Improved Generalized Family of Estimators of Population Mean Using Information on Transformed Auxiliary Variables

Housila P. Singh
School of studies in Statistics, Vikram University, Ujjain- 456010, M.P., India
hpsujn@gmail.com

Anita Yadav
School of studies in Statistics ,VikramUniversity,Ujjain- 456010, M.P., India
yadavd.anita@gmail.com

Abstract

This paper addresses the problem of estimating the population mean $\bar{Y}$ of the study variable y using information on transformed auxiliary variables. In addition to many, Yasmeen et al(2015) estimator shown to be the members of the suggested classes of estimators. We have derived the bias and mean squared error (*MSE*) of the suggested classes of estimators to the first degree of approximation. We have obtained the optimum conditions for which the suggested classes of estimators have minimum mean squared errors. It has been shown that the proposed classes of estimators are more efficient than the estimators recently envisaged by Yasmeen et al (2015)and other existing estimators.

Keywords: Study variate, Mean, Variance, Auxiliary variate, Efficiency Comparison.

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1. Introduction

Consider a finite population $U=(U_1,U_2,...U_N)$ of N units. Let us consider a study variate y and two auxiliary variates (x,z) taking values y_i and (x_i,z_i) respectively on the unit $U_i(i=1,2,...,N)$, where x is positively correlated with y and z is negatively correlated with y. We denote by

$\bar{Y}=\frac{1}{N}\sum_{i=1}^{N}y_i$: The population mean of the study variate y,

$\bar{X}=\frac{1}{N}\sum_{i=1}^{N}x_i$: The population mean of the auxiliary variate x,

$\bar{Z}=\frac{1}{N}\sum_{i=1}^{N}z_i$: The population mean of the auxiliary variate z,

$S_y^2=\frac{1}{(N-1)}\sum_{i=1}^{N}(y_i-\bar{Y})^2$: The population mean square / variance of the study variate y,

$S_x^2=\frac{1}{(N-1)}\sum_{i=1}^{N}(x_i-\bar{X})^2$: The population mean square / variance of the auxiliary variate x,

$S_z^2=\frac{1}{(N-1)}\sum_{i=1}^{N}(z_i-\bar{Z})^2$: The population mean square / variance of the auxiliary variate z,

$S_{xy}=\frac{1}{(N-1)}\sum_{i=1}^{N}\left(x_i-\overline{X}\right)\left(y_i-\overline{Y}\right)$: The population covariance between x and y,

$S_{xz}=\frac{1}{(N-1)}\sum_{i=1}^{N}\left(x_i-\overline{X}\right)\left(z_i-\overline{Z}\right)$: The population covariance between x and z,

$S_{yz}=\frac{1}{(N-1)}\sum_{i=1}^{N}\left(y_i-\overline{Y}\right)\left(z_i-\overline{Z}\right)$: The population covariance between y and z,

$\rho_{xy}=\frac{S_{xy}}{S_xS_y}$: The population correlation coefficient between x and y,

$\rho_{xz}=\frac{S_{xz}}{S_xS_z}$: The population correlation coefficient between x and z,

$\rho_{yz}=\frac{S_{yz}}{S_yS_z}$: The population correlation coefficient between y and z,

$C_x=\frac{S_x}{\overline{X}}$: The population coefficient of variation of the auxiliary variable x,

$C_y=\frac{S_y}{\overline{Y}}$: The population coefficient of variation of the auxiliary variable y,

$C_z=\frac{S_z}{\overline{Z}}$: The population coefficient of variation of the auxiliary variable z.

A simple random sample of size n is drawn without replacement from U to estimate the population mean $\overline{Y}$ of the study variate y assuming the knowledge of the population means $\overline{X}$ and $\overline{Z}$ of the auxiliary variates x and z respectively,

It is well known under simple random sampling without replacement (SRSWOR) that the usual unbiased estimator (which does not utilize auxiliary information) is

$$t_0=\bar{y}=\frac{1}{n}\sum_{i=1}^{n}y_i \quad (1.1)$$

The mean squared error (*MSE)* / variance of t_0 is given by

$$MSE(\bar{y})=Var(\bar{y})=\frac{(1-f)}{n}\overline{Y}^2C_y^2=\frac{(1-f)}{n}S_y^2, \quad (1.2)$$

where $f=\frac{n}{N}$ is the sampling fraction.

The classical ratio estimator due to Cochran (1940) for $\overline{Y}$ is defined by

$$t_1=\bar{y}\left(\frac{\overline{X}}{\bar{x}}\right) \quad (1.3)$$

where the population mean $\overline{X}$ of x is known in advance.

The MSE of the ratio estimator t_1 to the first degree of approximation is given by

$$MSE(t_1)=\frac{(1-f)}{n}\overline{Y}^2\left[C_y^2+C_x^2\left(1-2K_{yx}\right)\right], \quad (1.4)$$

where $K_{yx} = \rho_{yx}\left(\frac{C_y}{C_x}\right)$.

The classical product estimator due to Robson (1957) and revisited by Murthy (1964) for $\bar{Y}$ is defined by

$$t_2 = \bar{y}\left(\frac{\bar{z}}{\bar{Z}}\right), \quad (1.5)$$

where the population mean $\bar{Z}$ of is known in advance.

The MSE of the product estimator t_2 to the first degree of approximation is given by

$$MSE(t_2) = \frac{(1-f)}{n}\bar{Y}^2\left[C_y^2 + C_z^2(1-2K_{yz})\right] \quad (1.6)$$

where $K_{yz} = \rho_{yz}\left(\frac{C_y}{C_z}\right)$.

Using transformations:

$x_{ti} = (1+g)\bar{X} - gx_i \quad ; \; i = 1,2,...,N$;

And $z_{ti} = (1+g)\bar{Z} - gz_i \quad ; \; i = 1,2,...,N$;

With $g = \frac{n}{N-n} = \frac{f}{(1-f)}$, Srivenkataramana (1980) and Bandyopadhyaya (1980) suggested a dual to ratio estimator for $\bar{Y}$ as

$$t_3 = \bar{y}\left(\frac{\bar{x}_t}{\bar{X}}\right) \quad (1.7)$$

and a dual to product estimator for $\bar{Y}$ as

$$t_4 = \bar{y}\left(\frac{\bar{Z}}{\bar{z}_t}\right) \quad (1.8)$$

where $\bar{x}_t = \{(1+g)\bar{X} - g\bar{x}\} = \frac{(N\bar{X} - n\bar{x})}{(N-n)}$ and $\bar{z}_t = \{(1+g)\bar{Z} - g\bar{z}\} = \frac{(N\bar{Z} - n\bar{z})}{(N-n)}$

To the first degree of approximation, the MSEs of the estimators t_3 and t_4 are respectively, given by

$$MSE(t_3) = \frac{(1-f)}{n}\bar{Y}^2\left[C_y^2 + gC_x^2(g-2K_{yx})\right] \quad (1.9)$$

and

$$MSE(t_4) = \frac{(1-f)}{n}\bar{Y}^2\left[C_y^2 + gC_z^2(g+2K_{yz})\right] \quad (1.10)$$

1.1 Reviewing Some Existing Estimators Based on Two Auxiliary Variates

When the population means $(\bar{X}, \bar{Z})$ of auxiliary variates (x, z) respectively are known, Singh (1967) suggested a ratio-cum-product estimator for $\bar{Y}$ of y as

$$t_5 = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)\left(\frac{\bar{z}}{\bar{Z}}\right) \quad (1.11)$$

The *MSE* of t_5 to the first degree of approximation is given by

$$MSE(t_5)=\frac{(1-f)}{n}\bar{Y}^2\left[C_y^2+C_x^2(1-2K_{yx})+C_z^2(1+2K_{yz})-2\rho_{xz}C_xC_z\right] \quad (1.12)$$

A generalized version of t_5 on the line of Srivastava (1967) due to Abu-Dayyeh et al (2003) is given by

$$t_6=\bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)^{\alpha 1}\left(\frac{\bar{z}}{\bar{Z}}\right)^{\alpha 2}, \quad (1.13)$$

where (α_1,α_2) are suitably chosen constants.

