First-Order Fractionally Integrated Non-Separable Spatial Autoregressive (FINSSAR(1,1)) Model and Some of its Properties

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Abstract

Spatial modelling has its applications in many fields. There exist in the literature a class of models known as the fractionally integrated separable spatial autoregressive (FISSAR) model. In this paper the objective of our research is to develop a non-separable counterpart of the FISSAR model. We term this model as the fractionally integrated non-separable spatial autoregressive (FINSSAR) model. The FINSSAR model is a more general model as it encompasses the FISSAR and the standard separable autoregressive (SSAR) models. The theoretical autocovariance function and the spectral function of the model are obtained and some numerical results is presented. This model may be able to model many type of real phenomena.

Keywords: Spatial autoregressive model, Separable, Non-separable, Long memory, Fractional autoregressive, FISSAR model, Regression estimation, Whittle's estimation

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1. Introduction

Spatial modelling has its applications in many fields like geology, geography, agriculture, meteorology, etc. Some examples are i) ore, gas and petroleum reserves (Cressie 1993), ii) sea surface temperature (Lim *et al.* 2002), iii) wheat yield in agricultural field trials (Whittle 1956 and 1962; Lambert *et al.* 2003), iv) rainfall (Nunez *et al.* 2008), v) spread of infectious disease (Marshall 1991) are some examples which the observed value of them may depend on the location.

There exist class of spatial models that are known as separable models which its correlation structure can be expressed as a product of correlations (Martin 1979, 1996**;** Basawa, Brockwell and Mandrekar, 1991). In some cases spatial data may exhibit a long memory structure (Hurst, 1951) where their autocorrelation function decays rather slowly. Spatial long memory model in general is defined as, (Guo *et al.*, 2009)

$$\Phi(B_1, B_2)(1-B_1)^{d_1}(1-B_2)^{d_2}Y_{ij} = \Theta(B_1, B_2)Z_{ij}$$

where B_1 and B_2 are the usual backward shift operators acting in the i^{th} and j^{th} direction, respectively. Parameters d_1 and d_2 are called memory parameters which

$-0.5 < d_1, d_2 < 0.5$ and $\{Z_{ij}\}$ is the two-dimensional white noise process with mean zero and variance σ^2. and

$$\Phi(z_1,z_2) = 1 - \sum_{k=1}^{p_1}\sum_{l=1}^{p_2}\phi_{kl} z_1^k z_2^l,$$

$$\Theta(z_1,z_2) = 1 + \sum_{k=0}^{q_1}\sum_{l=0}^{q_2}\theta_{kl} z_1^k z_2^l,$$

and the roots of Φ and Θ $(i = 1,2)$ are outside the unit circle.

Guo *et al.* (2009) showed that the Whittle's estimator of the memory parameters of this model are consistent and asymptotically normal.

In the separable case of the spatial long memory model we have,

$$\Phi(z_1,z_2) = \Phi_1(z_1)\Phi_2(z_2),$$

$$\Theta(z_1,z_2) = \Theta_1(z_1)\Theta_2(z_2),$$

where

$$\Phi_1(z) = 1 - \sum_{j=1}^{p_1}\phi_{1j} z^j, \quad \Phi_2(z) = 1 - \sum_{j=1}^{p_2}\phi_{2j} z^j,$$

$$\Theta_1(z) = 1 + \sum_{j=1}^{q_1}\theta_{1j} z^j, \quad \Theta_2(z) = 1 + \sum_{j=1}^{q_2}\theta_{2j} z^j,$$

and the roots of the polynomials Φ_i and Θ_i $(i = 1,2)$; are outside the unit circle (See Beran *et al.* 2009).

Sethuraman and Basawa (1995) established the asymptotic normality of the maximum likelihood estimators of the parameters of separable case in which $p_1 = q_1 = q_2 = d_2 = 0$. Boissy *et al.* (2005) showed the consistency of the Whittle's estimators of the parameters in the separable case of $p_1 = q_1 = 1$ and $p_2 = q_2 = 0$ and also established the asymptotic normality of them. Independently, Shitan (2008, 2009) termed this model as fractionally integrated separable spatial autoregressive (FISSAR(1,1)) model and derived the autocorrelation function of this model. The FISSAR(1,1) model is defined as,

$$(1 - \phi_{10}B_1 - \phi_{01}B_2 + \phi_{10}\phi_{01}B_1B_2)(1-B_1)^{d_1}(1-B_2)^{d_2} X_{ij} = Z_{ij} \tag{1}$$

or equivalently

$$(1 - \phi_{10}B_1)(1 - \phi_{01}B_2)(1-B_1)^{d_1}(1-B_2)^{d_2} X_{ij} = Z_{ij}$$

where $|\phi_{10}| < 1$ and $|\phi_{01}| < 1$. The autocovariance function of this model is given as

$$\gamma_X(k,l) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\phi_{10}^{r+m}\phi_{01}^{s+n}\gamma_W(k+r-m, l+s-n), \tag{2}$$

(see Shitan, 2008, 2009) where $k, l \in \mathrm{Z}$ and

$$\gamma_W(k,l) = \frac{\sigma_Z^2 (-1)^{h_1+h_2} \Gamma(1-2d_1)\Gamma(1-2d_2)}{\Gamma(k-d_1+1)\Gamma(1-k-d_1)\Gamma(l-d_2+1)\Gamma(1-l-d_2)} \quad (3)$$

is the autocovariance function of the fractional white noise process in two-dimensions as follows,

$$(1-B_1)^{d_1}(1-B_2)^{d_2} W_{ij} = Z_{ij}$$

Shitan (2008) also proposed the regression method of estimating the memory parameters of the FISSAR(1,1) model, which seems to be useful because requires less a priori known information about the model. This method (originally proposed by Geweke and Porter-Hudak (1983) in time series) is based on the periodogram of the model. Ghodsi and Shitan (2009) performed a simulation study to compare the properties of the regression estimators and Whittle's estimators (Whittle, 1954) of the memory parameters of this model and showed that the RMSE of the regression estimates is less than that of the Whittle's method. Beran *et al.* (2009) derived the asymptotic distribution of the least squares estimators of the parameters of the separable case of spatial long memory models.

