The New Odd Log-Logistic Generalised Half-Normal Distribution: Mathematical Properties and Simulations

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Abstract

The new distributions are very useful in describing real data sets, because these distributions are more flexible to model real data that present a high degree of skewness and kurtosis. The choice of the best-suited statistical distribution for modeling data is very important. In this paper, A new class of distributions called the New odd log-logistic generalized half-normal (NOLL-GHN) family with four parameters is introduced and studied. This model contains sub-models such as half-normal (HN), generalized half-normal (GHN)and odd log-logistic generalized half-normal (OLL-GHN) distributions. some statistical properties such as moments and moment generating function have been calculated. The Biases and MSE's of estimator methods such as maximum likelihood estimators , least squares estimators, weighted least squares estimators, Cramer-von-Mises estimators, Anderson-Darling estimators and right tailed Anderson-Darling estimators are calculated. The fitness capability of this model has been investigated by fitting this model and others based on real data sets. The maximum likelihood estimators are assessed with simulated real data from proposed model. We present the simulation in order to test validity of maximum likelihood estimators.

Keywords: Generalized half-normal distribution, Moments, Maximum likelihood estimator, Odd log-logistic generalized family, Mean square error.

1 Introduction

Fatigue is considered one of the most common causes of failures of mechanical components. The fatigue process starts with an imperceptible fissure, the initiation, growth and propagation of which produces a dominant crack in the specimen due to cyclic patterns of stress, whose ultimate extension causes the rupture or failure of this specimen (Pescim *et al.*, 2014). From experimental investigation, the fatigue process appears as a random process. In this sense, the extension of a crack produced by fatigue in each cycle is modeled by a random variable which depends on the magnitude of the stress, the type of material, the number of previous cycles, etc. In the literature, there is a large number of statistical models which allow to study the random variation of the failure times associated to materials exposed to fatigue as a result of different cyclic patterns. The most widely used models to describe the lifetime under fatigue process are the half-normal (HN) and Birnbaum-Saunders (BS) distributions. For fitting monotone hazard rates, the HN distribution may be initial choices because of their negatively and positively skewed density shapes. However, in some practical situations, it does not provide a reasonable parametric fit for modeling phenomenon with non-monotone failure rates such as the bathtub shaped and the unimodal failure rates, which are common in reliability and biological studies. To deal with part of this problem, Cooray and Ananda (2008) proposed the *generalized half-normal* (GHN) distribution derived from a model for static fatigue. They demonstrated that the GHN distribution modeling monotone failure rates (increasing and decreasing) and non-monotone failure rate (bathtub shaped) for certain values of its shape parameter, thus providing its greater applicability.

The GHN density function with shape parameter $\lambda > 0$ and scale parameter $\theta > 0$ (Cooray and Ananda, 2008) is given by (for $x > 0$)

$$g(x;\theta,\lambda) = \sqrt{\frac{2}{\pi}}\left(\frac{\lambda}{x}\right)\left(\frac{x}{\theta}\right)^{\lambda}\exp[-\frac{1}{2}\left(\frac{x}{\theta}\right)^{2\lambda}]. \tag{1}$$

Its cumulative distribution function (cdf) depends on the error function

$$G(x;\theta,\lambda) = 2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1 = \text{erf}\left[\frac{\left(\frac{x}{\theta}\right)^{\lambda}}{\sqrt{2}}\right], \tag{2}$$

where

$$\Phi(x) = \frac{1}{2}\left[1+\text{er}f\left(\frac{x}{\sqrt{2}}\right)\right] \quad \textit{and} \quad \text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt.$$

Its nth moment is given by (Cooray and Ananda, 2008) $E(X^n) = \sqrt{\frac{2^{\frac{n}{\lambda}}}{\pi}}\Gamma\left(\frac{n+\lambda}{2\lambda}\right)\theta^n$, where $\Gamma(\cdot)$ is the gamma function. The HN distribution is a sub-model when $\lambda = 1$.

Although this type of density function is asymmetric, the degrees of skewness and/or kurtosis in some cases are outside the distributional range defined by the GHN distribution. Moreover, this distribution is not suitable in situations where the hazard rate function (hrf) is unimodal. Hence, a more suitable distribution for adjusting such unexpected deviations is an important issue. Thus, a great number of extended distributions has been proposed in survival analysis in order to provide a greater flexibility for extended distributions and to allow that the hrf may describe monotone (increasing and decreasing) and non-monotone (bathtub shaped and unimodal) forms. Further, the extended distribution can be used to discriminate models, since it has as a special case, the distribution that generated it.

The GHN distribution has been widely modified and studied in recent years and various authors developed new generalizations from this lifetime model. Pescim et al. (2010) introduced the beta generalized half-Normal (BGHN) distribution with applications to myelogenous leukemia data. Cordeiro et al. (2012) defined the Kumaraswamy generalized half-normal (KwGHN) distribution for censored data. Recently, Pescim et al. (2013) proposed a log-linear regression model based on the BGHN distribution, while Ramires et al. (2013) defined the beta generalized half-normal geometric (BGHNG) distribution in order to achieve wider diversity among the density and failure rate functions.

Corderio *et al.* (2016) introduced and studied a three-parameter extension of the GHN distribution based on the OLL-G family refereed to as the odd log-logistic generalized half-normal (OLLGHN) distribution, which contains as sub-models the HN and GHN distributions.

In this paper, we introduce and study a four-parameter extension of the GHN distribution based on the T-X idea by Alzaatreh et.al, (2013), which contains as sub-models the HN, GHN and OLLGHN distributions. The new distribution due to its flexibility in accommodating bathtub and unimodal shape forms of the hrf could be an important model in a variety of problems in survival analysis. It is also suitable for testing goodness-of-fit of the special cases.

The cdf and pdf of new family are given by

$$F(x;\alpha,\beta,\theta,\lambda)=\int_0^{\frac{G(x;\theta,\lambda)^{\alpha}}{\bar{G}(x;\theta,\lambda)^{\beta}}}\frac{dt}{(1+t)^2}=\frac{G(x;\theta,\lambda)^{\alpha}}{G(x;\theta,\lambda)^{\alpha}+\bar{G}(x,;\theta,\lambda)^{\beta}}$$

$$=\frac{\left\{2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right\}^{\alpha}}{\left\{2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right\}^{\alpha}+\left\{2-2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]\right\}^{\beta}}, \quad (3) \qquad (3)$$

and

$$f(x;\alpha,\beta,\theta,\lambda)=\frac{g(x;\theta,\lambda)G(x;\theta,\lambda)^{\alpha-1}\bar{G}(x;\theta,\lambda)^{\beta-1}}{\left[G(x;\theta,\lambda)^{\alpha}+\bar{G}(x;\theta,\lambda)^{\beta}\right]^2}\times[\alpha+(\beta-\alpha)G(x;\theta,\lambda)]$$

$$=\frac{\sqrt{\frac{2}{\pi}}\left(\frac{\lambda}{x}\right)\left(\frac{x}{\theta}\right)^{\lambda}e^{-\frac{1}{2}\left(\frac{x}{\theta}\right)^{2\lambda}}\left\{2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right\}^{\alpha-1}\left\{2-2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]\right\}^{\beta-1}}{\left\{\left\{2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right\}^{\alpha}+\left\{2-2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]\right\}^{\beta}\right\}^2}$$

$$\times\left(\alpha+(\beta-\alpha)\left(2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right)\right) \quad (4)$$

where $\alpha,\beta>0$ are two shape parameters. These parameters can provide great flexibility to model the skewness and kurtosis of the generated distribution.

Henceforth, we denote by $X \sim NOLLGHN(\lambda,\theta,\alpha,\beta)$ a random variable having pdf (4). The new model contains some important sub-models. For $\alpha=\beta=1$, it gives the GHN distribution. For $\alpha=\beta$, it gives the OLLGHN distribution. If $\lambda=1$ and $\alpha=\beta$ it yields the odd log-logistic half-normal (OLLHN) distribution.

Further, if $\alpha=\beta=1$, in addition to $\lambda=1$, it reduces to the HN distribution. The hrf corresponding to (4) is given by

$$h(x)=\frac{\sqrt{\frac{2}{\pi}}\left(\frac{\lambda}{x}\right)\left(\frac{x}{\theta}\right)^{\lambda} \mathrm{e}^{-\frac{1}{2}\left(\frac{x}{\theta}\right)^{2\lambda}}\left\{2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right\}^{\alpha-1}}{\left\{2-2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]\right\}\left\{\left[2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right]^{\alpha}+\left[2-2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]\right]^{\beta}\right\}}$$
$$\times\left(\alpha+(\beta-\alpha)\left(2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right)\right), \qquad (5)$$

respectively. Plots of density functions and hazard rate functions for selected parameter values, are displayed in Figures 1, 2, 3 and 4, respectively. It is evident that the new distribution is much more flexible than the GHN and OLLGHN distribution. Further, it allows four major hazard shapes: increasing, bathtub and unimodal hazard rates.