The *MSE* of t_6 to the first degree of approximation is given by

$$MSE(t_6)=\frac{(1-f)}{n}\bar{Y}^2\left[C_y^2+\alpha_1C_x^2(\alpha_1-2K_{yx})+\alpha_2C_z^2(\alpha_2+2K_{yz})-2\alpha_1\alpha_2\rho_{xz}C_xC_z\right] \quad (1.14)$$

The $MSE(t_6)$ at (1.14) is minimized for

$$\left.\begin{aligned}\alpha_1&=\frac{(\rho_{yx}-\rho_{xz}\rho_{yz})C_y}{(1-\rho_{xz}^2)C_x}=\alpha_{10}(say)\\ \alpha_2&=\frac{(\rho_{xy}\rho_{xz}-\rho_{yz})C_y}{(1-\rho_{xz}^2)C_z}=\alpha_{20}(say)\end{aligned}\right\} \quad (1.15)$$

Thus the resulting minimum *MSE* of t_6 is given by

$$min.\,MSE(t_6)=\frac{(1-f)}{n}\bar{Y}^2C_y^2(1-\rho_{y.xz}^2), \quad (1.16)$$

where $\rho_{y.xz}^2=\frac{(\rho_{xy}^2+\rho_{yz}^2-2\rho_{xy}\rho_{yz}\rho_{xz})}{(1-\rho_{xz}^2)}$ (1.17)

Kadilar and Cingi (2005) suggested another class of ratio-cum-product estimators based on two auxiliary variables (x,z) for population mean $\bar{Y}$ as

$$t_7=\bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)^{\alpha 1}\left(\frac{\bar{Z}}{\bar{z}}\right)^{\alpha 2}+b_{yx}(\bar{X}-\bar{x})+b_{yz}(\bar{Z}-\bar{z}) \quad (1.18)$$

where (α_1,α_2) are suitably chosen constants,

$$b_{yx}=\frac{s_{yx}}{s_x^2},\, b_{yz}=\frac{s_{yz}}{s_z^2},\, s_{yx}=\frac{1}{(n-1)}\sum_{i=1}^{n}(y_i-\bar{y})(x_i-\bar{x}),$$

$$s_{yz}=\frac{1}{(n-1)}\sum_{i=1}^{n}(y_i-\bar{y})(z_i-\bar{z}),\, s_x^2=\frac{1}{(n-1)}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

and $s_z^2=\frac{1}{(n-1)}\sum_{i=1}^{n}(z_i-\bar{z})^2$.

The *MSE* of t_7 to the first degree of approximation is given by

$$MSE(t_7)=\frac{(1-f)}{n}\begin{bmatrix} S_y^2+(\alpha_1 R_x+\beta_{yx})^2 S_x^2+(\alpha_2 R_z+\beta_{yz})^2 S_z^2 \\ -2\,(\alpha_1 R_x+\beta_{yx})S_{yx}-2\,(\alpha_2 R_z+\beta_{yz})S_{yz} \\ +2\,(\alpha_1 R_x+\beta_{yx})(\alpha_2 R_z+\beta_{yz})S_{xz} \end{bmatrix}, \qquad (1.19)$$

where $R_x=\frac{\bar{Y}}{\bar{X}}$, $R_z=\frac{\bar{Y}}{\bar{Z}}$, $\beta_{yx}=\frac{S_{yx}}{S_x^2}$ and $\beta_{yz}=\frac{S_{yz}}{S_z^2}$.

The $MSE(t_7)$ at (1.19) is minimized for

$$\left.\begin{aligned} \alpha_1&=\frac{S_y}{R_x S_x}\rho_1^*=\alpha_{10}^*\ (say) \\ \alpha_2&=\frac{S_y}{R_x S_z}\rho_2^*=\alpha_{10}^*\ (say) \end{aligned}\right\}, \qquad (1.20)$$

where $\rho_1^*=\frac{\rho_{xz}(\rho_{yx}\rho_{xz}-\rho_{yz})}{(1-\rho_{xz}^2)}$ and $\rho_2^*=\frac{\rho_{xz}(\rho_{yz}\rho_{xz}-\rho_{yx})}{(1-\rho_{xz}^2)}$.

Thus the minimum *MSE* of t_7 is given by

$$min.MSE(t_7)=\frac{(1-f)}{n}S_y^2(1-\rho_{y.xz}^2) \qquad (1.21)$$

Singh et al (2005) suggested a dual to ratio-cum-product estimator for population mean $\bar{Y}$ as

$$t_8=\bar{y}\left(\frac{\bar{x}_t}{\bar{X}}\right)\left(\frac{\bar{Z}}{\bar{z}_t}\right). \qquad (1.22)$$

The *MSE* of t_8 to the first degree of approximation is given by

$$MSE(t_8)=\frac{(1-f)}{n}\bar{Y}^2\left[C_y^2+gC_x^2(g-2K_{yx})+gC_z^2(g+2K_{yz})-2g^2\rho_{xz}C_xC_z\right] \qquad (1.23)$$

The generalized version of the estimator t_7 due to Singh et al (2011) is given by

$$t_9=\bar{y}\left(\frac{\bar{x}_t}{\bar{X}}\right)^{\alpha 1}\left(\frac{\bar{Z}}{\bar{z}_t}\right)^{\alpha 2} \qquad (1.24)$$

To the first degree of approximation, the *MSE* of t_9 is given by

$$MSE(t_9)=\frac{(1-f)}{n}\bar{Y}^2\begin{bmatrix} C_y^2+\alpha_1 gC_x^2(\alpha_1 g-2K_{yx})+\alpha_2 gC_z^2(\alpha_2 g+2K_{yz}) \\ -2\alpha_1\alpha_2 g^2\rho_{xz}C_xC_z \end{bmatrix} \qquad (1.25)$$

which is minimized for

$$\left.\begin{aligned} \alpha_1&=\frac{(\rho_{yx}-\rho_{xz}\rho_{yz})C_y}{g(1-\rho_{xz}^2)C_x}=\alpha_{10}^{**}(say) \\ \alpha_2&=\frac{(\rho_{yx}\rho_{xz}-\rho_{yz})C_y}{g(1-\rho_{xz}^2)C_z}=\alpha_{20}^{**}(say) \end{aligned}\right\} \qquad (1.26)$$

Thus the resulting minimum *MSE* of t_9 is given by

$$min\,.\,MSE(t_9)=\frac{(1-f)}{n}\bar{Y}^2C_y^2\left(1-\rho_{y.xz}^2\right). \qquad (1.27)$$

Assuming that the correlation coefficient ρ_{xz} between x and z is known, Singh and Tailor (2005) suggested a ratio-cum-product estimator for $\bar{Y}$ as

$$t_{10}=\bar{y}\left(\frac{\bar{X}+\rho_{xz}}{\bar{x}+\rho_{xz}}\right)\left(\frac{\bar{z}+\rho_{xz}}{\bar{Z}+\rho_{xz}}\right). \qquad (1.28)$$

To the first degree of approximation, the MSE of t_{10} is given by

$$MSE(t_{10})=\frac{(1-f)}{n}\bar{Y}^2\left[C_y^2+\tau_1C_x^2\left(\tau_1-2K_{yx}-2\tau_1^*K_{zx}\right)+\tau_1^*C_z^2\left(\tau_1^*+2K_{yz}\right)\right], \qquad (1.29)$$

where $\tau_1=\dfrac{\bar{X}}{\left(\bar{X}+\rho_{xz}\right)}, \tau_1^*=\dfrac{\bar{Z}}{\left(\bar{Z}+\rho_{xz}\right)}$ and $K_{zx}=\rho_{xz}\left(\dfrac{C_z}{C_x}\right)$.

Tailor et al (2012) envisaged a dual to Singh and Tailor (2005) estimator for $\bar{Y}$ as

$$t_{11}=\bar{y}\left(\frac{\bar{x}_t+\rho_{xz}}{\bar{X}+\rho_{xz}}\right)\left(\frac{\bar{Z}+\rho_{xz}}{\bar{z}_t+\rho_{xz}}\right) \qquad (1.30)$$

The MSE of t_{11} to the first degree of approximation is given by

$$MSE(t_{11})=\frac{(1-f)}{n}\bar{Y}^2\left[\begin{array}{r}C_y^2+\tau_1g\,C_x^2\left(\tau_1g-2K_{yx}-2\tau_1^*gK_{zx}\right)\\+\tau_1^*gC_z^2\left(\tau_1^*g+2K_{yz}\right)\end{array}\right]. \qquad (1.31)$$

Vishwakarma et al (2014) developed a class of dual to ratio-cum-product estimators for $\bar{Y}$ as

$$t_{12}=\bar{y}\left(\frac{\bar{x}_t+\rho_{xz}}{\bar{X}+\rho_{xz}}\right)^{\alpha 1}\left(\frac{\bar{Z}+\rho_{xz}}{\bar{z}_t+\rho_{xz}}\right)^{\alpha 2}, \qquad (1.32)$$

where (α_1,α_2) are suitably chosen constants.

The MSE of t_{12} to the first degree of approximation is given by

$$MSE(t_{12})=\frac{(1-f)}{n}\bar{Y}^2\left[\begin{array}{r}C_y^2+\alpha_1\tau_1gC_x^2\left(\alpha_1\tau_1g-2K_{yx}-2\alpha_2\tau_1^*gK_{zx}\right)\\+\alpha_2\tau_1^*gC_z^2\left(\alpha_2\tau_1^*g+2K_{yz}\right)\end{array}\right] \qquad (1.33)$$

which is minimized for

$$\left.\begin{array}{l}\alpha_1=\dfrac{\left(\rho_{yx}-\rho_{yz}\rho_{xz}\right)C_y}{\tau_1g\left(1-\rho_{xz}^2\right)C_x}=\alpha_{10}^{***}(say)\\ \alpha_2=\dfrac{\left(\rho_{yx}\rho_{xz}-\rho_{yz}\right)C_y}{\tau_1^*g\left(1-\rho_{xz}^2\right)C_z}=\alpha_{20}^{***}(say)\end{array}\right\}. \qquad (1.34)$$

Thus the resulting minimum MSE of t_{12} is given by

$$min.MSE(t_{12})=\frac{(1-f)}{n}\bar{Y}^2C_y^2\left(1-\rho_{y.xz}^2\right). \qquad (1.35)$$

Further Vishwakarma and Kumar (2015) suggested a class of estimators $\bar{Y}$ as

$$t_{13} = \bar{y}\left(\frac{a\bar{x}_t + b}{a\bar{X} + b}\right)^{\alpha_1} \left(\frac{a\bar{Z} + b}{a\bar{z}_t + b}\right)^{\alpha_2}, \tag{1.36}$$

where $a\,(\neq 0)$ and b are either real numbers or functions of some known population parameters
associated with auxiliary variables (x, z) and the study variable y such as coefficients of variation $(C_y C_x, C_z)$ and correlation coefficient $(\rho_{yx}, \rho_{yz}, \rho_{xz})$ etc and (α_1, α_2) are suitably chosen constants.

