A General Transmuted Family of Distributions

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Abstract

In this article we have proposed a general transmuted family of distributions with emphasis on the cubic transmuted (CT) family of distributions. This new class of distributions provide additional flexibility in modeling the bi-modal data. The proposed cubic transmuted family of distributions has been linked with the $T-X$ family of distributions proposed by Alzaatreh et al. (2013). Some members of the proposed family of distributions have been discussed. The cubic transmuted exponential distribution has been discussed in detail and various statistical properties of the distribution have been explored. The maximum likelihood estimation for parameters of cubic transmuted exponential distribution has also been discussed alongside Monte Carlo simulation study to assess the performance of the estimation procedure. Finally, the cubic transmuted exponential distribution has been fitted to real datasets to investigate it's applicability.

Keywords: Cubic transmuted distribution, Exponential distribution, General transmutation, Maximum likelihood estimation, Reliability analysis

1. Introduction

Probability distributions has been popularly used in many areas of life. The standard probability distributions has been used in practice from long. In recent years the extension and generalization of probability distributions have attracted several statisticians. A simple extension of the probability distributions is proposed by Gupta et al. (1998) and is known as the *exponentiated family of distributions*. The exponentiated family of distributions provides flexibility by adding one more parameter to the distribution and has attracted several authors, see for example Gupta and Kundu (1999, 2007), among others.

Eugene et al. (2002) have proposed the *Beta-G* distributions using logit of the Beta distributions. The *Beta-G* distributions extends the distribution of order statistics. Another family of distributions, known as *Kum-G* distributions, has been proposed by Cordeiro and Castro (2010) by using *cdf* of the Kumaraswamy distribution. The

Beta-G and *Kum-G* families of distributions uses *cdf* of bounded random variables and thus has limited extension. Alzaatreh et al. (2013), have proposed a general method of extending probability distributions known as *T-X* family of distributions which used *cdf* of any random variable.

The *Beta-G*, *Kum-G* and *T-X* families of distributions uses some baseline distribution. Another method of generating families of distributions has been proposed by Shaw and Buckley (2007), which used quadratic transmutation map to generate new probability distribution using any baseline distribution. The quadratic transmuted family of distributions of Shaw and Buckley (2007) has *cdf*

$$F(x) = (1+\lambda)G(x) - \lambda G^2(x),$$

for $\lambda \in [-1, 1]$, where $G(x)$ is the *cdf* of the baseline distribution. Observe that at $\lambda = 0$, we get the baseline *cdf*.

The quadratic transmuted family of distributions by Shaw and Buckley (2007), has opened doors to extend the existing probability models to capture the quadratic behavior of the data. The quadratic transmuted class has been further studied and extended by Aryal and Tsokos (2009, 2011); Nofal et al. (2017); Alizadeh et al. (2016); Merovci et al. (2016) and Yousof et al. (2015). At this time, quadratic transmuted distributions are familiar in the literature, several quadratic transmuted distributions are provided by Tahir and Cordeiro (2016). Alizadeh et al. (2017), have noted that the quadratic transmuted family of distribution can be linked with the *T-X* family of distributions for suitable choice of a bounded density function between $[0, 1]$. In order to capture the complexity of the data and increases the flexibility, new classes of cubic transmuted distributions have been developed by Granzotto et al. (2017); AL-Kadim and Mohammed (2017).

In this paper, we have proposed a family of general transmuted distributions and have developed the cubic transmuted family of distributions as well. This family of distributions captures the complexity of the data arising in engineering, environmental, financial and other areas of life. The layout of the paper is given below.

1.1 Organization of the Article

The paper is structured as follows. The new family of general transmuted distributions is provided in Section 2. Section 3 provides a new class of cubic transmuted distributions. Some examples related to the cubic transmuted distributions are presented in Section 4. The cubic transmuted exponential distribution has been explored in detail in Section 5. Section 6 describes the expressions for moments, quantiles, reliability function and random number generation for the proposed cubic transmuted exponential distribution. In Section 7, we have presented estimation of the parameters and two real-life applications along with simulation study is given in Section 8. Finally, in Section 9, some concluding remarks are given.

2. General Transmuted Family of Distributions

In this section, we have proposed a general transmuted family of distributions that can be used to generate new families. The family is defined below.

Let X be a random variable with *cdf* $G(x)$, then a general transmuted family; called $k-$ transmuted family; is defined as

$$F(x) = G(x) + [1 - G(x)] \sum_{i=1}^{k} \lambda_i [G(x)]^i, \tag{1}$$

with $\lambda_i \in [-1, 1]$ for $i = 1, 2, \cdots, k$ and $-k \leq \sum_{i=1}^{k} \lambda_i \leq 1$. The general transmuted family reduces to the base distribution for $\lambda_i = 0$ for $i = 1, 2, \cdots, k$. The density function corresponding to (1) is

$$f(x) = g(x) \left[1 - \sum_{i=1}^{k} \lambda_i G^i(x) + \{1 - G(x)\} \sum_{i=1}^{k} i \lambda_i G^{i-1}(x) \right]$$

or

$$f(x) = g(x) \left[1 - \sum_{i=1}^{k} \lambda_i G^i(x) \left\{ 1 - \frac{i(1 - G(x))}{G(x)} \right\} \right] \tag{2}$$

We now give the cubic transmuted family of distributions in the following.

3. Cubic Transmuted Family of Distributions

In this section we have discussed the cubic transmuted family of distributions. The cubic transmuted family of distributions is obtained by setting $k = 2$ in (1) and is given as

$$F(x) = G(x) + \lambda_1 G(x)[1 - G(x)] + \lambda_2 G^2(x)[1 - G(x)],$$

which can also be written as

$$F(x) = (1 + \lambda_1)G(x) + (\lambda_2 - \lambda_1)G^2(x) - \lambda_2 G^3(x), \tag{3}$$

where $\lambda_1 \in [-1, 1]$ and $\lambda_2 \in [-1, 1]$ and $-2 \leq \lambda_1 + \lambda_2 \leq 1$. It can be easily seen that the cubic transmuted family of distributions proposed by AL-Kadim and Mohammed (2017), turned out to be a special case of (3) for $\lambda_2 = -\lambda_1$. The cubic family of transmuted distributions reduces to the quadratic transmuted family of distributions of Shaw and Buckley (2007), for $\lambda_2 = 0$. Also, the family (3) is different as compared with the family proposed by Granzotto et al. (2017).

We now present the genesis of cubic transmuted family of distributions given in (3) in context with distribution of order statistics and with $T - X$ family of distributions in following two theorems.

