The Alpha-Beta Skew Generalized t Distribution: Properties and Applications

Hamid Esmaeili
Department of Statistics
Persian Gulf University, Bushehr, Iran
esmailihamid@ymail.com

Fazlollah Lak
Department of Statistics
Persian Gulf University, Bushehr, Iran
fazlollahlak@gmail.com

Mohammad Esmaeil Dehghanmonfard
Department of Statistics
Persian Gulf University, Bushehr, Iran
monfard2@gmail.com

Morad Alizadeh
Department of Statistics
Persian Gulf University, Bushehr, Iran
m.alizadeh@pgu.ac.ir

Abstract
In this paper, we propose a new distribution, namely alpha-beta-skew generalized t distribution. The proposed distribution is really flexible and includes as special models some important distributions like Normal, t-student, Cauchy and etc as its marginal component distributions. It features a probability density function with up to three modes. The moment generating function as well as the main moments are provided. Inference is based on a usual maximum-likelihood estimation approach and a small Monte Carlo simulation is conducted for studying the asymptotic properties of the maximum-likelihood estimate. The usefulness of the new model is illustrated in a real data.

Keywords: t distribution, Maximum likelihood, Moments method, Skew distribution.

Introduction
Traditionally, the normality assumption is one of the conditions in statistical procedures. However, in most cases of real-life problems, the normality assumption has not established and non-normal distributions for modeling data sets having skewness and/or kurtosis are more prevalent, see for example (Tiku and eatl,2011) and (Celik and eatl , 2015). Therefore, the construction of the non-normal distributions has been an enormous interest and attracted the attention of researchers.
The Generalized t distribution was defined by (McDonald and Newey,1998) to develop a partially adaptive M-regression procedure. The procedure includes many other estimation methods such as least squares, least absolute deviation and L_p. It has been followed up more recently by (Theodossiou,1998) and (Arslan and Genç,2003).
Suppose the random variable X have the generalized t distribution, then the probability density function (pdf) is,

$$f_{GT}(x,\sigma,p,q) = \frac{p}{2\sigma q^{1/2} B(1/p,q)} \left(1 + \frac{|x|^p}{q\sigma^p}\right)^{-(q+1/p)}, \qquad x \in R, \tag{1}$$

where $\sigma,\ p,\ q$ are distributional parameters, σ corresponds to the standard deviation, p and q are parameters corresponding to the shape of distribution, $B(\cdot,\cdot)$is the Beta function and denoted by $X \sim GT(\sigma, p, q)$. The shape parameters p and q control the tails of the distribution. Larger values of p and q are associated with thinner tails of the distribution. Similarly, smaller values of the shape parameters correspond to thicker tails. Thus, the GT distribution is useful in accommodating both leptokurtic and platykurtic symmetric unimodal distributions.
The GT distribution has several important subdistributions as special or limiting cases of the shape parameters. For example, for $p = 2$, we get the usual t distribution with $2q$ degrees of freedom, and for $p \to \infty$ and $q \to \infty$, we get the uniform and power exponential distributions, respectively. Figure 0 shows the distribution tree of the GT distribution family.

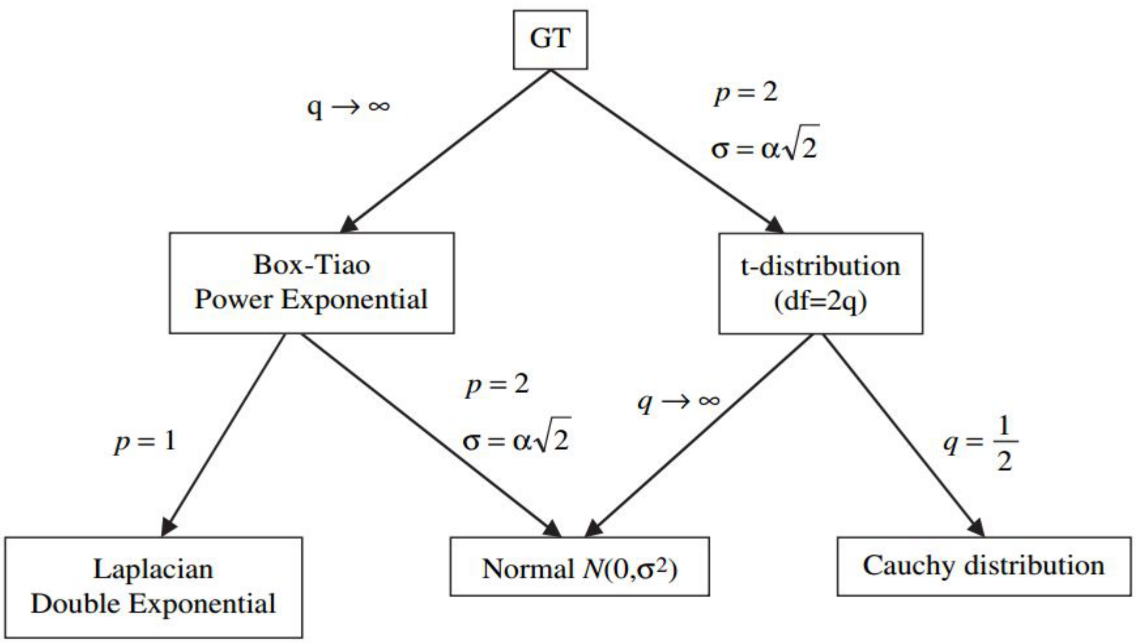


Figure 1: Distribution tree of the GT distribution family

Skew symmetric distributions is a rapidly growing and they have attracted significant attention in climatology, economics, finance and other areas of the sciences. A random variable Y is said to have the skew symmetric distribution if its pdf is given by

