

A New Extension of Lindley Geometric Distribution and its Applications

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Abstract

A new four-parameter distribution called the beta Lindley-geometric distribution is proposed. The hazard rate function of the new model can be constant, decreasing, increasing, upside down bathtub or bathtub failure rate shapes. Various structural properties including of the new distribution are derived. The estimation of the model parameters is performed by maximum likelihood method. We present simulation results to assess the performance of the maximum likelihood estimation. The usefulness of the new distribution is illustrated using a real data set.

Keywords: Lindley-geometric distribution, Moments, Moment generating function, Maximum Likelihood Estimation.

1. Introduction

The Lindley distribution (Lindley, 1958) is important for studying stress-strength reliability modeling. Besides, some researchers have proposed new classes of distributions based on modifications of the Lindley distribution. The Lindley distribution specified by the probability density function (PDF)

$$f_L(x, \theta) = \frac{\theta^2}{\theta+1} (1+x)e^{-\theta x}, x > 0, \theta > 0.$$

The corresponding cumulative distribution function (CDF) is given by

$$F_L(x, \theta) = 1 - (1 + \frac{\theta x}{\theta+1})e^{-\theta x}, x > 0, \theta > 0.$$

Sankaran (1970) introduced the discrete Poisson-Lindley distribution by combining the Poisson and Lindley distributions. Ghitany et al.(2008) investigated most of the statistical properties of the Lindley distribution, showing this distribution may provide a better fitting than the exponential distribution. Recently a new extension of the Lindley distribution, called extended Lindley distribution, which offers a more flexible model for lifetime data is introduced by Bakouch et al. (2012). Adamidis and Loukas (1998) introduced a two-parameter lifetime distribution, called exponential geometric distribution, with decreasing failure rate by compounding the exponential and geometric

distributions. Zakerzadeh and Mahmoudi (2012) introduced the Lindley-geometric (LGc) distribution with CDF and PDF given by

$$F_{LG}(x, \theta, p) = \frac{1 - (1 + \frac{\theta x}{\theta + 1})e^{-\theta x}}{1 - p(1 + \frac{\theta x}{\theta + 1})e^{-\theta x}}, x > 0, \theta > 0, 0 < p < 1 \quad (1)$$

and

$$f_{LG}(x, \theta, p) = \frac{\theta^2}{\theta + 1} (1 - p)(1 + x)e^{-\theta x} \left[1 - p \left(1 + \frac{\theta x}{\theta + 1} \right) e^{-\theta x} \right]^{-2}, \quad (2)$$

respectively.

Let $G(x; \phi)$ be the baseline CDF of an absolutely continuous random variable, where ϕ is a $p \times 1$ parameter vector. A general class generated from the logit of a beta random variable is introduced by Eugene et al. (2002) and it is called the beta-G (B-G) family with the CDF

$$F(x; a, b, \phi) = \frac{1}{B(a, b)} \int_0^{G(x; \phi)} w^{a-1} (1 - w)^{b-1} dw = \frac{B(G(x; \phi); a, b)}{B(a, b)} = I_{G(x; \phi)}(a, b), \quad (3)$$

where $a > 0$ and $b > 0$ are two additional shape parameters whose role is to introduce skewness and to vary tail weight, $B(y; a, b) = \int_0^y w^{a-1} (1 - w)^{b-1} dw$ is the incomplete beta function with $B(a, b) = B_1(a, b)$ and $I_y(a, b) = \frac{B(y; a, b)}{B(a, b)}$ is the incomplete beta function ratio. One major benefit of this class of distributions is its ability of fitting skewed data that cannot be properly fitted by existing distributions. If $b = 1$, $F(x) = G(x)^a$ and then F is usually called the exponentiated G distribution (or the Lehmann type-I distribution).

Some special cases of BG distributions are given below:

(i) If $G(x; \phi)$ is the CDF of a standard uniform distribution, then the CDF given in Equation (3) yields the CDF of a beta distribution with parameters a and b .

(ii) If a is an integer value and $b = n - a + 1$, then the CDF (3) becomes

$$\begin{aligned} F(x; a, b, \phi) &= \frac{1}{B(a, n-a+1)} \int_0^{G(x; \phi)} w^{a-1} (1 - w)^{b-1} dw \\ &= \sum_{i=a}^n \binom{n}{i} [G(x; \phi)]^i [1 - G(x; \phi)]^{n-i}, \end{aligned}$$

which is really the CDF of the a_{th} order statistic of a random sample of size n from distribution $G(x; \phi)$.

(iii) If $a = b = 1$, then the CDF (3) reduces to $F(x; \phi) = G(x; \phi)$.

(iv) If $a = 1$, then the CDF (3) reduces to $F(x; b, \phi) = [1 - G(x; \phi)]^b$.

(v) If $b = 1$, then the CDF (3) reduces to $F(x; a, \phi) = [G(x; \phi)]^a$.

The two classes given in (iv) and (v) are called, respectively, the frailty parameter and resilience parameter families with underlying distribution $G(x; \phi)$ (Marshall and Olkin, 2007). Clearly, for positive integer values of b (or a), the CDF given in (iv) ((v)) above is the CDF of a series (parallel) system with b (or a) independent components all having the CDF $G(x; \phi)$. Some well-known distributions belonging to the resilience parameter family are the exponentiated Weibull distribution (Mudholkar et al., 1995), the generalized exponential distribution proposed by Gupta and Kundu (1999), the exponentiated type distributions introduced by Nadarajah and Kotz (2006).