Theorem 3.1. *Let X_1, X_2 and X_3 be iid random variables each with distribution* *function $G(x)$, then the cubic transmuted family of distributions (3) can be obtained as a weighted sum of three order statistics.*

Proof. Consider three order statistics as

$$X_{3:3} = max(X_1, X_2, X_3),\ X_{2:3} \text{ and } X_{1:3} = min(X_1, X_2, X_3),$$

and the random variable Y as

$$Y \stackrel{d}{=} X_{3:3}, \quad \text{with probability } p_1,$$

$$Y \stackrel{d}{=} X_{2:3}, \quad \text{with probability } p_2,$$

$$Y \stackrel{d}{=} X_{1:3}, \quad \text{with probability } p_3,$$

such that $\sum_{i=1}^{3} p_i = 1 \Rightarrow p_3 = 1 - p_1 - p_2$. So the distribution function, $F_Y(x)$, of Y is given as

$$\begin{aligned} F_Y(x) &= p_1 F_{3:3}(x) + p_2 F_{2:3}(x) + p_3 F_{1:3}(x) \\ &= (3 - 3p_1 - 3p_2)G(x) + (3p_1 + 6p_2 - 3)G^2(x) - (3p_2 - 1)G^3(x). \end{aligned} \tag{4}$$

Setting $2 - 3p_1 - 3p_2 = \lambda_1$ and $3p_2 - 1 = \lambda_2$, the distribution given in (4) reduces to the cubic transmuted family of distributions given (3). □

Theorem 3.2. *Let $G(x)$ be cdf of a random variables X and $p(t)$ be density function of a bounded random variable with support on $[0, 1]$, then the cubic transmuted family given in (3) can be obtained by using $T - X$ family of distributions introduced by Alzaatreh et al. (2013), for suitable choice of $p(t)$. Also, the density $p(t)$ can be written as weighted sum of three bounded densities $p_1(t), p_2(t)$ and $p_3(t)$ with support on $[0, 1]$.*

Proof. The $T - X$ family introduced by Alzaatreh et al. (2013), is given as

$$F(x) = \int_0^{G(x)} p(t)\ \mathrm{d}t,\ x \in \mathbb{R}, \tag{5}$$

where $p(t)$ indicates the *pdf* with support on $[0, 1]$. Using

$$p(t) = 1 + \lambda_1 - 2\lambda_1 t + 2\lambda_2 t - 3\lambda_2 t^2,$$

in (5), we obtain the cubic transmuted family (3). Now, suppose that the density $p(t)$ is written as weighted sum of three densities as

$$p(t) = (1 - \lambda_1 - \lambda_2)p_1(t) + \lambda_1 p_2(t) + \lambda_2 p_3(t). \tag{6}$$

Hence, using $p_1(t) = 1,\ p_2(t) = 2(1 - t)$ and $p_3(t) = 1 + 2t - 3t^2$ in (6) we have

$$F(x) = \int_0^{G(x)} [(1 - \lambda_1 - \lambda_2)p_1(t) + \lambda_1 p_2(t) + \lambda_2 p_3(t)]\ \mathrm{d}t. \tag{7}$$

On simplification, the *cdf* (7) reduces to the *cdf* given in (3). □

We now give the density function of cubic transmuted family in the following.

Definition 3.1 (Cubic Transmuted Distribution)**.** *A random variable X is said to have a cubic transmuted distribution, with parameters $\lambda_1 \in [-1,1]$, $\lambda_2 \in [-1,1]$ and $-2 \leq \lambda_1 + \lambda_2 \leq 1$, if its pdf is given as*

$$f(x) = g(x)\left[1 + \lambda_1 + 2(\lambda_2 - \lambda_1)G(x) - 3\lambda_2 G^2(x)\right],\ x \in \mathbb{R}, \tag{8}$$

where $g(x)$ is the pdf associated with baseline cdf $G(x)$.

Observe that for $\lambda_2 = 0$, the *pdf* of cubic transmuted distribution reduces to the *pdf* of quadratic transmuted distribution. Also, observe that, at $\lambda_1 = \lambda_2 = 0$, we have the *pdf* of the baseline random variable, as it should be. The density function of cubic transmuted family of distributions given in (8) can be obtained by using $k = 2$ in the density function of general transmuted family of distributions given in (2).

Lemma 3.1. *The pdf $f(x)$ given in (8) is well defined.*

Proof. The proof is simple. □

4. Some Examples of Cubic Transmuted Distributions

In this section, we have discussed some members of the cubic transmuted family of distributions given in (3) for different choices of baseline $G(x)$. These are discussed in the following subsections.

4.1 Cubic Transmuted Normal Distribution

Suppose that the random variable X has normal distribution with *cdf* $\Phi(x)$, then the cubic transmuted normal distribution has *cdf*

$$F(x) = (1 + \lambda_1)\Phi(x) + (\lambda_2 - \lambda_1)\Phi^2(x) - \lambda_2\Phi^3(x),\ x \in \mathbb{R},$$

The density function of cubic transmuted normal distribution can be similarly written.

4.2 Cubic Transmuted Gamma Distribution

Let X is a gamma random variable with shape parameter α and scale parameter $1/\beta$, then the *cdf* of cubic transmuted gamma distribution is given as

$$G(x) = \frac{\Gamma(\alpha) - \Gamma(\alpha, \beta x)}{\Gamma(\alpha)},\ x \in \mathbb{R}^+,$$

where $\alpha, \beta \in \mathbb{R}^+$, $\Gamma(\alpha)$ is complete gamma function and $\Gamma(\alpha, \beta x)$ is the upper incomplete gamma function. Using (3), the *cdf* of the cubic transmuted gamma distribution is given by

$$F(x) = \frac{[\Gamma(\alpha)^2 + \lambda_1\Gamma(\alpha)\eta(x) + \lambda_2\eta(x)\{\Gamma(\alpha) - \eta(x)\}]}{[\Gamma(\alpha) - \eta(x)]^{-1}\,\Gamma(\alpha)^3},\ x \in \mathbb{R}^+,$$

where $\eta(x) = \Gamma(\alpha, \beta x)$.