$$g(y) = 2f(y)F(\gamma y), \tag{2}$$

where $-\infty < \gamma < \infty$ and f and F are valid pdf and cdf (respectively) of a distribution symmetric around 0 (see, for example, (Gupta and eatl,2002)). If one takes f in Equation (2) to be given as Equation (1) then one would have the skew generalized t distribution. The calculation of the mathematical properties of this distribution and their applications could be find in (Theodossiou,1998) and (Nadarajah,2008).
Recently (Acitas,2015), introduced a new class of skew generalized t distributions called alpha-skew generalized t distribution, with the pdf,

$$g_{ASGT}(x; \alpha, p, q) = \frac{(1-\alpha x)^2+1}{2+\alpha^2 c(p,q)} f_{GT}(x; p, q), \qquad x \in R, pq > 2 \tag{3}$$

where,

$$c(p,q)=\frac{q^{\frac{2}{p}}\Gamma\left(\frac{3}{p}\right)\Gamma\left(q-\frac{2}{p}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma(q)}$$

The aim of this paper is to introduce a new family of distributions as an extension of alpha skew generalized t distribution with the pdf (3). The rest of this paper is organized as follows: In Section 2, the new family of distributions is defined and studied in details. In Section 3, the random variable generating method has been investigated. The usual maximum-likelihood estimation approach are derived in in Section 4 and the simulation study is performed using Monte Carlo method. Finally, in Section 5, to illustrate the applicability of the proposed model, two real data sets are analysed.

Alpha-Beta-Skew Generalized t Distribution

Hazarika and Chakraborty(2014) defined the alpha skew logistic distribution and Shafaei et.al (2016) defined alpha beta skew normal distribution. In this Section, the new family of distributions is introduced. This new family of skew distributions include alpha beta skew normal distribution as especial case. it can use also for modeling heavy tailed data.

Definition 2.1 (ABSGT distribution) The random variable Z follows the alpha-beta-skew generalized t distribution, denoted by $ABSGT(\alpha,\beta,p,q)$, if it has the pdf,

$$g_{ABSGT}(z;\alpha,\beta,p,q)=\frac{(1-\alpha z-\beta z^3)^2+1}{2+d(\alpha,\beta,p,q)}$$
$$\times\frac{p}{2\sigma q^{1/2}B(1/p,q)}\left(1+\frac{|z|^p}{q\sigma^p}\right)^{-(q+1/p)}, \qquad (4)$$

where,

$$d(\alpha,\beta,p,q)=\alpha^2\frac{q^{2/p}\Gamma(\frac{3}{p})\Gamma(q-\frac{2}{p})}{\Gamma(\frac{1}{p})\Gamma(q)}+2\alpha\beta\left(\frac{q^{4/p}\Gamma(\frac{5}{p})\Gamma(q-\frac{4}{p})}{\Gamma(\frac{1}{p})\Gamma(q)}\right)$$
$$+\beta^2\frac{q^{6/2}\Gamma(\frac{7}{p})\Gamma(q-\frac{6}{p})}{\Gamma(\frac{1}{p})\Gamma(q)}.$$

Theorem 2.2 The presented density function in definition 2.1, is a proper probability density function.

Proof. Let,

$$f_{GT}(x;p,q)=\frac{p}{2\sigma q^{1/2}B(1/p,q)}\left(1+\frac{|z|^p}{q\sigma^p}\right)^{-(q+1/p)},$$

So we have,

$$\int\left(\left(1-\alpha x-\beta x^3\right)^2+1\right)f_{GT}(x;p,q)dx=\int\left(\left[(1-\alpha x)^2+1\right]-2\beta x^3\left[(1-\alpha x)^2+1\right]\right.$$
$$\left.+(2\beta x^3+2\beta\alpha^2x^5+\beta^2x^6)\right)f_{GT}(x;p,q)dx$$
$$=c(p,q)\int\quad g_{ASGT}(x;p,q)dx-2\beta E_{ASGT}(X^3)$$
$$+2\beta E_{GT}(X^3)+2\beta\alpha^2E_{GT}(X^5)+\beta^2E_{GT}(X^6)$$

Where $c(p,q)$ is defined by equation (3).

and because of symmetry of GT distribution, for odd values of n we have $E_{GT}(X^n)=0$, and for even values of n,

$$E_{GT}(X^n)=\frac{q^{n/2}\Gamma(\frac{n+1}{p})\Gamma(q-\frac{n}{p})}{\Gamma(\frac{1}{p})\Gamma(q)},$$

Also,

$$E_{ASGT}(X^n) = \begin{cases} \frac{q^{n/2}\Gamma(\frac{n+1}{p})\Gamma(q-\frac{n}{p})}{\Gamma(\frac{1}{p})\Gamma(q)}, & \text{nisodd}, \\ 0, & \text{niseven}. \end{cases}$$

And by some simple computation, the proof is completed.

Definition 2.3 If $Z \sim ABSLG(\alpha,\beta,p,q)$, then $Y = \mu + \sigma Z$ is said to be the location (μ) and scale extension (σ) of Z and has the pdf,

$$f_Z(z;\alpha,\beta,p,q,\mu,\sigma) = \frac{\left(1-\alpha\left(\frac{y-\mu}{\sigma}\right)-\beta\left(\frac{y-\mu}{\sigma}\right)^3\right)^2+1}{2+d(\alpha,\beta,p,q)}$$
$$\times \frac{p}{2\sigma q^{1/2}B(1/p,q)}\left(1+\frac{|\frac{y-\mu}{\sigma}|^p}{q\sigma^p}\right)^{-(q+1/p)}.$$

This fact is denoted by $Y \sim ABSGT(\alpha,\beta,p,q,\mu,\sigma)$.