For general a and b , we can express the CDF (3) in terms of the well-known hypergeometric function defined by

$$F(x; a, b, \phi) = \frac{G(x; \phi)^a}{aB(a, b)_2} {}_2F_1(a, 1 - b, a + 1; G(x; \phi)),$$

where

$${}_2F_1(\alpha, \theta, \gamma; x) = \sum_{i=0}^{\infty} \frac{(\alpha)_i (\theta)_i}{(\gamma)_i i!} x^i, |x| < 1,$$

where $(\alpha)_i = \frac{\Gamma(\alpha+i)}{\Gamma(\alpha)} = \alpha(\alpha+1)\dots(\alpha+i-1)$ denotes the ascending factorial of α . We obtain the properties of $F(x)$ for any beta- G distribution defined from a parent $G(x; \phi)$ in (3) could, in principle, follow from the properties of the hypergeometric function which are well established in the literature; see, for example, Section 9.1 of Gradshteyn and Ryzhik (2000). The PDF and hazard rate function (HRF) of the B-G family are given by

$$f(x; a, b, \phi) = \frac{g(x; \phi)}{B(a, b)} G(x; \phi)^{a-1} [1 - G(x; \phi)]^{b-1} \quad (4)$$

and

$$h(x; a, b, \phi) = \frac{g(x; \phi) G(x; \phi)^{a-1} \{1 - G(x; \phi)\}^{b-1}}{B(a, b) I_{[1-G(x; \phi)]}(a, b)},$$

respectively, where $I_{[1-G(x; \phi)]}(a, b) = 1 - I_{G(x; \phi)}(a, b) = \bar{F}(x; a, b, \phi)$ is the survival function of the B-G class.

The distributions that have been explored based on the B-G class are: the beta normal (Eugene et al., 2002), the beta exponential (Nadarajah and Kotz, 2006), the beta weibull geometric (Cordeiro et al., 2013), the beta exponential-geometric (Bidram, 2012) and the beta Weibull geometric distributions (Bidram et al., 2013). The beta transmuted-H and beta WeibullG families due to Afify et al. (2017) and Yousof et al. (2017), respectively.

In this article, we propose a new extension of the LGc distribution of Zakerzadeh and Mahmoudi (2012) by taking $G(x; \phi)$ in (3) to the CDF of the LGc distribution. The new model is referred to as the beta Lindley geometric (BLGc) distribution. We also study some of its mathematical properties and its applications to real data.

The reminder of the paper is organized as follows. In Section 2, we define the BLGc distribution and provide some plots for its PDF and HRF to show its flexibility. The expansion for the CDF and PDF of the BLGc distribution and some other properties are discussed in Section 3. In Section 4, the maximum likelihood estimation of the model

parameters is performed. In Section 5, the potentiality of the new model is illustrated via two applications to real data.

2. The BLGc distribution

Replacing $G(x; \phi)$ in Equation (3) by the CDF (1) yields the CDF of the BLGc model as

$$F_{BLGc}(x; \varphi) = I_{G(x; \theta, p)}(a, b) = \frac{1}{B(a, b)} B\left(\frac{1 - (1 + \frac{\theta x}{\theta + 1})e^{-\theta x}}{1 - p(1 + \frac{\theta x}{\theta + 1})e^{-\theta x}}; a, b\right), x > 0, \quad (5)$$

where $a > 0$, $b > 0$ and $0 < p < 1$ are shape parameters and $\theta > 0$ is a scale parameter. A random variable X with the CDF (5) is said to have a BLGc distribution and will be denoted by $X \sim \text{BLGc}(\varphi)$ where $\varphi = (\theta, p, a, b)$.

The PDF of the BLGc distribution takes the form

$$f_{BLGc}(x; \varphi) = \frac{\theta^2(1-p)^b(1+x)e^{-b\theta x}(1+\frac{\theta x}{\theta+1})^{b-1}\left(1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}\right)^{a-1}}{B(a, b)(\theta+1)\left[1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}\right]^{a+b}}. \quad (6)$$

One of the characteristic in reliability analysis is the hazard rate function (HRF) defined by

$$h_{BLGc}(x; \varphi) = \frac{\theta^2(1-p)^b(1+x)e^{-b\theta x}(1+\frac{\theta x}{\theta+1})^{b-1}\left(1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}\right)^{a-1}}{B(a, b)(\theta+1)\left[1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}\right]^{a+b} I_{[1-G(x; \theta, p)]}(a, b)}.$$

The BLGc distribution reduces to the exponentiated LGc model for $b = 1$ and reduces to the LGc model for $a = b = 1$. Figure 1 displays some plots of the PDF of the BLGc distribution for some selected values of the parameters. Its HRF plots are provided in Figure 2.