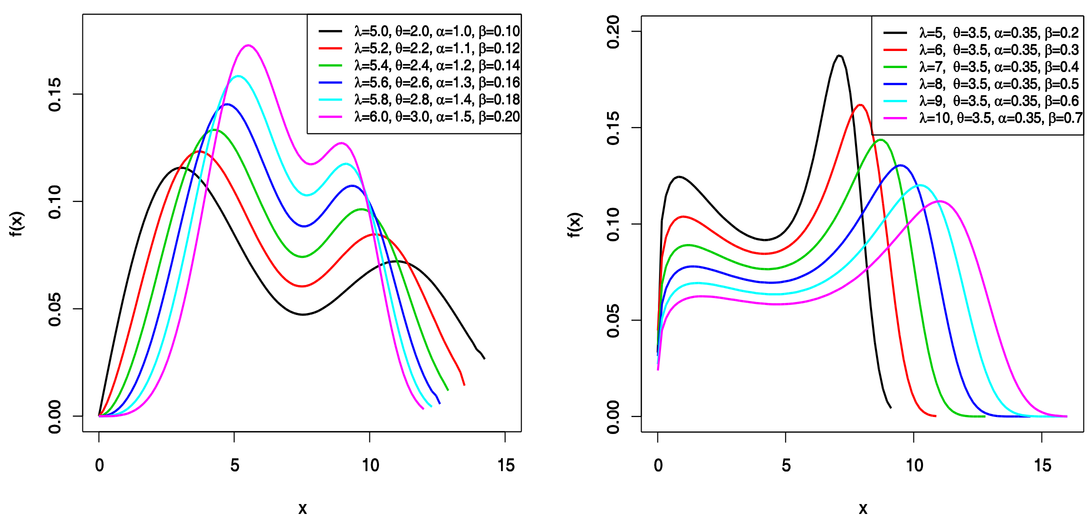

Figure 1: Plots of the NOLLGHN pdf for some parameter values.

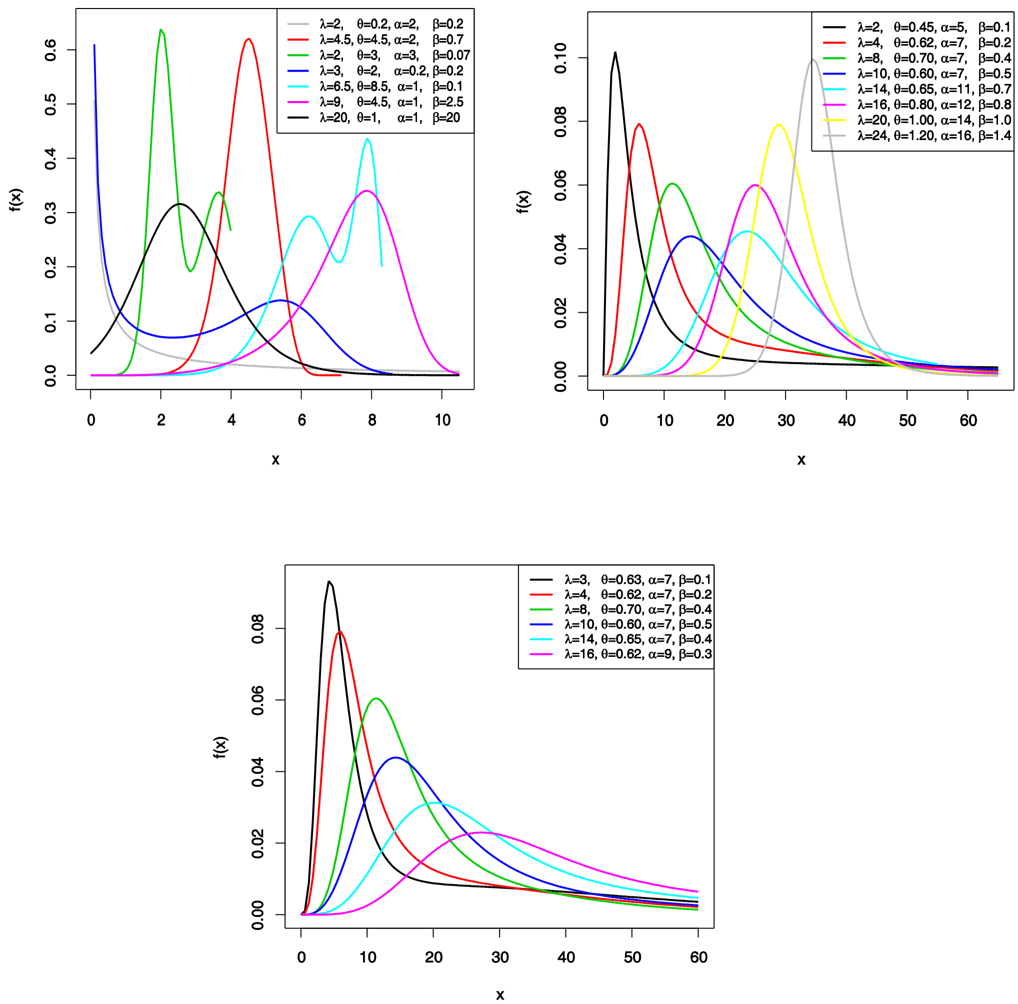

Figure 2: Plots of the NOLLGHN pdf for some parameter values.

2 Useful expansions

Useful expansions for equations (3) and (4) can be derived using the idea of the exponentiated-G ("EG" for short) family of distributions. Its properties have been reported by several authors in recent years, see Mudholkar and Srivastava (1993) and Mudholkar *et al.* (1995) for exponentiated Weibull, Gupta *et al.* (1998) for exponentiated Pareto, Gupta and Kundu (2001) for exponentiated exponential, Nadarajah (2005) for exponentiated Gumbel, Kakade and Shirke (2006) for exponentiated log-normal, Nadarajah and Gupta (2007) for exponentiated gamma and Cordeiro *et al.* (2011) for exponentiated generalized gamma distributions.

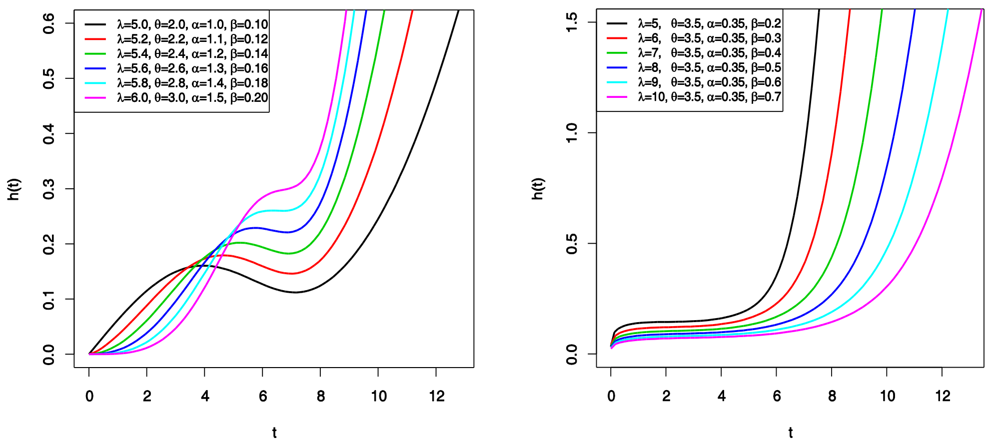

Figure 3: Plots of the NOLLGHN hazard rate for some parameter values.

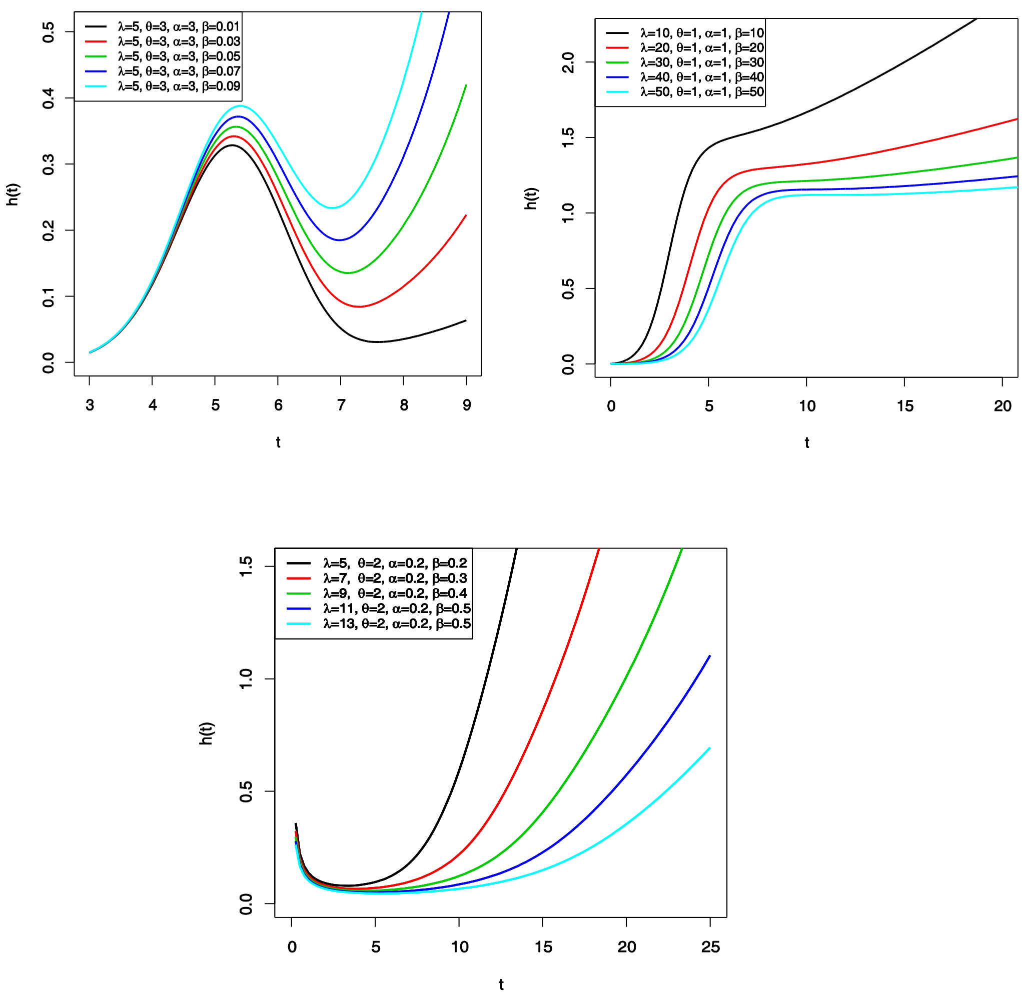

Figure 4: Plots of the NOLLGHN hazard rate for some parameter values.