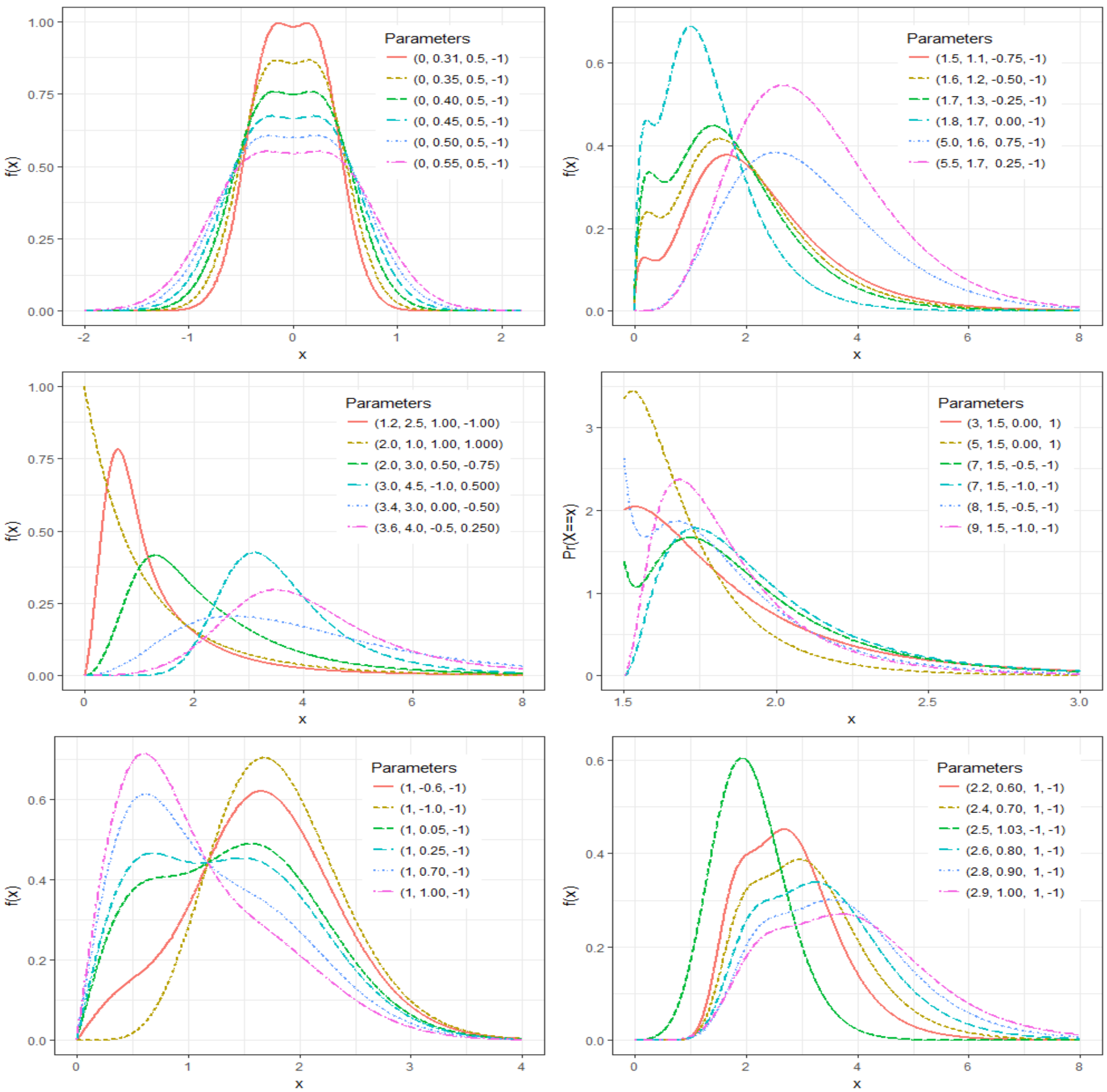

Figure 1: Density functions; Upper panels: CT normal $(\mu, \theta, \lambda_1, \lambda_2)$ and CT gamma $(\alpha, \beta, \lambda_1, \lambda_2)$; Middle panels: CT log-logistic $(\alpha, \beta, \lambda_1, \lambda_2)$ and CT Pareto $(\theta, k, \lambda_1, \lambda_2)$; Lower panels: CT Rayleigh $(\sigma, \lambda_1, \lambda_2)$ and CT Gumbel $(\mu, \beta, \lambda_1, \lambda_2)$, all are plotted for different values of model parameters.

4.3 Cubic Transmuted Log-logistic Distribution

Let X has Log-logistic distribution with *cdf*

$$G(x) = \frac{x^\beta}{\alpha^\beta + x^\beta}, \; x \in [0, \infty),$$

where $\alpha, \beta \in \mathbb{R}^+$ are the scale and shape parameters respectively. Using (3), the *cdf* of the cubic transmuted log-logistic distribution is given by

$$F(x) = \frac{x^\beta \left[\lambda_1 \alpha^\beta \left(\alpha^\beta + x^\beta\right) + \lambda_2 \alpha^\beta x^\beta + \left(\alpha^\beta + x^\beta\right)^2\right]}{\left(\alpha^\beta + x^\beta\right)^3}, \; x \in [0, \infty).$$

4.4 Cubic Transmuted Pareto Distribution

Suppose X has Pareto distribution with *cdf*

$$G(x) = 1 - \left(\frac{k}{x}\right)^{\theta},\ x \in [k, \infty),$$

where $k, \theta \in \mathbb{R}^+$ are the scale and shape parameters. Using (3), the *cdf* of cubic transmuted Pareto distribution is given by

$$F(x) = \left[\left(\frac{k}{x}\right)^{\theta} - 1\right]\left[-\lambda_1\left(\frac{k}{x}\right)^{\theta} + \lambda_2\left\{\left(\frac{k}{x}\right)^{\theta} - 1\right\}\left(\frac{k}{x}\right)^{\theta} - 1\right],\ x \in [k, \infty).$$

4.5 Cubic Transmuted Rayleigh Distribution

Let X has Rayleigh distribution with *cdf*

$$G(x) = 1 - e^{-\frac{x^2}{2\sigma^2}},\ x \in [0, \infty),$$

where $\sigma \in \mathbb{R}^+$ is the scale parameter of the distribution. Using (3), the *cdf* of cubic transmuted Rayleigh distribution is given by

$$F(x) = e^{-\frac{3x^2}{2\sigma^2}}\left(e^{\frac{x^2}{2\sigma^2}} - 1\right)\left[\lambda_1 e^{\frac{x^2}{2\sigma^2}} + \lambda_2\left(e^{\frac{x^2}{2\sigma^2}} - 1\right) + e^{\frac{x^2}{\sigma^2}}\right],\ x \in [0, \infty).$$

4.6 Cubic Transmuted Gumbel Distribution

Let random variable X has Gumbel distribution with *cdf*

$$G(x) = 1 - e^{-e^{-\frac{x-\mu}{\beta}}},\ x \in \mathbb{R},$$

where $\mu \in \mathbb{R}$, $\beta \in \mathbb{R}^+$ are the location and scale parameters respectively. Using (3), the *cdf* of the cubic transmuted Gumbel distribution is

$$F(x) = e^{-3e^{\frac{\mu-x}{\beta}}}\left(e^{e^{\frac{\mu-x}{\beta}}} - 1\right)\left[e^{2e^{\frac{\mu-x}{\beta}}} + \lambda_1 e^{e^{\frac{\mu-x}{\beta}}} + \lambda_2\left(e^{e^{\frac{\mu-x}{\beta}}} - 1\right)\right],\ x \in \mathbb{R}.$$

The plot of density function for above mentioned distributions is given in Figure 1. We now discuss the cubic transmuted exponential distribution in detail in the following section.

