Estimation of Stress-Strength Reliability For Weibull Distribution Based on Type-II Right Censored Ranked Set Sampling Data

Fatma Gül Akgül
Department of Computer Engineering
Artvin Çoruh University, Turkey
ftm.gul.fuz@artvin.edu.tr

Birdal Şenoğlu
Department of Statistics
Ankara University, Turkey
senoglu@science.ankara.edu.tr

Abstract

In this paper, we consider the estimation of stress-strength reliability $R = P(X < Y)$ under the type-II right censored data when the distributions of both the stress and the strength are Weibull. First, we discuss the estimation of R based on simple random sampling (SRS). Then, we use the effective and the efficient alternative of SRS which is known to be the ranked set sampling (RSS) to estimate R. In the estimation procedure of R, we use two different approaches they are i) maximum likelihood (ML) which requires an iterative method and ii) modified maximum likelihood (MML) which has an explicit form. Monte-Carlo simulation study is performed to identify the efficient sampling method (i.e., SRS or RSS) and the efficient estimation method (i.e., ML or MML). Finally, strength and wind speed data sets are analyzed to illustrate the proposed methods in practice.

Keywords: Stress-strength model; Ranked set sampling; Type-II right censoring; Modified maximum likelihood; Weibull distribution, Monte-Carlo simulation.

1. Introduction

In the literature, considerable attention has been raised to estimate the stress-strength reliability $R = P(X < Y)$. Here, X and Y represent the stress and the strength of the system, respectively. The reliability of the system is defined as the probability of X non-exceeding Y. Since, if $X > Y$, the system fails, otherwise it continues to work. Therefore, R is also called as system reliability. There is a vast literature on estimation of R, see for example Downtown (1973), Tong (1977), Kundu and Gupta (2006), Rezaei et al. (2010) and Rao et al. (2016). For more detailed information, one may refer to Kotz et al. (2003). Structures and deterioration of rocket motors fatigue failure of aircraft structures and aging concrete pressure vessels are the some practical examples of R, see Dey et al. (2015).

Traditionally, R is estimated by using the complete simple random sampling (SRS) data. However, in many life-testing and reliability studies, complete information may not always be obtained on failure times of experimental units. This type of data is called as censored data. In recent years, most of the works concerning with the estimation of R have been done under the assumption of censored SRS data. In this context, Krishnanmoorthy and Lin (2010) considered the interval estimation of the stress-strength reliability involving two independent Weibull distribution under complete and censored data. Saraçoğlu et al. (2012) considered the estimation of R based on progressively type-II censored data. They assume that both the stress and the strength have exponential distribution. In some other studies, the maximum likelihood (ML) estimator of R is obtained under progressively first failure censoring when the distributions of both X and Y are Burr XII, see Lio and Tsai (2012). Asgharzadeh et al. (2011) and Valiollahi et al. (2013) derive the estimators of R under the progressively type-II censored data when both X and Y are Weibull.

It should also be noted that the usage of the ranked set sampling (RSS) method, originated by McIntyre (1952), is brought a new insight for the estimation of the system reliability R. In some experimental situations, sample sizes may be large therefore, the cost of the measurements for these sampling units may be expensive. In this case, RSS provides an opportunity to determine the sampling units in a cost effective and inexpensive way. See Patil et al. (1994), Kaur et al. (1995) and Chen et al. (2004) for complete review of the applications and the theoretical studies about RSS. RSS is very feasible to different areas such as for environmental studies, see Yu and Lam (1997), Barnett (1999) and Bocci et al. (2010). There are lots of studies in literature for estimating the system reliability R based on RSS data, see Sengupta and Mukhuti (2008a, 2008b), Muttlak et al. (2010), Dong et al. (2013), Mahdizadeh and Zamanzade (2016, 2017, 2018a, 2018b) and Akgül and Şenoğlu (2017, 2018).

In contrast to SRS, there has been few studies concerning with the censored RSS data in the literature, for example, Yu and Tam (2002) considered the estimation of the population mean and standard deviation based on left censored RSS data with fixed censoring times in the context of ML and Kaplan-Meier (KM) methodologies. He and Naharaja (2012) developed the Fisher information matrix in censored samples from Downton's bivariate exponential distribution based on RSS. Strzalkowska-Kominiak and Mahdizadeh (2014) derived the KM estimator based on the right censored RSS data with random censoring times. Mahdizadeh and Strzalkowska-Kominiak (2017) dealt with constructing the confidence intervals for a distribution function based on censored ranked set sampling data.