To the first degree of approximation, the *MSE* of t_{13} is given by

$$MSE(t_{13}) = \frac{(1-f)}{n}\bar{Y}^2\left[\begin{array}{l} C_y^2 + \alpha_1\lambda g C_x^2\left(\alpha_1\lambda g - 2K_{yx} - 2\alpha_2\lambda^* g K_{zx}\right) \\ \qquad + \alpha_2\lambda^* g C_z^2\left(\alpha_2\lambda^* g + 2K_{yz}\right)\end{array}\right] \tag{1.37}$$

where $\lambda = \dfrac{a\bar{X}}{(a\bar{X} + b)}$ and $\lambda^* = \dfrac{a\bar{Z}}{(a\bar{Z} + b)}$.

The *MSE* of the class of estimators t_{13} is minimized for

$$\left.\begin{array}{l} \alpha_1 = \dfrac{(\rho_{yx} - \rho_{yz}\rho_{xz})C_y}{\lambda g(1-\rho_{xz}^2)C_x} = \alpha_{10}^{****}(say) \\ \alpha_2 = \dfrac{(\rho_{yx}\rho_{xz} - \rho_{yz})C_y}{\lambda^* g(1-\rho_{xz}^2)C_z} = \alpha_{20}^{****}(say)\end{array}\right\} \tag{1.38}$$

Thus the resulting minimum *MSE* of t_{13} is given by

$$min.MSE(t_{13}) = \frac{(1-f)}{n}\bar{Y}^2 C_y^2\left(1 - \rho_{y.xz}^2\right) \tag{1.39}$$

Motivated by Singh (1967) and Bahl and Tuteja (1991), Singh et al (2009) suggested a ratio-cum-product-type exponential estimator for $\bar{Y}$ as

$$t_{14} = \bar{y}\,exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right) exp\left(\frac{\bar{z} - \bar{Z}}{\bar{z} + \bar{Z}}\right) \tag{1.40}$$

The *MSE* of t_{14} to the first degree of approximation is given by

$$MSE(t_{14}) = \frac{(1-f)}{n}\bar{Y}^2\left[C_y^2 + \frac{C_x^2}{4}\left(1 - 4K_{yx} - 2K_{zx}\right) + \frac{C_z^2}{4}\left(1 + 4K_{yz}\right)\right] \tag{1.41}$$

A generalized version of the estimator t_{15} due to Singh et al (2009) is given by

$$t_{15} = \bar{y}\,exp\left\{\frac{\alpha_3(\bar{X} - \bar{x})}{(\bar{X} + \bar{x})}\right\} exp\left\{\frac{\alpha_4(\bar{z} - \bar{Z})}{(\bar{z} + \bar{Z})}\right\} \tag{1.42}$$

where α_3 and α_4 are suitably chosen constants.

To the first degree of approximation, the *MSE* of the class of estimators t_{14} is given by

$$MSE(t_{15}) = \frac{(1-f)}{n}\bar{Y}^2\left[C_y^2 + \frac{\alpha_3 C_x^2}{4}\left(\alpha_3 - 4K_{yx}\right) + \frac{\alpha_4 C_z^2}{4}\left(\alpha_4 + 4K_{yz}\right) - \frac{1}{2}\alpha_3\alpha_4 K_{zx} C_x^2\right] \tag{1.43}$$

The *MSE* of the class of estimators t_{15} at (1.43) is minimized for

$$\left.\begin{aligned}\alpha_3 &= \frac{2\left(\rho_{yx}-\rho_{xz}\rho_{yz}\right)C_y}{\left(1-\rho_{xz}^2\right)C_x} = \alpha_{30}(say)\\ \alpha_4 &= \frac{2\left(\rho_{yx}\rho_{xz}-\rho_{yz}\right)C_y}{\left(1-\rho_{xz}^2\right)C_z} = \alpha_{40}(say)\end{aligned}\right\} \quad (1.44)$$

Thus the resulting minimum *MSE* of t_{15} is given by

$$min.MSE(t_{15}) = \frac{(1-f)}{n}\bar{Y}^2 C_y^2\left(1-\rho_{y.xz}^2\right). \quad (1.45)$$

Yasmeen et al (2015) suggested the following classes of estimators for the population mean $\bar{Y}$ as

$$t_{16} = w\,\bar{y} + (1-w)\bar{y}\left(\frac{\bar{x}_t}{\bar{X}}\right) exp\left\{\frac{\eta_4\left(\bar{z}_t-\bar{Z}\right)}{\left(\bar{z}_t+\bar{Z}\right)}\right\}, \quad (1.46)$$

$$t_{17} = \delta\bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)^{\alpha_1} exp\left\{\frac{\alpha_4\left(\bar{Z}-\bar{z}\right)}{\left(\bar{z}+\bar{Z}\right)}\right\} + (1-\delta)\,\bar{y}\left(\frac{\bar{x}_t}{\bar{X}}\right)^{\eta_1} exp\left\{\frac{\eta_4\left(\bar{z}_t-\bar{Z}\right)}{\left(\bar{z}_t+\bar{Z}\right)}\right\} \quad (1.47)$$

where $(\alpha_1, \alpha_4, \eta_1)$ are generalized constants and (w, δ) are optimization constants.

We should add here that the class of estimators t_{16} is revisited by Adichwal et al (2015). The biases and mean squared errors of the estimators t_{16} and t_{17} will be given later.
It is observed from (1.16), (1.21), (1.27), (1.35), (1.38) and (1.44) that the estimators $t_6, t_7, t_9, t_{12}, t_{13}$ and t_{15} respectively defined in (1.13), (1.18), (1.24), (1.32), (1.36) and (1.42) have common minimum *MSE* at their optimum conditions, that is,

$$\begin{aligned}&min.MSE(t_6) = min.MSE(t_7) = min.MSE(t_9)\\ &= min.MSE(t_{12}) = min.MSE(t_{13}) = min.MSE(t_{15}) = \frac{(1-f)}{n}S_y^2\left(1-\rho_{y.xz}^2\right).\end{aligned} \quad (1.48)$$

It can be easily seen from (1.2), (1.4), (1.6), (1.9), (1.10), (1.12), (1.23), (1.29), (1.37) and (1.48) that the unbiased estimator $t_0 = \bar{y}$, ratio estimator t_1, the product estimator t_2, dual to ratio estimator t_3, dual to product estimator t_4, ratio-cum-product estimator t_5, dual to ratio-cum-product estimator t_8, Singh and Tailor's (2005) ratio-cum-product estimator t_{10}, Tailor et al's (2012) estimators t_{11} and Singh et al's (2009) estimator t_{14} are less efficient than the estimators $t_6, t_7, t_9, t_{12}, t_{13}$ and t_{15} at their optimum conditions. The quest in this paper is to obtain an estimator better than $t_6, t_7, t_9, t_{12}, t_{13}$ and t_{15}. So we have made an effort to suggest a class of estimators based on two auxiliary variables (x, z) which is more efficient than the estimators discussed here.

2.The Suggested class of estimators

Taking motivation from the estimators discussed in section-1, we propose the class of estimators for $\bar{Y}$ as

$$T = \bar{y}\begin{bmatrix} \delta_1\left(\dfrac{a\bar{x}+b}{a\bar{X}+b}\right)^{\alpha_1}\left(\dfrac{c\bar{z}+d}{c\bar{Z}+d}\right)^{\alpha_2} exp\left\{\dfrac{\alpha_3 a\left(\bar{x}-\bar{X}\right)}{a\left(\bar{X}+\bar{x}\right)+2b}\right\} exp\left\{\dfrac{\alpha_4 c\left(\bar{z}-\bar{Z}\right)}{c\left(\bar{z}+\bar{Z}\right)+2d}\right\} \\ +\delta_2\left(\dfrac{a\bar{x}_t+b}{a\bar{X}+b}\right)^{\eta_1}\left(\dfrac{c\bar{z}_t+d}{c\bar{Z}+d}\right)^{\eta_2} exp\left\{\dfrac{\eta_3 a\left(\bar{x}_t-\bar{X}\right)}{a\left(\bar{x}_t+\bar{X}\right)+2b}\right\} exp\left\{\dfrac{\eta_4 c\left(\bar{z}_t-\bar{Z}\right)}{c\left(\bar{z}_t+\bar{Z}\right)+2d}\right\}\end{bmatrix} \quad (2.1)$$

where $(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\eta_1,\eta_2,\eta_3,\eta_4)$ are constants take real values, $(a \neq o,b,c \neq o,d)$ are either real numbers or the functions of the known parameters of the auxiliary variable *(x,z)*, such as standard variations (S_x,S_z),coefficients of variation (C_x,C_z) ,coefficient of skewness $(\beta_1(x),\beta_1(z))$, coefficients of kurtosis $(\beta_2(x),\beta_2(z))$, and correlation coefficients $(\rho_{xy},\rho_{xz},\rho_{yz})$ etc. and (δ_1,δ_2) are suitably chosen constants such that the MSE of *T* is minimum. We mention that for various values of parameters in (2.1), one can get several estimators.