In the following, without loss of generality, we assume μ and σ equals 0 and 1 respectively. When α tends to $\pm\infty$ in equation (4), the pdf becomes

$$g_{BGT}(z;p,q) = z^2 f_{GT}(z;p,q), \qquad -\infty < z < \infty, \quad pq > 2. \tag{5}$$

It is easy to see that the function given in equation (5) is also a pdf. Thus, this limiting case of ABSGT is called a bimodal generalized t (BGT) distribution. BGT is symmetric and bimodal distribution. It can be seen as an extension of Elal-Olivero's bimodal normal distribution.

Proposition 2.4 expresses the cdf of Eqution 4 in terms of the incomplete beta function ratio. The cdf of generalized t distribution is obtained by (Nadarajah,2008) and (Acitas,2015) by following the lines of (Nadarajah,2008), obtained the cdf of ASGT distribution. In the proposition 2.4 we get the cdf of ABSGT by following the similar lines.

Proposition 2.4 Let $Z \sim ABSGT(\alpha,\beta,p,q)$ then the cdf of random variable Z is given as,

$$G_{ABSGT}(z) = \frac{1}{2+d(\alpha,\beta,p,q)}\{2F_{GT}(z) - 2\alpha G_1(z) + \alpha^2 G_2(z)$$
$$-2\beta G_3(z) + 2\alpha\beta G_4(z) + \beta^2 G_6(z)\}$$

Such that,

$$F_{GT}(z) = \begin{cases} \frac{1}{2} - \frac{1}{2} I_{(1-\frac{1}{1+(-z)^{p/q}})}\left(\frac{1}{p},q\right), & z < 0, \\ \frac{1}{2} + \frac{1}{2} I_{(1-\frac{1}{1+z^{p/q}})}\left(\frac{1}{p},q\right), & z \geq 0. \end{cases}$$

and,

$$G_i(z) = \frac{q^{i/p}\Gamma(\frac{i+1}{p})\Gamma(q-\frac{i}{p})}{\Gamma(\frac{1}{p})\Gamma(q)} \begin{cases} \frac{1}{2} - \frac{1}{2} I_{(1-\frac{1}{1+(-z)^{p/q}})}\left(\frac{i+1}{p},q-\frac{i}{p}\right), & z < 0, \\ \frac{1}{2} + \frac{1}{2} I_{(1-\frac{1}{1+z^{p/q}})}\left(\frac{i+1}{p},q-\frac{i}{p}\right), & z \geq 0. \end{cases}$$

where $I_y(a,b)$ denotes the incomplete beta function.

Proof. The cdf of ABSGT is obtained as shown below.

$$G_{ABSGT}(z) = \int_{-\infty}^{z} g_{ABSGT}(t)dt = \int_{-\infty}^{z} \frac{(1-\alpha t-\beta t^3)^2+1}{2+d(\alpha,\beta,p,q)} f_{GT}(t;p,q)dt$$
$$= \frac{1}{2+d(\alpha,\beta,p,q)}\int_{-\infty}^{z} (2-2\alpha t+\alpha^2 t^2 - 2\beta t^3 + 2\alpha\beta t^4 + \beta^2 t^6) f_{GT}(t;p,q)dt$$
$$= \frac{1}{2+d(\alpha,\beta,p,q)}\Big(2\int_{-\infty}^{z} f_{GT}(t;p,q)dt - 2\alpha\int_{-\infty}^{z} t f_{GT}(t;p,q)dt$$

$$+\alpha^2 \int_{-\infty}^{z} t^2 f_{GT}(t;p,q)dt - 2\beta \int_{-\infty}^{z} t^3 f_{GT}(t;p,q)dt$$
$$+2\alpha\beta \int_{-\infty}^{z} t^4 f_{GT}(t;p,q)dt + \beta^2 \int_{-\infty}^{z} t^6 f_{GT}(t;p,q)dt.$$

Let

$$F_{GT}(z) = \int_{-\infty}^{z} f_{GT}(t;p,q)dt, \quad G_i(z) = \int_{-\infty}^{z} t^i f_{GT}(t;p,q)dt,$$

then the main idea for obtaining $F_{GT}(\cdot)$and $G_i(\cdot)$is $u = 1 - \frac{1}{1-\frac{t^p}{q}}$ transformation for $p > 0$, see (Nadarajah,2008) and (Acitas,2015) for details.

Some basic properties of the ABSGT distribution are stated next.

Proposition 2.5 Let $Z \sim ABSGT(\alpha,\beta,p,q)$ then,

1. if $\beta = 0$, then $Z \sim ASGT(\alpha,p,q)$,
2. if $\alpha = 0$, the pdf (4) is simplified as the following:

$$f(z;\beta,p,q) = \frac{(1-\beta z^3)+1}{2+\frac{q^{6/2}\Gamma(\frac{7}{p})\Gamma(q-\frac{6}{p})}{\Gamma(\frac{1}{p})\Gamma(q)}\beta^2} \times f_{GT}(z;p,q), \qquad z \in R.$$

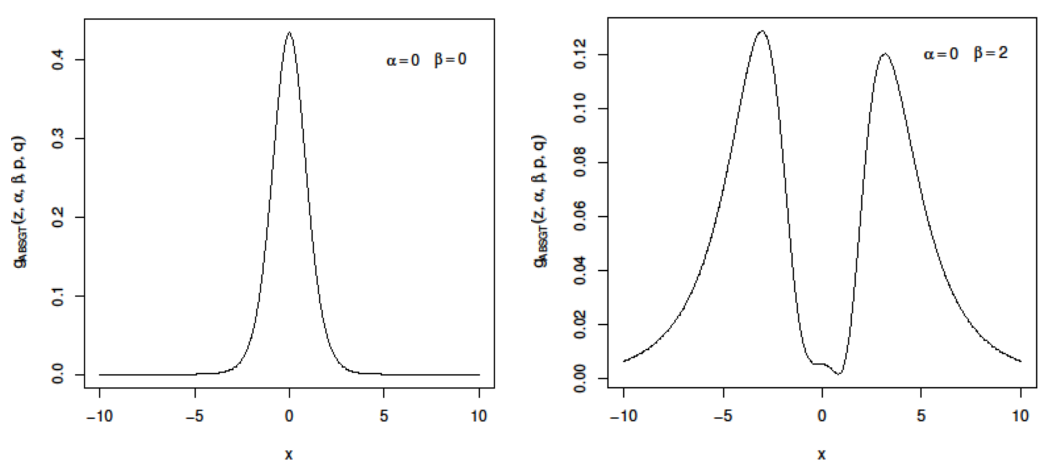
The above equation is referred to as the beta skew generalized t distribution.