First, we define the EG family for an arbitrary parent distribution $G(x)$, say $Y \sim EG(\phi)$, $\phi > 0$, if its cdf and pdf (for $\phi > 0$) are given by

$$H_\phi(x) = G(x)^\phi \qquad \text{and} \qquad h_\phi(x) = \phi g(x) G(x)^{\phi-1},$$

respectively. This transformed model is also called the Lehmann type I distribution, say $EG(\phi)$.

Next, we obtain an expansion for $F(x)$. First, we use a power series for $\{2\Phi[(\frac{x}{\theta})^\lambda]-1\}^\alpha$, for $\alpha > 0$ real, given by

$$\left\{2\Phi\left[\left(\frac{x}{\theta}\right)^\lambda\right]-1\right\}^\alpha = \sum_{k=0}^{\infty} a_k \left\{2\Phi\left[\left(\frac{x}{\theta}\right)^\lambda\right]-1\right\}^k, \tag{6}$$

where

$$a_k = a_k(\alpha) = \sum_{j=k}^{\infty} (-1)^{k+j} \binom{\alpha}{j}\binom{j}{k}.$$

For any real $\alpha > 0$, we consider the generalized binomial expansion

$$\left\{2-2\Phi\left[\left(\frac{x}{\theta}\right)^\lambda\right]\right\}^\beta = \sum_{k=0}^{\infty} (-1)^k \binom{\beta}{k} \left\{2\Phi\left[\left(\frac{x}{\theta}\right)^\lambda\right]-1\right\}^k. \tag{7}$$

Inserting (6) and (7) in equation (3), we obtain

$$F(x) = \frac{\sum_{k=0}^{\infty} a_k \left\{2\Phi\left[\left(\frac{x}{\theta}\right)^\lambda\right]-1\right\}^k}{\sum_{k=0}^{\infty} b_k \left\{2\Phi\left[\left(\frac{x}{\theta}\right)^\lambda\right]-1\right\}^k},$$

where $b_k = a_k + (-1)^k \binom{\beta}{k}$.

The ratio of the two power series can be expressed as

$$F(x) = \sum_{k=0}^{\infty} c_k \left\{2\Phi\left[\left(\frac{x}{\theta}\right)^\lambda\right]-1\right\}^k, \tag{8}$$

where the coefficients c_k 's (for $k \geq 1$) are determined from the recurrence equation

$$c_k = b_0^{-1}\left(a_k - b_0^{-1} \sum_{r=1}^{k} b_r c_{k-r}\right).$$

and $c_0 = \frac{a_0}{b_0}$

By differentiating (8), the pdf of X can be expressed as

$$f(x) = \sum_{k=0}^{\infty} c_{k+1} h_{k+1}(x), \tag{9}$$

where

$$h_{k+1}(x)=(k+1)\sqrt{\frac{2}{\pi}}\left(\frac{\lambda}{x}\right)\left(\frac{x}{\theta}\right)^{\lambda}\exp\left[-\frac{1}{2}\left(\frac{x}{\theta}\right)^{2\lambda}\right]\left\{2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right\}^{k}$$

is the *exponentiated generalized half-normal* (EGHN) density function with power parameter $(k+1)$. Equation (9) reveals that the density function of X is a linear combination of EGHN densities. Thus, some structural properties of the NOLLGHN distribution such as the ordinary and incomplete moments and generating function can be determined from well-established properties of the EGHN distribution. Equations (8) and (9) are the main results of this section.

3 Statistical properties

In this section some statistical properties of the proposed distribution are investigated.

3.1 Moments

By setting $u=(x/\theta)^{\lambda}$ and considering the error function as the cdf of the GHN distribution, the nth moment of X can be obtained from equation (9) as

$$E(X^{n})=\theta^{n}\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}c_{r+1}I\left(\frac{n}{\lambda},k\right),$$

where

$$I\left(\frac{n}{\lambda},k\right)=\int_{0}^{\infty}u^{\frac{n}{\lambda}}\exp\left(-\frac{u^{2}}{2}\right)\left[\mathrm{e}rf\left(\frac{u}{\sqrt{2}}\right)\right]^{k}du.$$

Inserting the power series for the error function

$$\mathrm{e}rf(x)=\frac{2}{\sqrt{\pi}}\sum_{m=0}^{\infty}\frac{(-1)^{m}x^{2m+1}}{(2m+1)m!},$$

in the last equation and computing the integral, we have (for any real $k+n/\lambda$)

$$E(X^{n})=\theta^{n}\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}c_{k+1}I\left(\frac{n}{\lambda},k\right), \tag{10}$$

where

$$I\left(\frac{n}{\lambda},k\right)=\pi^{-\frac{k}{2}}2^{k+\frac{n}{2\lambda}-\frac{1}{2}}\sum_{m_1,\ldots,m_k=0}^{\infty}\frac{(-1)^{m_1+\ldots+m_k}\Gamma\left(m_1+\ldots+m_k+\frac{k+\frac{n}{\lambda}+1}{2}\right)}{(m_1+1/2)\ldots(m_r+1/2)m_1!\ldots m_k!}. \tag{11}$$

Further, for the very special case when $k+\frac{n}{\lambda}$ is even, the integral $I\left(\frac{n}{\lambda},k\right)$ can be expressed in terms of the Lauricella function of type A (Exton, 1978; Aarts, 2000) defined by

$$F_A^{(n)}(a;b_1,\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n)=\sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c_1)_{m_1}\cdots(c_n)_{m_n}}\frac{x_1^{m_1}\ldots x_n^{m_n}}{m_1!\ldots m_{n!}},$$

where $(a)_k = a(a+1)\ldots(a+k-1)$ is the ascending factorial (with the convention that $(a)_0 = 1$). Numerical routines for the direct computation of the Lauricella function of type A are available, see Exton (1978) and Mathematica (Trott, 2006). Hence, $E(X^n)$ can be expressed in terms of the Lauricella functions of type A

$$E(X^n)=\theta^n\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}B_k\,F_A^{(k)}\left(\frac{k+\frac{n}{\lambda}+1}{2};\frac{1}{2},\ldots,\frac{1}{2};\frac{3}{2},\ldots,\frac{3}{2};-1,\ldots,-1\right),$$

where

$$B_k=\pi^{-\frac{k}{2}}2^{2k+\frac{n}{2\lambda}-\frac{1}{2}}\Gamma\left(\frac{k+\frac{n}{\lambda}+1}{2}\right)c_{k+1}.$$

The p th descending factorial moment of X is

$$E(X^{(p)})=E[X(X-1)\times\cdots\times(X-p+1)]=\sum_{n=0}^{p}s(p,n)E(X^n),$$

where $s(r,n)=(n!)^{-1}[d^n n^{(r)}/dx^n]_{x=0}$ is the Stirling number of the first kind. Other kinds of moments may also be obtained in closed-form, but we consider only the previous moments for reasons of space.

The measures of skewness and kurtosis of the NOLLGHN distribution can be obtained as follows:

$$Skewness(X)=\frac{\mu_3^{'}-3\mu_2^{'}\mu_1^{'}+2(\mu_1^{'})^3}{\left(\mu_2^{'}-(\mu_1^{'})^2\right)^{\frac{3}{2}}}=\frac{\kappa_3}{\kappa_2^{(3/2)}}, \tag{12}$$

and

$$Kurtosis(X)=\frac{\mu_4^{'}-4\mu_1^{'}\mu_3^{'}+6(\mu_1^{'})^2\mu_3^{'}-3(\mu_1^{'})^4}{\kappa_2}, \tag{13}$$

Plots of skewness and kurtosis of the NOLLGHN distribution are displayed in Figure 5.