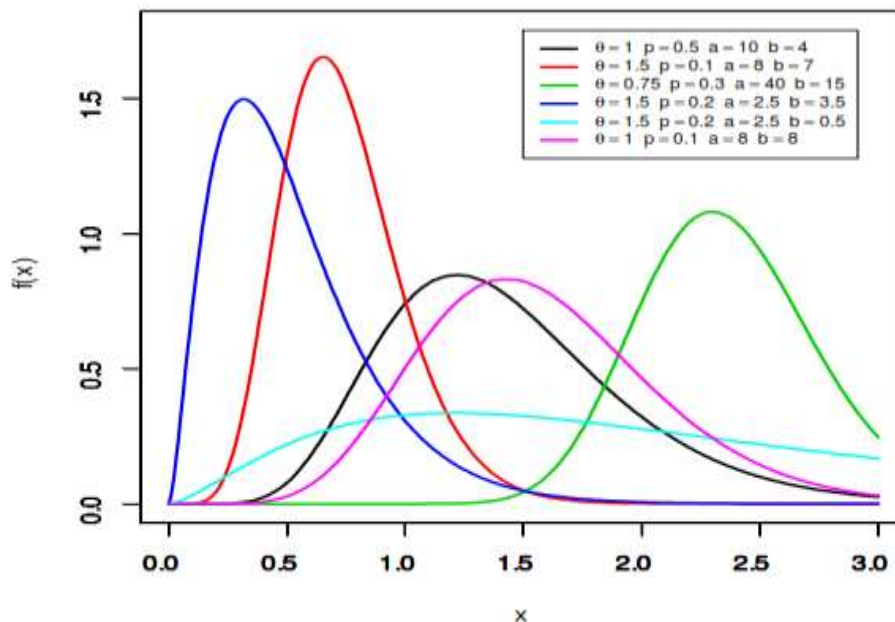


Figure 1: The PDF plots of the BLGc model for some parameter values

The plots in Figure 1 reveal that the BLGc can provide concave down, symmetric, unimodal, left skewed or right skewed shapes for its PDF. One can see, from Figure 2, that the BLGc HRF can be constant, decreasing, increasing, upside down bathtub or bathtub failure rate shapes. Then, it seems to be a useful model which has the ability to provide all important shapes of failure rate which are quite common in reliability and biological studies.

There are following motivations of the BLGc distribution:

- (i) To obtain a generalized version of LGc distribution that includes different other sub-models useful for explaining typical types of uncertainties.
- (ii) To obtain an improvement on the hazard rate function that may accommodate constant, decreasing, increasing, upside down bathtub or bathtub failure rate shapes.
- (iii) To find specific characteristic, if any, of the proposed density function.
- (iv) To identify better applicability of the proposed distribution and to establish competency over the other popular lifetime models.

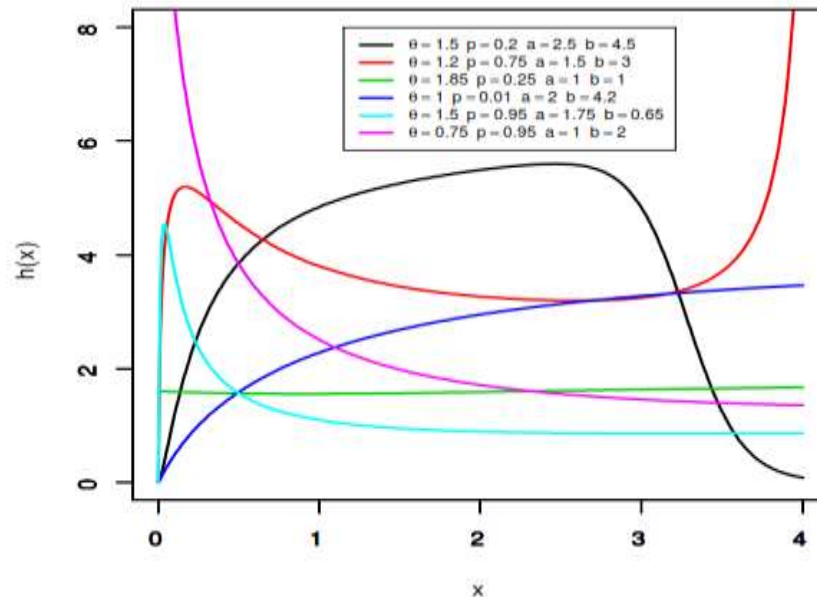


Figure 2: The HRF plots of the BLGc model for some parameter values

3. Statistical properties

In this section, we discuss some properties of the BLGc distribution.

3.1 Expansion for the BLGc CDF and PDF

In this subsection, we present some representations of CDF and PDF of the BLGc distribution. The mathematical relation given below will be useful in this subsection.

Here and henceforth, let X be a random variable having the BLGc(φ), we can obtain some alternative expressions for Equations (5) and (6).

If b is a positive real non-integer and $|z| < 1$ then

$$(1-z)^{b-1} = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(b)}{\Gamma(b-i)!} z^i \quad (7)$$

Using the expansion (7), we can write (5) as

$$\begin{aligned} F_{BLGc}(x, \varphi) &= \frac{1}{B(a, b)} \int_0^{\frac{1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}}{1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}}} w^{(a-1)}(1-w)^{b-1} dw \\ &= \frac{\Gamma(b)}{B(a, b)} \sum_{i=0}^{\infty} \left(\frac{(-1)^i}{\Gamma(b-i)!} \right) \int_0^{\frac{1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}}{1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}}} w^{a+j-1} dw \\ &= \frac{\Gamma(a+b)}{\Gamma(a)} \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(b)}{\Gamma(b-i)!(a+i)} \left(\frac{1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}}{1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}} \right)^{a+i}. \end{aligned}$$

Using the series representation

$$(1-z)^{-k} = \sum_{i=0}^{\infty} \frac{\Gamma(k+i)}{\Gamma(k)!} z^i, \quad |z| < 1, k > 0.$$