In practice we would like to fit the model with less parameters in which $p_1 = p_2 = 1$ and $q_1 = q_2 = 0$. We refer to this model as First-Order *Fractionally Integrated Non-Separable Spatial Autoregressive* (FINSSAR(1,1)) model.

In Section 2 of this paper we introduce the FINSSAR(1,1) model. We also provide some realisations of this model in this section. The usefulness of this paper is an explicit expression form of the autocorrelation function for the FINSSAR(1,1) model is derived in Section 3. Some numerically evaluated values of the autocorrelation function is also discussed in Section 3. The spectral function and some numerical results of this function are provided in section 4. We will discuss the estimation procedures for this model in section 5. Finally in Section 5 the conclusions are drawn.

2. The FINSSAR(1,1) Model

The Fractionally Integrated Non-Separable Spatial Autoregressive (FINSSAR(1,1)) model is defined as

$$(1-\phi_{10}B_1-\phi_{01}B_2-\phi_{11}B_1B_2)(1-B_1)^{d_1}(1-B_2)^{d_2} Y_{ij} = Z_{ij}, \quad (4)$$

where $Y_{ij} : i, j \in \mathrm{Z}$ is a spatial process defined on a two-dimensional *regular* lattice, $\{Z_{ij}\}$ is a two-dimensional white noise process with mean zero and variance σ^2, $|\phi_{10}|<1$, $|\phi_{01}|<1, |\phi_{10}+\phi_{01}|<1-\phi_{11}, |\phi_{10}-\phi_{01}|<1+\phi_{11}$, $|d_1|<0.5$ and $|d_2|<0.5$.

This model can be used for modelling spatial processes with long-range dependence. Ground water flow and contaminant transport and essential physical properties like hydraulic conductivity (Guo *et al.* 2009), ocean temperature (Kim and Kim, 2002; Lim *et al.*, 2002), aquifers (Benson, 2006), and climate (Lin *et al.*, 2007) are some spatial processes that can be modelled by fractional spatial ARIMA models.

Note that model (4) (FINSSAR(1,1) model) is to be distinguished for model (1) (FISSAR(1,1) model). When ϕ_{11} in the FINSSAR(1,1) model equals to $-\phi_{10}\phi_{01}$ the expression $1-\phi_{10}B_1-\phi_{01}B_2-\phi_{11}B_1B_2$ in (4) factorises as $(1-\phi_{10}B_1)(1-\phi_{01}B_2)$ giving rise to FISSAR(1,1) model. As such the FINSSAR(1,1) model is more general than the FISSAR(1,1) model.

Another special case of FINSSAR(1,1) model is the first-order autoregressive, AR(1,1) model (also known as PK process, Pickard 1980) when $d_1 = d_2 = 0$. The autocorrelation structure and the explicit form of the likelihood function of this model has been derived by Basu and Reinsel (1993). They showed that $\rho(k,l) = \lambda^k \mu^{-l}$, $k \geq 0$ and $l \leq 0$, where λ and μ satisfy $\alpha_1\lambda + \alpha_2\mu^{-1} + \alpha_3\lambda\mu^{-1} = 1 = \alpha_1\lambda^{-1} + \alpha_2\mu + \alpha_3\lambda^{-1}\mu$.

The FINSSAR(1,1) model also reduces to the SSAR(1,1) model when $\phi_{11} = -\phi_{10}\phi_{01}$ and $d_1 = d_2 = 0$.

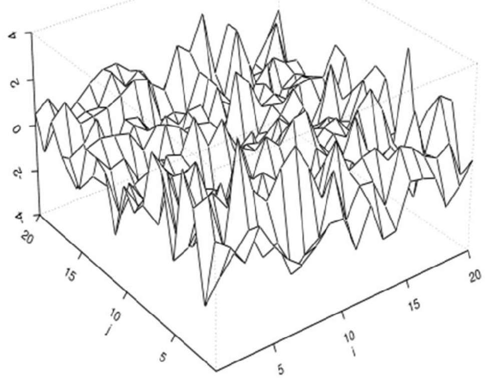
To simulate the FINSSAR(1,1) model note that (4) can equivalently be represented by the following two equations,