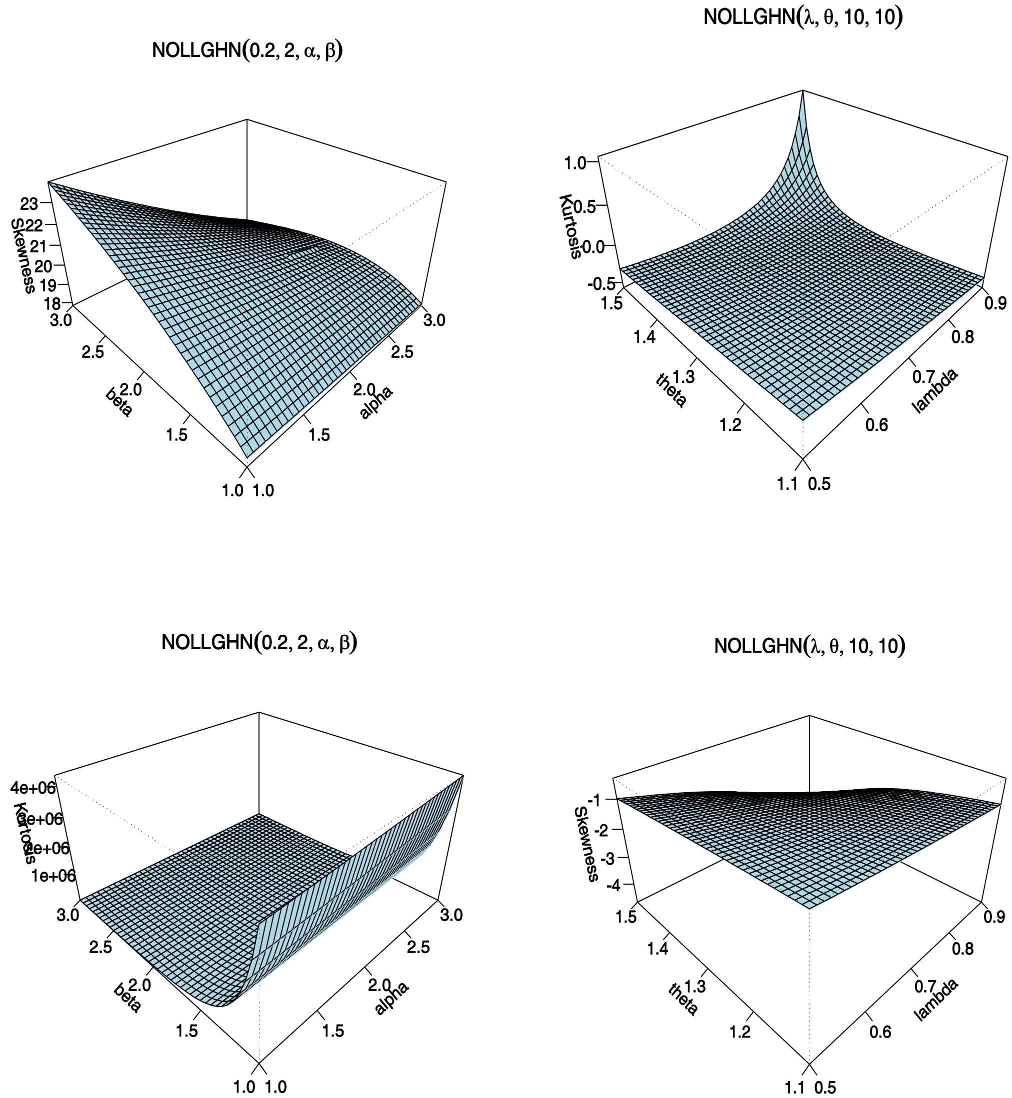

Figure 5: Skewness and Kurtosis for NOLLGHN.

3.2 Generating Function

By setting $u=\left(\frac{x}{\theta}\right)^{\lambda}$, the moment generating function (mgf) of X can be obtained from (9) as

$$M(t)=\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}(k+1)c_{k+1}\sum_{m=0}^{\infty}\frac{\theta^m t^m}{m!}\int_0^{\infty}u^{\frac{m}{\lambda}}\exp\left(-\frac{1}{2}u^2\right)\left[erf\left(\frac{u}{\sqrt{2}}\right)\right]^k du,$$

$M(t)$ can be reduced to

$$M(t) = \sum_{m=0}^{\infty} \frac{A_m t^m}{m!}, \tag{14}$$

where

$$A_m = \sqrt{\frac{2}{\pi}} \theta^m \sum_{k=0}^{\infty} (k+1) c_{k+1} I\left(\frac{m}{\lambda}, k\right),$$

and $I\left(\frac{m}{\lambda}, k\right)$ is defined by (11).

4 Estimation and inference

In this section, we determine the maximum likelihood estimates (MLE's) of the model parameters of the new family from complete samples only. Let $x_1, \dots, x_n$ be observed values from the NOLLGHN family with parameters λ, θ, α and β. Let $\gamma = (\alpha, \beta, \lambda, \theta)^{\mathrm{T}}$ be the $r \times 1$ parameter vector. The total log-likelihood function for γ is given by

$$\begin{aligned}
\ell_n(\gamma) &== n \log\left(\sqrt{\frac{2}{\pi}}\right) + n \log(\lambda) - n\lambda \log(\theta) \\
&+ (\lambda - 1) \sum_{i=1}^{n} \log(x_i) - \frac{1}{2} \sum_{i=1}^{n} \left(\frac{x_i}{\theta}\right)^{2\lambda} + (\alpha - 1) \sum_{i=1}^{n} \log(q_i - 1) \\
&+ (\beta - 1) \sum_{i=1}^{n} \log(2 - q_i) - 2 \sum_{i=1}^{n} \log\{(q_i - 1)^{\alpha} + (2 - q_i)^{\beta}\} \\
&+ \sum_{i=1}^{n} \log\left(\alpha + (\beta - \alpha)(q - i - 1)\right),
\end{aligned} \tag{15}$$

where $x_i > 0$ and $q_i = 2\Phi\left[\left(\frac{x_i}{\theta}\right)^{\lambda}\right]$.

$U_n(\gamma) = \left(\partial \ell_n / \partial \lambda, \partial \ell_n / \partial \theta, \partial \ell_n / \partial \alpha, \partial \ell_n / \partial \beta\right)^{\mathrm{T}}$ are given by

$$\begin{aligned}
\frac{\partial \ell_n}{\partial \lambda} &= \frac{n}{\lambda} + \sum_{i=1}^{n} \log\left(\frac{x_i}{\theta}\right) - \sum_{i=1}^{n} \left(\frac{x_i}{\theta}\right)^{2\lambda} \log\left(\frac{x_i}{\theta}\right) \\
&+ (\alpha - 1) \sum_{i=1}^{n} \frac{q_i^{(\lambda)}}{q_i - 1} - (\beta - 1) \sum_{i=1}^{n} \frac{q_i^{(\lambda)}}{2 - q_i} \\
&- 2 \sum_{i=1}^{n} \frac{q_i^{(\lambda)} \left[\alpha (q_i - 1)^{\alpha - 1} - \beta (2 - q_i)^{\beta - 1}\right]}{(q_i - 1)^{\alpha} + (2 - q_i)^{\beta}} \\
&+ (\beta - \alpha) \sum_{i=1}^{n} \frac{q_i^{(\lambda)}}{\alpha + (\beta - \alpha)(q_i - 1)},
\end{aligned}$$

$$\frac{\partial \ell_n}{\partial \theta} = -\frac{n\lambda}{\theta} + \frac{\lambda}{\theta} \sum_{i=1}^{n} \left(\frac{x_i}{\theta}\right)^{2\lambda}$$

$$+(\alpha-1)\sum_{i=1}^{n}\frac{q_i^{(\theta)}}{q_i-1}-(\beta-1)\sum_{i=1}^{n}\frac{q_i^{(\theta)}}{2-q_i}$$
$$-2\sum_{i=1}^{n}\frac{q_i^{(\theta)}\left[\alpha(q_i-1)^{\alpha-1}-\beta(2-q_i)^{\beta-1}\right]}{(q_i-1)^{\alpha}+(2-q_i)^{\beta}}$$
$$+(\beta-\alpha)\sum_{i=1}^{n}\frac{q_i^{(\theta)}}{\alpha+(\beta-\alpha)(q_i-1)},$$

$$\frac{\partial\ell_n}{\partial\alpha}=\sum_{i=1}^{n}\log(q_i-1)-2\sum_{i=1}^{n}\frac{(q_i-1)^{\alpha}\log(q_i-1)}{(q_i-1)^{\alpha}+(2-q_i)^{\beta}}+\sum_{i=1}^{n}\frac{2-q_i}{\alpha+(\beta-\alpha)(q_i-1)},$$

$$\frac{\partial\ell}{\partial\beta}=\sum_{i=1}^{n}\log(2-q_i)-2\sum_{i=1}^{n}\frac{(2-q_i)^{\beta}\log(2-q_i)}{(q_i-1)^{\alpha}+(2-q_i)^{\beta}}+\sum_{i=1}^{n}\frac{q_i-1}{\alpha+(\beta-\alpha)(q_i-1)},$$

where

$$q_i^{(\lambda)}=\frac{\partial q_i}{\partial\lambda}=2\left(\frac{x_i}{\theta}\right)^{\lambda}\log\left(\frac{x_i}{\theta}\right)\phi\left[\left(\frac{x_i}{\theta}\right)^{\lambda}\right],$$

$$q_i^{(\theta)}=\frac{\partial q_i}{\partial\theta}=-2\left(\frac{\lambda}{\theta}\right)\left(\frac{x_i}{\theta}\right)^{\lambda}\phi\left[\left(\frac{x_i}{\theta}\right)^{\lambda}\right].$$

The MLE γ of γ is obtained by solving the nonlinear likelihood equations $U_{\lambda}(\gamma)=0$, $U_{\theta}(\gamma)=0$, $U_{\alpha}(\gamma)=0$ and $U_{\beta}(\gamma)=0$. These equations cannot be solved analytically and statistical software can be used to solve them numerically. We can use iterative techniques such as a Newton-Raphson type algorithm to obtain the estimate γ. We employ the numerical procedures in R.

For interval estimation and hypothesis tests on the parameters in γ, we obtain the observed information matrix since the expected information matrix is very complicated and requires numerical integration. The 4×4 observed information matrix $J(\gamma)$, becomes as follows:

$$J(\gamma)=-\begin{pmatrix} L_{\alpha\alpha} & L_{\alpha\beta} & L_{\alpha\lambda} & L_{\alpha\theta} \\ . & L_{\beta\beta} & L_{\beta\lambda} & L_{\beta\theta} \\ . & . & L_{\lambda\lambda} & L_{\lambda\theta} \\ . & . & . & L_{\theta\theta} \end{pmatrix}.$$

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the estimated approximate multivariate normal $N_4(0,n^{-1}J(\gamma)^{-1})$ can be used to construct approximate confidence intervals for the model parameters.