The PDF (6) can be expressed as

$$\begin{aligned} f_{BLGc}(x, \varphi) &= \sum_{i,j=0}^{\infty} \sum_{k=0}^{i+j} (-1)^i \binom{a-1}{i} \binom{i+j}{k} \frac{\Gamma(a+b+j)}{\Gamma(a+b)j!} \\ &\times \frac{\theta^2(1-p)^b p^j}{B(a, b)(\theta+1)} \left(\frac{\theta}{1+\theta} \right)^k x^k (1+x) e^{-\theta(b+i+j)x} \\ &= \sum_{i,j=0}^{\infty} \sum_{k=0}^{i+j} \omega_{i,j,k} (x^k + x^{k+1}) e^{-\theta(b+i+j)x}. \end{aligned} \quad (8)$$

where

$$\omega_{i,j,k} = (-1)^i \binom{a-1}{i} \binom{i+j}{k} \frac{\Gamma(a+b+j)}{\Gamma(a+b)j!} \frac{\theta^2(1-p)^b p^j}{B(a, b)(\theta+1)} \left(\frac{\theta}{1+\theta} \right)^k.$$

3.2 Moments

The r th moment of X follows from (8) as

$$\mu_r' = \sum_{i,j=0}^{\infty} \sum_{k=0}^{i+j} \omega_{i,j,k} \left[\frac{\Gamma(r+k+1)}{[\theta(b+i+j)]^{r+k+1}} \left(1 + \frac{(r+k+1)}{\theta(b+i+j)} \right) \right].$$

The moment generating function (mgf) of the BLGc distribution is given by

$$M_X(t) = \sum_{i,j=0}^{\infty} \sum_{k=0}^{i+j} \omega_{i,j,k} \left[\frac{\Gamma(k+1)}{[\theta(b+i+j)-t]^{k+1}} \left(1 + \frac{(k+1)}{\theta(b+i+j)-t} \right) \right].$$

For lifetime models, it is also of interest to obtain the conditional moments, the mean residual lifetime (MRL) and mean inactivity time (MIT). Further, it is of interest to know the r th lower and upper incomplete moments of X defined (for $s > 0$) by $v_s(t) = E(X^s | X < t) = \int_0^t x^s f(x, \phi) dx$ and $\eta_s(t) = E(X^s | X > t) = \int_0^{\infty} x^s f(x, \phi) dx$, respectively.

The r th lower incomplete moment of the BLGc distribution is

$$v_s(t) = \int_0^t x^s f(x) dx = \omega_{i,j,k} \int_0^t (x^{s+k} + x^{s+k+1}) e^{-\theta(b+i+j)x} dx.$$

Or

$$v_s(t) = \sum_{i,j=0}^{\infty} \sum_{k=0}^{i+j} \omega_{i,j,k} \{(\gamma(s+k+1, \theta(b+i+j)t)) + \gamma(s+k+2, \theta(b+i+j)t)\}, \quad (9)$$

where $\gamma(s, t) = \int_0^t x^{s-1} e^{-x} dx$ is the lower incomplete gamma function.

The r th upper incomplete moment of the BLGc distribution is

$$\begin{aligned} \eta_s(t) &= \int_t^{\infty} x^s f(x) dx = \omega_{i,j,k} \int_t^{\infty} (x^{s+k} + x^{s+k+1}) e^{-\theta(b+i+j)x} dx \\ &= \sum_{i,j=0}^{\infty} \sum_{k=0}^{i+j} \omega_{i,j,k} \{(\Gamma(s+k+1, \theta(b+i+j)t)) \\ &\quad + \Gamma(s+k+2, \theta(b+i+j)t)\}, \end{aligned}$$

where $\Gamma(s, t) = \int_t^{\infty} x^{s-1} e^{-x} dx$ is the upper incomplete gamma function.

The MRL has many applications in biomedical sciences, life insurance, maintenance and product quality control, economics and social studies, demography and product technology (see Lai and Xie, 2006). Guess and Proschan (1988) gave an extensive coverage of possible applications of the mean residual life. The MRL (or the life expectancy at age t) represents the expected additional life length for a unit, which is alive at age t . The MRL is given by

$$m_x(t) = E(X|X > t) = \frac{\eta_1(t)}{\bar{F}(t)} - t,$$

where $\eta_1(t)$ is the first incomplete moment of X and by setting $s = 1$ in equation (9), we get

$$\begin{aligned} m_x(t) &= \sum_{i,j=0}^{\infty} \sum_{k=0}^{i+j} \frac{\omega_{i,j,k}}{\bar{F}(t)} \{(\Gamma(k+2, \theta(b+i+j)t)) \\ &\quad + \Gamma(k+3, \theta(b+i+j)t)\} - t \end{aligned}$$

The MIT represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0, t)$. The MIT of X is defined (for $t > 0$) by

$$\begin{aligned} \tau_x(t) &= E(X|X < t) = t - \frac{v_1(t)}{F(t)} \\ &= t - \sum_{i,j=0}^{\infty} \sum_{k=0}^{i+j} \frac{\omega_{i,j,k}}{F(t)} (\Gamma(k+2, \theta(b+i+j)t) + \Gamma(k+3, \theta(b+i+j)t)). \end{aligned}$$

The mean, variance, skewness and kurtosis of the BLGc distribution are computed numerically for

some selected values of θ, p, a and b . The numerical values displayed in Table I indicate that the skewness of the BLGc distribution can range in the interval (1.52,253.35). The spread for its kurtosis is much larger ranging from 6.08 to 4896.44.

