical results of a simulation study to compare the performance of the various estimates for different combinations of (n,m,k). We consider different sampling schemes $R=(R_1,\dots,R_m)$ as follow

Table 1: Censoring schemes of progressively censored

(k,n,m)	Scheme	R_i
(2,50,30)	I	$(20,29^0)$
	II	$(29^0,20)$
	III	$(10,28^0,10)$
(2,50,40)	I	$(10,39^0)$
	II	$(39^0,10)$
	III	$(5,38^0,5)$
(4,75,45)	I	$(30,44^0)$
	II	$(44^0,30)$
	III	$(15,43^0,15)$
(4,75,55)	I	$(20,54^0)$
	II	$(54^0,20)$
	III	$(10,53^0,10)$

To carry out this comparative study, we follow the following steps:

- In this study the following parameters values were used $(\alpha_1,\alpha_2,p)=(1,2,0.40)$ along with $(\theta_1,\theta_2)=(0.5,1.5)$. The values chosen for the constants q are $(0.5,-0.5)$ for the LINEX and for h are $(1,-1)$ for general entropy loss functions. The following values are used for the hyper parameters $(a_1=3,a_2=2,b_1=2,b_2=1,c=2\ and\ d=4)$ for informative prior. In case of non-informative prior, we take $\{(a_1=a_2=b_1=b_2=0),(c=d=1)\}$.
- Appling the algorithms of Balakrishnan and Sandhu (1995) to generate a progressive first-failure censored sample from mixture Weibull and Lomax distributions for different values of (n,m,k,R).
- Based on the progressive first-failure censored data, maximum likelihood estimates and Bayes estimates of the parameters are calculated according to Section 3 and Section 5, respectively.
- The above steps are repeated 1000 times, and computation off the average of the estimates and mean square error for different values (n,m,k,R) are presented in Tables (2-8). The computations are done using Mathematica 10.0.

Table (2-8) indicates that the Bayes estimates perform better under informative prior than non-informative prior for all different loss functions. In most cases, notice that the performance of the Bayes estimates under informative prior are smaller than the maximum likelihood estimates in terms of MSE. Also, the Bayesian estimates under general entropy loss function in case of the value $(h=-1)$ are almost the same as the estimates under squared error loss function. The mean square error of all estimates decreases when the sample size n and effective sample size m increase. Also, when the value of the group size k increases, the mean square error decreases.

Table 2: Average values and the corresponding MSE based MLEs

(k,n,m)	SC	Maximum Likelihood Estimation		
		α_1	α_2	p
(2,50,30)	I	1.03676 (0.02118)	2.20610 (0.19323)	0.40994 (0.00065)
	II	0.74133 (0.11755)	1.70425 (0.26606)	0.48890 (0.01020)
	III	0.75646 (0.11778)	1.73760 (0.27541)	0.47448 (0.00804)
(2,50,40)	I	1.04959 (0.01516)	2.25190 (0.16074)	0.40806 (0.00037)
	II	0.76587 (0.11302)	1.73385 (0.27612)	0.46558 (0.00654)
	III	0.86449 (0.07537)	1.90287 (0.22749)	0.44234 (0.00369)
(4,75,45)	I	0.65021 (0.15266)	1.43521 (0.41863)	0.41540 (0.00031)
	II	0.81885 (0.07292)	1.76575 (0.16383)	0.41292 (0.00043)
	III	0.78610 (0.08594	1.70020 (0.20451)	0.41297 (0.00036)
(4,75,55)	I	0.64870 (0.14971)	1.42957 (0.41584)	0.41550 (0.00029)
	II	0.77324 (0.08521)	1.67761 (0.19722)	0.41426 (0.00037)
	III	0.73617 (0.10280)	1.60063 (0.25517)	0.41424 (0.00034)

Table 3: Average values and corresponding MSE of parameter α_1 based informative prior