The likelihood ratio (LR) statistics are useful for comparing the new distribution with some special models. For example, we may use the LR statistic to check if the fit using the NOLLGHN distribution is statistically "superior" to a fit using the GHN and OLLGHN distribution for a given data set. In any case, considering the partition $\gamma = (\gamma_1^{\mathrm{T}}, \gamma_2^{\mathrm{T}})^{\mathrm{T}}$, tests of hypotheses of the type $H_0 : \gamma_1 = \gamma_1^{(0)}$ versus $H_A : \gamma_1 \neq \gamma_1^{(0)}$ can be performed using the LR statistic $w = 2\{\ell(\gamma) - \ell(\gamma)\}$, where γ and γ are the estimates of γ under H_A and H_0, respectively. Under the null hypothesis H_0, $w \xrightarrow{d} \chi_q^2$, where q is the dimension of the parameter vector γ_1 of interest. The LR test rejects H_0 if $w > \xi_\gamma$, where ξ_γ denotes the upper 100γ% point of the χ_q^2 distribution.

5 Simulation study

5.1 The Maximum Likelihood Estimator

In this section, the Maximum likelihood estimators of parameters of purpose density function has been assessed by simulating: $(\lambda, \theta, \alpha, \beta) = (4, 4, 2, 0.5)$. The density function has been indicated in Figure 6.

To verify the validity of the maximum likelihood estimator, the bias of MLE and the mean square error of MLE have been used. For example, as described in Section 4, for $(\lambda, \theta, \alpha, \beta) = (4, 4, 2, 0.5)$, $r = 1000$ times have been simulated samples of $n = 10, 11, ..., 70$ of $\mathrm{NOLLGHN}(4, 4, 2, 0.5)$. To estimate the numerical value of the maximum likelihood, the *optim* function (in the *stat* package) and *Nelder – Mead* method in R software has been used. If $\mathrm{q} = (\lambda, \theta, \alpha, \beta)$, for any simulation by n volume and $i = 1, 2, ..., r$, the maximum likelihood estimates are obtained as $\mathrm{q}_i = (\lambda_i, \theta_i, \alpha_i, \beta_i)$.

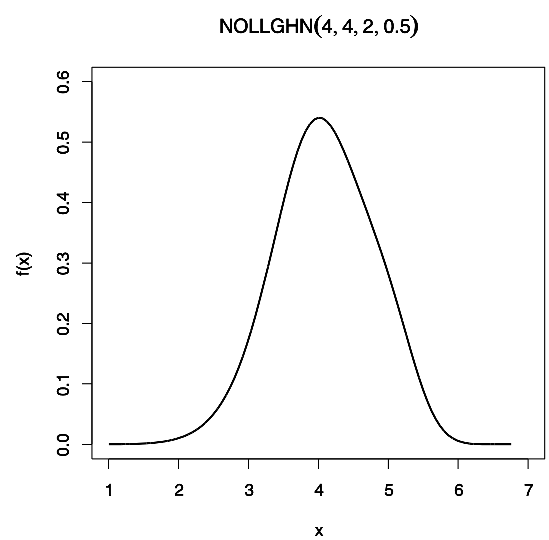

Figure 6: The density function for simulation study.

To examine the performance of the MLE's for the NOLLGHN distribution, we perform a simulation study:

1. Generate r samples of size n from equation (4).
2. Compute the MLE's for the r samples, say $(\lambda_i, \theta_i, \alpha_i, \beta_i)$ for $i = 1, 2, \ldots, r$.
3. Compute the standard errors of the MLE's for r samples, say $(s_{\hat{\lambda}}, s_{\hat{\theta}}, s_{\hat{\alpha}}, s_{\hat{\beta}})$

for $i = 1, 2, \ldots, r$.

4. Compute the biases and mean squared errors given by

$$Bias_{\mathrm{q}}(n) = \frac{1}{r}\sum_{i=1}^{r}(\mathrm{q}_i - \mathrm{q}_i),$$

$$MSE_{\mathrm{q}}(n) = \frac{1}{r}\sum_{i=1}^{r}(\mathrm{q}_i - \mathrm{q}_i)^2,$$

for $\theta = (\lambda, \theta, \alpha, \beta)$.

We repeat these steps for $r = 1000$ and $n = 10, 11, \ldots n^*$ (n^* is different in each issue) with different values of $(\lambda, \theta, \alpha, \beta)$, so computing $Bias_{\mathrm{q}}(n)$, $MSE_{\mathrm{q}}(n)$.

Figure 7, 8 respectively reveals how the four biases, mean squared errors vary with respect to n. As expected, the Biases and MSEs of estimated parameters converges to zero while n growing.

5.2 The other estimation methods

There are several approaches to estimate the parameters of distributions that each of them has its characteristic features and benefits. In this subsection five of those methods are briefly introduced and will be numerically investigated in the simulation study Figure 6. A useful summary of these methods can be seen in Dey et al., (2017). Here $\{t_i; i = 1, 2, ..., n\}$ is the associated order statistics and F is the distribution function of NOLLGHN.

5.2.1 Least squares and weighted least squares estimators

The Least Squares (LSE) and weighted Least Squares Estimators (WLSE) are introduced by Swain et al., (1988). The LSE's and WLSE's are obtained by minimizing the following functions:

$$S_{\mathrm{LSE}}(\alpha, \theta, \mu, \sigma) = \sum_{i=1}^{n}\left(F(t_i; \alpha, \theta, \mu, \sigma) - \frac{i}{n+1}\right)^2,$$

$$S_{\mathrm{WLSE}}(\alpha, \theta, \mu, \sigma) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(F(t_i; \alpha, \theta, \mu, \sigma) - \frac{i}{n+1}\right)^2.$$

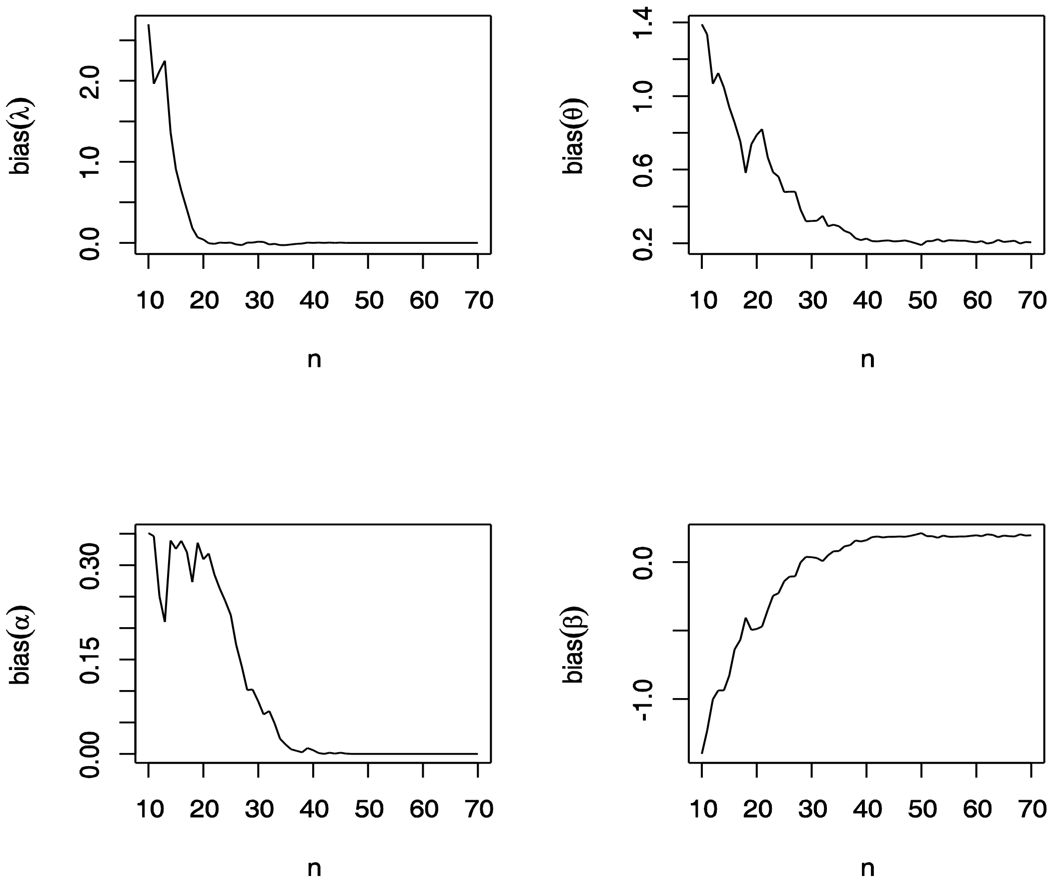

Figure 7: Bias of $\hat{\lambda},\hat{\theta},\hat{\alpha},\hat{\beta}$ versus n when $(\lambda,\theta,\alpha,\beta)=(4,4,2,0.5)$.

5.2.2 Cramér– von– Mises estimator

Cramér– von– Mises Estimator (CME) is introduced by Choi and Bulgren (1968). The CMEs is obtained by minimizing the following function:

$$S_{\text{CME}}(\alpha,\theta,\mu,\sigma)=\frac{1}{12n}+\sum_{i=1}^{n}\left(F(t_i;\alpha,\theta,\mu,\sigma)-\frac{2i-1}{2n}\right)^2.$$

5.2.3 Anderson– Darling and right-tailed Anderson– Darling

The Anderson–Darling (ADE) and Right-Tailed Anderson–Darling Estimators (RTADE) are introduced by Anderson and Darling (1952). The ADE's and RTADE's are obtained by minimizing the following functions:

$$S_{\text{ADE}}(\alpha,\theta,\mu,\sigma) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\{\log F(t_i;\alpha,\theta,\mu,\sigma) + \log \overline{F}(t_{n+1-i};\alpha,\theta,\mu,\sigma)\}$$

$$S_{\text{RTADE}}(\alpha,\theta,\mu,\sigma) = \frac{n}{2} - 2\sum_{i=1}^{n}F(t_i;\alpha,\theta,\mu,\sigma) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log \overline{F}(t_{n+1-i};\alpha,\theta,\mu,\sigma)$$

where $\overline{F}(\cdot) = 1 - F(\cdot)$.