3. if $\alpha = 0,\ \beta = 0, Z \sim GT(p,q)$,
4. $\beta = 0$ and $p = 2$, we get the pdf of alpha-Skew Student's t distribution,
5. if $\beta = 0, p = 2$ and $q \to \infty$, we get the pdf of alpha skew normal distribution,
6. if $p = 2$ and $q \to \infty$, we get the pdf of alpha beta normal distribution,
7. if $\beta = 0$ and $q \to \infty$, we get the pdf of alpha-skew power exponentialdistribution,
8. if $\beta = 0, p = 1$ and $q \to \infty$, we get the pdf of alpha-skew Laplace distribution

Proposition 2.6 Let $Z \sim ABSGT(\alpha,\beta,p,q)$ then,

1. if $Z \sim ABSGT(\alpha,\beta,p,q)$, then $-Z \sim ABSGT(-\alpha,-\beta,p,q)$,
2. if $Z \sim ABSGT(\alpha,\beta,p,q)$, then $aZ \sim ABSGT(a\alpha,a\beta,p,q)$,

Figure 5.1 presents the ABSGT pdf for different choices of the parameters α and β. It can be seen from Figure 5.1 that the proposed model has at most three modes, also the effects on the skewness can be seen.



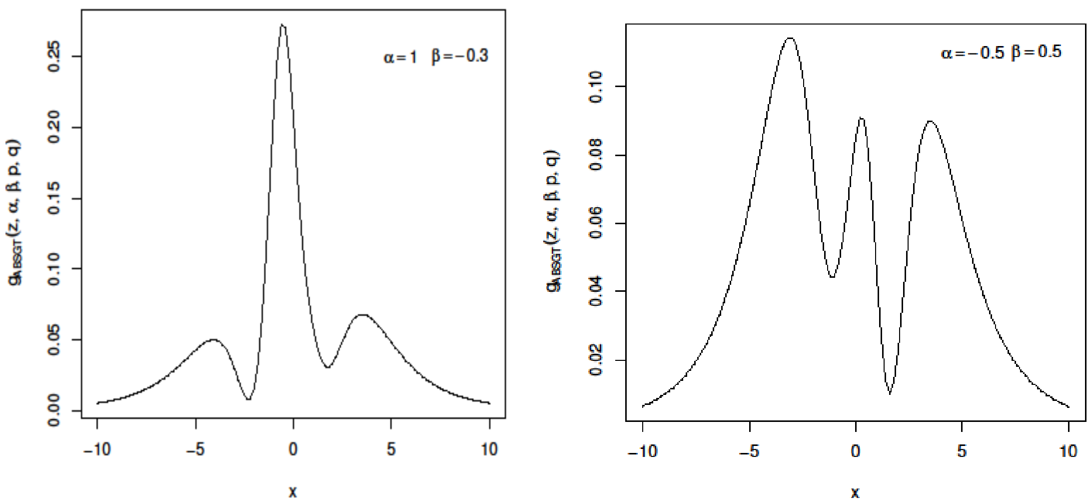


Figure 2: The pdf of the $ABSGT(\alpha,\beta)$ distribution for some selected values of the parameters.

Moments

In this subsection, moments of ABSGT distribution are derived. As a result, the skewness and the kurtosis measures are given. Here, it should be realized that the even and the odd moments of ABSGT distribution can be obtained by using the moments of GT distribution. This is because of the fact that the pdf of ABSGT distribution includes the pdf of GT distribution. The following proposition gives for both even and odd n values of $E(Z^n)$ based on this phenomena where $Z \sim ABSGT(\alpha,\beta,p,q)$.

Proposition 3.1 Let $Z \sim ABSGT(\alpha,\beta,p,q)$ then for $k \in N$,

$$\mu_{2k} = E(Z^{2k}) = \frac{1}{2+d(\alpha,\beta,p,q)}\left[\frac{2q^{2k/p}\Gamma(\frac{2k+1}{p})\Gamma(q-\frac{2k}{p})}{\Gamma(\frac{1}{p})\Gamma(q)} + \frac{\alpha^2 q^{2(k+1)/p}\Gamma(\frac{2k+3}{p})\Gamma(q-\frac{2k+2}{p})}{\Gamma(\frac{1}{p})\Gamma(q)}\right]$$
$$+\frac{2\beta}{2+d(\alpha,\beta,p,q)}\left[\frac{\alpha^2 q^{2(k+1)+1/p}\Gamma(\frac{2(k+1)+1}{p})\Gamma(q-\frac{2k}{p})}{\Gamma(\frac{1}{p})\Gamma(q)}\right]$$
$$+\beta^2\frac{1}{2+d(\alpha,\beta,p,q)}\left[\frac{2q^{2(k+3)/p}\Gamma(\frac{2k+7}{p})\Gamma(q-\frac{2(k+3)}{p})}{\Gamma(\frac{1}{p})\Gamma(q)} + \frac{\alpha^2 q^{2(k+4)/p}\Gamma(\frac{2k+9}{p})\Gamma(q-\frac{2(k+4)}{p})}{\Gamma(\frac{1}{p})\Gamma(q)}\right], \quad (6)$$