$$Y_{ij} = (1-\phi_{10}B_1-\phi_{01}B_2-\phi_{11}B_1B_2)^{-1}W_{ij}, \tag{5}$$

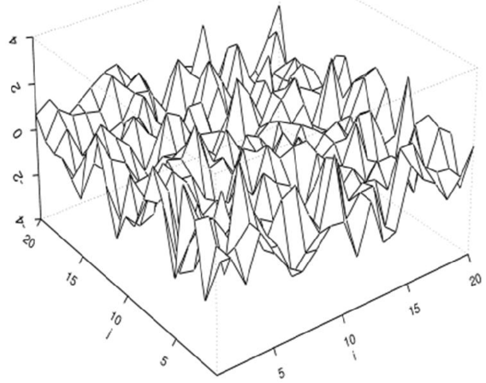
and

$$W_{ij} = (1-B_1)^{-d_1}(1-B_2)^{-d_2}Z_{ij}. \tag{6}$$

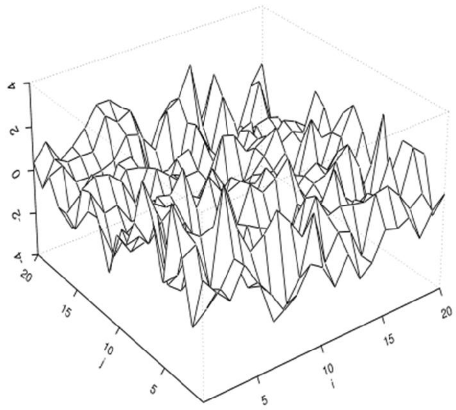
We simulated the FINSSAR(1,1) process in two stages. First the two dimensional white noise $\{Z_{ij}\}$ was generated and by using equation (6) we obtained $\{W_{ij}\}$. Then $\{Y_{ij}\}$ was obtained by using equation (5). Three typical realisations of the FINSSAR(1,1) process on a grid size of 20×20 are shown in Figure 1. In Figure 1(a) the parameter values are $\phi_{10} = -0.1, \phi_{01} = -0.1, \phi_{11} = -0.1, d_1 = 0.3$ and $d_2 = 0.1$ while in Figure 1(b) the parameter values are $\phi_{10} = 0.2, \phi_{01} = 0.2, \phi_{11} = 0.1, d_1 = -0.1$ and $d_2 = -0.1$ and the parameter values in Figure 1(c) are $\phi_{10} = 0.1, \phi_{01} = 0.2, \phi_{11} = -\phi_{10}\phi_{01} = -0.02, d_1 = 0.2$ and $d_2 = 0.1$. In the next section we discuss the autocovariance function (ACVF) of the FINSSAR(1,1) model.



(a)



(b)



(c)

Figure 1: Three typical Realisations of the FINSSAR(1,1) Process for selected parameter values.

(a) $\phi_{10} = -0.1, \phi_{01} = -0.1, \phi_{11} = -0.1, d_1 = 0.3$ and $d_2 = 0.1$

(b) $\phi_{10} = 0.2 \phi_{01} = 0.2, \phi_{11} = 0.1, d_1 = -0.1$ and $d_2 = -0.1$

(c) $\phi_{10} = 0.1, \phi_{01} = 0.2, \phi_{11} = -0.02, d_1 = 0.2$ and $d_2 = 0.1$

3. Autocovariance Function (ACVF) of the FINSSAR(1,1) Model

In this section, we derive the autocovariance function at spatial lags (h_1, h_2) for the FINSSAR(1,1) model.

Proposition 1. *For the process defined in equation* (4) *the autocovariance function of the process* $\gamma_Y(h_1, h_2)$ *is given as,*

$$\begin{aligned}\gamma_Y(h_1,h_2) &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(k+l+r)!}{k!l!r!}.\frac{(m+n+p)!}{m!n!p!}\phi_{10}^{k+m}\phi_{01}^{l+n}\phi_{11}^{r+p} \\ &\times \gamma_W(h_1+k+r-m-p, h_2+l+r-n-p)\end{aligned} \tag{7}$$

where $h_1, h_2 \in \mathrm{Z}$ *and* $\gamma_W(\cdot,\cdot)$ *is the autocovariance function of the fractional white noise process in two-dimensions as defined in* (3).

Proof. The multinomial expansion for $(1-\phi_{10}B_1-\phi_{01}B_2-\phi_{11}B_1B_2)^{-1}$ is

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty}\frac{(k+l+r)!}{k!l!r!}\phi_{10}^{k}\phi_{01}^{l}\phi_{11}^{r}B_1^{k+r}B_2^{l+r}$$

hence (4) can be written as

$$Y_{ij} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty}\frac{(k+l+r)!}{k!l!r!}\phi_{10}^{k}\phi_{01}^{l}\phi_{11}^{r}W_{i-k-r,j-l-r}$$

where W_{ij} is given as (see Shitan, 2008)

$$W_{ij} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(k+d_1)}{\Gamma(k+1)\Gamma(d_1)}\frac{\Gamma(l+d_2)}{\Gamma(l+1)\Gamma(d_2)}Z_{i-k,j-l}.$$

Clearly $E(Y_{ij}) = 0$ and therefore

$$\begin{aligned}\gamma_Y(h_1,h_2) &= E(Y_{i+h_1,j+h_2}Y_{ij}) \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(k+l+r)!}{k!l!r!}\frac{(m+n+p)!}{m!n!p!}\phi_{10}^{k+m}\phi_{01}^{l+n}\phi_{11}^{r+p} \\ &\times E(W_{i+h_1-m-p,j+h_2-n-p}W_{i-k-r,j-l-r}) \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(k+l+r)!}{k!l!r!}\frac{(m+n+p)!}{m!n!p!}\phi_{10}^{k+m}\phi_{01}^{l+n}\phi_{11}^{r+p} \\ &\times \gamma_W(h_1+k+r-m-p, h_2+l+r-n-p)\end{aligned}$$

which completes the proof.

In Figure 2, we have plotted some theoretical values of ACF for selected parameter values. In general we observe that the autocorrelation function will be symmetric with respect to the origin (i.e. $\rho_{h_1,h_2} = \rho_{-h_1,-h_2}$). However, in the special case when $\phi_{10} = \phi_{01}$ and $d_1 = d_2$ then the ACF will be diagonally symmetric (i.e. $\rho_{h_1,h_2} = \rho_{-h_1,-h_2} = \rho_{h_2,h_1} = \rho_{-h_2,-h_1}$, see Figure 2 (b)). Further, if $\phi_{11} = -\phi_{10} \times \phi_{01}$ then the ACF will be reflection or axially symmetric (i.e. $\rho_{h_1,h_2} = \rho_{-h_1,-h_2} = \rho_{h_1,-h_2} = \rho_{-h_1,h_2}$, see Figure 2 (c)).

In Figure 3, we include some further plots of the ACF for a variety of parameter values.