Table 1: Mean, variance, skewness and kurtosis of the BLGc distribution

θ	p	a	b	Mean	Variance	Skewness	Kurtosis
0.75	0.50	0.50	1.50	1.448297	1.777968	1.521557	6.089516
0.75	0.50	1.50	1.50	0.194821	0.630747	4.927939	31.00787
0.75	0.50	3.50	1.50	0.029752	0.127514	13.73352	215.7578
0.75	0.50	5.00	1.50	0.012576	0.059556	21.72350	527.4776
0.75	0.50	10.0	1.50	0.002162	0.012230	55.35067	3317.415

0.75	0.50	0.50	3.50	0.021575	0.031670	11.19085	156.3427
0.75	0.50	1.50	5.00	0.000037	0.000060	253.3699	74163.69
2.75	0.75	2.50	1.50	0.019083	0.018668	8.422746	84.57181
2.75	0.75	2.50	2.50	0.000232	0.000172	65.55413	4896.447
1.50	0.95	1.50	1.50	0.003761	0.006918	26.36931	814.0115

3.3 Mean deviation

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and median. One can derive the mean deviations about the mean $\mu = E(X)$ and the mean deviations about the median M which are defined by

$$\delta_1(x) = \int_0^\infty |x - \mu|f(x)dx$$

and

$$\delta_2(x) = \int_0^\infty |x - M|f(x)dx,$$

respectively. The measures $\delta_1(x)$ and $\delta_2(x)$ can be calculated using the relationships

$$\delta_1(x) = \int_0^\infty |x - \mu|f(x)dx = 2[\mu F(\mu) - J(\mu)]$$

and

$$\delta_2(x) = \int_0^\infty |x - M|f(x)dx = \mu - 2J(M),$$

where

$$J(d) = \sum_{i,j=0}^\infty \sum_{k=0}^{i+j} \omega_{i,j,k}(\Gamma(k+2, \theta(b+i+j)d)) + \Gamma(k+3, \theta(b+i+j)d).$$

3.4 Bonferroni and Lorenz curves

The Bonferroni and Lorenz curves (Bonferroni 1930) and the Bonferroni and Gini indices have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine.

The Bonferroni and Lorenz curves are given, respectively, by

$$B(\pi) = \frac{J(q)}{\pi\mu} = \sum_{i,j=0}^\infty \sum_{k=0}^{i+j} \frac{\omega_{i,j,k}(\Gamma(k+2, \theta(b+i+j)q)) + \Gamma(k+3, \theta(b+i+j)q)}{\pi\mu}$$

and

$$L(p) = \sum_{i,j=0}^\infty \sum_{k=0}^{i+j} \frac{\omega_{i,j,k}(\Gamma(k+2, \theta(b+i+j)q)) + \Gamma(k+3, \theta(b+i+j)q)}{\mu}.$$

4. Estimation and simulation

In this section, we determine the maximum likelihood estimates (MLEs) of the parameters of the BLGc distribution from complete samples only. Let X_1, X_2, \dots, X_n be a random sample of size n from this distribution with the parameter vector $\varphi = (\theta, p, a, b)^T$. The log likelihood function for φ can be written as

$$\begin{aligned} \ell &= n\log(1-p) + 2n\log(\theta) - n\log[B(a, b)] - n\log(1+\theta) - \\ &b\theta \sum_{i=1}^n x_i \\ &+ (b-1) \sum_{i=1}^n \log\left(1 + \frac{\theta x_i}{\theta+1}\right) + (a-1) \sum_{i=1}^n \log\left(1 - \left(1 + \frac{\theta x_i}{\theta+1}\right)e^{-\theta x_i}\right) \\ &- (a+b) \sum_{i=1}^n \log\left[1 - p\left(1 + \frac{\theta x_i}{\theta+1}\right)e^{-\theta x_i}\right] + \sum_{i=1}^n \log(1-x_i). \end{aligned}$$

The associated score function is given by

$$U_n(\varphi) = \left[\frac{\partial \ell}{\partial \theta}, \frac{\partial \ell}{\partial p}, \frac{\partial \ell}{\partial a}, \frac{\partial \ell}{\partial b} \right]^T.$$

The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating the log-likelihood function. The components of the score vector are given by

$$\begin{aligned} \frac{\partial \ell}{\partial \theta} &= \frac{2n}{\theta} - \frac{n}{1-\theta} - b \sum_{i=1}^n x_i + (a-1) \sum_{i=1}^n \frac{e^{-\theta x_i} \left(z_i - \frac{1}{(1+\theta)^2} \right)}{(1-z_i e^{-\theta x_i})} \\ &+ (b-1) \sum_{i=1}^n \frac{x_i}{z_i(1+\theta)^2} + p(a+b) \sum_{i=1}^n \frac{e^{-\theta x_i} \left(z_i - \frac{1}{(1+\theta)^2} \right)}{[1-pz_i e^{-\theta x_i}]}, \end{aligned}$$

$$\frac{\partial \ell}{\partial p} = (a+b) \sum_{i=1}^n \frac{z_i e^{-\theta x_i}}{[1-pz_i e^{-\theta x_i}]} - \frac{nb}{1-p},$$

$$\frac{\partial \ell}{\partial a} = n\psi(a+b) - n\psi(a) + \sum_{i=1}^n \log(1 - z_i e^{-\theta x_i}) - \sum_{i=1}^n \log[1 - pz_i e^{-\theta x_i}]$$

and

$$\begin{aligned} \frac{\partial \ell}{\partial b} &= n\psi(a+b) - n\psi(b) + \frac{n}{\log(1-p)} - \theta \sum_{i=1}^n x_i \\ &+ \sum_{i=1}^n \log z_i - \sum_{i=1}^n \log[1 - pz_i e^{-\theta x_i}], \end{aligned}$$