and

$$\mu_{2k-1} = E(Z^{2k-1}) = -\frac{1}{2+d(\alpha,\beta,p,q)}\left[\frac{2\alpha^2 q^{2k/p}\Gamma(\frac{2k+1}{p})\Gamma(q-\frac{2k}{p})}{\Gamma(\frac{1}{p})\Gamma(q)}\right]$$
$$-2\beta\frac{1}{2+d(\alpha,\beta,p,q)}\left[\frac{2q^{2(k+1)/p}\Gamma(\frac{2k+3}{p})\Gamma(q-\frac{2(k+1)}{p})}{\Gamma(\frac{1}{p})\Gamma(q)} + \frac{\alpha^2 q^{2(k+1)/p}\Gamma(\frac{2k+3}{p})\Gamma(q-\frac{2k+2}{p})}{\Gamma(\frac{1}{p})\Gamma(q)}\right]$$

$$+2\beta\frac{1}{2+d(\alpha,\beta,p,q)}\left[\frac{2q^{2(k+1)/p}\Gamma(\frac{2k+3}{p})\Gamma(q-\frac{2(k+1)}{p})}{\Gamma(\frac{1}{p})\Gamma(q)}\right]$$

$$+2\alpha^2\beta\frac{1}{2+d(\alpha,\beta,p,q)}\left[\frac{2q^{2(k+2)/p}\Gamma(\frac{2k+5}{p})\Gamma(q-\frac{2(k+2)}{p})}{\Gamma(\frac{1}{p})\Gamma(q)}\right]. \quad (7)$$

Proof. We have,

$$\mu_{2k-1}=E(Z^{2k-1})=\int z^{2k-1}f_{ABSGT}(z;\alpha,\beta,p,q)dz$$

$$=\int z^{2k-1}\frac{(1-\alpha z-\beta z^3)^2+1}{d(\alpha,\beta,p,q)}f_{GT}(z;p,q)dz$$

$$=\frac{1}{2+d(\alpha,\beta,p,q)}\int (2z^{2k-1}-2\alpha z^{2k+1}+\alpha^2 z^{2k+1}$$

$$-2\beta z^{2k+2}+2\alpha\beta z^{2k+3}+\beta^2 z^{2k+5})f_{GT}(z;p,q)dz$$

$$=\frac{1}{2+d(\alpha,\beta,p,q)}(E_{ASGT}(Z^{2k-1})-2\beta E_{ASGT}(Z^{2k+2})$$

$$+2\beta E_{GT}(Z^{2k+2})+2\alpha^2\beta E_{GT}(Z^{2k+4})+\beta^2 E_{GT}(Z^{2k+5}))$$

According to (Acitas,2015), results are obtained. Similarly,

$$\mu_{2k}=E(Z^{2k})=\frac{1}{2+d(\alpha,\beta,p,q)}(E_{ASGT}(Z^{2k})-2\beta E_{ASGT}(Z^{2k+3})$$

$$+2\beta E_{GT}(Z^{2k+3})+2\alpha^2\beta E_{GT}(Z^{2k+5})+\beta^2 E_{GT}(Z^{2k+6}))$$

By some simple computation, results are obtained.

In Table 1, we give the skewness and the kurtosis values of ABSGT distribution for some selected values of the shape parameters. It is obvious that Table 1 provides extra information for modeling performance of ABSGT distribution.

Table 1: The skewness and the kurtosis values of the ABSGT distribution based on some selected values of the shape parameters α,β,p and q.

α	β	p	q	skewness $\sqrt{\beta_1}$	kurtosis β_2
.1	0	10	0.5	0	3.5
1	0	4	8	0.4	2.7
1	0	10	2	0.6	2.5
3	0	2	5	0.6	3.9
.01	0.25	10	5	-0.012	14.11
-0.1	0.1	10	5	-28.07	74.84
.05	-1	6	10	-0.42	62.37

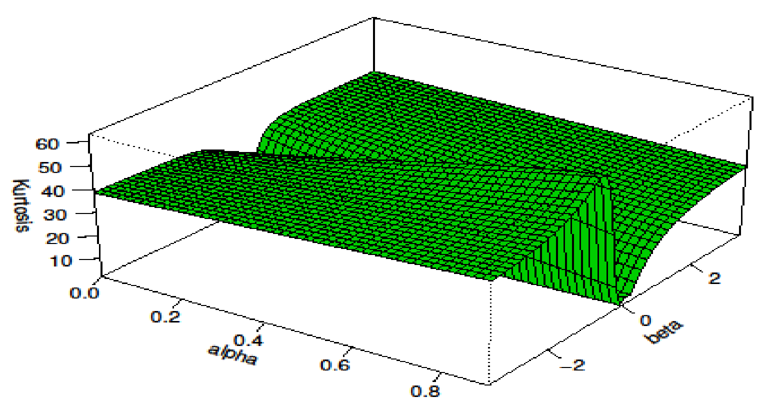


Figure 3: The kurtosis function of $ABSGT$ distribution function with p = 2 and q = 10.

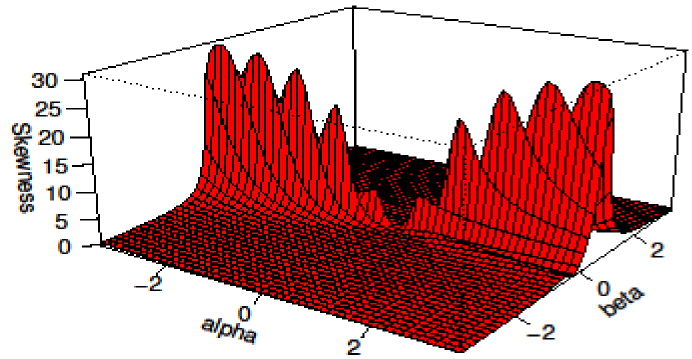


Figure 4: The skewness function of $ABSGT$ distribution function with p = 4 and q = 25.