Expressing *T* in terms of $e_i(i=0,1,2)$, we can write (2.1) as

$$T = \bar{Y}(1+e_0)\begin{bmatrix} \delta_1(1+\tau_1 e_1)^{\alpha_1}(1+\tau_2 e_1)^{\alpha_2} exp\left\{\dfrac{\alpha_3 e_1\tau_1}{2}\left(1+\dfrac{\tau_1 e_1}{2}\right)^{-1}\right\} \\ exp\left\{\dfrac{\alpha_4 e_2\tau_2}{2}\left(1+\dfrac{\tau_2 e_2}{2}\right)^{-1}\right\}+\delta_2(1-g\tau_1 e_1)^{\eta_1}(1-g\tau_2 e_2)^{\eta_2} \\ exp\left\{\dfrac{-\eta_3\tau_1 g e_1}{2}\left(1-\dfrac{g\tau_1 e_1}{2}\right)^{-1}\right\} exp\left\{\dfrac{-\eta_4\tau_2 g e_2}{2}\right\}\left(1-\dfrac{g\tau_2 e_2}{2}\right)^{-1}\end{bmatrix} \quad (2.2)$$

where $\tau_1 = \dfrac{a\bar{X}}{(a\bar{X}+b)}$ and $\tau_2 = \dfrac{c\bar{Z}}{(c\bar{Z}+d)}$.

Suppose $|\tau_1 e_1|<1$, $|\tau_2 e_2|<1$, $|g\tau_1 e_1|<1$and $|g\tau_2 e_2|<1$ so that $(1+\tau_1 e_1)^{\alpha}1$ $(1+\tau_2 e_2)^{\alpha}2$, $(1-g\tau_1 e_1)^{\eta}1$ and $(1-g\tau_2 e_2)^{\eta}2$ are expandable. Expanding the right hand side of (2.2) to first order of approximation, we obtain

$$(T-\bar{Y})=\bar{Y}\begin{bmatrix} \delta_1\begin{Bmatrix} 1+e_0+\dfrac{(2\alpha_1+\alpha_3)}{2}\tau_1 e_1+\dfrac{(2\alpha_2+\alpha_4)}{2}\tau_2 e_2+\dfrac{(2\alpha_1+\alpha_3)}{2}\tau_1 e_0 e_1 \\ +\dfrac{(2\alpha_2+\alpha_4)}{2}\tau_2 e_0 e_2+\dfrac{(2\alpha_1+\alpha_3)(2\alpha_2+\alpha_4)}{4}\tau_1\tau_2 e_1 e_2 \\ +\dfrac{(2\alpha_1+\alpha_3)(2\alpha_1+\alpha_3-2)}{8}\tau_1^2 e_1^2+\dfrac{(2\alpha_2+\alpha_4)(2\alpha_2+\alpha_4-2)}{8}\tau_2^2 e_2^2 \end{Bmatrix}+ \\ \delta_2\begin{Bmatrix} 1+e_0-\dfrac{(2\eta_1+\eta_3)}{2}g\tau_1 e_1-\dfrac{(2\eta_2+\eta_4)}{2}g\tau_2 e_2-\dfrac{(2\eta_1+\eta_3)}{2}g\tau_1 e_0 e_1 \\ -\dfrac{(2\eta_2+\eta_4)}{2}g\tau_2 e_0 e_2+\dfrac{(2\eta_1+\eta_3)(2\eta_2+\eta_4)}{4}g^2\tau_1\tau_2 e_1 e_2 \\ +\dfrac{(2\eta_1+\eta_3)(2\eta_1+\eta_3-2)}{8}g^2\tau_1^2 e_1^2+\dfrac{(2\eta_2+\eta_4)(2\eta_2+\eta_4-2)}{8}g^2\tau_2^2 e_2^2 \end{Bmatrix}-1 \end{bmatrix} \quad (2.3)$$

Taking expectation of both sides of (2.3), we get the bias of *T* to the first degree of approximation as

$$B(T)=\bar{Y}[\delta_1 A_4+\delta_2 A_5-1] \quad (2.4)$$

where

$$A_4=\left[1+\frac{(1-f)}{8n}\left\{\begin{array}{l}4(2\alpha_1+\alpha_3)\tau_1\rho_{xy}C_xC_y+4(2\alpha_2+\alpha_4)\tau_2\rho_{yz}C_yC_z\\+2(2\alpha_1+\alpha_3)(2\alpha_2+\alpha_4)\tau_1\tau_2\rho_{xz}C_xC_z\\+(2\alpha_1+\alpha_3)(2\alpha_1+\alpha_3-2)\tau_1^2C_x^2+(2\alpha_2+\alpha_4)(2\alpha_2+\alpha_4-2)\tau_2^2C_z^2\end{array}\right\}\right] \quad (2.5)$$

$$A_5=\left[1+\frac{(1-f)}{8n}\left\{\begin{array}{l}2(2\eta_1+\eta_3)(2\eta_2+\eta_4)g^2\tau_1\tau_2\rho_{xz}C_xC_z-4(2\eta_1+\eta_3)g\tau_1\rho_{xy}C_xC_y\\-4(2\eta_2+\eta_4)g\tau_2\rho_{yz}C_yC_z+(2\eta_1+\eta_3)(2\eta_1+\eta_3-2)g^2\tau_1^2C_x^2\\+(2\eta_2+\eta_4)(2\eta_2+\eta_4-2)g^2\tau_2^2C_z^2\end{array}\right\}\right] \quad (2.6)$$

Squaring both sides of (2.3) and neglecting terms of $e's$ having power greater than two have

$$(T-\bar{Y})^2=\bar{Y}^2\left[\begin{array}{l}1+\delta_1^2\left\{\begin{array}{l}1+2e_0+(2\alpha_1+\alpha_3)\tau_1e_1+(2\alpha_2+\alpha_4)\tau_2e_2\\+e_0^2+2(2\alpha_1+\alpha_3)\tau_1e_0e_1+2(2\alpha_2+\alpha_4)\tau_2e_0e_2\\+(2\alpha_1+\alpha_3)(2\alpha_2+\alpha_4)\tau_1\tau_2e_1e_2+\frac{(2\alpha_1+\alpha_3)(2\alpha_1+\alpha_3-1)}{2}\tau_1^2e_1^2\\+\frac{(2\alpha_2+\alpha_4)(2\alpha_2+\alpha_4-1)}{2}\tau_2^2e_2^2\end{array}\right\}+\\ \delta_2^2\left\{\begin{array}{l}1+2e_0-(2\eta_1+\eta_3)g\tau_1e_1-(2\eta_2+\eta_4)g\tau_2e_2\\+e_0^2-2(2\eta_1+\eta_3)g\tau_1e_0e_1-2(2\eta_2+\eta_4)g\tau_2e_0e_2\\+(2\eta_1+\eta_3)(2\eta_2+\eta_4)g^2\tau_1\tau_2e_1e_2+\frac{(2\eta_1+\eta_3)(2\eta_1+\eta_3-1)}{2}g^2\tau_1^2e_1^2\\+\frac{(2\eta_2+\eta_4)(2\eta_2+\eta_4-1)}{2}g^2\tau_2^2e_2^2\end{array}\right\}+\\ 2\delta_1\delta_2\left\{\begin{array}{l}1+2e_0+\frac{[(2\alpha_1+\alpha_3)-g(2\eta_1+\eta_3)]}{2}\tau_1e_1\\+\frac{[(2\alpha_2+\alpha_4)-g(2\eta_2+\eta_4)]}{2}\tau_2e_2+[(2\alpha_1+\alpha_3)-g(2\eta_1+\eta_3)]\tau_1e_0e_1\\+[(2\alpha_2+\alpha_4)-g(2\eta_2+\eta_4)]\tau_2e_0e_2+e_0^2\\+\frac{[\{(2\alpha_1+\alpha_3)-g(2\eta_1+\eta_3)\}\{(2\alpha_2+\alpha_4)-g(2\eta_2+\eta_4)\}]}{4}\tau_1\tau_2e_1e_2\\+\frac{\left[\begin{array}{l}(2\eta_1+\eta_3)(2\eta_1+\eta_3-2)g^2-2(2\alpha_1+\alpha_3)(2\eta_1+\eta_3)g\\+(2\alpha_1+\alpha_3)(2\alpha_1+\alpha_3-2)\end{array}\right]}{8}\tau_1^2e_1^2\\+\frac{\left[\begin{array}{l}(2\eta_2+\eta_4)(2\eta_2+\eta_4-2)g^2-2(2\alpha_2+\alpha_4)(2\eta_2+\eta_4)g\\+(2\alpha_2+\alpha_4)(2\alpha_2+\alpha_4-2)\end{array}\right]}{8}\tau_2^2e_2^2\end{array}\right\}\\ -2\delta_1\left\{\begin{array}{l}1+e_0+\frac{(2\alpha_1+\alpha_3)}{2}\tau_1e_1+\frac{(2\alpha_2+\alpha_4)}{2}\tau_2e_2+\frac{(2\alpha_1+\alpha_3)}{2}\tau_1e_0e_1\\+\frac{(2\alpha_2+\alpha_4)}{2}\tau_2e_0e_2+\frac{(2\alpha_1+\alpha_3)(2\alpha_2+\alpha_4)}{4}\tau_1\tau_2e_1e_2\\+\frac{(2\alpha_1+\alpha_3)(2\alpha_1+\alpha_3-2)}{8}\tau_1^2e_1^2+\frac{(2\alpha_2+\alpha_4)(2\alpha_2+\alpha_4-2)}{8}\tau_2^2e_2^2\end{array}\right\}\\ -2\delta_2\left\{\begin{array}{l}1+e_0-\frac{(2\eta_1+\eta_3)}{2}g\tau_1e_1-\frac{(2\eta_2+\eta_4)}{2}g\tau_2e_2-\frac{(2\eta_1+\eta_3)}{2}g\tau_1e_0e_1\\-\frac{(2\eta_2+\eta_4)}{2}g\tau_2e_0e_2+\frac{(2\eta_1+\eta_3)(2\eta_2+\eta_4)}{4}g^2\tau_1\tau_2e_1e_2\\+\frac{(2\eta_1+\eta_3)(2\eta_1+\eta_3-2)}{8}g^2\tau_1^2e_1^2+\frac{(2\eta_2+\eta_4)(2\eta_2+\eta_4-2)}{8}g^2\tau_2^2e_2^2\end{array}\right\}\end{array}\right] \quad (2.7)$$