where $z_i = (1 + \frac{\theta x_i}{\theta+1})$. The maximum likelihood estimation (MLE) of φ , say $\hat{\varphi}$, is obtained by solving the nonlinear system $U_n(\varphi) = 0$. These equations cannot be solved analytically, and statistical software can be used to solve them numerically via iterative methods. We can use iterative techniques such as a Newton–Raphson type algorithm to obtain the estimate $\hat{\varphi}$. Applying the usual large sample approximation, MLE of φ , i.e $\hat{\varphi}$ can be treated as being approximately $N_4(\varphi, J_n(\varphi)^{-1})$, where $J_n(\varphi) = E[I_n(\varphi)]$. Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\hat{\varphi} - \varphi)$ is $N_4(0, J(\varphi)^{-1})$, where $J(\varphi) = \lim_{n \rightarrow \infty} n^{-1} I_n(\varphi)$ is the unit information matrix. This asymptotic behavior remains valid if $J(\varphi)$ is replaced by the average sample information matrix evaluated at $\hat{\varphi}$, say $n^{-1} I_n(\hat{\varphi})$. The estimated asymptotic multivariate normal $N_4(\varphi, I_n(\hat{\varphi})^{-1})$ distribution of $\hat{\varphi}$ can be used to construct approximate confidence intervals for the parameters and for the hazard rate and survival functions. An $100(1 - \xi)$ asymptotic confidence interval for each parameter φ_r is given by

$$ACI_r = \left(\hat{\varphi}_r - z_{\frac{\xi}{2}} \sqrt{\widehat{I}_{rr}}, \hat{\varphi}_r + z_{\frac{\xi}{2}} \sqrt{\widehat{I}_{rr}} \right)$$

where \widehat{I}_{rr} is the (r, r) diagonal element of $I_n(\hat{\varphi})^{-1}$ for $r = 1, 2, 3, 4$, and $z_{\frac{\xi}{2}}$ is the quantile $1 - \frac{\xi}{2}$ of the standard normal distribution.

We now perform a small Monte Carlo simulation study to verify the finite sample behavior of the MLEs of the parameters. All simulation results are obtained from 1 000 Monte Carlo replications.

Table 2 lists the bias and mean square errors (MSE) of the MLEs of the model parameters by taking sample sizes $n = 25, 75, 150, 250$ and 400 . From Table 2, it is clear that if the sample size increases, the empirical biases and MSEs tend to 0.

Table 2: The bias and MSE values for BLGc($\theta=0.5, p=0.7, a=0.9, b=0.4$)

n	25		75		150		250		400	
	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
θ	0.357	0.187	0.288	0.177	0.199	0.128	0.105	0.019	0.055	0.005
p	0.217	0.112	0.199	0.107	0.187	0.087	0.123	0.059	0.075	0.019
a	0.231	0.103	0.126	0.057	0.100	0.046	0.019	0.330	0.012	0.320
b	0.119	0.105	0.112	0.101	0.103	0.074	0.088	0.066	0.076	0.054

5. Application

In this section, the flexibility of the BLGc distribution is illustrated using a real data set. The data contain 128 observations and represents the remission times (in months) of a random sample of bladder cancer patients (Lee and Wang, 2003). These data have been analyzed by Mead and Afify (2017), Aldahlan and Afify (2018) and Cordeiro et al. (2019).

The fitted models are compared using $-2\hat{\ell}$ (where $\hat{\ell}$ is the maximized log-likelihood), W^* (Cramér-Von Mises), A^* (Anderson-Darling) and KS (Kolmogorov Smirnov with its p-value (PV)) statistics.

We compare the fits of the BLGc distribution to the fits of some related distributions:

The Lindley geometric (LGc) due to Zakerzadeh and Mahmoudi (2012) with the PDF

$$f(x) = \frac{\theta^2}{\theta+1} (1-p)(1+x)e^{-\theta x} \left[1 - p \left(1 + \frac{\theta x}{\theta+1} \right) e^{-\theta x} \right]^{-2}$$

for $x > 0$, $\theta > 0$ and $0 < p < 1$; the five-parameter Lindley (FPL) distribution due to Al-Babtain et al. (2015) with the PDF

$$f(x) = \frac{\theta^2}{\eta+\theta k} \left[\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\eta(\theta x)^{\beta-1}}{\theta\Gamma(\beta)} \right] e^{-\theta x}$$

for $x > 0$, $\theta > 0$, $\alpha > 0$, $\beta > 0$, $\eta \geq 0$ and $k \geq 0$; the transmuted two-parameter Lindley (TTL) due to Kemaloglu and Yilmaz (2017) with the PDF

$$f(x) = \frac{a^2}{\alpha+a} (1+\alpha x) \exp(-ax) \left[1 - \lambda + 2\lambda \frac{\alpha+a+\alpha ax}{\alpha+a} \exp(-ax) \right]$$

for $x > 0$, $\theta > 0$, $\alpha > 0$ and $|\lambda| \leq 1$; the Weibull Lindley (WL) due to Asgharzadeh et al. (2018) with the PDF

$$f(x) = \frac{\exp[-\lambda x - (\beta x)^\alpha]}{1+\lambda} [\alpha\lambda(\beta x)^\alpha + \alpha\beta(1+\lambda)(\beta x)^{\alpha-1} + \lambda^2(1+x)]$$

for $x > 0$, $\alpha > 0$, $\beta > 0$ and $\lambda > 0$; the Quasi Lindley (QL) due to Shanker and Mishra (2013) with the PDF

$$f(x) = \frac{a}{\alpha+1} (\alpha + ax) \exp(-ax)$$

for $x > 0$, $a > 0$ and $\alpha > -1$; the complementary geometric transmuted Lindley (CGcTL) due to Afify et al. (2018) with the PDF