(k,n,m)	SC	Square Error	LINEX		General Entropy	
			$q=0.5$	$q=-0.5$	$h=1$	$h=-1$
(2,50,30)	I	1.11252 (0.09090)	1.09040 (0.07975)	1.13601 (0.10455)	1.03598 (0.06846)	1.11252 (0.09090)
	II	1.15429 (0.11326)	1.13055 (0.09904)	1.17962 (0.13049)	1.07533 (0.08309)	1.15429 (0.11326)
	III	1.12124 (0.10132)	1.09866 (0.08865)	1.14529 (0.11683)	1.04441 (0.07510)	1.12124 (0.10132)
(2,50,40)	I	1.10453 (0.07751)	1.08770 (0.06990)	1.12217 (0.08644)	1.04578 (0.06132)	1.10453 (0.07751)
	II	1.12562 (0.08511)	1.10799 (0.07638)	1.14411 (0.09528)	1.06522 (0.06573)	1.12562 (0.08511)
	III	1.10145 (0.08140)	1.08427 (0.07307)	1.11947 (0.09126)	1.04173 (0.06437)	1.10145 (0.08140)
(4,75,45)	I	1.07658 (0.06134)	1.06246 (0.05645)	1.09126 (0.06610)	1.02571 (0.05110)	1.07658 (0.06134)
	II	1.09767 (0.06848)	1.08266 (0.06243)	1.1133 (0.07549)	1.04457 (0.05519)	1.09767 (0.06848)
	III	1.07333 (0.06380)	1.05884 (0.05857)	1.0884 (0.06989)	1.02105 (0.05318)	1.07333 (0.06380)
(4,75,55)	I	1.06722 (0.05298)	1.0553 (0.04918)	1.06695 (0.05463)	1.02382 (0.04493)	1.06722 (0.05298)
	II	1.08769 (0.06191)	1.0754 (0.05740)	1.10038 (0.06700)	1.04378 (0.05185)	1.08769 (0.06191)
	III	1.0921 (0.05829)	1.07964 (0.05374)	1.10498 (0.06344)	1.04769 (0.04796)	1.0921 (0.05829)

Table 4: Average values and corresponding MSE of parameter α_2 based informative prior

(k,n,m)	SC	Square Error	LINEX		General Entropy	
			$q=0.5$	$q=-0.5$	$h=1$	$h=-1$
(2,50,30)	I	2.03588 (0.19947)	1.98295 (0.17784)	2.09315 (0.23221)	1.93307 (0.18304)	2.03588 (0.19947)
	II	2.11918 (0.23750)	2.06157 (0.20304)	2.18172 (0.28599)	2.01164 (0.20143)	2.11918 (0.23750)
	III	2.05643 (019529)	2.00280 (0.17155)	2.11441 (0.23068)	1.95296 (0.17461)	2.05643 (0.19529)
(2,50,40)	I	2.04134 (0.16174)	2.00055 (0.14688)	2.08452 (0.18227)	1.96233 (0.14898)	2.04134 (0.16174)
	II	2.07451 (0.15337)	2.03293 (0.13690)	2.11866 (0.17599)	1.99493 (0.13651)	2.07451 (0.15337)
	III	2.06282 (0.15368)	2.02129 (0.13824)	2.10691 (0.17497)	1.98290 (0.13874)	2.06282 (0.15368)
(4,75,45)	I	1.98372 (0.13120)	1.94923 (0.12471)	2.0201 (0.14132)	1.91489 (0.12977)	1.98372 (0.13120)
	II	2.05776 (0.14037)	2.02084 (0.12754)	2.09658 (0.15757)	1.98651 (0.12785)	2.05776 (0.14041)
	III	2.03646 (0.13702)	2.00075 (0.12757)	2.07408 (0.14790)	1.96694 (0.12822)	2.03648 (0.13702)
(4,75,55)	I	2.01132 (0.10798)	1.98235 (0.10203)	2.04153 (0.11639)	1.95408 (0.10426)	2.01132 (0.10798)
	II	2.07333 (0.13141)	2.04255 (0.12020)	2.10543 (0.14565)	2.01449 (0.11908)	2.07333 (0.13141)
	III	2.0087 (0.11706)	1.97964 (0.11031)	2.03899 (0.12639)	1.95149 (0.11251)	2.0087 (0.11706)

Table 5: Average values and corresponding MSE of parameter p based informative prior