As shown in fihures 3 ,4, onecan see that this model is useful for modeling various type of data(right skew, symmetric and left skew) with wide range of skewness and kurtosis.

Parameter Estimation

Let the data set $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ be modelled by the $ABSGT(\alpha, \beta, p, q)$ distribution. In this section, the problem of estimating the parameters α, β, p and q based on $\boldsymbol{x}$ is studied. From Equation 4, the associated log-likelihood function (LLF) reads

$$\ell = ln\, L\,(\alpha, \beta, p, q; \boldsymbol{x}) = -n(ln(\,2 + \alpha^2 + 40\beta^2 + 8\alpha\beta)$$
$$-\, ln(\,p) + ln(\,2\sigma q^{1/2} B(1/p, q))\big) \quad (8)$$
$$+ \sum_{i=1}^{n} \quad ln(\,(1 - \alpha x_i - \beta x_i^3)^2 + 1) - (q + 1/p)\, ln\left(1 + \frac{|x_i|^p}{q\sigma^p}\right)$$

We will refer to a maximum-likelihood estimate as any point in the parameter space at which the likelihood function has the global maximum. Taking the derivatives of equation (8) with respect to the parameters and setting them to zero yield the following likelihood equations:

$$\frac{\partial \ell}{\partial \alpha} = -\frac{n(2\alpha+8\beta)}{\alpha^2+8\beta\alpha+40\beta^2+2} - \sum_{i=1}^{n} \quad \frac{2x_i\left(-x_i\alpha-\beta x_i^3+1\right)}{\left(-x_i\alpha-\beta x_i^3+1\right)^2+1} = 0$$

$$\frac{\partial \ell}{\partial \beta} = -\frac{n(80\beta+8\alpha)}{40\beta^2+8\alpha\beta+\alpha^2+2} - \sum_{i=1}^{n} \quad \frac{2x_i^3\left(-x_i^3\beta-\alpha x_i+1\right)}{\left(-x_i^3\beta-\alpha x_i+1\right)^2+1} = 0$$

$$\frac{\partial \ell}{\partial p} = -\frac{n}{p} - \frac{\Psi_0\left(\frac{1}{p}+q\right)}{p^2} + \frac{\Psi_0\left(\frac{1}{p}\right)}{p^2}$$
$$+ \sum_{i=1}^{n} \quad \frac{ln\left(\frac{|x_i|^p}{q\sigma^p}+1\right)}{p^2} - \frac{\left(\frac{|x_i|^p\, ln(|x_i|)}{q\sigma^p} - \frac{ln(\sigma)|x_i|^p}{q\sigma^p}\right)\left(\frac{1}{p}+q\right)}{\frac{|x_i|^p}{q\sigma^p}+1} = 0$$

$$\frac{\partial \ell}{\partial q} = \Psi_0\left(q + \frac{1}{p}\right) - \Psi_0(q) + \frac{1}{2q}$$
$$+ \sum_{i=1}^{n} \quad \frac{|x_i|^p\left(q+\frac{1}{p}\right)}{\sigma^p\left(\frac{|x_i|^p}{\sigma^p q}+1\right)q^2} - ln\left(\frac{|x_i|^p}{\sigma^p q} + 1\right) = 0$$

Note that the likelihood equations are not in a simple closed form. Therefore, the maximumlikelihood estimators have to be calculated through some numerical procedures, for example, the Newton–Raphson and the L-BFGS-B (limited memory Broyden-Fletcher-Goldfarb-Shanno method to solve the bound constrained optimization problem) methods. The programs such as R provide computing routines for solving such nonlinear optimization problems. A few words can be said about the initial guesses of the parameters by visual inspection as follows. We first draw the histogram of the data set under study and then draw some pdfs over it. We then search the best, that is, the most likelihood, superimposed fit since it is important to choose near values to the true values. The parameter values in the best fit can be chosen as the initial guesses.

In the rest of this section, a small Monte Carlo simulation experiment is conducted to evaluate the maximum likelihood estimation of the $ABSGT(\alpha,\beta,p,q)$ distribution parameters. The sample sizes and true values of the parameters considered were $n = 20,40,70,100,140,200,270,350,450,600$, $\alpha = -1,0,1$, and $\beta = -0.5,0,0.5$, $p = 2$ and $q = 5$ while the location and the scale parameters were set to $\mu = 0$ and $\sigma = 1$.

All results are obtained from 5000 Monte Carlo replications and the simulations were carried out using the package 'HI' and 'stats' in statistical software R.