5. Cubic Transmuted Exponential Distribution

The exponential distribution is a widely used lifetime distribution. Several researchers have attempted to generalize exponential distribution for the limited applicability and they have proposed beta exponential (Nadarajah and Kotz, 2006), generalized exponential (Gupta and Kundu, 1999, 2007), exponentiated exponential (Gupta and Kundu, 2001) and quadratic transmuted exponential (Owoloko et al., 2015) distribu-

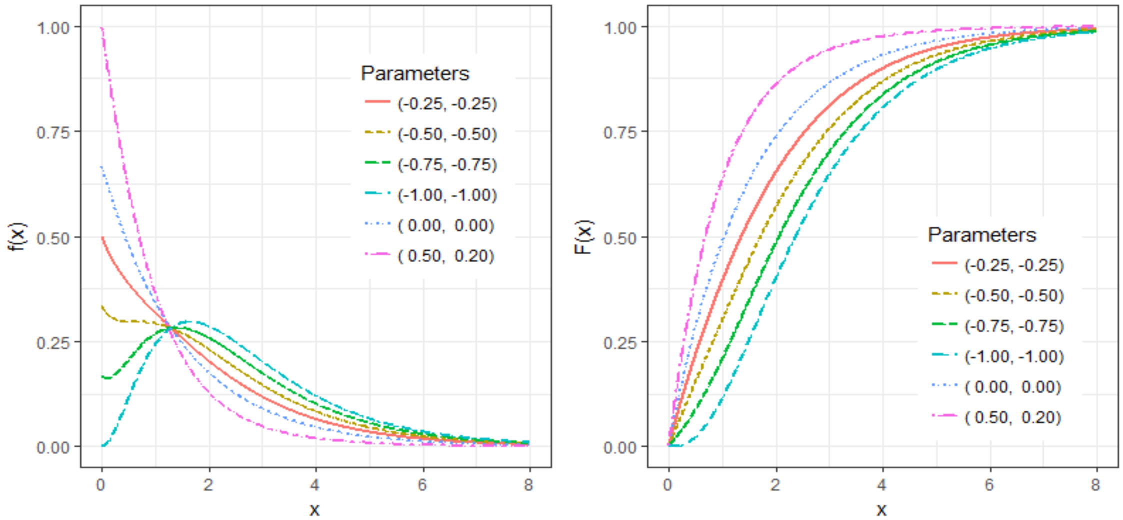

Figure 2: Density and distribution functions are plotted for the CT exponential distribution for different values of model parameters λ_1 and λ_2, setting $\theta = 1.5$.

tions which are more flexible than the exponential distribution. The *cdf* of exponential distribution is given by

$$G(x) = 1 - \mathrm{e}^{-\frac{x}{\theta}}, \; x \in [0, \infty),$$

where $\theta \in [0, \infty)$ is the scale parameter. Owoloko et al. (2015), have developed the quadratic transmuted exponential distribution with *cdf* given as

$$F(x) = \left[1 - e^{-\frac{x}{\theta}}\right]\left[1 + \lambda e^{-\frac{x}{\theta}}\right], \; x \in [0, \infty),$$

where $\lambda \in [-1, 1]$. Using the *pdf* and *cdf* of exponential distribution in (8), the *pdf* of cubic transmuted exponential distribution is given in the following.

Proposition 5.1. *Let X has exponential distribution with parameter $\theta \in [0, \infty)$, then the pdf of cubic transmuted exponential distribution with parameters $\theta \in [0, \infty)$, $\lambda_1 \in [-1, 1]$, $\lambda_2 \in [-1, 1]$ and $-2 \leq \lambda_1 + \lambda_2 \leq 1$, is given by*

$$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} \left[1 + \lambda_1 + 2(\lambda_2 - \lambda_1)\left(1 - e^{-\frac{x}{\theta}}\right) - 3\lambda_2 \left(1 - e^{-\frac{x}{\theta}}\right)^2\right], \; x \in [0, \infty). \quad (9)$$

Proof. Using the *cdf* of exponential distribution in (3), the *cdf* of cubic transmuted exponential distribution is

$$F(x) = \frac{(1 + \lambda_1)}{\left(1 - e^{-\frac{x}{\theta}}\right)^{-1}} + \frac{(\lambda_2 - \lambda_1)}{\left(1 - e^{-\frac{x}{\theta}}\right)^{-2}} - \lambda_2 \left(1 - e^{-\frac{x}{\theta}}\right)^3, \; x \in [0, \infty). \quad (10)$$

The *pdf* of cubic transmuted exponential distribution is easily obtained by differentiating (10) and is given in (9). □

Figure 2 shows some of the possible shapes for density and distribution functions of cubic transmuted exponential distribution for various values of λ_1 and λ_2 at $\theta = 1.5$.

Table 1: Mean of the CT exponential distribution

		$\lambda_2=-1$	$\lambda_2=-0.5$	$\lambda_2=0$	$\lambda_2=0.5$	$\lambda_2=1$
	$\lambda_1=-1$	1.833	1.667	1.500	1.333	1.167
	$\lambda_1=-0.5$	1.583	1.417	1.250	1.083	0.917
$\theta=1$	$\lambda_1=0$	1.333	1.167	1.000	0.833	0.667
	$\lambda_1=0.5$	1.083	0.917	0.750	0.583	–
	$\lambda_1=1$	0.833	0.667	0.500	–	–
	$\lambda_1=-1$	3.667	3.333	3.000	2.667	2.333
	$\lambda_1=-0.5$	3.167	2.833	2.500	2.167	1.833
$\theta=2$	$\lambda_1=0$	2.667	2.333	2.000	1.667	1.333
	$\lambda_1=0.5$	2.167	1.833	1.500	1.167	–
	$\lambda_1=1$	1.667	1.333	1.000	–	–
	$\lambda_1=-1$	5.500	5.000	4.500	4.000	3.500
	$\lambda_1=-0.5$	4.750	4.250	3.750	3.250	2.750
$\theta=3$	$\lambda_1=0$	4.000	3.500	3.000	2.500	2.000
	$\lambda_1=0.5$	3.250	2.750	2.250	1.750	–
	$\lambda_1=1$	2.500	2.000	1.500	–	–
	$\lambda_1=-1$	7.333	6.667	6.000	5.333	4.667
	$\lambda_1=-0.5$	6.333	5.667	5.000	4.333	3.667
$\theta=4$	$\lambda_1=0$	5.333	4.667	4.000	3.333	2.667
	$\lambda_1=0.5$	4.333	3.667	3.000	2.333	–
	$\lambda_1=1$	3.333	2.667	2.000	–	–
	$\lambda_1=-1$	9.167	8.333	7.500	6.667	5.833
	$\lambda_1=-0.5$	7.917	7.083	6.250	5.417	4.583
$\theta=5$	$\lambda_1=0$	6.667	5.833	5.000	4.167	3.333
	$\lambda_1=0.5$	5.417	4.583	3.750	2.917	–
	$\lambda_1=1$	4.167	3.333	2.500	–	–

6. Statistical Properties

In this section, we have discussed some distributional properties of the cubic transmuted exponential distribution given in (9). These properties include expressions for moment, quantile, reliability and hazard functions. We have also discussed the random number generation from cubic transmuted exponential distribution.