(k,n,m)	SC	Square Error	LINEX		General Entropy	
			$q=0.5$	$q=-0.5$	$h=1$	$h=-1$
(2,50,30)	I	0.38863 (0.00553)	0.38708 (0.00555)	0.39018 (0.00551)	0.37137 (0.00655)	0.38863 (0.00553)
	II	0.39062 (0.00610)	0.38908 (0.00611)	0.39218 (0.00609)	0.37344 (0.00707)	0.39062 (0.00610)
	III	0.38925 (0.00606)	0.38771 (0.00607)	0.39080 (0.00604)	0.37208 (0.00605)	0.38925 (0.00606)
(2,50,40)	I	0.39290 (0.00438)	0.39167 (0.00439)	0.39413 (0.00437)	0.37961 (0.00494)	0.39290 (0.00438)
	II	0.39150 (0.00473)	0.39027 (0.00474)	0.39279 (0.00472)	0.37804 (0.00536)	0.39150 (0.00473)
	III	0.38871 (0.00481)	0.38749 (0.00483)	0.38993 (0.00479)	0.37535 (0.00550)	0.38871 (0.00481)
(4,75,45)	I	0.39352 (0.00393)	0.39240 (0.00394)	0.39464 (0.00392)	0.38148 (0.00439)	0.39352 (0.00393)
	II	0.39256 (0.00412)	0.39145 (0.00413)	0.39368 (0.00412)	0.38061 (0.00460)	0.39256 (0.00412)
	III	0.39140 (0.00405)	0.39029 (0.00406)	0.39252 (0.00404)	0.37936 (0.00456)	0.39140 (0.00405)
(4,75,55)	I	0.38898 (0.00358)	0.38805 (0.00359)	0.38991 (0.00356)	0.37894 (0.00402)	0.38898 (0.00358)
	II	0.39275 (0.00367)	0.39181 (0.00368)	0.39369 (0.00367)	0.38272 (0.00405)	0.39175 (0.00367)
	III	0.39222 (0.00363)	0.39128 (0.00364)	0.39316 (0.00362)	0.38217 (0.00401)	0.39222 (0.00363)

Table 6: Average values and corresponding MSE of parameter α_1 based non-informative prior

(k,n,m)	SC	Square Error	LINEX		General Entropy	
			$q=0.5$	$q=-0.5$	$h=1$	$h=-1$
(2,50,30)	I	1.06434 (0.11318)	1.03748 (0.09795)	1.09373 (0.13197)	0.96989 (0.09849)	1.06434 (0.11318)
	II	1.11986 (0.15991)	1.09008 (0.13418)	1.15247 (0.19673)	1.02246 (0.11777)	1.11968 (0.15986)
	III	1.08625 (0.13244)	1.05873 (0.11403)	1.11633 (0.15731)	0.99166 (0.10380)	1.08625 (0.13244)
(2,50,40)	I	1.06124 (0.08708)	1.04194 (0.0777)	1.08166 (0.09873)	0.99239 (0.07124)	1.06124 (0.08708)
	II	1.07316 (0.09799)	1.05355 (0.08794)	1.09388 (0.11019)	1.0040 (0.08007)	1.07316 (0.09799)
	III	1.07992 (0.09728)	1.05973 (0.08659)	1.10142 (0.11057)	1.00927 (0.07843)	1.07992 (0.09728)
(4,75,45)	I	1.03387 (0.06658)	1.01782 (0.06128)	1.0507 (0.07308)	0.97434 (0.05856)	1.03387 (0.06658)
	II	1.06191 (0.07635)	1.04502 (0.06956)	1.07963 (0.08453)	1.00096 (0.06410)	1.06191 (0.07635)
	III	1.0442 (0.07317)	1.02785 (0.06716)	1.06137 (0.08047)	0.98452 (0.06305)	1.0442 (0.07317)
(4,75,55)	I	1.02123 (0.05676)	1.00857 (0.05311)	1.03437 (0.06113)	0.97383 (0.05103)	1.02123 (0.05676)
	II	1.07138 (0.06791)	1.05751 (0.06232)	1.08581 (0.07443)	1.02165 (0.05711)	1.07138 (0.06791)
	III	1.04105 (0.05995)	1.02818 (0.05590)	1.05439 (0.06471)	0.99337 (0.05307)	1.04105 (0.05995)

Table 7: Average values and corresponding MSE of parameter α_2 based non-informative prior