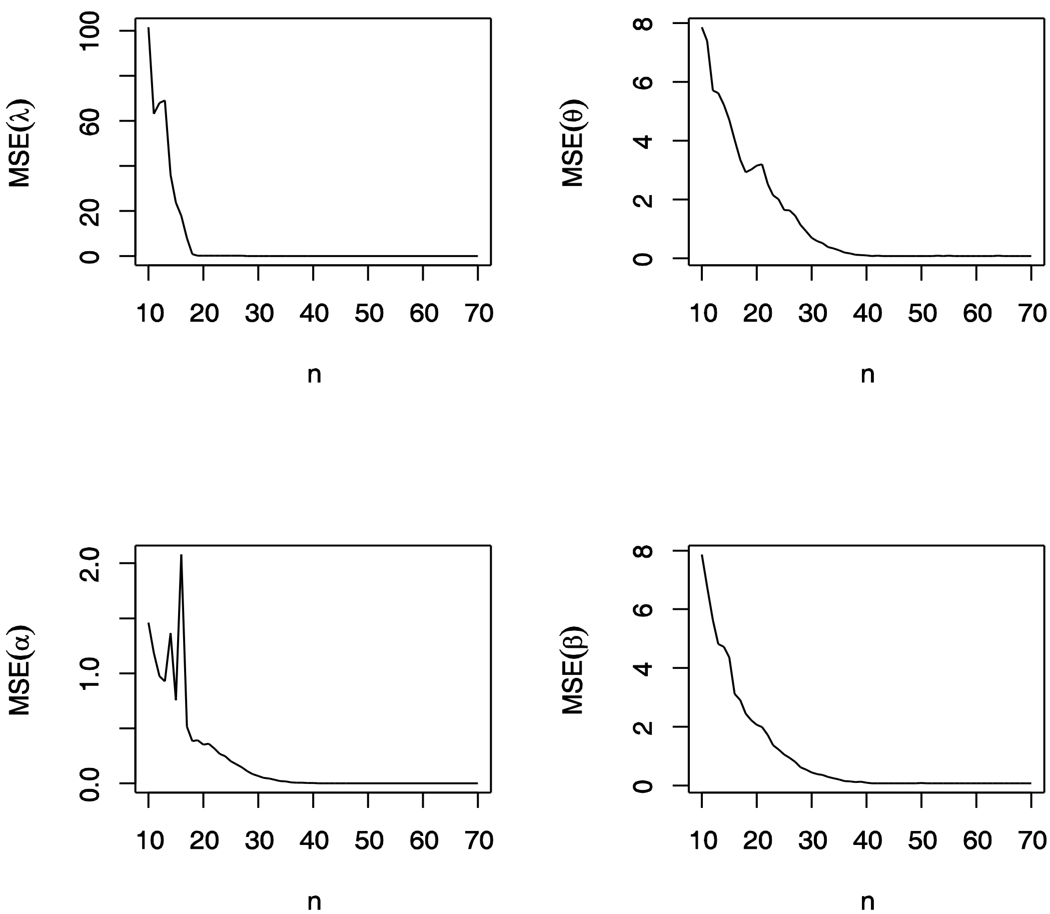

Figure 8: MSE of $\hat{\lambda},\hat{\theta},\hat{\alpha},\hat{\beta}$ versus n when $(\lambda,\theta,\alpha,\beta) = (4,4,2,0.5)$.

In order to explore the estimators introduced above we consider the one model that have been used in this section , and investigate MSE of those estimators for different samples. For instance according to what has been mentioned above, for $(\lambda,\theta,\alpha,\beta) = (4,4,2,0.5)$. we have simulated r = 1000 times with sample size of the $n = 50,55,60,\cdots 600$ and then the MSE formula that are mentioned in the subsection 5.1 are calculated for them. To obtain the value of the estimators, we have used the optima function and $Nelder-Mead$ method in R.

The result of the simulations of this subsection is shown in Figure 9. As it is clear from the MSE plot for two parameters with the increase in the volume of the sample all methods will approach to zero and this verifies the validity of the these estimation methods and numerical calculations for the distribution parameters NOLLGHN.

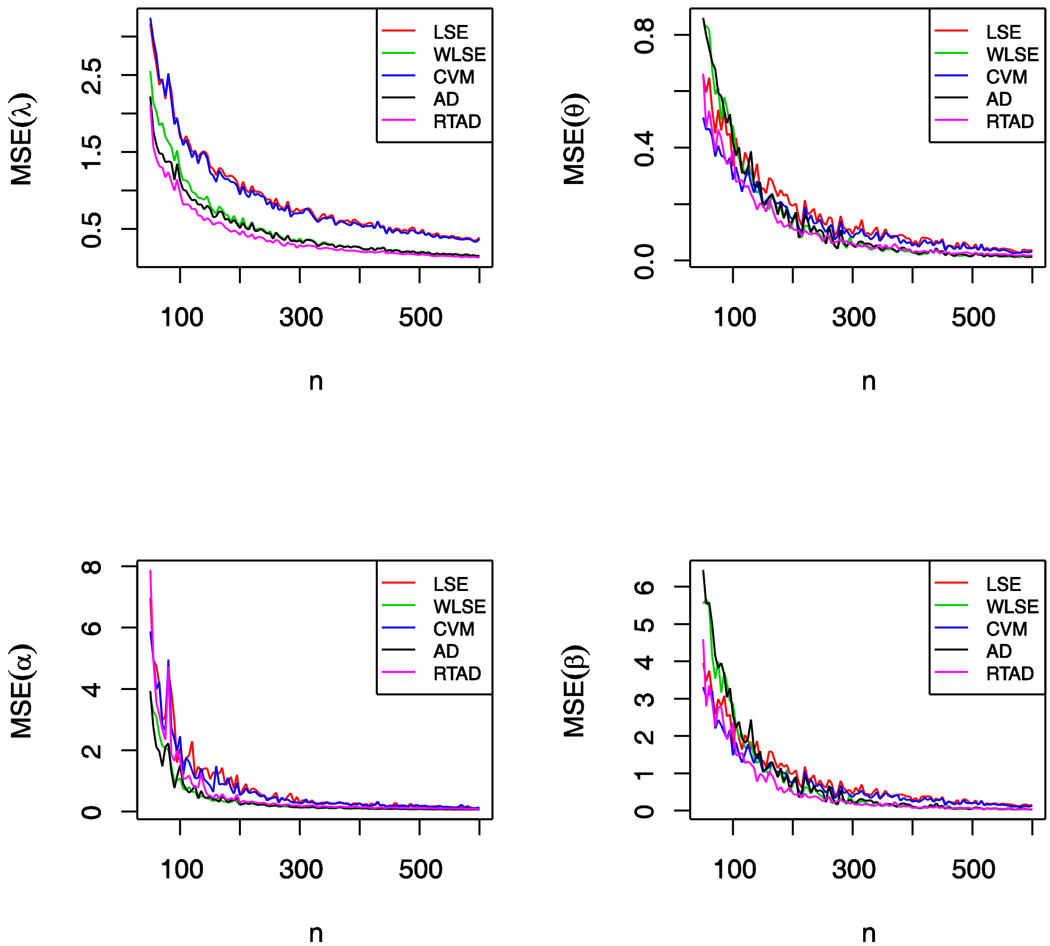

Figure 9: MSE of $\hat{\lambda},\hat{\theta},\hat{\alpha},\hat{\beta}$ versus n when $(\lambda,\theta,\alpha,\beta)=(4,4,2,0.5)$.

6 Application

Here, for the purpose of illustration, we analyze four data sets. We choose these data because they really show in different fields that it is necessary to have non-negative support.

Description of the data sets.

Data Set 1: Survival times: The data analyzed by Kundu et al. (2008) and Leiva et al. (2009) correspond to 72 survival times of guinea pigs injected with different doses of tubercle bacilli.

Data Set 2: OTIS IQ Scores: The data set concerns OTIS IQ Scores for 52 minority (non-white) males hired by the company. One of the key questions of the study was the predictability of job performance when the OTIS test was applied.

This data were given by Sharfi and Behboodian (2008), which were compiled in 1971 by a large insurance company in order to investigate its selection procedures for claims adjusters.

Data Set 3: USS Halfbeak Diesel Engine: The data set presented by Ascher and Feingold (1984) from a USS Halfbeak (submarine) diesel engine. The data denote 73 failure times (in hours) of unscheduled maintenance actions for the USS Halfbeak number 4 main propulsion diesel engine over 25.518 operating hours.

Data Set 4: failure times:

The data set relates to one hundred and one data points subjected to constant sustained pressure at the ninety percent stress level until all had failed, so the data are complete. The failure times are in hours are shown in Andrews and Herzberg (1985).

Table 1 gives a descriptive summary for these data showing different degrees of skewness and kurtosis.

Table 1: Descriptive statistics for six data set.

Data Set	Mean	Median	SD	Skewness	Kurtosis	Min.	Max.
Survival times	99.8194	70	81.1179	1.8347	2.8937	12	376
OTIS IQ Scores	106.6538	105	8.3099	0.3861	-0.5039	91	123
USS Halfbeak Diesel Engine	19.3997	21.461	5.8165	-1.5764	1.6525	1.382	25.518
failure times	1.0248	0.8	1.1194	3.0472	14.4745	0.01	7.89

In the following, we compare the proposed model with some other lifetime distributions, namely:

(i) Generalized half-normal distribution (GHN) (Cooray and Ananda (2008)).