Taking expectation of both sides (2.7) we get the *MSE* of *T* to the first degree of approximation as

$$MSE(T)=\bar{Y}^2\left[1+\delta_1^2A_1+\delta_2^2A_2+2\delta_1\delta_2A_3-2\delta_1A_4-2\delta_2A_5\right] \qquad (2.8)$$

$$A_1=\left[1+\frac{(1-f)}{n}\left\{\begin{array}{l}C_y^2+\dfrac{(2\alpha_1+\alpha_3)(2\alpha_1+\alpha_3-1)}{2}\tau_1^2C_x^2\\+\dfrac{(2\alpha_2+\alpha_4)(2\alpha_2+\alpha_4-1)}{2}\tau_2^2C_z^2\\+2(2\alpha_1+\alpha_3)\tau_1\rho_{xy}C_xC_y+2(2\alpha_2+\alpha_4)\tau_2\rho_{yz}C_yC_z\\ \qquad+(2\alpha_1+\alpha_3)(2\alpha_2+\alpha_4)\tau_1\tau_2\rho_{xz}C_xC_z\end{array}\right\}\right] \qquad (2.9)$$

$$A_2=\left[1+\frac{(1-f)}{n}\left\{\begin{array}{l}C_y^2-2(2\eta_1+\eta_3)g\tau_1\rho_{xy}C_xC_y-2(2\eta_2+\eta_4)g\tau_2\rho_{yz}C_yC_z\\+(2\eta_1+\eta_3)(2\eta_2+\eta_4)g^2\tau_1\tau_2\rho_{xz}C_xC_z\\+\dfrac{(2\eta_1+\eta_3)(2\eta_1+\eta_3-1)}{2}g^2\tau_1^2C_x^2+\dfrac{(2\eta_2+\eta_4)(2\eta_2+\eta_4-1)}{2}g^2\tau_2^2C_z^2\end{array}\right\}\right] \qquad (2.10)$$

$$A_3=\left[1+\frac{(1-f)}{n}\left\{\begin{array}{l}C_y^2+[(2\alpha_1+\alpha_3)-g(2\eta_1+\eta_3)]\tau_1\rho_{xy}C_xC_y\\+[(2\alpha_2+\alpha_4)-g(2\eta_2+\eta_4)]\tau_2\rho_{yz}C_yC_z\\+\dfrac{[\{(2\alpha_1+\alpha_3)-g(2\eta_1+\eta_3)\}\{(2\alpha_2+\alpha_4)-g(2\eta_2+\eta_4)\}]}{4}\tau_1\tau_2\rho_{xz}C_xC_z\\+\dfrac{\begin{bmatrix}(2\eta_1+\eta_3)(2\eta_1+\eta_3-2)g^2-2(2\alpha_1+\alpha_3)(2\eta_1+\eta_3)g\\+(2\alpha_1+\alpha_3)(2\alpha_1+\alpha_3-2)\end{bmatrix}}{8}\tau_1^2C_x^2\\+\dfrac{\begin{bmatrix}(2\eta_2+\eta_4)(2\eta_2+\eta_4-2)g^2-2(2\alpha_2+\alpha_4)(2\eta_2+\eta_4)g\\+(2\alpha_2+\alpha_4)(2\alpha_2+\alpha_4-2)\end{bmatrix}}{8}\tau_2^2C_z^2\end{array}\right\}\right] \qquad (2.11)$$

and A_4 and A_5 are respectively defined in (2.8) and (2.9).

Differentiating $MSE(T)$ at (2.8) partially with respect to δ_1 and δ_2 and equating them to zero we get

$$\begin{bmatrix}A_1 & A_3\\ A_3 & A_2\end{bmatrix}\begin{bmatrix}\delta_1\\ \delta_1\end{bmatrix}=\begin{bmatrix}A_4\\ A_5\end{bmatrix} \qquad (2.12)$$

Solving (2.12) we get the optimum values of δ_1 and δ_2 as

$$\left.\begin{array}{l}\delta_1=\dfrac{(A_2A_4-A_3A_5)}{(A_1A_2-A_3^2)}=\delta_{10}\quad(say)\\ \delta_2=\dfrac{(A_1A_5-A_3A_4)}{(A_1A_2-A_3^2)}=\delta_{20}\quad(say)\end{array}\right\} \qquad (2.13)$$

Thus the resulting minimum *MSE* of the class of estimators T is given by

$$min.MSE(T)=\bar{Y}^2\left[1-\frac{(A_2A_4^2-2A_3A_4A_5+A_1A_5^2)}{(A_1A_2-A_3^2)}\right] \qquad (2.14)$$

Now we established the following theorem.

Theorem 2.1- To the first degree of approximation,

$$MSE(T) \geq \bar{Y}^2\left[1-\frac{\left(A_2A_4^2-2A_3A_4A_5+A_1A_5^2\right)}{\left(A_1A_2-A_3^2\right)}\right]$$

with equality holding if

$$\delta_1=\delta_{10},$$

and $\delta_2=\delta_{20}$ **.**

Special Cases:

Case-I-Putting $(\alpha_1,\alpha_2,\alpha_3,\alpha_4,a,b,c,d,\eta_1,\eta_2,\eta_3,\eta_4)=(0,0,0,0,1,0,1,0,1,0,0,\eta_4)$ and $(\delta_1,\delta_2)=(w_1,w_2)$ we get a class of estimators for $\bar{Y}$ as

$$T_1=w_1\bar{y}+w_2\left(\frac{\bar{x}_t}{\bar{X}}\right)exp\left\{\frac{\eta_4(\bar{z}_t-\bar{Z})}{\bar{z}_t+\bar{Z}}\right\} \tag{2.15}$$

where (w_1,w_2) are suitably chosen constants such that *MSE* of (2.15) is minimum.
Putting $(\alpha_1,\alpha_2,\alpha_3,\alpha_4,a,b,c,d,\eta_1,\eta_2,\eta_3,\eta_4)=(0,0,0,0,1,0,1,0,1,0,0,\eta_4)$ in (2.4) and (2.8) we get the bias and *MSE* of T respectively as

$$B(T_1)=\bar{Y}\left[w_1A_{4(1)}+w_2A_{5(1)}-1\right] \tag{2.16}$$

$$MSE(T_1)=\bar{Y}^2\left[1+w_1^2A_{1(1)}+w_2^2A_{2(1)}+2w_1w_2A_{3(1)}-2w_1A_{4(1)}-2w_2A_{5(1)}\right], \tag{2.17}$$

where

$$A_{1(1)}=\left[1+\frac{(1-f)}{n}C_y^2\right],$$

$$A_{2(1)}=\left[1+\frac{(1-f)}{n}\left\{\begin{aligned}&C_y^2-4g\tau_1\rho_{xy}C_xC_y-2\eta_4g\tau_2\rho_{yz}C_yC_z\\&+2\eta_4g^2\tau_1\tau_2\rho_{xz}C_xC_z+g^2\tau_1^2C_x^2+\frac{\eta_4(\eta_4-1)}{2}g^2\tau_2^2C_z^2\end{aligned}\right\}\right],$$

$$A_{3(1)}=\left[1+\frac{(1-f)}{n}\left\{\begin{aligned}&C_y^2-2g\tau_1\rho_{xy}C_xC_y-g\eta_4\tau_2\rho_{yz}C_yC_z\\&+g^2\left(\frac{\eta_4}{2}\right)\tau_1\tau_2\rho_{xz}C_xC_z+\frac{\eta_4(\eta_4-1)}{8}g^2\tau_2^2C_z^2\end{aligned}\right\}\right],$$

$$A_{4(1)}=1,$$

$$A_{5(1)}=\left[1+\frac{(1-f)}{2n}\left\{\begin{aligned}&g^2\eta_4\tau_1\tau_2\rho_{xz}C_xC_z-2g\tau_1\rho_{xy}C_xC_y\\&-g\eta_4\tau_2\rho_{yz}C_yC_z+\frac{\eta_4(\eta_4-1)}{4}g^2\tau_2^2C_z^2\end{aligned}\right\}\right].$$

The $MSE(T_1)$ at (2.17) is minimum when

$$\left.\begin{aligned} w_1 &= \frac{\left(A_{2(1)} - A_{3(1)}A_{5(1)}\right)}{\left(A_{1(1)}A_{2(1)} - A_{3(1)}^2\right)} = w_{10} \quad (say) \\ w_2 &= \frac{\left(A_{1(1)}A_{5(1)} - A_{3(1)}\right)}{\left(A_{1(1)}A_{2(1)} - A_{3(1)}^2\right)} = w_{20} \quad (say) \end{aligned}\right\} \tag{2.18}$$

Thus the resulting minimum *MSE* of T_1 is given by

$$min.MSE(T_1) = \bar{Y}^2\left[1 - \frac{\left(A_{2(1)} - 2A_{3(1)}A_{5(1)} + A_{1(1)}A_{5(1)}^2\right)}{\left(A_{1(1)}A_{2(1)} - A_{3(1)}^2\right)}\right] \tag{2.19}$$

Thus we state the following theorem.