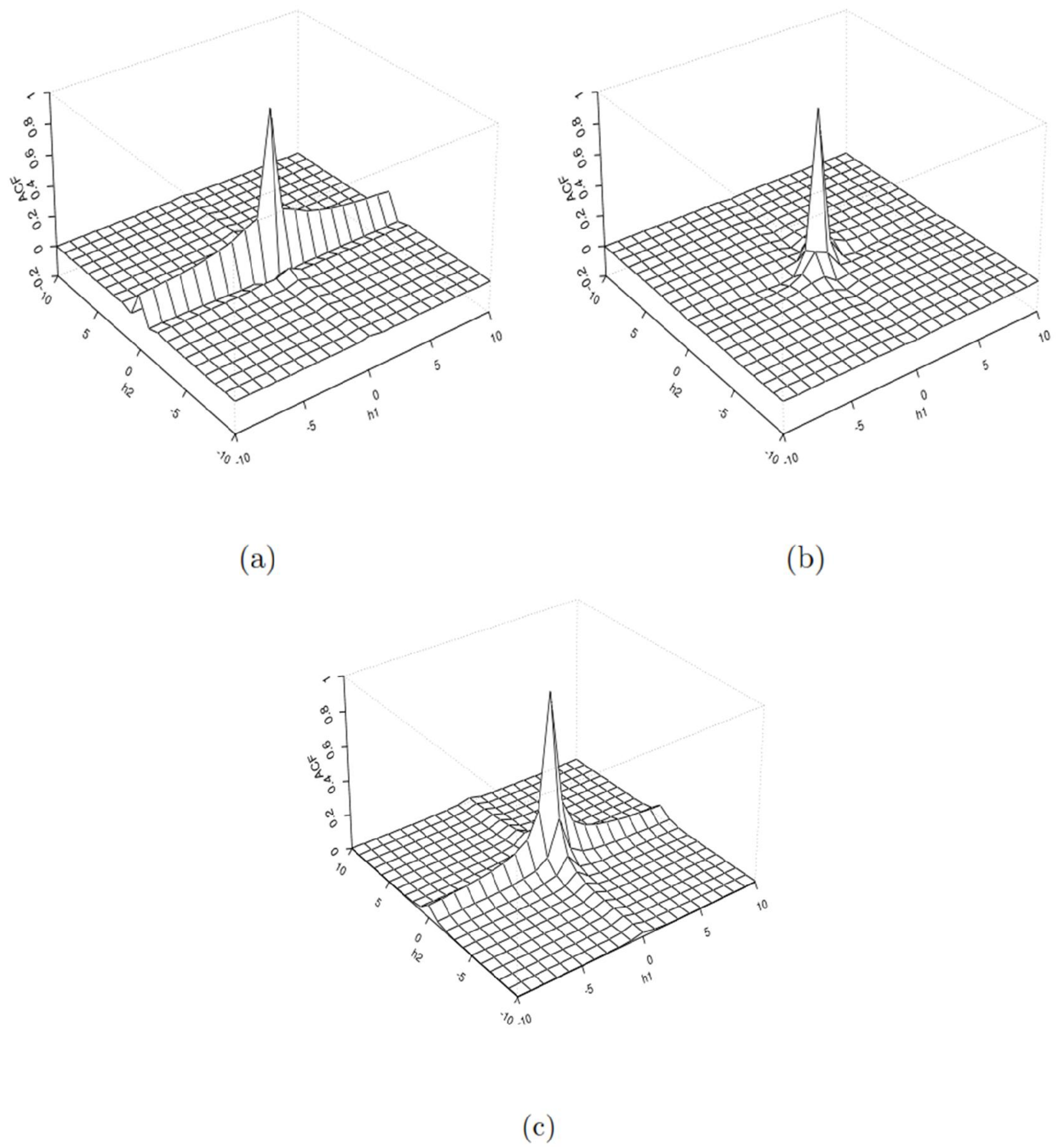


Figure 2: Autocorrelation function of the FINSSAR(1,1) model for selected parameter values.

(a) $\phi_{10} = -0.1, \phi_{01} = -0.1, \phi_{11} = -0.1, d_1 = 0.3$ and $d_2 = 0.1$

(b) $\phi_{10} = 0.2, \phi_{01} = 0.2, \phi_{11} = 0.1, d_1 = -0.1$ and $d_2 = -0.1$

(c) $\phi_{10} = 0.1, \phi_{01} = 0.2, \phi_{11} = -0.02, d_1 = 0.2$ and $d_2 = 0.1$