6.1 Moments

The rth moment of the cubic transmuted exponential distribution is given as

$$E(X^r)=\int_0^\infty x^r f(x)\mathrm{d}x.$$

Using $f(x)$ as given in (9) and simplifying, the rth moment for cubic transmuted exponential distribution is given as

$$E(X^r)=\frac{\theta^r}{6^r}r!\left[6^r-3^r(2^r-1)\lambda_1-(2^r-2\cdot 3^r+6^r)\lambda_2\right]. \tag{11}$$

The mean and variance can be obtained by using (11), and are given as

$$E(X) = \frac{\theta}{6}\left(6 - 3\lambda_1 - 2\lambda_2\right) \text{ and}$$

$$V(X) = \frac{\theta^2}{36}\left(36 - 18\lambda_1 - 20\lambda_2 - 9\lambda_1^2 - 12\lambda_1\lambda_2 - 4\lambda_2^2\right).$$

The higher moments can also be obtained from (11). Table 1 provides the mean for various combinations of the parameters.

6.2 Quantile Function

The quantile function is obtained by solving (10) for x and is given as

$$x_q = \theta\,[-ln(y)], \tag{12}$$

where

$$\left.\begin{array}{l} y = -\frac{b}{3a} - \frac{2^{1/3}\xi_1}{3a\left(\xi_2+\sqrt{4\xi_1^3+\xi_2^2}\right)^{1/3}} + \frac{\left(\xi_2+\sqrt{4\xi_1^3+\xi_2^2}\right)^{1/3}}{3(2^{1/3})a}, \\ \xi_1 = -b^2 + 3ac,\ \xi_2 = -2b^3 + 9abc - 27a^2d, \\ a = \lambda_2,\ b = -\lambda_1 - 2\lambda_2,\ c = \lambda_1 + \lambda_2 - 1 \text{ and } d = 1 - q. \end{array}\right\} \tag{13}$$

The three quartiles can be obtained by using $q = 0.25,\ 0.50$ and 0.75 in (12), respectively.

6.3 Reliability Analysis

The reliability function describes the probability of an element not failing prior to some time t, and is defined by $R(t) = 1 - F(t)$. The reliability function of cubic transmuted exponential distribution can be obtained as

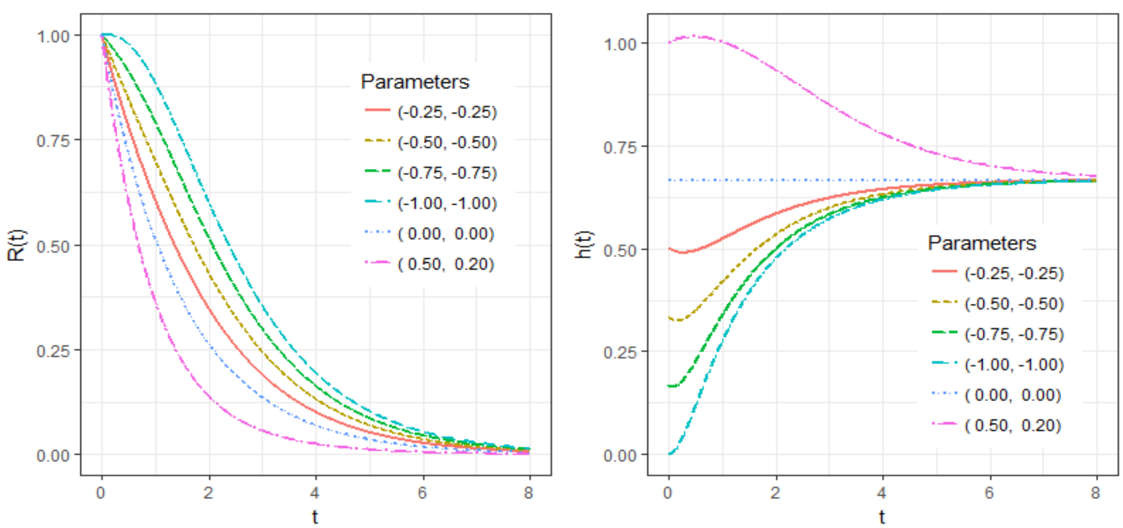

Figure 3: Reliability and hazard functions are plotted for the CT exponential distribution for different values of model parameters λ_1 and λ_2, setting $\theta = 1.5$.

$$R(t) = 1 - \eta(t)\left[(1+\lambda_1) + (\lambda_2 - \lambda_1)\eta(t) - \lambda_2\eta^2(t)\right],\ t \in \mathbb{R}^+.$$

The hazard function, $h(t)$, is defined by

$$h(t) = \frac{f(t)}{1 - F(t)},$$

which, for cubic transmuted exponential distribution, is given as

$$h(t) = \frac{\left(\frac{1}{\theta}e^{-\frac{t}{\theta}}\right)\left[1 + \lambda_1 + 2(\lambda_2 - \lambda_1)\eta(t) - 3\lambda_2\eta^2(t)\right]}{1 - \eta(t)\left[(1+\lambda_1) + (\lambda_2 - \lambda_1)\eta(t) - \lambda_2\eta^2(t)\right]},\ t \in \mathbb{R}^+,$$

where $\eta(t) = \left(1 - e^{-\frac{t}{\theta}}\right)$. The hazard function indicates the instantaneous rate of failure at time t, given that the component has survived up to time t. Some of the possible shapes of the reliability and hazard functions for the selected values of model parameters λ_1 and λ_2 keeping $\theta = 1.5$ are presented in Figure 3. We have observed that the reliability and distribution functions behaved complement to each other. In order to failure time t, we observed increasing, increasing then decreasing and decreasing then increasing and constant hazard rates from the shapes.