Theorem- 2.2 – To the first degree of approximation,

$$MSE(T_1) \geq \bar{Y}^2\left[1 - \frac{\left(A_{2(1)} - 2A_{3(1)}A_{5(1)} + A_{1(1)}A_{5(1)}^2\right)}{\left(A_{1(1)}A_{2(1)} - A_{3(1)}^2\right)}\right]$$

with equality holding if

$w_1 = w_{10}$,

$w_2 = w_{20}$.

For $w_1 = w$ and $w_2 = (1-w)$ in (2.15) the class of estimators T_1 at (2.15) reduces to the estimator $t_{16} = T_{(1)}$ at (1.46) due to Yasmeen et al (2015). Putting $w_1 = w$ and $w_2 = (1-w)$ in (2.16) and (2.17) respectively we get the bias and $MSET_{1(1)}$ as

$$B(T_{1(1)}) = \bar{Y}(1-w)\left(A_{5(1)} - 1\right) \tag{2.20}$$

$$MSE(T_{1(1)}) = \bar{Y}^2\begin{bmatrix} 1 + A_{2(1)} - 2A_{5(1)} + w^2\left(A_{1(1)} + A_{2(1)} - 2A_{3(1)}\right) \\ -2w\left(A_{2(1)} - A_{3(1)} + 1 - A_{5(1)}\right) \end{bmatrix} \tag{2.21}$$

The $MSE(T_1)$ is minimized for

$$w = \frac{\left(A_{2(1)} - A_{3(1)} + 1 - A_{5(1)}\right)}{\left(A_{1(1)} + A_{2(1)} - 2A_{3(1)}\right)} = w_0 \quad (say) \tag{2.22}$$

Thus the resulting minimum *MSE* of the class of estimators $T_{1(1)}$ is given by

$$min.MSE(T_{1(1)}) = \bar{Y}^2\left[1 + A_{2(1)} - 2A_{5(1)} - \frac{\left(A_{2(1)} - A_{3(1)} + 1 - A_{5(1)}\right)^2}{\left(A_{1(1)} + A_{2(1)} - 2A_{3(1)}\right)}\right] \tag{2.23}$$

from (2.19) and (2.23) we have

$$min.MSE(T_{1(1)}) - min.MSE(T_1) = \bar{Y}^2\frac{\left[A_{1(1)}\left(A_{2(1)} - A_{5(1)}\right) + \left(A_{3(1)} - A_{2(1)}\right) + A_{3(1)}\left(A_{5(1)} - A_{3(1)}\right)\right]^2}{\left(A_{1(1)} + A_{2(1)} - 2A_{3(1)}\right)\left(A_{1(1)} + A_{2(1)} - 2A_{3(1)}^2\right)} \tag{2.24}$$

which is positive.

It follows that the proposed class of estimators T_1 is more efficient than the class of estimators $T_{1(1)} = t_{16}$ due to Yasmeen et al (2015).

Case-II-For $(\alpha_1,\alpha_2,\alpha_3,\alpha_4,a,b,c,d,\eta_1,\eta_2,\eta_3,\eta_4)=(-\alpha_1,0,0,-\alpha_4,1,0,1,0,\eta_1,0,0,\eta_4)$ in (2.1), the proposed class of estimators T reduces to the class of estimators T_2 as

$$T_2=\bar{y}\left[\delta_1\bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)^{\alpha 1}exp\left\{\frac{\alpha_4(\bar{Z}-\bar{z})}{(\bar{z}+\bar{Z})}\right\}+\delta_2\left(\frac{\bar{x}_t}{\bar{X}}\right)^{\eta 1}exp\left\{\frac{\eta_4(\bar{z}_t-\bar{Z})}{(\bar{z}_t+\bar{Z})}\right\}\right] \tag{2.25}$$

Putting $(\alpha_1,\alpha_2,\alpha_3,\alpha_4,a,b,c,d,\eta_1,\eta_2,\eta_3,\eta_4)=(-\alpha_1,0,0,-\alpha_4,1,0,1,0,\eta_1,0,0,\eta_4)$ in (2.4) and (2.8) we get the bias and *MSE* of the subclass of estimators T_2 to the first degree of approximation respectively as

$$B(T_2)=\bar{Y}\left[\delta_1A_{4(2)}+\delta_2A_{5(2)}-1\right] \tag{2.26}$$

$$MSE(T_2)=\bar{Y}^2\left[1+\delta_1^2A_{1(2)}+\delta_2^2A_{2(2)}+2\delta_1\delta_2A_{3(2)}-2\delta_1A_{4(2)}-2\delta_2A_{5(2)}\right] \tag{2.27}$$

where

$$A_{1(2)}=\left[1+\frac{(1-f)}{n}\left\{\begin{array}{l}C_y^2+\alpha_1(2\alpha_1+1)C_x^2+\dfrac{\alpha_4(\alpha_4+1)}{2}C_z^2-4\alpha_1\rho_{xy}C_xC_y\\-2\alpha_4\rho_{yz}C_yC_z+2\alpha_1\alpha_4\rho_{xz}C_xC_z\end{array}\right\}\right],$$

$$A_{2(2)}=\left[1+\frac{(1-f)}{n}\left\{\begin{array}{l}C_y^2-4\eta_1g\rho_{xy}C_xC_y-2\eta_4g\rho_{yz}C_yC_z\\-2\eta_1\eta_4g^2\rho_{xz}C_xC_z+\eta_1(2\eta_1-1)g^2C_x^2+\dfrac{\eta_4(\eta_4-1)}{2}g^2C_z^2\end{array}\right\}\right],$$

$$A_{3(2)}=\left[1+\frac{(1-f)}{n}\left\{\begin{array}{l}C_y^2-2(\alpha_1+g\eta_1)\rho_{xy}C_xC_y-(\alpha_4+g\eta_4)\rho_{yz}C_yC_z\\+\dfrac{(\alpha_1+g\eta_1)(\alpha_4+g\eta_4)}{2}\rho_{xz}C_xC_z+\dfrac{\left\{(\eta_1g+\alpha_1)^2-(\eta_1g^2-\alpha_1)\right\}}{2}C_x^2\\+\dfrac{\left\{(\eta_4g+\alpha_4)^2-2(\eta_4g^2-\alpha_4)\right\}}{8}C_z^2\end{array}\right\}\right]$$

$$A_{4(2)}=\left[1+\frac{(1-f)}{8n}\left\{\begin{array}{l}4\alpha_1\rho_{xz}C_{xz}-8\alpha_1\rho_{yz}C_yC_z-4\alpha_4\rho_{yz}C_yC_z\\+4\alpha_1(\alpha_1+1)C_x^2+\alpha_4(\alpha_4+2)C_z^2\end{array}\right\}\right],$$

$$A_{5(2)}=\left[1+\frac{(1-f)}{8n}\left\{\begin{array}{l}4\eta_1\eta_4g^2\rho_{xz}C_xC_z-8\eta_1g\rho_{xy}C_xC_y-4\eta_4g\rho_{yz}C_yC_z\\+4\eta_1(\eta_1-1)g^2C_x^2+\eta_4(\eta_4-2)g^2C_z^2\end{array}\right\}\right].$$

The $MSE(T_2)$ at (2.27) is minimized for

$$\left.\begin{array}{l}\delta_1=\dfrac{\left(A_{2(2)}A_{4(2)}-A_{3(2)}A_{5(2)}\right)}{\left(A_{1(2)}A_{2(2)}-A_{3(2)}^2\right)}=\delta_{10}^*\quad(say)\\[2ex]\delta_2=\dfrac{\left(A_{1(1)}A_{5(2)}-A_{3(2)}A_{4(2)}\right)}{\left(A_{1(2)}A_{2(2)}-A_{3(2)}^2\right)}=\delta_{20}^*\quad(say)\end{array}\right\} \tag{2.28}$$

Thus the resulting minimum $MSE(T_2)$ is given by

$$min.MSE(T_2)=\bar{Y}^2\left[1-\frac{\left(A_{2(2)}A_{4(2)}^2-2A_{3(2)}A_{4(2)}A_{5(2)}+A_{1(2)}A_{5(2)}^2\right)}{\left(A_{1(2)}A_{2(2)}-A_{3(2)}^2\right)}\right] \quad (2.29)$$

Thus we established the following theorem.