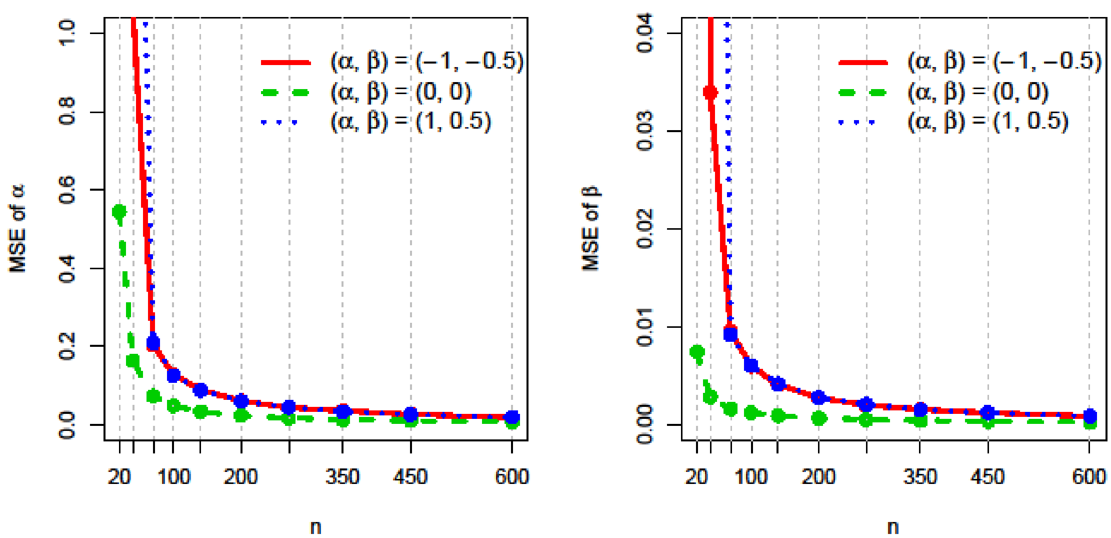


Figure 5: The mean square error of $\hat{\alpha}$ and $\hat{\beta}$ vesus n.

Figures 5 represents the MSE of the parameters α and β for fixed values of another parameters in the model. As can be seen in the Figures5 , by increasing the sample size, the MSE of estimated parameters decreases.

Real Data

In this section, we consider two real data sets to model with the ABSGT distribution. The first data set is an example of unimodal data called Roller data. Faithful geyser data is the second one which is bimodal. All data analyses were conducted in R program. The maximum-likelihood estimators for all the parameters were calculated using the optim function with the L-BFGS-B method in R program. Then, we find the inverse of the resulting Hessian matrix using the solve function. The square roots of the diagonal

elements of this inverse matrix give the standard errors of the estimates. We compared the proposed distribution $ABSGT$ with alpha-skew generalized t distribution of (Wahed and ali,2001) and ASN of (Elal-Olivero,2010). Akaike Information Criterion (AIC) is used for model comparison. Further, since are nested models the likelihood ratio (LR) test is used to discriminate between them. The LR test is carried out to test the following hypothesis:
$H_0: \beta = 0$, that is the sample is drawn from $ASGT(\alpha, p, q, \mu, \sigma)$
against the alternative
$H_1: \beta \neq 0$, that is the sample is drawn from $ABSGT(\alpha, \beta, p, q, \mu, \sigma)$.

Aplication 5.1 In this study, we model roller data by using ABSGT distribution. This data set has 1,150 observations which are available at http://lib.stat.cmu.edu/jasadata/laslett website, see also (Gómez,2011) and (Acitas,2015) who analyzed the same data. ML estimates of the parameters and AIC values are given in Table 2. By comparing the AICs, one can see the ABSLG distribution is the best among the fitted models.

Table 2: Estimates of the parameters and AIC for fitted model for real data.

	α	β	μ	σ	p	q	AIC	BIC
ASN	0.0025	—	3.5363	0.6497	—	—	2277.73	2292.87
ASGT	0.3877	—	3.7838	0.6210	1.5208	5.8424	2154.36	2179.60
ASLG	0.2291	—	3.802	0.331	—	—	2165.062	2180.20
ABSN	-0.8117	0.1437	3.3337	0.6298	—	—	2129.778	2149.96
ABSGT	0.3642	0.1421	3.5171	0.6321	1.5321	4.8710	2032.65	2062.94

AIC values suggest that ABSGT distribution is more reliable than ASN and ASGT distributions for roller data. It is also obvious from Figure 3 that ASGT provides a substantially good fitting than ASGT distribution.

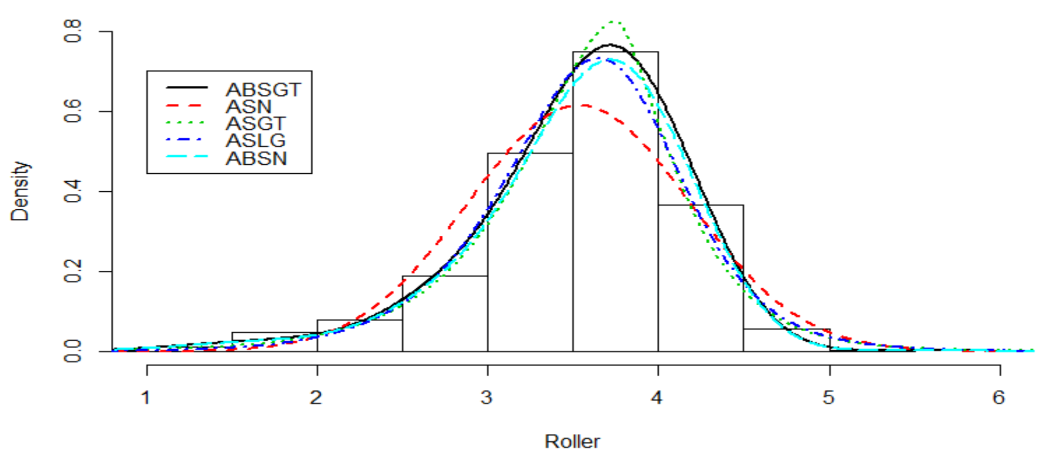


Figure 6: The histogram and the fitted $ASGT$ and $ABSGT$ densities for Roller data.

Results obtained from AIC are also supported by the following likelihood ratio test (LRT) given below. The value of LR test statistics is 42.1 which exceed the 95% critical value. Thus there is evidence in favors of the alternative hypothesis that the sampled data comes from $ABSLG$ not from $ASGT$. Note that (Acitas,2015) conclude that the ASGT distribution yields a better than the ASN.