(k,n,m)	SC	Square Error	LINEX		General Entropy	
			$q=0.5$	$q=-0.5$	$h=1$	$h=-1$
(2,50,30)	I	2.05088 (0.25008)	1.99093 (0.21861)	2.11627 (0.29671)	1.93594 (0.22475)	2.05088 (0.25008)
	II	2.10759 (0.31001)	2.04345 (0.26424)	2.17814 (0.37649)	1.98835 (0.26617)	2.10759 (0.31001)
	III	2.10062 (0.27298)	2.03715 (0.22998)	2.17026 (0.33621)	1.98188 (0.23176)	2.10062 (0.27298)
(2,50,40)	I	2.06909 (0.20625)	2.02334 (0.18368)	2.11791 (0.23678)	1.98215 (0.18473)	2.06909 (0.20625)
	II	2.10569 (0.23132)	2.05883 (0.20762)	2.15573 (0.27302)	2.01818 (0.20652)	2.10569 (0.23570)
	III	2.08595 (0.21484)	2.0393 (0.18857)	2.13576 (0.26746)	1.99824 (0.18837)	2.08595 (0.21484)
(4,75,45)	I	2.01609 (0.17404)	1.9777 (0.16088)	2.05673 (0.19249)	1.94115 (0.16490)	2.01609 (0.17404)
	II	2.06383 (0.18269)	2.02361 (0.16494)	2.10634 (0.20625)	1.98697 (0.16564)	2.06383 (0.18269)
	III	2.0391 (0.16029)	2.0006 (0.14681)	2.07976 (0.17880)	1.96445 (0.14903)	2.0391 (0.16029)
(4,75,55)	I	2.02569 (0.12196)	1.99425 (0.11356)	2.05854 (0.13338)	1.96426 (0.11492)	2.02569 (0.12196)
	II	2.06601 (0.13954)	2.03336 (0.12752)	2.1001 (0.15496)	2.00333 (0.12710)	2.06601 (0.13954)
	III	2.04398 (0.13454)	2.01167 (0.12416)	2.07784 (0.14837)	1.98154 (0.12468)	2.04398 (0.13454)

Table 8: Average values and corresponding MSE of parameter p based non-informative prior

(k,n,m)	SC	Square Error	LINEX		General Entropy	
			$q=0.5$	$q=-0.5$	$h=1$	$h=-1$
(2,50,30)	I	0.40647 (0.00763)	0.40472 (0.00759)	0.40823 (0.00678)	0.38753 (0.00823)	0.40647 (0.00763)
	II	0.41368 (0.00714)	0.41191 (0.00708)	0.41545 (0.00721)	0.39487 (0.00743)	0.41368 (0.00714)
	III	0.40721 (0.00687)	0.40546 (0.00683)	0.40897 (0.00692)	0.38829 (0.00742)	0.40721 (0.00687)
(2,50,40)	I	0.40503 (0.00533)	0.40367 (0.00531)	0.40639 (0.00536)	0.39058 (0.00567)	0.40503 (0.00533)
	II	0.40356 (0.00527)	0.40220 (0.00526)	0.40492 (0.00530)	0.38911 (0.00565)	0.40356 (0.00527)
	III	0.40329 (0.00541)	0.40194 (0.00539)	0.40464 (0.00543)	0.38897 (0.00577)	0.40329 (0.00541)
(4,75,45)	I	0.40534 (0.00491)	0.40411 (0.00489)	0.40658 (0.00494)	0.392411 (0.005153)	0.40534 (0.00491)
	II	0.40310 (0.00476)	0.40278 (0.00473)	0.40521 (0.00477)	0.39121 (0.00502)	0.40310 (0.00475)
	III	0.402959 (0.00450)	0.40174 (0.00448)	0.40418 (0.00452)	0.39016 (0.00477)	0.40296 (0.00450)
(4,75,55)	I	0.40293 (0.00429)	0.40193 (0.00427)	0.403941 (0.00430)	0.39243 (0.00448)	0.40293 (0.00429)
	II	0.40366 (0.00409)	0.40265 (0.00408)	0.40466 (0.00411)	0.39316 (0.00427)	0.40366 (0.00409)
	III	0.40375 (0.00413)	0.40274 (0.00411)	0.40476 (0.00414)	0.39322 (0.00431)	0.40375 (0.00413)