The GHN density function is given by

$$f(x;\theta,\lambda)=\sqrt{\frac{2}{\pi}}\left(\frac{\lambda}{x}\right)\left(\frac{x}{\theta}\right)^{\lambda}e^{-\frac{1}{2}\left(\frac{x}{\theta}\right)^{2\lambda}},$$

where $x>0$, $\theta>0$ and $\lambda>0$.

(ii) Odd log-logistic generalized half-normal distribution (OLLGHN) (Cordeiro et al. (2016)).

The OLLGHN density function is given by

$$f(x;\theta,\lambda,\alpha)=\frac{\alpha\sqrt{\frac{2}{\pi}}\left(\frac{\lambda}{x}\right)\left(\frac{x}{\theta}\right)^{\lambda}e^{-\frac{1}{2}\left(\frac{x}{\theta}\right)^{2\lambda}}\left[2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right]^{\alpha-1}\left[2-2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]\right]^{\alpha-1}}{\left\{\left(2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right)^{\alpha}+\left(2-2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]\right)^{\alpha}\right\}^{2}},$$

where $x>0$, $\theta>0$, $\lambda>0$ and $\alpha>0$, and

$$\Phi(z)=\frac{1}{2}\left[1+erf\left(\frac{z}{\sqrt{2}}\right)\right] \quad and \quad erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt.$$

(iii) Exponentiated generalized half-normal distribution (EGHN).

The EGHN density function is given by

$$f(x;\theta,\lambda,\alpha)=\alpha\sqrt{\frac{2}{\pi}}\left(\frac{\lambda}{x}\right)\left(\frac{x}{\theta}\right)^{\lambda} e^{-\frac{1}{2}\left(\frac{x}{\theta}\right)^{2\lambda}}\left[2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right]^{\alpha-1},$$

where $x>0$, $\theta>0$, $\lambda>0$ and $\alpha>0$.

(iv) Kumaraswamy generalized half-normal lifetime distribution (KWGHN) (Cordeiro et al. (2012)).

The KWGHN density function is given by

$$f(x;\theta,\lambda,a,b)=ab\sqrt{\frac{2}{\pi}}\left(\frac{\lambda}{x}\right)\left(\frac{x}{\theta}\right)^{\lambda} e^{-\frac{1}{2}\left(\frac{x}{\theta}\right)^{2\lambda}}\left[2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right]^{a-1}\left[1-\left(2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right)^{a}\right]^{b-1},$$

where $x>0$, $\theta>0$, $\lambda>0$, $a>0$ and $b>0$.

(v) Beta generalized half-normal lifetime distribution (betaGHN) (Pescim et al. (2010)).

The betaGHN density function is given by

$$f(x;\theta,\lambda,a,b)=\frac{\sqrt{\frac{2}{\pi}}\left(\frac{\lambda}{x}\right)\left(\frac{x}{\theta}\right)^{\lambda} e^{-\frac{1}{2}\left(\frac{x}{\theta}\right)^{2\lambda}}}{B(a,b)}2^{b-1}\left[2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right]^{a-1}\left[1-\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]\right]^{b-1}$$

where $x>0$, $\theta>0$, $\lambda>0$, $a>0$ and $b>0$, and

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad and \quad \Gamma(\alpha)=\int_0^{\infty}u^{\alpha-1}e^{-t}dt.$$

(vi) Beta Marshal-Olkin generalized half-normal lifetime distribution (betaMOGHN) (Alizadeh et al. (2015)).

The betaMOGHN density function is given by

$$f(x;\theta,\lambda,a,b,c)=\frac{c^{b}\sqrt{\frac{2}{\pi}}\left(\frac{\lambda}{x}\right)\left(\frac{x}{\theta}\right)^{\lambda} e^{-\frac{1}{2}\left(\frac{x}{\theta}\right)^{2\lambda}}\left[2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right]^{a-1}\left[2-2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]\right]^{b-1}}{B(a,b)\left[c-(c-1)\left[2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]-1\right]\right]^{a+b}}$$

where $x>0$, $\theta>0$, $\lambda>0$, $a>0$, $b>0$ and $c>0$.

(vii) Extended generalized half-normal lifetime distribution (EXGHN) (Sanchez et al. (2016)).

The EXGHN density function is given by

$$f(x;\theta,\lambda,a,b)=ab\sqrt{\frac{2}{\pi}}\left(\frac{\lambda}{x}\right)\left(\frac{x}{\theta}\right)^{\lambda}e^{-\frac{1}{2}\left(\frac{x}{\theta}\right)^{2\lambda}}\left[2-2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]\right]^{a-1}\left[1-\left(2-2\Phi\left[\left(\frac{x}{\theta}\right)^{\lambda}\right]\right)^{a}\right]^{b-1},$$

where $x>0$, $\theta>0$, $\lambda>0$, $a>0$ and $b>0$.

We fit the above models to the current data sets and compute the MLEs, their standard errors (given in parentheses) and the following statistics: negative log-likelihood ($-\log L$) value, Kolmogorov-Smirnov statistic (K-S) and their p-values, Crámer-von Mises (W^*) and Anderson–Darling (A^*). The computations are performed using the AdequacyModel script in R package. Tables 2-5 lists the above values.

In all four tables, the better the fit of the model, the smaller the values of the corresponding statistics. The results indicate that the NOLLGHN model has the smallest values of these statistics and largest p-values among all fitted models. So, it could be chosen as the more suitable model.

In order to assess if the model is appropriate, we show in Figures 9-12 the histograms of the data sets, the plots of the fitted NOLLGHN, GHN, OLLGHN, EGHN, KWGHN, betaGHN betaMOGHN and EXGHN density functions and their estimated survival functions and the plots of the empirical distributions. We can conclude that the new distribution is a very suitable model to fit the four data sets.

Table 2: MLEs , their standard errors, and some statistics for the fitted models to data set 1.

Model	Parameter Estimation					-log(L)	W^*	A^*	K-S	p-value
	θ	λ	α	β	c					
NOLLGHN	39.174 (11.374)	0.641 (0.161	5.809 (2.349)	0.394 (0.177)		388.132	0.102	0.561	0.086	0.666
OLLGHN	232.222 (86.853)	0.353 (0.121)	4.047 (1.512)			391.393	0.193	1.061	0.094	0.552
GHN	129.196 (11.894)	1.0161 (0.091)				401.738	0.580	3.197	0.168	0.035
EGHN	2.398 (5.072)	0.250 (0.093)	39.677 (53.075)			390.107	0.141	0.768	0.099	0.487
KWGHN	4.348 (12.003)	0.3033 (0.290)	28.449 (51.911)	0.735 (1.178)		390.094	0.144	0.782	0.099	0.484
betaGHN	4.796 (6.731)	0.338 (0.199)	23.857 (29.464)	0.536 (0.665)		390.035	0.143	0.780	0.099	0.471
betaMOGHN	0.576 (0.002)	0.324 (0.002)	1.043 (0.339)	0.066 (0.008)	0.940 (0.448)	429.795	0.273	1.484	0.354	2.927e-08
EXGHN	0.788 (0.003)	0.339 (0.004)	0.198 (0.023)	9.179 (2.384)		390.558	0.188	1.015	0.105	0.4081

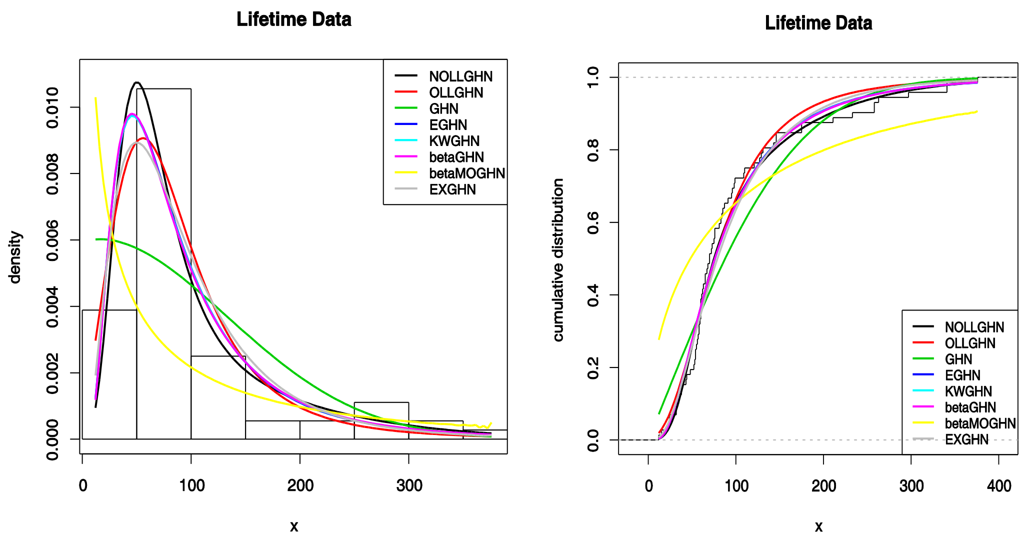

Figure 10: Estimated densities and Estimated cdf for data set 1.

Table 3: MLEs , their standard errors, and some statistics for the fitted models to data set 2.