Aplication 5.2 Faithful geyser data includes 272 observations which denote the waiting-time between eruptions and the duration-time of these eruptions for Old Faithful geyser in Yellow National Park, Wyoming, USA. This popular data is available at Rsystem and see also (Arellano-Valle,2010) in which it is indicated that data is negatively skewed and bimodal. In this study, we use ABSGT distribution to model this popular data and compare results with ASGT distribution. ML estimates of the parameters and AIC values for ABSGT, ASGT and ASN distributions are given in Table 3. The histogram and the fitted density for Faithful geyser data are given in Figure 4.

Table 3: Estimates of the parameters and AIC for fitted model for real data.

	α	β	μ	σ	p	q	AIC	BIC
ASN	-6.0772	—	3.2344	0.6857	—	—	633.90	644.71
ASGT	-7.7342	—	.2586	1.5169	16.2196	1.5262	549.31	567.34
ASLG	1000	—	3.1687	0.3461	—	—	719.60	730.42
ABSN	1534.5479	762.3240	3.1643	0.4954	—	—	617.99	632.41
ABSGT	-10.9143	-3.2509	3.2541	1.3427	12.2041	1.9124	509.87	531.50

The value of LR test statistics is 13.61 which exceed the 95% critical value. Thus, there is evidence in favors of the alternative hypothesis that the sampled data comes from $ABSGT$ not from $ASGT$.

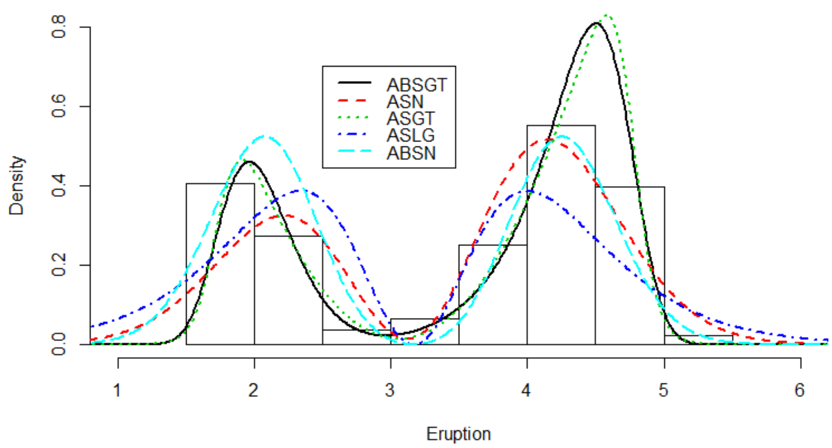


Figure 7: The histogram and the fitted $ASGT$ and $ABSGT$ densities for Eruption data.

Conclusion

In this paper, we proposed a new family of distributions with one extra generator parameters, which includes as special cases of GT, ASGT, BSLG and BLG distributions and some of its basic properties are investigated which include moments, Skewness and Kurtosis functions, mean deviations. Also the below zero truncated version of the proposed distribution was presented as a potential life time distribution. The application of the new family is straightforward. The model parameters are estimated by maximum likelihood and two real examples are used for illustration, where the new family does fit well both data sets.

Acknowledgment
The authors would like to thank the referees and editor for supplying extremely helpful comments and suggestions.

References

1. Acitas, S., Senoglu, B., & Arslan, O. (2015). Alpha-skew generalized t distribution. Revista Colombiana de Estadística, 38(2), 353-370.
2. Arellano-Valle, R. B., Cortés, M. A., & Gómez, H. W. (2010). An extension of the epsilon-skew-normal distribution. *Communications in Statistics—Theory and Methods*, *39*(5), 912-922.
3. Arslan, O., & İ. Genç, A. (2003). Robust location and scale estimation based on the univariate generalized t (GT) distribution. *Communications in Statistics-Theory and Methods*, *32*(8), 1505-1525.
4. Celik, N., Senoglu, B., & Arslan, O. (2015). Estimation and testing in one-way ANOVA when the errors are skew-normal. *Revista Colombiana de Estadística*, *38*(1), 75-91.
5. Elal-Olivero, D. (2010). Alpha-skew-normal distribution. *Proyecciones (Antofagasta)*, *29*(3), 224-240.
6. Gómez, H. W., Elal-Olivero, D., Salinas, H. S., & Bolfarine, H. (2011). Bimodal extension based on the skew-normal distribution with application to pollen data. *Environmetrics*, *22*(1), 50-62.
7. Gupta, A. K., Chang, F. C., & Huang, W. J. (2002). Some skew-symmetric models. *Random Operators and Stochastic Equations*, *10*(2), 133-140.
8. Hazarika, P., & Chakraborty, S. (2014). Alpha-Skew-Logistic Distribution. IOSR Journal of Mathematics, 10(4), 36-46.
9. McDonald, J. B., & Newey, W. K. (1988). Partially adaptive estimation of regression models via the generalized t distribution. *Econometric theory*, *4*(3), 428-457.
10. Nadarajah, S. (2008). On the generalized t (GT) distribution. *Statistics*, *42*(5), 467-473.
11. Theodossiou, P. (1998). Financial data and the skewed generalized t distribution. *Management Science*, *44*(12-part-1), 1650-1661.
12. Tiku, M. L., Islam, M. Q., & Selçuk, A. S. (2001). Nonnormal regression. II. Symmetric distributions. *Communications in Statistics-Theory and Methods*, *30*(6), 1021-1045.
13. Wahed, A., & Ali, M. M. (2001). The skew-logistic distribution. *J. Statist. Res*, *35*(2), 71-80.
14. Wang, D., & Romagnoli, J. A. (2005). Generalized T distribution and its applications to process data reconciliation and process monitoring. *Transactions of the Institute of Measurement and Control*, *27*(5), 367-390.
15. Shafiei, S., Doostparast, M., & Jamalizadeh, A. (2016). The alpha–beta skew normal distribution: Properties and applications. *Statistics*, *50*(2), 338-349.