6.4 Random Numbers Generation

The random sample from cubic transmuted exponential distribution can be obtained by setting the distribution function (10) equal to u, where u is a uniform random variable, that is the random number from cubic transmuted exponential distribution is obtained by solving

$$(1+\lambda_1)\left(1 - e^{-\frac{x}{\theta}}\right) + (\lambda_2 - \lambda_1)\left(1 - e^{-\frac{x}{\theta}}\right)^2 - \lambda_2\left(1 - e^{-\frac{x}{\theta}}\right)^3 = u,$$

where $u \sim U(0,1)$. On simplification, this can be presented as

$$X = \theta\ [-ln(y)], \tag{14}$$

where y is given in (13) with $d = 1 - u$. The random sample from cubic transmuted exponential distribution can be obtained by using (14) for various values of the parameters θ, λ_1 and λ_2.

7. Parameter Estimation and Inference

In this section, we have obtained maximum likelihood estimators ($MLEs$) for parameters of the cubic transmuted exponential distribution. For this, let $X_1, X_2, \cdots, X_n$ be a random sample of size n from cubic transmuted exponential distribution. The likelihood function is, then, given by

$$L = \left(\frac{1}{\theta}\right)^n e^{-\sum_{i=1}^n \frac{x_i}{\theta}} \prod_{i=1}^n \left[1 + \lambda_1 + \frac{2(\lambda_2 - \lambda_1)}{\left(1 - e^{-\frac{x_i}{\theta}}\right)^{-1}} - 3\lambda_2\left(1 - e^{-\frac{x_i}{\theta}}\right)^2\right].$$

The log-likelihood function $l = ln(L)$ is given as

$$\begin{aligned} l = -n \cdot ln(\theta) - \sum_{i=1}^{n} \left(\frac{x_i}{\theta}\right) \\ + \sum_{i=1}^{n} ln\left[1 + \lambda_1 + 2(\lambda_2 - \lambda_1)\left(1 - e^{-\frac{x_i}{\theta}}\right) - 3\lambda_2\left(1 - e^{-\frac{x_i}{\theta}}\right)^2\right]. \end{aligned} \tag{15}$$

The $MLEs$ of θ, λ_1 and λ_2 are obtained by maximizing (15). The derivatives of (15) wrt the unknown parameters are given as

$$\frac{\partial l}{\partial \theta} = -\frac{n}{\theta} + \sum_{i=1}^{n}\left(\frac{x_i}{\theta^2}\right) + \sum_{i=1}^{n} \frac{\frac{6e^{-x_i/\theta}\left(1-e^{-x_i/\theta}\right)x_i\lambda_2}{\theta^2} - \frac{2e^{-x_i/\theta}x_i(\lambda_2-\lambda_1)}{\theta^2}}{1 + \lambda_1 - 3\left(1 - e^{-x_i/\theta}\right)^2\lambda_2 + \frac{2\left(1-e^{-x_i/\theta}\right)}{(\lambda_2-\lambda_1)^{-1}}},$$

$$\frac{\partial l}{\partial \lambda_1} = \sum_{i=1}^{n} \frac{1 - 2\left(1 - e^{-x_i/\theta}\right)}{1 + \lambda_1 - 3\left(1 - e^{-x_i/\theta}\right)^2\lambda_2 + 2(1 - e^{-x_i/\theta})(\lambda_2 - \lambda_1)},$$

$$\frac{\partial l}{\partial \lambda_2} = \sum_{i=1}^{n} \frac{2\left(1 - e^{-x_i/\theta}\right) - 3\left(1 - e^{-x_i/\theta}\right)^2}{1 + \lambda_1 - 3\left(1 - e^{-x_i/\theta}\right)^2\lambda_2 + 2\left(1 - e^{-x_i/\theta}\right)(\lambda_2 - \lambda_1)}.$$

The likelihood equations are given as

$$\frac{\partial l}{\partial \theta} = 0, \ \frac{\partial l}{\partial \lambda_1} = 0, \text{ and } \frac{\partial l}{\partial \lambda_2} = 0,$$

gives the maximum likelihood estimator $\hat{\Theta} = \left(\hat{\theta}, \hat{\lambda_1}, \hat{\lambda_2}\right)'$ of $\Theta = (\theta, \lambda_1, \lambda_2)'$. As $n \to \infty$ the asymptotic distribution of the MLE $\left(\hat{\theta}, \hat{\lambda_1}, \hat{\lambda_2}\right)$ for the cubic transmuted exponential distribution is given as, see for examples, transmuted Weibull distribution (Aryal and Tsokos, 2011), transmuted exponential distribution (Owoloko et al., 2015) and modified Weibull distribution (Zaindin and Sarhan, 2009),

$$\begin{pmatrix} \hat{\theta} \\ \hat{\lambda_1} \\ \hat{\lambda_2} \end{pmatrix} \sim N\left[\begin{pmatrix} \theta \\ \lambda_1 \\ \lambda_2 \end{pmatrix}, \begin{pmatrix} \hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13} \\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23} \\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33} \end{pmatrix}\right],$$

where

$$V^{-1} = -E\begin{bmatrix} \frac{\partial^2 l}{\partial \theta^2} & \frac{\partial^2 l}{\partial \theta \cdot \partial \lambda_1} & \frac{\partial^2 l}{\partial \theta \cdot \partial \lambda_2} \\ \frac{\partial^2 l}{\partial \theta \cdot \partial \lambda_1} & \frac{\partial^2 l}{\partial \lambda_1^2} & \frac{\partial^2 l}{\partial \lambda_1 \cdot \partial \lambda_2} \\ \frac{\partial^2 l}{\partial \theta \cdot \partial \lambda_2} & \frac{\partial^2 l}{\partial \lambda_1 \cdot \partial \lambda_2} & \frac{\partial^2 l}{\partial \lambda_2^2} \end{bmatrix}. \tag{16}$$

The asymptotic variance and covariance matrix of the estimates $\hat{\theta}$, $\hat{\lambda_1}$ and $\hat{\lambda_2}$; see Appendices 1, is obtained by inverting (16). An approximate $100(1-\alpha)\%$ asymptotic confidence intervals for θ, λ_1 and λ_2 are, respectively, given by

$$\hat{\theta} \pm Z_{\alpha/2}\sqrt{\hat{V}_{11}}, \quad \hat{\lambda}_1 \pm Z_{\alpha/2}\sqrt{\hat{V}_{22}} \quad \text{and} \quad \hat{\lambda}_2 \pm Z_{\alpha/2}\sqrt{\hat{V}_{33}},$$

where Z_α is the αth upper percentile of the standard normal distribution.

8. Numerical Studies

In this section, an extensive Monte Carlo simulation study is carried out to assess the performance of estimation method. We have also considered two real-life datasets to investigate the applicability of the proposed model.