$$f(x) = \frac{\frac{\theta\alpha^2}{1+\alpha}(1+x)\exp(-\alpha x)\left[1-\lambda+2\lambda\frac{1+\alpha+\alpha x}{1+\alpha}\exp(-\alpha x)\right]}{\left\{1-(1-\theta)\left[1-\frac{1+\alpha+\alpha x}{1+\alpha}\exp(-\alpha x)\right]\left[1+\lambda\frac{1+\alpha+\alpha x}{1+\alpha}\exp(-\alpha x)\right]\right\}^2}$$

for $x > 0$, $\alpha > 0$, $\theta \in (0,1)$ and $|\lambda| \leq 1$; the new weighted Lindley (NWL) due to Asgharzadeh et al. (2016) with the PDF

$$f(x) = \frac{a^2(1+\alpha)^2(1+x)[1-\exp(-\alpha x)]\exp(-\alpha x)}{a\alpha(1+\alpha)+\alpha(2+\alpha)}$$

for $x > 0$, $\alpha > 0$ and $a > 0$; the Lindley distribution with the PDF

$$f(x) = \frac{\theta^2}{1+\theta}(1+x)\exp(-\theta x)$$

for $x > 0$ and $\theta > 0$.

Table 3 lists the values of $-2\hat{\ell}$, W^* , A^* , KS, PV based on the KS statistic (in parentheses), the parameter estimates and standard errors (SEs) (in parentheses) for the fitted BLGc distribution and other fitted models.

The fitted PDFs of the fitted distributions and the empirical histogram are given in Figure 3. The corresponding probability (PP) plots are given in Figure 4. Figure 5 displays the estimated CDF and estimated survival function (SF) of the BLGc distribution.

Table 3: The $-\hat{\ell}$, W^* , A^* , KS and estimates for cancer data

Distribution	$-2\hat{\ell}$	W^*	A^*	KS	Estimates
BLGc	818.476	0.013	0.083	0.028 (0.999)	θ 0.0496(0.0404)
					p 0.9330(0.0845)
					a 1.0160(0.2092)
					b 1.2107(0.3282)
LGc	819.186	0.015	0.1037	0.040 (0.984)	θ 0.0741(0.0351)
					p 0.8899(0.0991)
FPL	820.007	0.026	0.166	0.039 (0.988)	θ 0.1792(0.0348)
					α 7.6843(2.0177)
					β 1.4085(0.1853)
					k 0.1935(14.434)
					η 0.7703(57.446)
TTL	825.882	0.117	0.687	0.063 (0.677)	α 0.1578(0.1670)
					a 0.1170(0.0293)
					λ 0.7128(0.2068)
WL	828.176	0.131	0.786	0.070 (0.556)	α 1.0478(0.0675)
					β 0.1045(0.0093)
					λ 0.0010(0.0176)
QL	828.686	0.119	0.716	0.084 (0.318)	α 117.887(1483)
					a 0.1076(0.0146)
CGcTL	830.326	0.104	0.617	0.089 (0.252)	α 0.1558(0.0203)
					θ 0.9990(0.2547)
					λ 0.6169(0.1695)
NWL	838.928	0.169	1.012	0.116 (0.061)	α 235.080(558.73)
					a 0.1961(0.0123)
L	839.058	0.171	1.025	0.116 (0.062)	θ 0.1960(0.0123)

From Table 2, we conclude that the BLGc distribution has the lowest values for all goodness-of-fit statistics among all fitted distributions. So, it can be chosen as the best model to fit this data set.