Theorem 2.3- To the first degree of approximation,

$$MSE(T_2)\geq\bar{Y}^2\left[1-\frac{\left(A_{2(2)}A_{4(2)}^2-2A_{3(2)}A_{4(2)}A_{5(2)}+A_{1(2)}A_{5(2)}^2\right)}{\left(A_{1(2)}A_{2(2)}-A_{3(2)}^2\right)}\right]$$

with equality holding if

$\delta_1=\delta_{10}^*$,

and $\delta_2=\delta_{20}^*$ **.**

For $\delta_1=\delta$ and $\delta_2=(1-\delta)$ in (2.25), the class of estimators T_2 reduces to the class of estimators $T_{2(1)}=t_{17}$ defined at (1.47) due to Yasmeen et al (2015). Putting $\delta_1=\delta$ and $\delta_2=(1-\delta)$ in (2.26) and (2.27) we get the bias and *MSE* of the class of estimators $T_{2(1)}=t_{17}$ to the first degree of approximation, respectively as

$$B(T_{2(1)}=t_{17})=\bar{Y}\left[\delta\left(A_{4(2)}-A_{5(2)}\right)+A_{5(2)}-1\right] \quad (2.30)$$

$$MSE(T_{(2)1}=t_{17})=\bar{Y}^2\begin{bmatrix}1+A_{2(2)}-2A_{5(2)}+\delta^2\left(A_{1(2)}+A_{2(2)}-2A_{3(2)}\right)\\ -2\delta\left(A_{2(2)}-A_{3(2)}+A_{4(2)}-A_{5(2)}\right)\end{bmatrix} \quad (2.31)$$

The $MSE\left(T_{2(1)}=t_{17}\right)$ at (2.31) is minimized for

$$\delta=\frac{\left(A_{2(2)}-A_{3(2)}+A_{4(2)}-A_{5(2)}\right)}{\left(A_{1(2)}A_{2(2)}-2A_{3(2)}\right)}=\delta_0 \quad (say) \quad (2.32)$$

Thus the resulting minimum *MSE* of $T_{2(1)}=t_{17}$ is given by

$$min.MSE(T_{2(1)}=t_{17})=\bar{Y}^2\left[1+A_{2(2)}-2A_{5(2)}-\frac{\left(A_{2(2)}-A_{3(2)}+A_{4(2)}-A_{5(2)}\right)^2}{\left(A_{1(2)}+A_{2(2)}-2A_{3(2)}\right)}\right] \quad (2.33)$$

From (2.29) and (2.33) we have

$$min.MSE(T_{2(1)}=t_{17})-min.MSE(T_2)=\frac{\bar{Y}^2\begin{bmatrix}A_{1(2)}\left(A_{2(2)}-A_{5(2)}\right)+A_{4(2)}\left(A_{3(2)}-A_{2(2)}\right)\\ +A_{3(2)}\left(A_{5(2)}-A_{3(2)}\right)\end{bmatrix}^2}{\left(A_{1(2)}+A_{2(2)}-2A_{3(2)}\right)\left(A_{1(2)}+A_{2(2)}-A_{3(2)}^2\right)} \quad (2.34)$$

which is positive.

It follows from (2.34) that the proposed class of estimators T_2 is more efficient than the class of estimators $T_{2(1)}=t_{17}$ due to Yasmeen et al (2015).

Case- III- We consider the case when sum of the weights δ_1 and δ_2 is unity $(i.e.\ \delta_1+\delta_2=1)$**.** Putting $\delta_2=(1-\delta_1)$ in (2.1) we get a class of estimators for $\bar{Y}$ as

$$T_3 = \bar{y}\begin{bmatrix} \delta_1 \left(\frac{a\bar{x}+b}{a\bar{X}+b}\right)^{\alpha_1}\left(\frac{c\bar{z}+d}{c\bar{Z}+d}\right)^{\alpha_2} exp\left\{\frac{\alpha_3 a\left(\bar{x}-\bar{X}\right)}{a\left(\bar{X}+\bar{x}\right)+2b}\right\} exp\left\{\frac{\alpha_1 c\left(\bar{z}-\bar{Z}\right)}{c\left(\bar{z}+\bar{Z}\right)+2d}\right\} \\ +\left(1-\delta_1\right)\left(\frac{a\bar{x}_t+b}{a\bar{X}+b}\right)^{\eta_1}\left(\frac{c\bar{z}_t+d}{c\bar{Z}+d}\right)^{\eta_2} exp\left\{\frac{\eta_3 a\left(\bar{x}_t-\bar{X}\right)}{a\left(\bar{x}_t+\bar{X}\right)+2b}\right\} exp\left\{\frac{\eta_4 c\left(\bar{z}_t-\bar{Z}\right)}{c\left(\bar{z}_t+\bar{Z}\right)+2d}\right\}\end{bmatrix} \quad (2.35)$$

Putting $\delta_2 = (1-\delta_1)$ in (2.4) and (2.8) we get the bias and *MSE* of the class of estimators T_3 to the first degree of approximation respectively as

$$B(T_3) = \bar{Y}\left[\delta(A_4 - A_5) + A_5 - 1\right] \quad (2.36)$$

$$MSE(T_3) = \bar{Y}^2\left[1 + A_2 - 2A_5 + \delta_1^2(A_1 + A_2 - 2A_3) - 2\delta_1(A_4 - A_3 + A_2 - A_5)\right] (2.37)$$

The $MSE(T_3)$ at (2.37) is minimized for

$$\delta_1 = \frac{(A_2 - A_3 + A_4 - A_5)}{(A_1 + A_2 - 2A_3)}. \ (2.38)$$

Thus the resulting minimum *MSE* of the class of estimators T_3 is given by

$$min.MSE(T_3) = \bar{Y}^2\left[1 + A_2 - 2A_5 - \frac{(A_2 - A_3 + A_4 - A_5)^2}{(A_1 + A_2 - 2A_3)}\right] (2.39)$$

From (2.8) and (2.39) we have

$$min.MSE(T_3) - min.MSE(T) = \bar{Y}^2 \frac{\left[A_1(A_2 - A_5) + A_4(A_3 - A_2) + A_3(A_5 - A_3)\right]^2}{(A_1 + A_2 - 2A_3)\left(A_1 A_2 - A_3^2\right)} (2.40)$$

which is positive.

Thus the T-class of estimators is more efficient than the T_3-class of estimators.

Case- IV-If we set $\delta_2 = 0$, in (2.1) we get a Searls (1964), Singh et al (1973) and Searls and Intarapanich(1990) -type class of estimators for $\bar{Y}$ as

$$T_4 = \bar{y}\delta_1\left(\frac{a\bar{x}+b}{a\bar{X}+b}\right)^{\alpha_1}\left(\frac{c\bar{z}+d}{c\bar{Z}+d}\right)^{\alpha_2} exp\left\{\frac{\alpha_3 a\left(\bar{x}-\bar{X}\right)}{a\left(\bar{X}+\bar{x}\right)+2b}\right\} exp\left\{\frac{\alpha_4 c\left(\bar{z}-\bar{Z}\right)}{c\left(\bar{z}+\bar{Z}\right)+2d}\right\} (2.41)$$

Inserting $\delta_2 = 0$ in (2.4) and (2.8) we get the bias and *MSE* of the suggested class of estimators T as

$$B(T_4) = \bar{Y}(\delta_1 A_4 - 1) \quad (2.42)$$

$$MSE(T_4) = \bar{Y}^2\left[1 + \delta_1^2 A_1 - 2\delta_1 A_4\right] \quad (2.43)$$

The $MSE(T_4)$ at (2.43) is minimum when

$$\delta_1 = \frac{A_4}{A_1} = \delta_{10}^{**} \text{ (say)} \quad (2.44)$$

Thus the resulting minimum *MSE* of T_4 is given by

$$min.MSE(T_4) = \bar{Y}^2\left(1 - \frac{A_4^2}{A_1}\right) \quad (2.45)$$

Thus we state the following theorem.

Theorem 2.4- To the first degree of approximation,

$$MSE(T_4) \geq \bar{Y}^2\left(1-\frac{A_4^2}{A_1}\right)$$

with equality holding if

$\delta_1 = \delta_{10}^{**}$.

From (2.14) and (2.45) we have

$$min.MSE(T_4) - min.MSE(T) = \frac{\bar{Y}^2 (A_1 A_5 - A_3 A_4)^2}{A_1(A_1 A_2 - A_3^2)} \quad (2.46)$$

which is positive.

Thus the proposed *T*- class of estimators is more efficient than T_4-class of estimators.