In this study, we obtain the estimators of R based on SRS and RSS sampling methods under the assumption of Type-II right censoring. If the observation's lifetime is greater than the lifetime of the predetermined largest observation, it will be censored. This type of censoring is called as Type-II right censoring. It is assumed that the stress $X \sim Weibull(p, \sigma_1)$ and the strength $Y \sim Weibull(p, \sigma_2)$ are both independent. The main reason for using the Weibull distribution is its flexibility for modeling the asymmetric data and its extensive usage in engineering, life testing and reliability studies, see Lawless (1982) and Murthy et al. (2004).

The cumulative density function (cdf) and the probability density function (pdf) for the two-parameter Weibull distribution are given by

$$F_X(x; p, \sigma) = 1 - e^{-\frac{x^p}{\sigma}}, x > 0, p > 0, \sigma > 0 \quad (1)$$

and

$$f_X(x; p, \sigma) = \frac{p}{\sigma} x^{p-1} e^{-\frac{x^p}{\sigma}}, x > 0, p > 0, \sigma > 0 \quad (2)$$

respectively. Here, p is the shape parameter and σ is the scale parameter. Then, we obtain the system reliability R as

$$R = \int_0^{\infty} \left(1 - e^{-\frac{t^p}{\sigma_1}}\right) \frac{p}{\sigma_2} t^{p-1} e^{-\frac{t^p}{\sigma_2}} dt = \frac{\sigma_2}{\sigma_1 + \sigma_2}. \quad (3)$$

To derive the estimators of R, we use two different approaches. In the first approach, we use ML method and in the second approach non-iterative modified maximum likelihood (MML) method originated by Tiku (1967, 1968) is used. To the best of our knowledge, this is the first study applying MML methodology for estimating the system reliability R based on type-II right censored RSS data.

This paper is organized as follows. In Section 2, under the assumption of type-II right censored SRS data, we derive the ML estimator of R by using iterative methods. Then we propose to use the MML methodology for obtaining the estimator of R which has an explicit form. In Section 3, the ML and the MML estimators of R are obtained based on type-II right censored RSS data. In the following section, performances of the proposed estimators are compared via Monte-Carlo simulation study. Real data applications are given in Section 5. Finally, conclusions are presented in Section 6.

2. Estimators of *R* based on Type-II Right Censored SRS Data

In this section, the ML and the MML estimators of R based on type-II right censored SRS data are derived.

2.1 ML estimator of *R*

Let $X_1, X_2, \dots, X_n \sim Weibull(p, \sigma_1)$ and $Y_1, Y_2, \dots, Y_m \sim Weibull(p, \sigma_2)$ be two independent samples for the stress and the strength, respectively. Also, let r and r' be the number of censored observation(s) in the samples corresponding to X_i's and Y_j's, respectively. In the censoring procedure, if the observations $x_i \le x_{(n-r)}$ $(i = 1, \dots, n)$ and $y_j \le y_{(m-r')}$ $(j = 1, \dots, m)$, then we take them into the sample without changing their values, otherwise we reproduce them with the $x_{(n-r)}$th and $y_{(m-r')}$th ordered observations, respectively.

Then the likelihood function is then given by

$$\begin{aligned} L &= \prod_{i=1}^{n} f(x_i)^{\delta_i} [1 - F(x_i)]^{1-\delta_i} \prod_{j=1}^{m} f(y_j)^{\delta_j} [1 - F(y_j)]^{1-\delta_j} \\ &= \frac{p^{\sum_{i=1}^{n} \delta_i + \sum_{j=1}^{m} \delta_j}}{\sigma_1^{\sum_{i=1}^{n} \delta_i} \sigma_2^{\sum_{j=1}^{m} \delta_j}} \prod_{i=1}^{n} \left(x_i^{p-1}\right)^{\delta_i} e^{-x_i^p/\sigma_1} \prod_{j=1}^{m} \left(y_j^{p-1}\right)^{\delta_j} e^{-y_j^p/\sigma_2}. \end{aligned} \quad (4)$$

Here, δ_i $(i = 1, \dots, n)$ and δ_j $(j = 1, \dots, m)$ are the indicator functions defined by

$$\delta_i = \begin{cases} 1, x_i \le x_{(n-r)} \\ 0, x_i > x_{(n-r)} \end{cases} \quad \text{and} \quad \delta_j = \begin{cases} 1, y_j \le y_{(m-r')} \\ 0, y_j > y_{(m-r')} \end{cases}. \quad (5)$$

respectively. We obtain the likelihood equations by taking the derivatives of the log-likelihood function with respect to the unknown parameters σ_1, σ_2 and p. They are given as shown below:

$$\frac{\partial \ln L}{\partial \sigma_1} = -\frac{\sum_{i=1}^{n} \delta_i}{\sigma_1} + \frac{1}{\sigma_1^2}\sum_{i=1}^{n} x_i^p = 0, \qquad \frac{\partial \ln L}{\partial \sigma_2} = -\frac{\sum_{j=1}^{m} \delta_j}{\sigma_2} + \frac{1}{\sigma_2^2}\sum_{j=1}^{m} y_j^p = 0, \quad (6)$$