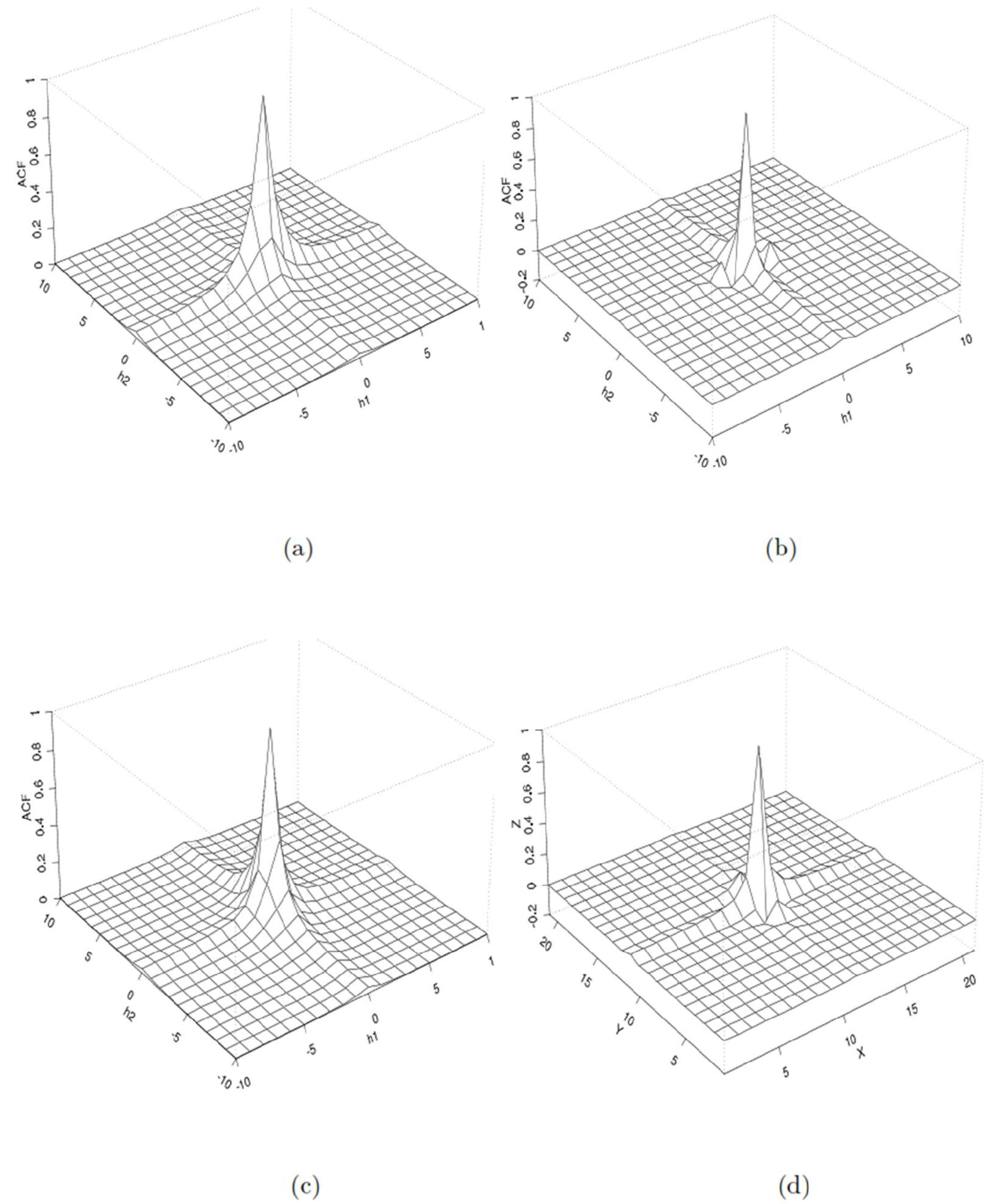


Figure 3: Autocorrelation function of the FINSSAR(1,1) model for selected parameter values.

(a) $\phi_{10} = 0.1, \phi_{01} = 0.1, \phi_{11} = 0.1, d_1 = 0.1$ and $d_2 = 0.1$.

(b) $\phi_{10} = 0.1, \phi_{01} = 0.1, \phi_{11} = 0.1, d_1 = -0.3$ and $d_2 = 0.1$

(c) $\phi_{10} = 0.3, \phi_{01} = 0.1, \phi_{11} = 0.1, d_1 = 0.1$ and $d_2 = 0.1$

(d) $\phi_{10} = -0.3, \phi_{01} = 0.1, \phi_{11} = 0.1, d_1 = 0.1$ and $d_2 = 0.1$

To compare the ACF of the FINSSAR(1,1) model with that of the AR(1,1) model, in table 1 we have included some values of $\rho(h_1, h_2)$ of these two models for $h_1 = 0$ and $h_2 = 0,1,2,3,4$ and 5. True autoregressive parameter values for both models are $\phi_{10} = -0.1, \phi_{01} = -0.1, \phi_{11} = -0.1$ and the true memory parameter values for the FINSSAR(1,1) model are $d_1 = 0.3$ and $d_2 = 0.1$. As it can be seen from this table the absolute values of ACF of the FINSSAR(1,1) model decays in slower rate than that of the AR(1,1) model.

Table 1: Autocorrelation $\rho(h_1, h_2)$ of the FINSSAR(1,1) model and AR(1,1) model for $h_1 = 0$ and varying h_2, (Model parameters are $\phi_{10} = -0.1, \phi_{01} = -0.1, \phi_{11} = -0.1, d_1 = 0.3$, $d_2 = 0.1$)

	h_2					
Model	0	1	2	3	4	5
FINSSAR(1,1)	1.000	0.328	0.299	0.246	0.221	0.202
AR(1,1)	1.000	-0.092	0.008	-0.001	0.000	0.000

In the following section we discuss the Spectral Function of the FINSSAR(1,1) Model.

4. Spectral Function of the FINSSAR(1,1) Model

Let $\Gamma(z_1, z_2)$ denote the *Autocovariance Generating Function (ACGF)* for the FINSSAR(1,1) model (4)

$$\Gamma(z_1, z_2) = \sum_{h_1=-\infty}^{\infty} \sum_{h_2=-\infty}^{\infty} \gamma_Y(h_1, h_2) z_1^{h_1} z_2^{h_2}$$

and $A(z_1, z_2) = (1 - \phi_{10} z_1 - \phi_{01} z_2 - \phi_{11} z_1 z_2)(1 - z_1)^{d_1}(1 - z_2)^{d_2}$, then it follows that

$$\Gamma(z_1, z_2) = \frac{\sigma^2}{A(z_1, z_2) A(z_1^{-1}, z_2^{-1})}.$$

The *spectral function* of the FINSSAR(1,1) model is given as

$$f(\omega_1, \omega_2) = \frac{\Gamma(e^{-i\omega_1}, e^{-i\omega_2})}{4\pi^2},$$

and if we denote $A(e^{i\omega_1}, e^{i\omega_2}) A(e^{-i\omega_1}, e^{-i\omega_2}) = | A(e^{-i\omega_1}, e^{-i\omega_2}) |^2$ then

$$f(\omega_1, \omega_2) = \frac{\sigma^2}{4\pi^2} \frac{|1 - e^{-i\omega_1}|^{-2d_1} |1 - e^{-i\omega_2}|^{-2d_2}}{|1 - \phi_{10} e^{-i\omega_1} - \phi_{01} e^{-i\omega_2} - \phi_{11} e^{-i(\omega_1 + \omega_2)}|^2}, \quad (8)$$

where $\omega_1, \omega_2 \in [-\pi, \pi] \setminus \{0\}$.