7. Bayesian Prediction For Future Observations

In this section, the Bayesian two sample prediction of a future order statistics is considered based on the observed progressive first failure censored data $\underline{x}$. Based on a random sample of size n drawn from a population with pdf (1), a future unobservable independent random sample of size m from the same population is under consideration. Let y_s represents the s^{th} ordered statistic in the future sample, $1 \le s \le m$. The s^{th} order statistic in a sample of size m represents the life length of a $(m-s+1)$ out of m system. The distribution function of y_s the ordered future sample is given, [See Arnold et al.(1992) and Jaheen (2003)], by

$$F_{Y_s}(p,\alpha_1,\alpha_2)=\sum_{l=s}^{m}\binom{m}{l}[F_X(y_s|p,\alpha_1,\alpha_2)]^l[1-F_X(y_s|p,\alpha_1,\alpha_2)]^{m-l}$$

$$=\sum_{l=s}^{m}\sum_{j_1=0}^{l}\binom{m}{l}\binom{l}{j_1}(-1)^{j_1}[R(y_s)]^{m-l+j_1} \tag{16}$$

where $F_{Y_s}(p,\alpha_1,\alpha_2)$ is the distribution function of the mixture model and $R(y_s)$ is the reliability function of the mixture model after replacing x by y_s.

Using the binomial expansion for $[R(y_s)]^{m-l+j_1}$, we get

$$[R(y_s)]^{m-l+j_1}=\left[p_1e^{-\alpha_1y_s^{\theta_1}}+p_2(1+\theta_2y_s)^{-\alpha_2}\right]^{m-l+j_1}$$

$$=\sum_{j_2=0}^{m-l+j_1}\binom{m-l+j_1}{j_2}p_1^{\delta_1}p_2^{j_2}\left(e^{-\alpha_1y_s^{\theta_1}}\right)^{\delta_1}(1+\theta_2y_s)^{-\alpha_2j_2}$$

where $\delta_1=m-l+j_1-j_2$

Therefore,

$$F_{Y_s}(p,\alpha_1,\alpha_2)=\sum_{l=s}^{m}\sum_{j_1=0}^{l}\sum_{j_2=0}^{m-l+j_1}\binom{m}{l}\binom{l}{j_1}\binom{m-l+j_1}{j_2}(-1)^{j_1}p_1^{\delta_1}p_2^{j_2}\left(e^{-\alpha_1y_s^{\theta_1}}\right)^{\delta_1}(1+\theta_2y_s)^{-\alpha_2j_2} \tag{17}$$

The Bayes predictive pdf of y_s given $\underline{x}$ is defined by:

$$f^*(y_s|\underline{x})=\int_0^1\int_0^\infty\int_0^\infty f(y_s|p,\alpha_1,\alpha_2)P(p,\alpha_1,\alpha_2|\underline{x})d\alpha_1d\alpha_2dp \tag{18}$$

where $P(p,\alpha_1,\alpha_2|\underline{x})$ is the joint posterior density function for parameters α_1,α_2 and p and $f(y_s|p,\alpha_1,\alpha_2)$ is the pdf of s^{th} component in a future sample.

Therefore, using the Bayesian predictive density of y_s, for a given value ν, we obtain

$$Pr\left[y_s \geq \nu|\underline{x}\right] = \int_{\nu}^{\infty} f^*\left(y_s|\underline{x}\right)dy_s$$
$$= 1 - \int_0^1\int_0^{\infty}\int_0^{\infty} F_{Y_s}(\nu|p,\alpha_1,\alpha_2)P\left(p,\alpha_1,\alpha_2|\underline{x}\right)d\alpha_1 d\alpha_2 dp \tag{19}$$

where $F_{Y_s}(\nu|p,\alpha_1,\alpha_2)$ is the cumulative distribution of the s^{th} component in a future sample as given by (17). This cannot be evaluated analytically. Thus, the MCMC sampling procedure described in Subsection 5.1 is applied.