$$\frac{\partial \ln L}{\partial p} = \frac{\sum_{i=1}^{n} \delta_i + \sum_{j=1}^{m} \delta_j}{p} + \sum_{i=1}^{n} \delta_i \ln x_i - \frac{1}{\sigma_1}\sum_{i=1}^{n} x_i^p \ln x_i + \sum_{j=1}^{m} \delta_j \ln y_j - \frac{1}{\sigma_2}\sum_{j=1}^{m} y_j^p \ln y_j = 0. \quad (7)$$

Because of the non-linear functions $h_1(x) = x^p$, $h_2(x) = \ln x$, $h_1(y) = y^p$ and $h_2(y) = \ln y$, we cannot obtain the explicit solutions of the equations (6)-(7). Therefore, we resort to iterative methods.

It is clear from the equations (6) that ML estimators of σ_1 and σ_2 are the functions of the shape parameters p. They are given below

$$\hat{\sigma}_1 = \frac{\sum_{i=1}^{n} x_i^{\hat{p}}}{\sum_{i=1}^{n} \delta_i} \text{ and } \hat{\sigma}_2 = \frac{\sum_{j=1}^{m} y_{(j)}^{\hat{p}}}{\sum_{j=1}^{m} \delta_j}, \quad (8)$$

respectively. If we incorporate these estimators into (7) and solve it with respect to p by using the iterative methods, the ML estimate of p is obtained. Then, we insert this estimate value of p in to the equation (8) and obtain the ML estimates of σ_1 and σ_2. The iterative process, such as Newton-Raphson method, should converge quickly to its maximum in this case, if the initial guess is reasonably close to the actual solution, see Kundu and Gupta (2006).

After obtaining the ML estimators of p, σ_1 and σ_2 represented by $\hat{p}_{ML,SRS}$, $\hat{\sigma}_{1_{ML,SRS}}$ and $\hat{\sigma}_{2_{ML,SRS}}$, respectively, the ML estimator of R is obtained using the invariance property of the ML estimators. It is shown below

$$\hat{R}_{ML,SRS} = \frac{\hat{\sigma}_{2_{ML,SRS}}}{\hat{\sigma}_{1_{ML,SRS}} + \hat{\sigma}_{2_{ML,SRS}}}. \quad (9)$$

2.2 MML estimator of *R*

In previous section, we observed that the likelihood equations have no explicit solutions and therefore we solved them by using numerical methods. However, solving them by iteration is indeed problematic for reasons of (i) multiple roots, (ii) non-convergence of iterations, or (iii) convergence to wrong values; see Barnett (1966), Vaughan (2002) and Tiku and Şenoğlu (2009). To overcome these difficulties, we use the MML methodology introduced by Tiku (1967,1968).

The MML methodology can be used for any location-scale distribution of the type $(1/\sigma)f((x-\mu)/\sigma)$. It is known that if the random variable X has two parameter Weibull distribution with the shape parameter p and the scale parameter σ, then the natural logarithm of X, i.e., $U = \ln X$ has an Extreme Value (EV) distribution with the location parameter μ and the scale parameter η. The EV distribution has the following pdf and cdf

$$f_U(u;\mu,\eta) = \frac{1}{\eta} e^{\left(\frac{u-\mu}{\eta} - e^{\frac{u-\mu}{\eta}}\right)}, -\infty < u < \infty \quad (10)$$

where, $\mu \in R$ and $\eta \in R^+$.

The MML estimators of the shape and the scale parameters of the Weibull distribution are obtained by using the following relationship between the parameters of the EV distribution and the parameters of Weibull distribution, i.e., $\mu = 1/p \ln \sigma$ and $\eta = 1/p$.

After deriving the estimators of the parameters of EV distribution, the scale and the shape parameters of the Weibull distribution are obtained by using the following inverse transformations

$$\sigma = e^{\mu p} \text{ and } p = \frac{1}{\eta}, \tag{11}$$

respectively.