Case- V- If we set $\delta_1 = 0$ in (2.1) we get a class of estimators for the population mean $\bar{Y}$ as

$$T_5 = \delta_2 \bar{y}\left(\frac{a\bar{x}_t + b}{a\bar{X} + b}\right)^{\eta 1}\left(\frac{c\bar{z}_t + d}{c\bar{Z} + d}\right)^{\eta 2} exp\left\{\frac{\eta_3 a\left(\bar{x}_t - \bar{X}\right)}{a\left(\bar{x}_t + \bar{X}\right) + 2b}\right\} exp\left\{\frac{\eta_4 c\left(\bar{z}_t - \bar{Z}\right)}{c\left(\bar{z}_t + \bar{Z}\right) + 2d}\right\} \quad (2.47)$$

Putting $\delta_1 = 0$ in (2.4) and (2.8) we get the bias and *MSE* of estimators T_5 to the first degree of approximation respectively as

$$B(T_5) = \bar{Y}(\delta_2 A_5 - 1) \quad (2.48)$$

$$MSE(T_5) = \bar{Y}^2\left[1 + \delta_2^2 A_2 - 2\delta_2 A_5\right] \quad (2.49)$$

The $MSE(T_5)$ at (2.49) is minimum when

$$\delta_2 = \frac{A_5}{A_2} = \delta_{20}^{**} \text{ (say)} \quad (2.50)$$

Thus the resulting minimum *MSE* of estimators T_5 is given by

$$min.MSE(T_5) = \bar{Y}^2\left(1 - \frac{A_5^2}{A_2}\right) \quad (2.51)$$

Now we state the following theorem.

Theorem 2.5 – To the first degree of approximation,

$$MSE(T_5) \geq \bar{Y}^2\left(1-\frac{A_5^2}{A_2}\right)$$

with equality holding if

$\delta_2 = \delta_{20}^{**}$.

From (2.8) and (2.51) we have

$$min.MSE(T_5) - min.MSE(T) = \frac{\bar{Y}^2 (A_2 A_4 - A_3 A_5)^2}{A_2(A_1 A_2 - A_3^2)} \quad (2.52)$$

which is positive.

It follows from (2.52) that the proposed *T*- class of estimators is more efficient than T_3-class of estimators.

3. Empirical Study

To illustrate our general results we have computed the percent relative efficiencies (*PREs*) of different estimators of population mean $\bar{Y}$ with respect to usual unbiased estimator $\bar{y}$ for two natural populations data sets earlier considered by Yasmeen et al (2015).

Population – I- Source : Anderson (1958).
y : Head length of second son.
x : Head length of first son.
z : Head breathe of first son.
The required parameters of the population are:

$$\bar{Y} = 183.84,\quad C_y = 0.0546,\quad \rho_{yx} = 0.7108,\quad N = 25.$$
$$\bar{X} = 185.72,\quad C_x = 0.0526,\quad \rho_{yz} = 0.6932,\quad n = 5.$$
$$\bar{Z} = 151.122,\quad C_z = 0.0488,\quad \rho_{xz} = 0.7346.$$

Population – II- Source :Cochran (1977).
y : Number of "placebo" children.
x : Number of paralytic polio cases in the "not inoculated" group.
z : Number of paralytic polio cases in the placebo group.
The required parameters of the population are:

$$\bar{Y} = 4.92,\quad C_y = 1.0123,\quad \rho_{yx} = 0.6430,\quad N = 34.$$
$$\bar{X} = 2.59,\quad C_x = 1.0720,\quad \rho_{yz} = 0.7326,\quad n = 5.$$
$$\bar{Z} = 2.91,\quad C_z = 1.2319,\quad \rho_{xy} = 0.6837.$$

We have computed the percent relative efficiencies (*PREs*) of different estimators of population mean $\bar{Y}$ with respect to the sample mean $\bar{y}$ by using the formulae:

$$PRE(t_1, \bar{y}) = \frac{C_y^2}{\left[C_y^2 + C_x^2(1 - 2K_{yx})\right]} * 100, \tag{3.1}$$

$$PRE(t_2, \bar{y}) = \frac{C_y^2}{\left[C_y^2 + C_z^2(1 + 2K_{yz})\right]} * 100, \tag{3.2}$$

$$PRE(t_3, \bar{y}) = \frac{C_y^2}{\left[C_y^2 + g\, C_x^2(g - 2K_{yx})\right]} * 100, \tag{3.3}$$

$$PRE(t_4, \bar{y}) = \frac{C_y^2}{\left[C_y^2 + g\, C_z^2(g + 2K_{yz})\right]} * 100, \tag{3.4}$$

$$PRE(t_5, \bar{y}) = \frac{C_y^2}{\left[C_y^2 + C_x^2(1 - 2K_{yx}) + C_z^2(1 + 2K_{yz}) - 2\rho_{xz}C_xC_z\right]} * 100, \tag{3.5}$$

$$PRE(t_j, \bar{y}) = \left(1 - \rho_{y.xz}^2\right)^{-1} * 100, \tag{3.6}$$

$j = 6,7,9,12,13,15.$

$$PRE(t_8,\bar{y})=\frac{C_y^2}{\left[C_y^2+g\,C_x^2\left(g-2K_{yx}\right)+g\,C_z^2\left(g+2K_{yz}\right)-2g^2\rho_{xz}C_xC_z\right]}*100, \tag{3.7}$$

$$PRE(t_{10},\bar{y})=\frac{C_y^2}{\left[C_y^2+\tau_1C_x^2\left(\tau_1-2K_{yz}-2\tau_1^*K_{zx}\right)+\tau_1^*C_z^2\left(\tau_1^*+2K_{yz}\right)\right]}*100, \tag{3.8}$$

$$PRE(t_{11},\bar{y})=\frac{C_y^2}{\left[C_y^2+\tau_1gC_x^2\left(\tau_1g-2K_{yx}-2\tau_1^*gK_{zx}\right)+\tau_1^*g\,C_z^2\left(\tau_1^*g+2K_{yz}\right)\right]}*100, \tag{3.9}$$

$$PRE(t_{14},\bar{y})=\frac{C_y^2}{\left[C_y^2+\left(C_x^2/4\right)\left(1-4K_{yx}-2K_{zx}\right)+\left(C_z^2/4\right)\left(1+4K_{yz}\right)\right]}*100 \tag{3.10}$$

Findings are shown in Table 3.1 **.**

Table 3.1- Percent relative efficiencies (*PREs)* of different estimators of population mean $\bar{Y}$ with respect to the usual unbiased estimator $\bar{y}$

Estimator	$PRE(.,\bar{y})$	
	Population	
	I	**II**
$\bar{y}$	100.00	100.00
t_1	179.0334	131.6532
t_2	32.9169	23.4536
t_3	139.7383	125.2287
t_4	73.5452	73.9960
t_5	75.1044	44.2232
t_j, $j=6,7,9,12,13,15$	231.9047	235.0896
t_8	102.1717	90.9437
t_{10}	79.5458	52.0964
t_{11}	100.4145	92.6497
t_{14}	95.2136	70.3928

We have computed the *PREs* of the proposed class of estimators *T* with respect to the usual unbiased estimator $\bar{y}$ by using the formula:

$$PRE(T,\bar{y})=\frac{\frac{(1-f)}{n}C_y^2}{\left[1-\frac{\left(A_2A_4^2-2A_3A_4A_5+A_1A_5^2\right)}{\left(A_1A_2-A_3^2\right)}\right]}*100 \tag{3.11}$$

for different values of the constants $(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\eta_1,\eta_2,\eta_3,\eta_4,a,b,c,d)$**.**

Findings are shown in Table 3.2 **.**

Table- 3.2- *PREs* of the proposed class of estimators *T* with respect to $\bar{y}$

S. no.	Values of constant												$PRE(T,\bar{y})$	
	α_1	α_2	α_3	α_4	η_1	η_2	η_3	η_4	a	b	c	d	population I	population II
1	1	2.5	2	2	1	1	2	2	1	1.5	1	2	2272.2860	254.0740
2	0.75	2.25	0	0	0.75	3.5	0.25	1.25	1.75	2.25	1.75	1.75	309.3122	247.1698
3	0.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5	419.9894	242.7776
4	0.25	1.25	2	0.25	3	2	0.5	2.5	0.5	0.5	1.25	1.25	426.0669	300.4216
5	1.25	2	1.25	1.25	0	3	2.25	1.25	4	0.25	4	4.5	515.2957	822.2755
6	0	3	2.5	1	1.5	1.5	1.5	1	1.5	2.5	2.5	1	677.0071	311.6904

Table 3.1 depicts that the PRE of the estimator t_j, *j=6,7,9,12,13,15* with respect to $\bar{y}$ is the largest i.e. 231.5017% (in population I) and 235.0896% (in population II). So the estimator t_j, *j=6,7,9,12,13,15;* is the best among the estimators $\bar{y}, t_1, t_2, t_3, t_4, t_5, t_8, t_{10}, t_{11}$ *and* t_{14} given in Table 3.1. Entries in Table 3.2 exhibit that the PREs of the proposed class of estimators *T* with respect to $\bar{y}$ are larger than that of the estimator t_j, *j=6,7,9,12,13,15;* for the selected values of the scalars $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \eta_1, \eta_2, \eta_3, \eta_4, a, b, c, d)$ for both the population data sets. Findings of the Tables 3.1 and 3.2 clearly indicate that the gain in efficiency by using the proposed class of estimators *T* over other existing estimators is substantial. So we infer that there is enough scope of selecting the values of scalars $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \eta_1, \eta_2, \eta_3, \eta_4, a, b, c, d)$ involved in the proposed class of estimators *T* for obtaining the estimators more efficient than those considered the Table 3.1. thus our recommendation is in the favor of the suggested class of estimators *T*.

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