8.1 Simulation Study

Table 2: Average estimates of model parameters and $MSEs$

Sample Size	Estimate θ	Estimate λ_1	Estimate λ_2	MSE θ	MSE λ_1	MSE λ_2
50	1.993	0.478	-0.516	0.808	0.591	0.748
100	2.015	0.506	-0.535	0.384	0.209	0.302
200	1.999	0.509	-0.553	0.157	0.086	0.138
500	2.009	0.510	-0.528	0.060	0.032	0.054
1000	2.009	0.512	-0.517	0.025	0.014	0.026

A Monte Carlo simulation study is carried out for samples of sizes 50, 100, 200, 500 and 1000, drawn from cubic transmuted exponential distribution. The samples have been drawn for $\theta = 2,\ \lambda_1 = 0.5$ and $\lambda_2 = -0.5$ and maximum likelihood estimators for the parameters $\theta,\ \lambda_1$ and λ_2 are obtained. The procedure has been repeated for 10000 and the mean and mean square error for the estimates are computed. The results are summarized in Table 2. We have found that the simulated estimates are closed to the true values of parameters and hence the estimation method is adequate. We have also observed that estimated mean square errors ($MSEs$) consistently decreases with increasing sample size.

8.2 Life Test Data

This dataset represents the lifetimes of 50 devices and has been used by Aarset (1987) and Lai et al. (2003). The summary statistics of the data presented in Table 3.

Table 3: Summary statistics for selected datasets

	Min.	Q_1	Median	Mean	Q_3	Max.
Life Test Data	0.10	13.50	48.50	45.69	81.25	86.00
Electronics Data	0.03	0.78	1.80	1.94	2.90	5.09

In order to assess the performance of the cubic transmuted exponential distribution we have computed various measures for quadratic transmuted exponential, beta exponential and exponentiated exponential distributions. The estimated values of

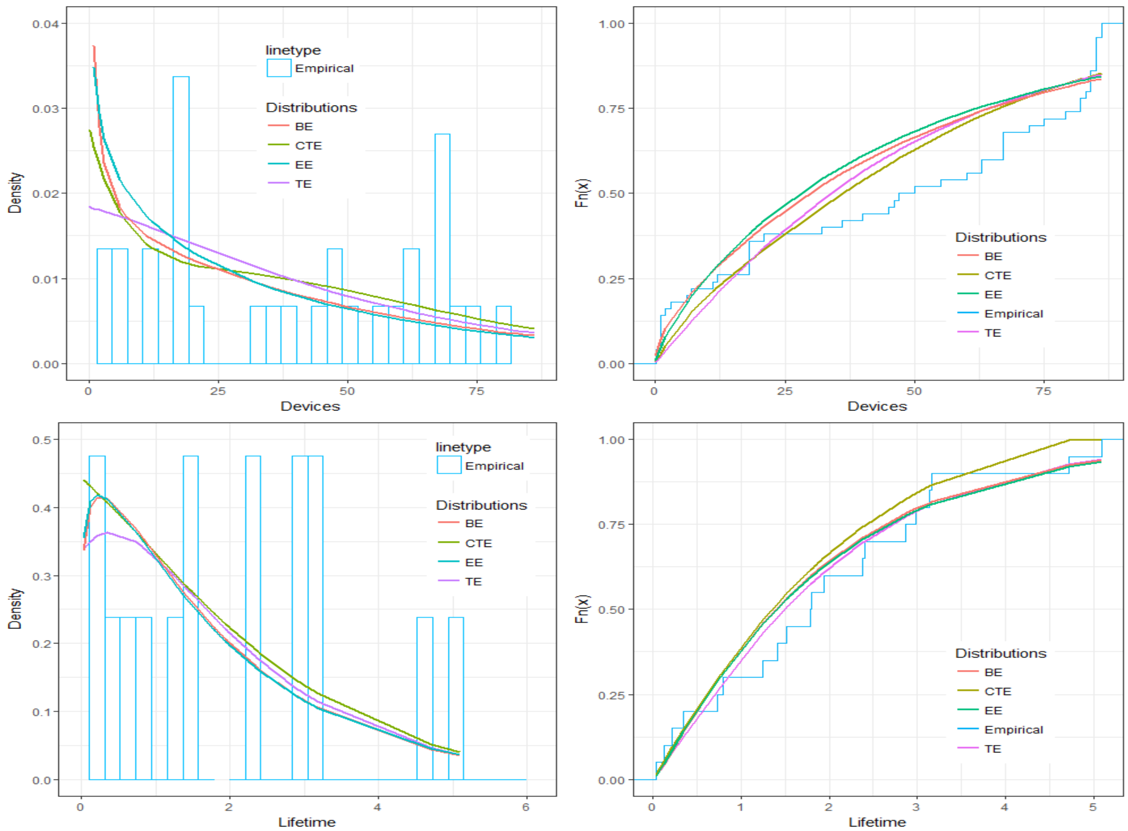

Figure 4: Upper panels: Estimated *pdf* and *cdf* for life test dataset; Lower panels: Estimated *pdf* and *cdf* for electronics dataset.

Table 4: $MLEs$ **of the parameters and respective** SEs **for selected models**

Distribution	**Parameter**	**Estimate**	SE
Cubic Transmuted Exponential	θ	33.765	5.044
	λ_1	-0.064	0.320
	λ_2	-0.971	0.739
Transmuted Exponential	θ	41.157	6.556
	λ	-0.243	0.242
Beta Exponential	θ	0.235	0.213
	λ_1	0.524	0.172
	λ_2	0.085	0.074
Exponentiated Exponential	θ	0.019	0.004
	λ	0.780	0.135

parameters alongside the standard errors (SEs) for various distributions are given in Table 4. Estimated *pdf* and *cdf* of the lifetimes of 50 devices are plotted over empirical density and distribution functions respectively and presented in the upper panels of Figure 4. Table 5 provides the log-likelihood, Akaike's information criterion (AIC), corrected Akaike's information criterion (AICc) and Bayesian information criterion (BIC). From Table 5, we can see that the cubic transmuted exponential distribution

Table 5: Selection criteria estimated for selected models

Distribution	LogLik	AIC	AICc	BIC
Cubic Transmuted Exponential	-236.018	478.036	478.558	483.772
Transmuted Exponential	-240.677	485.355	485.610	489.179
Beta Exponential	-238.120	482.240	482.762	487.976
Exponentiated Exponential	-239.995	483.990	484.246	487.814

is a good fit to the data as it has smallest values of the criterion.

8.3 Electronics Data

The dataset provides lifetimes of 20 electronic components and has been used by Murthy et al. (2004). The summary statistics are presented in Table 3.