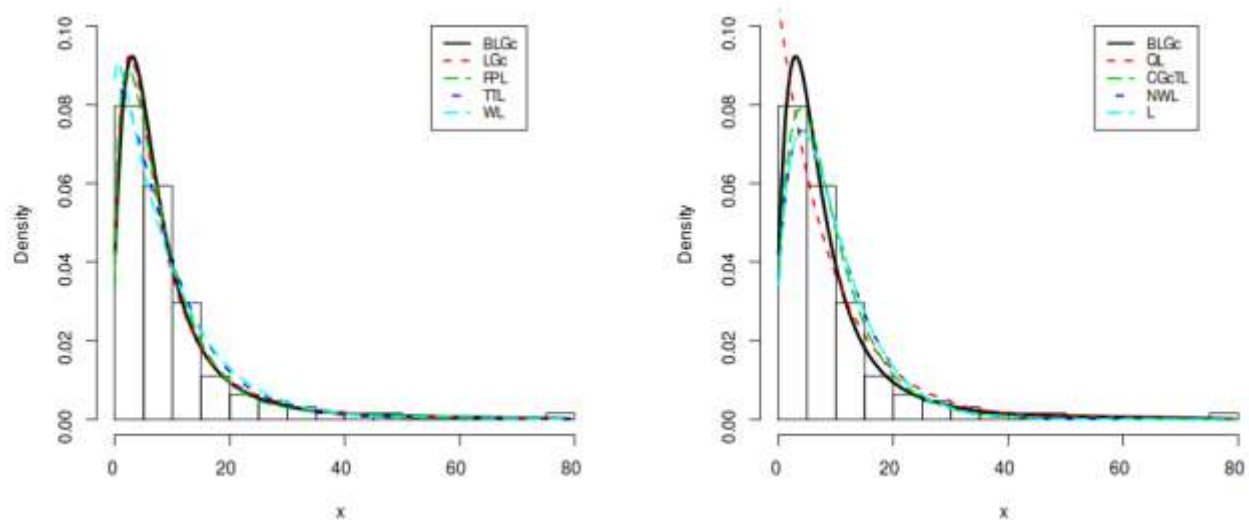


Figure 3: Fitted PDFs of the fitted distributions for cancer data

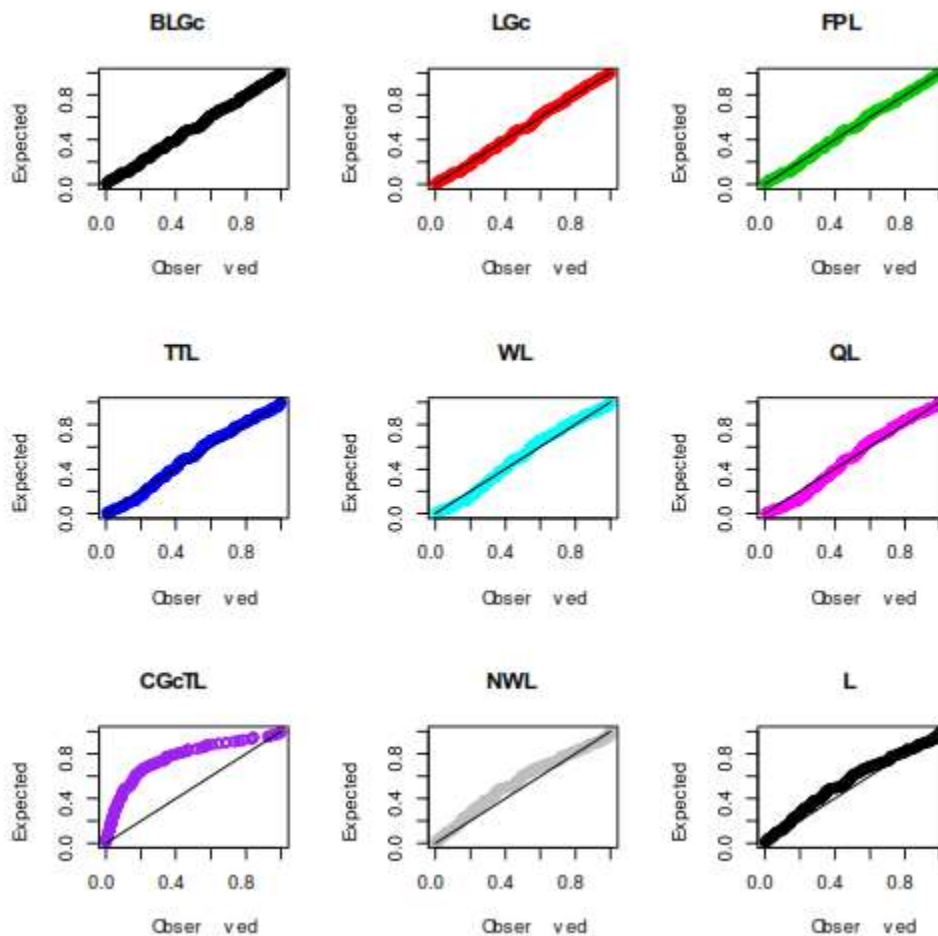


Figure 4: PP plots of the fitted distributions for cancer data

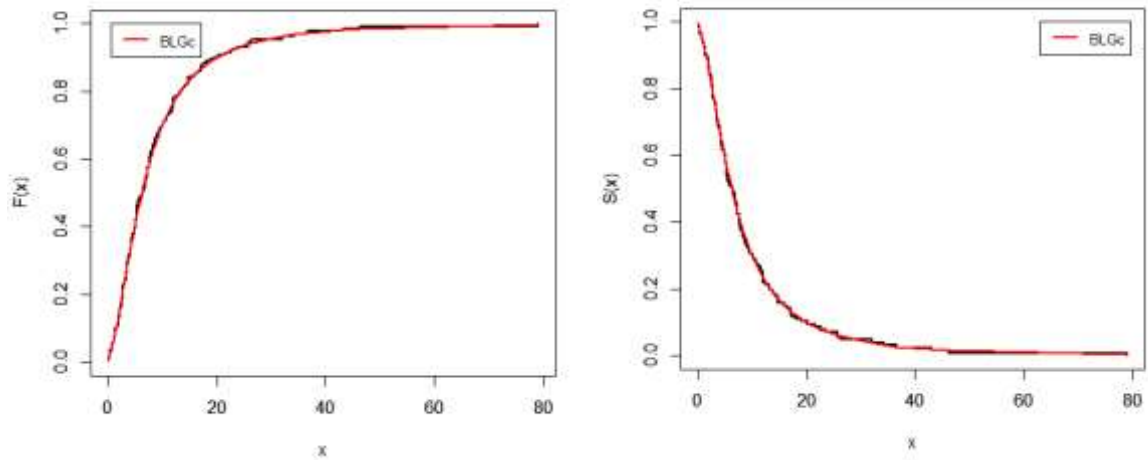


Figure 5: Estimated CDF of the BLGc distribution (left panel) and estimated SF of the BLGc distribution (right panel)

6. Conclusions

In this paper, we propose a new four-parameter model, called the beta Lindley-geometric (BLGc) distribution, which extends the Lindley-geometric distribution due to Zakerzadeh and Mahmoudi

(2012). We derive explicit expressions for the moments, mean deviation and Bonferroni and Lorenz curves. We discuss the maximum likelihood estimation of the model parameters and present a simulation results to assess the performance of the maximum likelihood estimation. An application illustrates that the BLGc model provides consistently better fit than other competitive

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