Figure 4 shows the spectral function of the FINSSAR(1,1) model for selected parameters values. In Figure 4(a) the parameter values are $\phi_{10} = -0.1, \phi_{01} = -0.1, \phi_{11} = -0.1, d_1 = 0.3$

and $d_2 = 0.1$ while in Figures 4(b) and 4(c) the parameter values are $\phi_{10} = 0.2, \phi_{01} = 0.2, \phi_{11} = 0.1, d_1 = -0.1, d_2 = -0.1$ and $\phi_{10} = 0.1, \phi_{01} = 0.2, \phi_{11} = -\phi_{10}\phi_{01} = -0.02, d_1 = 0.2, d_2 = 0.1$, respectively.

Some further plots of the spectral function for a variety of parameter values have been included in Figures 5.

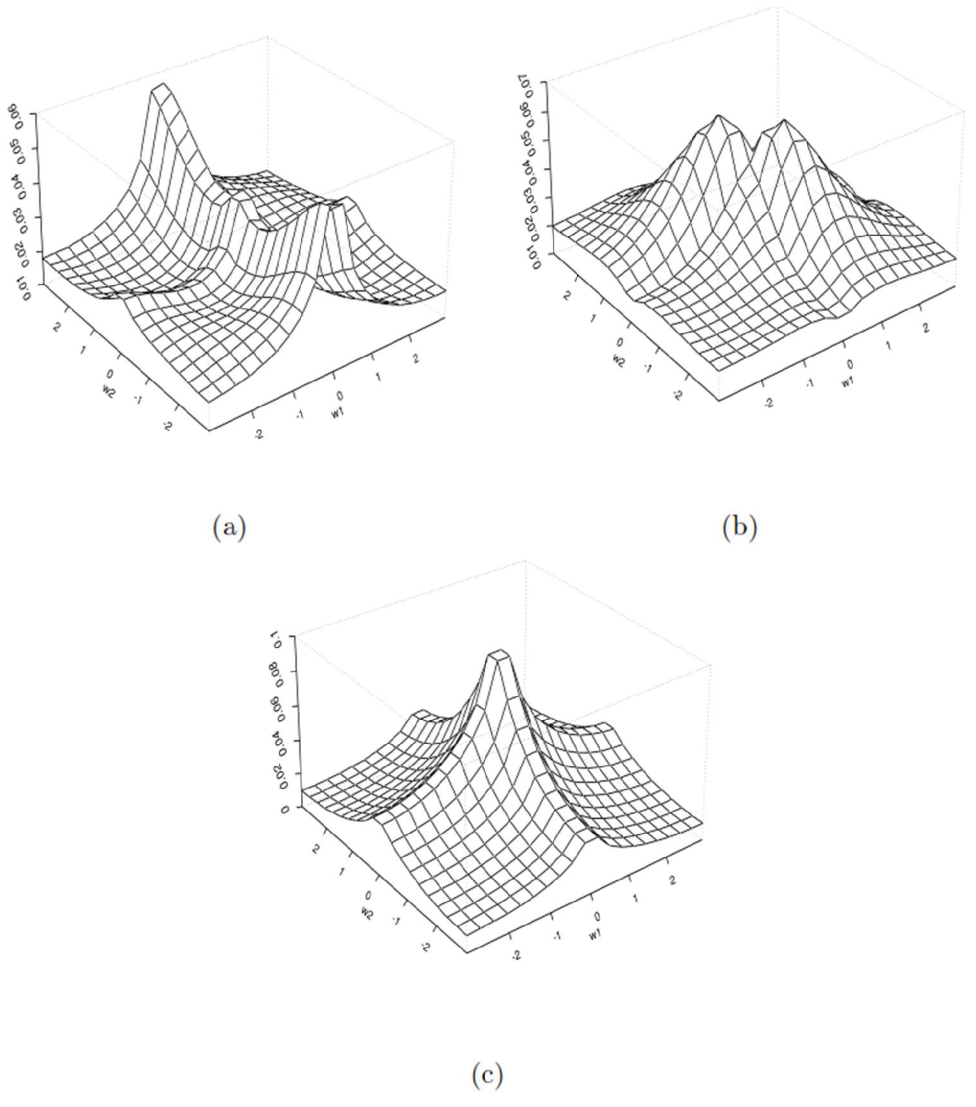


Figure 4: Spectral function of the FINSSAR(1,1) model for selected parameter values.

(a) $\phi_{10} = -0.1, \phi_{01} = -0.1, \phi_{11} = -0.1, d_1 = 0.3$ and $d_2 = 0.1$

(b) $\phi_{10} = 0.2, \phi_{01} = 0.2, \phi_{11} = 0.1, d_1 = -0.1$ and $d_2 = -0.1$

(c) $\phi_{10} = 0.1, \phi_{01} = 0.2, \phi_{11} = -0.02, d_1 = 0.2$ and $d_2 = 0.1$