Based on importance samples $\{(p_i,\alpha_{1i},\alpha_{2i}), i = 1,\dots,N\}$. A simulator of the predictive distribution of y_s, as

$$Pr\left[y_s \geq \nu|\underline{x}\right] = 1 - \sum_{i=1}^{N} F_{Y_s}(\nu|p,\alpha_1,\alpha_2)w_i \tag{20}$$

where

$$w_i = \frac{h(\alpha_{1i},\alpha_{2i},p_i)}{\sum_{i=1}^{N} h(\alpha_{1i},\alpha_{2i},p_i)}, i = 1,\dots,N$$

A 100 γ% prediction interval for y_s is given by

$$P\left[L\left(\underline{x}\right) < y_s < U\left(\underline{x}\right)\right] = \gamma$$

where $L\left(\underline{x}\right)$ and $U\left(\underline{x}\right)$ are obtained respectively by solving the following two nonlinear equations:

$$Pr\left[y_s > L\left(\underline{x}\right)|\underline{x}\right] = \frac{1+\gamma}{2} \qquad and \qquad Pr\left[y_s > U\left(\underline{x}\right)|\underline{x}\right] = \frac{1-\gamma}{2} \tag{21}$$

These equations cannot be solved analytically, using Mathematica software program.

7.1. Numerical Example

This section presents a numerical example to illustrate the methodology for the proposed estimates based on real data. The data set is an uncensored data set consisting of 66 observations on breaking stress of carbon fibers (in Gba). This data was presented by Al-Babtain et al.(2015) and is as follows: 3.70, 2.74, 2.73, 2.50, 3.60,3.11,3.27,2.87,1.47,3.11,3.56,4.42,2.41,3.19,3.22,1.69,3.28, 3.09,1.87,3.15,4.90,1.57,2.67,2.93,3.22,3.39,2.81,4.20,3.33,2.55,3.31,3.31,2.85,1.25,4.38,1.84,0.39,3.68,2.48,0.85,1.61,2.79,4.70,2.03,1.89,2.88,2.82,2.05,3.65.3.75,2.73,2.95,2.97,3.39,2.96,2.35,2.55,2.59,2.03,1.61,2.12,3.15,1.08,2.56,1.80,2.53. Now, we assume that the observations are randomly grouped into 22 groups with $k = 3$ observations within each group, with ($m = 16$, $n = 22$, $k = 3$), the grouped data set are as follows :{3.70,2.74,2.73},{2.50,3.60,3.11},{3.27,2.87,1.47},{3.11,3.56,4.42},{2.41,3.19,3.22}, {1.69,3.28,3.09},{1.87,3.15,4.90},{1.57,2.67,2.93},{3.22,3.39,2.81},{4.20,3.33,2.55},{3.31,3.31,2.85},{1.25,4.38,1.84},{0.39,3.68,2.48},{0.85,1.61,2.79},{4.70,2.03,1.89},{2.88,2.82,2.05},{3.65,3.75,2.43},{2.95,2.97,3.39},{2.96,2.35,2.55},{2.59,2.03,1.61},{2.12,3.15,1.08},{2.56,1.80,2.53}.The assumed censoring scheme $R = (1,1,1,10^0,1,1,1)$. The following progressive first failure censored data is given by: (0.39,1.08,1.47,1.61,1.69,1.80,1.87,1.89,2.05,2.35,2.41, 2.43,2.50,2.55,2.81,2.95). Taking $\theta_1 = 1.5$, $\theta_2 = 1.2$, $p = 0.60$, the following progressively first failure censored mixture real life data is

0.39, 1.08, 1.61, 1.80, 1.87, 1.89, 2.05, 2.35, 2.43, 2.55, 2.95 $(r_1 = 11)$

1.47, 1.69, 2.41, 2.50, 2.81 $(r_2 = 5)$

Based on these data, we compute predictive interval of future sample. According the algorithm in Subsection 5.1, we generate 1000 MCMC samples. The 90% and 95% predictive interval for the future observation Y_s are given by solving Equation (21) numerically. The results are presented in Table (9) and Table (10)

Table 9: Two sample prediction intervals for the future observation Y_s in case informative prior

90% prediction interval for Y_s		
s	(Lower, Upper)	Length
1	(0.01984,0.92570)	0.90586
2	(0.06398,1.20602)	1.14204
3	(0.28126,2.16251)	1.88125
4	(0.46879,3.13421)	2.66542
5	(0.76259,4.29607)	3.53348
6	(1.06686,6.3282)	5.26134
95% prediction interval for Y_s		
s	(Lower, Upper)	Length
1	(0.01413,1.06036)	1.04623
2	(0.05280,1.50413)	1.45133
3	(0.26863,2.46712)	1.89849
4	(0.43263,3.13654)	2.70391
5	(0.68720,5.66177)	4.97457
6	(0.95036,7.56534)	6.61498