Model	Parameter Estimation					-log(L)	W^*	A^*	K-S	p-value
	θ	λ	α	β	c					
NOLLGHN	99.288	8.114	4.767	0.193		179.886	0.049	0.313	0.093	0.762
	(3.051)	(2.059)	(2.253)	(0.099)						
OLLGHN	257.370	0.444	29.511			183.949	0.109	0.679	0.106	0.609
	(129.184)	(0.251)	(17.073)							
GHN	112.426	9.957	147.926			189.704	0.382	2.151	0.176	0.081
	(1.238)	(1.096)	(427.261)							
EGHN	57.808	1.719	37.712			182.396	0.069	0.453	0.094	0.746
	(32.373)	(1.044)	(55.246)							
KWGHN	78.940	3.962	33.793	0.228		181.942	0.077	0.481	0.096	0.726
	(9.108)	(1.531)	(64.621)	(0.237)						
betaGHN	77.743	3.906	0.973	0.214		181.921	0.075	0.470	0.096	0.728
	(12.827)	(2.145)	(0.515)	(0.234)						
betaMOGHN	0.638	0.399	77.367	0.030	0.6742	309.076	0.121	0.749	0.581	1.11e
	(0.001)	(0.001)	(586.665)	(0.004)	(0.547)					-15
EXGHN	246.979	3.040	4.767	87.686		182.393	0.070	0.456	0.095	0.738
	(553.123)	(2.250)	(2.253)	(285.52)						

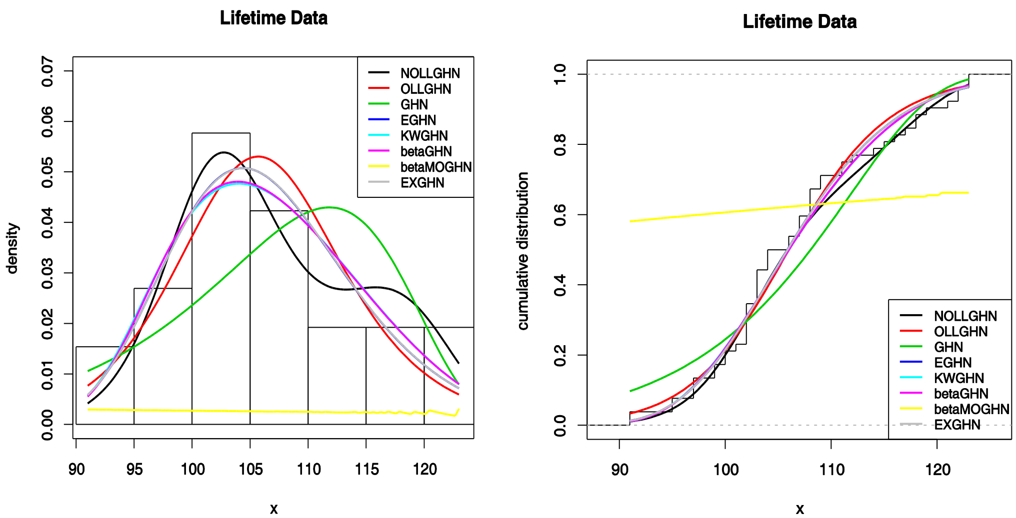

Figure 11: Estimated densities and Estimated cdf for data set 2.

Table 4: MLEs , their standard errors, and some statistics for the fitted models to data set 3.

Model	Parameter Estimation					-log(L)	W^*	A^*	K-S	p-value
	θ	λ	α	β	c					
NOLLGHN	37.960 (0.031)	9.211 (0.031)	0.166 (0.026)	233.860 (26.835)		194.778	0.095	0.551	0.081	0.707
OLLGHN	20.516 (0.573)	7.244 (0.963)	0.319 (0.065)			204.069	0.328	1.644	0.243	0.001
GHN	21.884 (0.497)	3.821 (0.415)				217.232	0.797	4.206	0.218	0.002
EGHN	24.487 (0.004)	12.141 (0.004)	0.240 (0.029)			206.734	0.486	2.481	0.218	0.002
KWGHN	17.459 (0.005)	5.490 (0.005)	0.150 (0.008)	0.099 (0.012)		195.821	0.107	0.621	0.116	0.273
betaGHN	17.902 (0.003)	5.935 (0.003)	0.262 (0.049)	0.092 (0.012)		195.775	0.136	0.729	0.114	0.289
betaMOGHN	0.498 (0.003)	0.534 (0.002)	1.051 (0.456)	0.035 (0.004)	0.650 (0.420)	281.348	1.589	8.105	0.381	8.7e-10
EXGHN	25.945 (0.003)	12.076 (0.003)	2.249 (0.007)	0.243 (0.029)		207.913	0.545	2.813	0.226	0.001

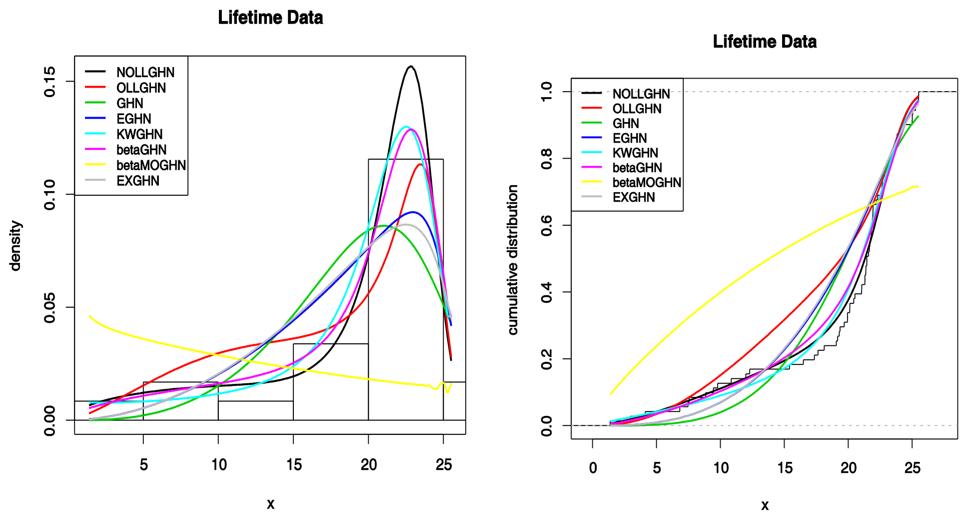

Figure 12: Estimated densities and Estimated cdf for data set 3.

Table 5: MLEs , their standard errors, and some statistics for the fitted models to data set 4.

Model	Parameter Estimation θ	λ	α	β	c	-log(L)	W^*	A^*	K-S	p-value
NOLLGHN	2.913	0.599	1.040	2.769		102.225	0.114	0.728	0.068	0.743
	(3.418)	(0.117)	(0.319)	(3.031)						
OLLGHN	1.319	0.614	1.241			102.697	0.156	0.917	0.077	0.584
	(0.178)	(0.106)	(0.255)							
GHN	1.224	0.711				103.335	0.125	0.804	0.080	0.544
	(0.132)	(0.055)								
EGHN	0.861	0.572	1.447			102.721	0.161	0.938	0.083	0.492
	(0.332)	(0.127)	(0.511)							
KWGHN	0.719	0.570	1.436	0.839		102.718	0.159	0.929	0.082	0.503
	(1.606)	(0.120)	(0.516)	(1.797)						
betaGHN	0.697	0.568	1.426	0.823		102.713	0.158	0.924	0.082	0.512
	(1.194)	(0.122)	(0.517)	(1.278)						
betaMOGHN	0.090	0.470	4.033	0.171	0.203	102.874	0.617	4.130	0.124	0.090
	(0.002)	(0.002)	(1.500)	(0.0189)	(0.092)					
EXGHN	0.447	0.541	0.605	1.430		102.685	0.153	0.902	0.080	0.543
	(1.104)	(0.152)	(1.127)	(0.503)						

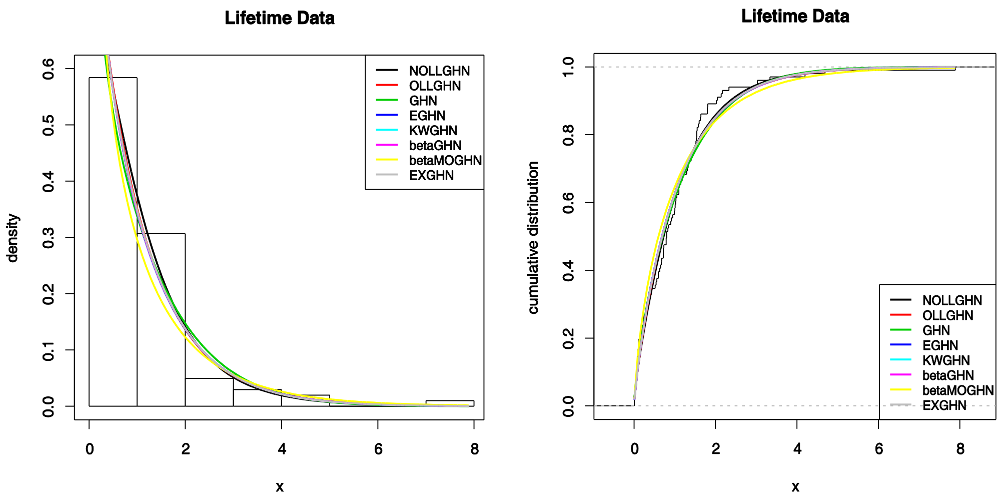

Figure 13: Estimated densities and Estimated cdf for data set 4.

7 Conclusions

In this paper, A new class of distributions called the *odd log-logistic generalized half-normal* (NOLL-GHN) family with four parameters is introduced and studied. Some of its various properties including explicit expansions, moment of residual life, reversed residual life, incomplete moments, order statistics, maximum likelihood estimator, are provided. The parameters of this model are estimated by the maximum likelihood estimators, least squares estimators, weighted least squares estimators, Cramer-von-Mises estimators, Anderson-Darling estimators and right tailed Anderson-Darling estimators. The NOLL-GHN is applied to fit four real data sets. applications demonstrate the importance of the NOLL-GHN family and show that the NOLL-GHN has the ability to fit the current data and it was always one of the best models. The results of tables and figures illustrate the importance of the new distribution to analyze of real data with respect to another well-known models.

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