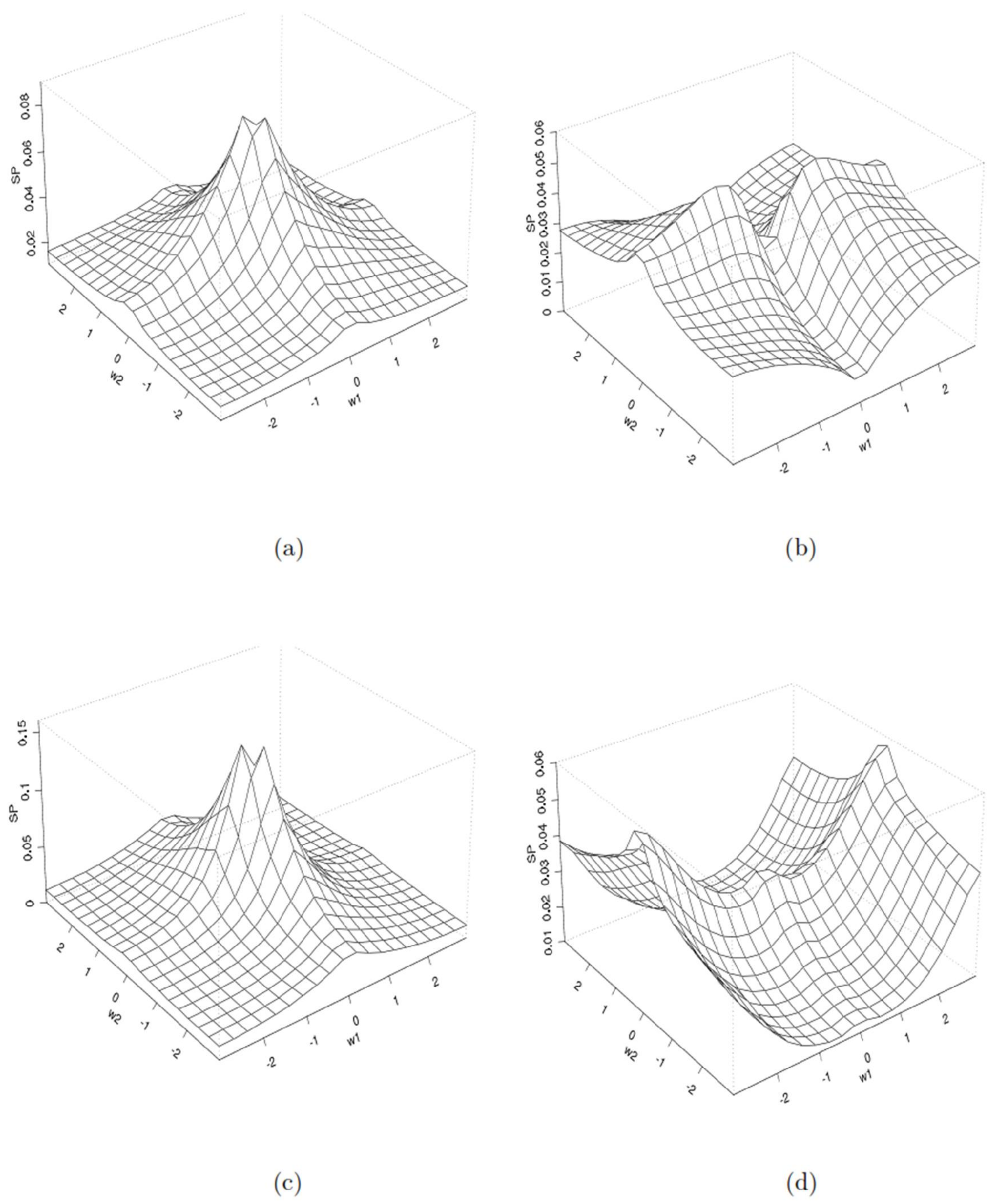


Figure 5: Spectral function of the FINSSAR(1,1) model for selected parameter values.

(a) $\phi_{10} = 0.1, \phi_{01} = 0.1, \phi_{11} = 0.1, d_1 = 0.1$ and $d_2 = 0.1$

(b) $\phi_{10} = 0.1, \phi_{01} = 0.1, \phi_{11} = 0.1, d_1 = -0.3$ and $d_2 = 0.1$

(c) $\phi_{10} = 0.3, \phi_{01} = 0.1, \phi_{11} = 0.1, d_1 = 0.1$ and $d_2 = 0.1$

(d) $\phi_{10} = -0.3, \phi_{01} = 0.1, \phi_{11} = 0.1, d_1 = 0.1$ and $d_2 = 0.1$

In the next section the estimation of the parameters of the FINSSAR(1,1) is briefly discussed.

5. Estimation of Parameters

Estimation of the memory parameters d_1 and d_2 can be done by two different methods, namely the regression method and Whittle's method as described below.

5.1 Regression Method for Estimating d_1 and d_2

Assume that $\{Y_{ij} : i, j \in Z\}$ is a FINSSAR(1,1) process as defined in (4). Denote $U_{ij} = (1-B_1)^{-d_1}(1-B_2)^{-d_2} Y_{ij}$, then from (8) the spectral function of this model can be written as

$$f(\omega_1,\omega_2) = |1-e^{-i\omega_1}|^{-2d_1}|1-e^{-i\omega_2}|^{-2d_2} f_U(\omega_1,\omega_2),$$

where

$$f_U(\omega_1,\omega_2) = \frac{\sigma^2}{4\pi^2 |1-\phi_{10}e^{-i\omega_1} - \phi_{01}e^{-i\omega_2} - \phi_{11}e^{-i(\omega_1+\omega_2)}|^2}$$

and $\omega_1, \omega_2 \in [-\pi,\pi]\setminus\{0\}$.

We can obtain estimates of d_1 and d_2 in terms of the natural logarithm of the periodogram $I(\omega_1,\omega_2)$ which is given as,

$$I(\omega_1,\omega_2) = \frac{1}{4\pi^2 n_1 n_2}\left|\sum_{s=1}^{n_1}\sum_{t=1}^{n_2} y_{s,t} e^{i(s\omega_1+t\omega_2)}\right|^2, \tag{9}$$

where $(y_{11}, y_{12}, \ldots, y_{1n_2}, y_{21}, y_{22}, \ldots, y_{2n_2}, \ldots, y_{n_1 1}, \ldots, y_{n_1 n_2})$ is an observed data set of grid size $n_1 \times n_2$ of the FINSSAR(1,1) process.