Table 6: $MLEs$ **of the parameters and respective** SEs **for selected models**

Distribution	Parameter	Estimate	SE
Cubic Transmuted Exponential	θ	4.063	0.991
	λ_1	0.799	0.623
	λ_2	0.940	1.487
Transmuted Exponential	θ	1.581	0.373
	λ	-0.472	0.382
Beta Exponential	θ	0.025	8.5×10^{-3}
	λ_1	1.163	0.328
	λ_2	24.373	5.9×10^{-6}
Exponentiated Exponential	θ	0.560	0.154
	λ	1.139	0.332

We have fitted quadratic transmuted exponential, cubic transmuted exponential, beta exponential and exponentiated exponential distributions to the data. The $MLEs$ with their corresponding SEs are given in Table 6. Estimated *pdf* and *cdf* of the lifetimes of 20 electronic components are plotted over empirical density and distribution functions respectively and presented in the lower panels of Figure 4. The computed Log-likelihood, AIC, AICc and BIC values are provided in Table 7. We have observed, on the basis of the criteria used, that the cubic transmuted exponential distribution is the most appropriate model for this data.

9. Concluding Remarks

In this article, we have introduced a general family of transmuted distributions with special reference to the cubic transmuted family of distributions. We have found the cubic transmuted distributions are flexible enough and are capable to capture the bi-modality of the data. In order to assess the performance of this new class of

Table 7: Selection criteria estimated for selected models

Distribution	LogLik	AIC	AICc	BIC
Cubic Transmuted Exponential	-31.070	68.139	69.639	71.126
Transmuted Exponential	-32.714	69.429	70.135	71.420
Beta Exponential	-33.070	72.140	73.640	75.128
Exponentiated Exponential	-33.110	70.220	70.926	72.212

distributions, we have focused on the exponential distribution and the cubic transmuted exponential distribution has been explored in detail. We have fitted the cubic transmuted exponential distribution for two datasets and have found that the cubic transmuted exponential distribution adequately fit the two datasets as compared with the other distributions used in the comparison.

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Appendices 1. The Hessian matrix for cubic transmuted exponential distribution

The Hessian matrix is given as

$$H=\begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix},$$

where the variance-covariance matrix V is obtained by

$$V=\begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix}=\begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix}^{-1},$$

with the elements of Hessian matrix are obtained as

$$H_{11}=-\frac{\delta^2 l}{\delta\theta^2}=\sum_{i=1}^{n}\frac{2x_i}{\theta^3}+\sum_{i=1}^{n}\left[\frac{n}{\theta^2}+\frac{\frac{\left\{6\lambda_2 x_i e^{-\frac{x_i}{\theta}}\eta_i-2(\lambda_2-\lambda_1)x_i e^{-\frac{x_i}{\theta}}\right\}^2}{\theta^4}}{\left\{-3\lambda_2\eta_i^2+2\left(\lambda_2-\lambda_1\right)\eta_i+\lambda_1+1\right\}^2}\right.$$
$$\left.-\frac{\frac{6\lambda_2 x_i^2 e^{-\frac{x_i}{\theta}}\eta_i}{\theta^4}-\frac{6\lambda_2 x_i^2 e^{-\frac{2x_i}{\theta}}}{\theta^4}-\frac{2(\lambda_2-\lambda_1)x_i^2 e^{-\frac{x_i}{\theta}}}{\theta^4}+\frac{4(\lambda_2-\lambda_1)x_i e^{-\frac{x_i}{\theta}}}{\theta^3}-\frac{12\lambda_2 x_i e^{-\frac{x_i}{\theta}}\eta_i}{\theta^3}}{-3\lambda_2\eta_i^2+2\left(\lambda_2-\lambda_1\right)\eta_i+\lambda_1+1}\right],$$

$$H_{12}=-\frac{\delta^2 l}{\delta\theta\cdot\delta\lambda_1}=-\sum_{i=1}^{n}\left[\frac{2x_i e^{-\frac{x_i}{\theta}}}{\theta^2\left\{-3\lambda_2\eta_i^2+2\left(\lambda_2-\lambda_1\right)\eta_i+\lambda_1+1\right\}}\right.$$
$$\left.-\frac{\left\{1-2\eta_i\right\}\frac{1}{\theta^2}\left\{6\lambda_2 x_i e^{-\frac{x_i}{\theta}}\eta_i-2\left(\lambda_2-\lambda_1\right)x_i e^{-\frac{x_i}{\theta}}\right\}}{\left\{-3\lambda_2\eta_i^2+2\left(\lambda_2-\lambda_1\right)\eta_i+\lambda_1+1\right\}^2}\right],$$

$$H_{13}=-\frac{\delta^2 l}{\delta\theta\cdot\delta\lambda_2}=-\sum_{i=1}^{n}\left[\frac{\left\{\frac{1}{\theta^2}6x_i e^{-\frac{x_i}{\theta}}\eta_i-\frac{1}{\theta^2}2x_i e^{-\frac{x_i}{\theta}}\right\}}{-3\lambda_2\eta_i^2+2\left(\lambda_2-\lambda_1\right)\eta_i+\lambda_1+1}\right.$$
$$\left.-\frac{\left\{2\eta_i-3\eta_i^2\right\}\left\{\frac{1}{\theta^2}6\lambda_2 x_i e^{-\frac{x_i}{\theta}}\eta_i-\frac{1}{\theta^2}2\left(\lambda_2-\lambda_1\right)x_i e^{-\frac{x_i}{\theta}}\right\}}{\left(-3\lambda_2\eta_i^2+2\left(\lambda_2-\lambda_1\right)\eta_i+\lambda_1+1\right)^2}\right],$$

$$H_{22} = -\frac{\delta^2 l}{\delta \lambda_1^2} = \sum_{i=1}^{n} \frac{\{1 - 2\eta_i\}^2}{\left(-3\lambda_2 \eta_i^2 + 2\left(\lambda_2 - \lambda_1\right)\eta_i + \lambda_1 + 1\right)^2},$$

$$H_{23} = -\frac{\delta^2 l}{\delta \lambda_1 \cdot \delta \lambda_2} = \sum_{i=1}^{n} \frac{\{1 - 2\eta_i\}\{2\eta_i - 3\eta_i^2\}}{\left(-3\lambda_2 \eta_i^2 + 2\left(\lambda_2 - \lambda_1\right)\eta_i + \lambda_1 + 1\right)^2},$$

$$H_{33} = -\frac{\delta^2 l}{\delta \lambda_2^2} = \sum_{i=1}^{n} \frac{\{2\eta_i - 3\eta_i^2\}^2}{\left(-3\lambda_2 \eta_i^2 + 2\left(\lambda_2 - \lambda_1\right)\eta_i + \lambda_1 + 1\right)^2},$$

where $\eta_i = \left(1 - e^{-\frac{x_i}{\theta}}\right)$.