Table 10: Two sample prediction intervals for the future observation Y_s in case non- informative prior

90% prediction interval for Y_s		
s	(Lower, Upper)	Length
1	(0.02533,0.97698)	0.95165
2	(0.08472,1.71576)	1.63104
3	(0.29504,2.34837)	2.05333
4	(0.62261,3.7236)	3.10099
5	(1.03315,5.41041)	4.37726
6	(1.15488,7.61791)	6.46303
95% prediction interval for Y_s		
s	(Lower, Upper)	Length
1	(0.02160,1.10856)	1.08696
2	(0.07490,1.98037)	1.90547
3	(0.28267,2.82536)	2.54269
4	(0.56385,4.30136)	3.73751
5	(0.89780,6.29315)	5.39535
6	(1.03551,8.91873)	7.88322

8. Conclusion

Based on progressively first-failure censored scheme, in this paper, we have addressed the estimation and prediction problems of the mixture of Weibull and Lomax distributions. The Bayes estimates cannot be obtained in explicit form so importance samples procedure is used to draw MCMC samples. Also, the same MCMC method is used for computing two sample predictive intervals. An example using real data set was used for illustration.

References

1. Abou-Elheggag, N.A. (2013). Estimation for Rayleigh distribution using progressive first failure censored data. Journal of Statistics Applications and Probability, 2, 171-182.

2. Abu- Zinadah, H. H. (2010). A study on mixture of Exponentiated Pareto and Exponential distributions. Journal of Applied Sciences Research, 6(4), 358-376.
3. Al-Babtain, A., Eid, A.M., Ahmed, A., H.N. and Merovci, F. (2015). The five parameter Lindley distribution. Pakistan Journal of Statistics, 31(4),363-384.
4. Al-Hussaini, E. K., Nigm, A. M., and Jaheen, Z., F. (2001). Bayesian prediction based on finite mixtures of Lomax components model and type I censoring,.Statistics, 35,259-268.
5. Arnold, B. C., Balakrishnan, N. and Nagaraja, H.,N. (1992). A First Course in Order Statistics. John Wiley and Sons, New York.
6. Balakrishnan, N. (2007). Progressive censoring methodology: an appraisal.Test,16, 211-259.
7. Balakrishnan, N. and Aggarwala,R. (2000). Progressive censoring: Theory, Methods and Application, Birkhauser, Boston.
8. Balakrishnan, N. and Sandhu, R. A. (1995). A simple simulation algorithm for generating progressive type-II censored samples. The American Statistician, 49(2),229-230.
9. Balasooriya, U. (1995). Failure-censored reliability sampling plans for the exponential distribution. Journal of Statistical Computation and Simulation, 52,337-349.
10. Balasooriya, U., Saw, S. L. C. and Gadag, V. (2000). Progressively censored reliability sampling plans for the Weibull distribution.Technometrics.42,160-167.
11. Basu, A. P. and Ebrahimi, N. (1991). Bayesian approach to life testing and reliability estimation using asymmetric loss function. Journal of Statistics Planning and Inference, 29, 21-31.
12. Calabria, R. and Pulcini, G. (1996).Point estimation under asymmetric loss function for left-truncated exponential samples. Communications in Statistics-Theory and Methods, 25(3), 585-600.
13. Cohen, A. C. (1963). Progressively censored samples in life testing. Technometrics, 5,327-329.
14. Daniyal, M. and Rajab, M. (2015). On some classical properties of the mixture of Burr XII and Lomax distributions. Journal of Statistics Applications and Probability, 4(1),173-181.
15. Dube, M., Garg, R. and Krishna, H. (2016). On progressively first failure censored Lindley distribution. Computational Statistics, 31,139-163.
16. Elshahat, M. A. T. and Mahmoud, A. A. M. (2016). A study on the mixture of exponentiated- weibull distribution part ii (The Method of Bayesian Estimation). Pakistan Journal of Statistics and Operation Research, Vol.XII(4).709-737.
17. Erisoglu, U., Erisoglu, M. and Erol, H. (2011). A mixture model of two different distributions approach to the analysis of heterogeneous survival data. World Academy of Science, Engineering Technology International Journal of Computer, Electrical, Automation, Control and Information Engineering, 5(6),544-548.
18. Everitt, B.S. and Hand, D. J. (1981). Finite Mixture Distributions. The University Press, Cambridge.
19. Feroze, N. and Aslam, M.(2014). Bayesian analysis of doubly censored lifetime data using two component mixture of Weibull distribution. Journal of the National Science Foundation of Sri Lanka, 42(4),325-334.
20. Huang, S.R and Wu, S.J. (2011). Estimation of Pareto distribution under progressive first-failure censoring with random removals. Journal of the Chinese Statistical Association,49,82-97.
21. Jaheen, Z. F.(2003). Bayesian prediction under a mixture of two- component Gompertz lifetime model. Sociedad de Estadistica e Investigacion Opertiva Test, 12(2),413-426.
22. Javadkhani, N., Azhdari, P. and Azimi, R. (2014). On Bayesian estimation from two parameters Bathtub-Shaped lifetime distribution based on progressive first-failure censored sampling. International Journal of Scientific World, 2(1),31-41.
23. Johnson, L.G. (1964). Theory and technique of variation research. Elsevier, Amsterdam.
24. Kim, C. and Han, K. (2015). Bayesian estimation of generalized exponential distribution under progressive first failure censored sample. Applied Mathematical Sciences, 9(41),2037-2047.
25. Lindsay, B. G.(1995).Mixture Models: Theory, Geometry and Applications. Institute of Mathematical Statistics, California.
26. Lomax,K.S.(1954). Business failure,another example of the analysis of failure data. Journal of the American Statistical Association,49,847-852.
27. Mahmoud, M., Nassar, M. M. and Aefa, M. A. (2017). Bayesian estimation and Prediction for a mixture of Weibull and Lomax distributions. International Journal of Innovative Research and Development,6 (5),33-46.
28. Mann, N.R. (1971). Best linear invariant estimation for Weibull parameters under progressive censoring. Technometrics,13,521-533.
29. Mclachlan, G.J. and Basford, K.E. (1988). Mixture Models: Inferences and Applications to Clustering. Marcel Dekker, New York.
30. Mclachlan, G.J. and Peel, D. (2000). Finite Mixture Models. John Wiley and Sons, New York.
31. Murthy,D.N.P. Xie, M. and Jiang,R. (2003). Weibull models. New York: Wiley.