Let $\omega_{1,j_1} = 2\pi j_1 / n_1$ and $\omega_{2,j_2} = 2\pi j_2 / n_2$ where $j_1 = 1, \ldots, m_1$ and $j_2 = 1, \ldots, m_2$ ($m_1 = \sqrt{n_1}$ and $m_2 = \sqrt{n_2}$ as suggested by Geweke and Porter-Hudak (1983)), and suppose I_{j_1,j_2} denote $I(\omega_1,\omega_2)$ evaluated at $\omega_1 = \omega_{1,j_1}$ and $\omega_2 = \omega_{2,j_2}$. The regression estimators for d_1 and d_2 can be obtained by the least squares estimates of the coefficients in the multiple regression equation given as,

$$\ln I_{j_1,j_2} = \beta_0 + \beta_1 x_{1,j_1} + \beta_2 x_{2,j_2} + \varepsilon_{j_1,j_2},$$

where $\beta_0 = \ln f_U(0,0), \beta_1 = -d_1, \beta_2 = -d_2$, $x_{1,j_1} = \ln|1-e^{-i\omega_{1,j_1}}|^2$, $x_{2,j_2} = \ln|1-e^{-i\omega_{2,j_2}}|^2$, $\varepsilon_{j_1,j_2} = \ln(I_{j_1,j_2} / f_{j_1,j_2})$ and $f_{j_1,j_2} = f(\omega_{1,j_1}, \omega_{2,j_2})$.

5.2 Whittle's Method

Whittle's method of estimation (originally proposed by Whittle, 1951) is another technique to estimate the parameters of the model. This method is based on the spectral density function and periodogram of the model. The Whittle's estimator $\hat{\boldsymbol{\theta}}$ is the value of $\boldsymbol{\theta} = (\phi_{10}, \phi_{01}, \phi_{11}, d_1, d_2)'$ that minimizes the Whittle's likelihood function given as (see for details Boissy, 2005),

$$L(\boldsymbol{\theta}) = \frac{1}{n_1 n_2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{I(\omega_1, \omega_2)}{g(\omega_1, \omega_2; \boldsymbol{\theta})} d\omega_1 d\omega_2, \qquad (10)$$

where $g(\cdot,\cdot) = \frac{4\pi^2}{\sigma^2} f(\cdot,\cdot)$ and $f(\cdot,\cdot)$ and $I(\cdot,\cdot)$ are defined in (8) and (9), respectively. An equivalent form of (10) for computational purposes is given by,

$$L_W(\boldsymbol{\theta}) = \frac{1}{n_1 n_2} \sum_{j_1} \sum_{j_2} \frac{I(\omega_{1j_1}, \omega_{2j_2})}{g(\omega_{1j_1}, \omega_{2j_2}; \boldsymbol{\theta})}, \qquad (11)$$

where $\omega_{1j_1} = \frac{2\pi j_1}{n_1}$, $\omega_{2j_2} = \frac{2\pi j_2}{n_2}$ ($j_1 = -\left[\frac{n_1 - 1}{2}\right], \ldots, \left[\frac{n_1}{2}\right]$, and

$j_2 = -\left[\frac{n_2 - 1}{2}\right], \ldots, \left[\frac{n_2}{2}\right]$). An estimator for σ^2 is $4\pi^2 L_W(\hat{\boldsymbol{\theta}})$.

6. A Real Data Example

To illustrate the fitting of the FINSSAR(1,1) model, we consider the data set from a regular grid of 728 presented by Kempton and Howes (1981) on the yield of barley (kg) from an agricultural uniformity trial experiment at Plant Breeding Institute, Cambrige, UK. Table 2 shows the sample spatial correlations of the data. Note from Table 2 that the data within columns are highly correlated.

We fitted the FINSSAR(1,1) model to the mean corrected data subjected to the constraints on the model parameters as in (4). The fitted model using Whittle's estimation is as follows,

$$(1 - 0.185B_1 - 0.850B_2 + 0.203B_1B_2)(1 - B_1)^{-0.499}(1 - B_2)^{-0.181} Y_{ij} = Z_{ij},$$

where $\{Z_{ij}\} \sim N(0, 0.023)$.

Table 2: Some values of the sample spatial autocorrelation for the barley data

	k			
l	0	1	2	3
4	0.470	0.220	0.053	0.035
3	0.570	0.231	0.037	0.015
2	0.677	0.241	0.041	0.026
1	0.796	0.253	0.035	0.015
0	1.000	0.264	0.013	-0.025
-1	0.796	0.190	-0.045	-0.060
-2	0.677	0.137	-0.061	-0.064
-3	0.570	0.088	-0.063	-0.065
-4	0.470	0.044	-0.079	-0.073

We do not claim that the FINSSAR(1,1) model is the best model for the agricultural uniformity trial experiment data, but it is merely included in our paper as an example of fitting the FINSSAR(1,1) model to a real data set.

7. Conclusion

In this paper the objective of our research was to develop a non-separable counterpart of the FISSAR(1,1) model. We termed this model as the first-order fractionally integrated non-separable spatial autoregressive (FINSSAR(1,1)) model.

The FINSSAR(1,1) model is a more general model as it encompasses the FISSAR(1,1) and the standard separable autoregressive (SSAR) models. The theoretical ACVF has been derived and its spectral density function has also been obtained. From the ACF and spectral density, it is clear that this model may be able to model many type of real phenomena which display long memory correlation structure like ground water flow, contaminant transport, hydraulic conductivity, ocean temperature, climatic data, etc. Both regression estimators and Whittle's estimation have been discussed. A real data example has also been provided to illustrate the fitting of FINSSAR(1,1) model. It is hoped that with this illustration of fitting the model to a real data set, more researchers would consider using the model for various applications.

The authors are currently working on other aspects of this model and shall be reported in future papers.

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