32. Rashad, M., Mahmoud, M. and Yusuf, M. (2017). Bayes estimation of the scale parameter of the Logistic distribution based on progressive first-failure censored scheme. Journal of Statistics Applications and Probability, 6(1), 49-63.
33. Shawky, A.I. and Bakoban, R.A. (2009). On finite mixture of two component exponentiated gamma distribution. Journal Applied Sciences Research, 5(10), 1351-1369.
34. Soliman, A.A., Abd-Ellah, A.H., Abou-Elheggag, N.A. and Abd-Elmougod, G.A. (2012).Estimation of the parameters of life for Gompertz distribution using progressive first-failure censored data. Computational Statistics and Data Analysis, 56, 2471-2485.
35. Titterington, D. M., Smith, A. F. M. and Makov, U. E. (1985). Statistical Analysis of Finite Mixture Distributions. John Wiley and Sons, New York.
36. Varian, H. R. (1975). A Bayesian approach to real estate assessment. ln: Stephen, E.F and Zellner, A.(Eds).Studies in Bayesian Econometrics and statistics in Honor of Leonard J. Savage. Amsterdam: North Holland,195-208.
37. Viveros, R. and Balakrishnan, N. (1994). Interval estimation of parameters of life from progressively censored data. Technometrics,36(1), 84-91.
38. Wu, J.W., Hung, W.L. and Tsai, C. H. (2003). Estimation of the parameters of the Gompertz distribution under the first failure censored sampling plan. Statistics, 37(6), 517-525.
39. Wu, J. W. and Yu, H.Y. (2005). Statistical inference about the shape parameter of the Burr type XII distribution under the failure- censored sampling plan. Applied Mathematics and Computation,163, 443-482.
40. Wu, S. J. and Kus, C. (2009).On estimation based on progressive first-failure censored sampling. Computational Statistics and Data Analysis, 53, 3659-3670