Let $X_1, \dots, X_n \sim Weibull(p, \sigma_1)$ and $Y_1, \dots, Y_m \sim Weibull(p, \sigma_2)$ be the SRS data for the stress and the strength, respectively. As mentioned above $U = \ln X$ and $V = \ln Y$ have the $EV(\mu_1, \eta)$ and $EV(\mu_2, \eta)$ distributions, respectively. Then the likelihood function for the type-II right censored data can be written as follows

$$L \propto \frac{1}{\eta^{\sum_{i=1}^{n}\delta_i + \sum_{j=1}^{m}\delta_j}} \prod_{i=1}^{n} f(z_{(i)})^{\delta_i}\left[1 - F(z_{(i)})\right]^{1-\delta_i} \prod_{j=1}^{m} f(w_{(j)})^{\delta_j}\left[1 - F(w_{(j)})\right]^{1-\delta_j} \tag{12}$$

since the complete sums are invariant to ordering, i.e., $\sum_{i=1}^{n} f(z_i) = \sum_{i=1}^{n} f(z_{(i)})$. Here, $z_{(i)} = (u_{(i)} - \mu_1)/\eta,\ i = 1, \dots, n$ and $w_{(j)} = (v_{(j)} - \mu_2)/\eta,\ j = 1, \dots, m$ are the standardized order statistics. Also, δ_i and δ_j are censoring indicators as defined earlier.

Derivatives of the log-likelihood function with respect to the unknown parameters μ_1, μ_2 and η i.e., $\frac{\partial \ln L}{\partial \mu_1} = 0$, $\frac{\partial \ln L}{\partial \mu_2} = 0$ and $\frac{\partial \ln L}{\partial \eta} = 0$, are the likelihood equations. However, these equations do not have explicit solutions, because of the following nonlinear functions of the parameters

$$g_1(z_{(i)}) = \frac{f'(z_{(i)})}{f(z_{(i)})}, g_2(z_{(i)}) = \frac{f(z_{(i)})}{1-F(z_{(i)})}; g_1(w_{(j)}) = \frac{f'(w_{(j)})}{f(w_{(j)})}, g_2(w_{(j)}) = \frac{f(w_{(j)})}{1-F(w_{(j)})}. \tag{13}$$

Therefore, we linearize them around the expected values of the standardized ordered statistics $t_{(i)}^{u} = E(z_{(i)})$ and $t_{(j)}^{v} = E(w_{(j)})$ using the first two terms of Taylor series expansion. Then, we get

$$g_1(z_{(i)}) = \alpha_{1i}^{u} - \beta_{1i}^{u} z_{(i)}, \quad g_2(z_{(i)}) = \alpha_{2i}^{u} + \beta_{2i}^{u} z_{(i)}, \quad i = 1, \dots n,$$
$$g_1(w_{(j)}) = \alpha_{1j}^{v} - \beta_{1j}^{v} w_{(j)}, \quad g_2(w_{(j)}) = \alpha_{2j}^{v} + \beta_{2j}^{v} w_{(j)}, \quad j = 1, \dots, m,$$

where

$$\alpha_{1i}^{u} = 1 - e^{t_{(i)}^{u}} + t_{(i)}^{u} e^{t_{(i)}^{u}}, \beta_{1i}^{u} = e^{t_{(i)}^{u}} \text{ and } \alpha_{2i}^{u} = e^{t_{(i)}^{u}} - t_{(i)}^{u} e^{t_{(i)}^{u}}, \beta_{2i}^{u} = e^{t_{(i)}^{u}},$$

$$t_{(i)}^{u} = \begin{cases} \ln\left(-\ln\left(1 - \frac{i}{n+1}\right)\right), & if\ z_{(i)} \le z_{(n-r)} \\ \ln\left(-\ln\left(1 - \frac{n-r}{n+1}\right)\right), & if\ z_{(i)} > z_{(n-r)} \end{cases}. \tag{14}$$

$(\alpha_{1j}^{v}, \beta_{1j}^{v})$ and $(\alpha_{2j}^{v}, \beta_{2j}^{v})$ coefficients and $t_{(j)}^{v}$ are obtained similarly as in (14). Therefore we do not reproduce it for the sake of brevity.

Modified likelihood equations are obtained by incorporating equations (14) in the likelihood equations. Solving these equations yield the following MML estimators of μ_1, μ_2 and η,

$$\hat{\mu}_{1_{MML,SRS}} = K_1 - \frac{\Delta_1}{m_1}\hat{\eta}_{MML,SRS}, \quad \hat{\mu}_{2_{MML,SRS}} = K_2 - \frac{\Delta_2}{m_2}\hat{\eta}_{MML,SRS} \text{ and}$$
$$\hat{\eta}_{MML,SRS} = \frac{-B+\sqrt{B^2+4AC}}{2A}, \tag{15}$$

where

$$\gamma_{1i}^u = \delta_i\beta_{1i}^u,\ \gamma_{2i}^u = (1-\delta_i)\beta_{2i}^u,\ \gamma_i^u = \gamma_{1i}^u + \gamma_{2i}^u,\ m_1 = \sum_{i=1}^n \gamma_i^u,\ K_1 = \frac{\sum_{i=1}^n \gamma_i^u u_{(i)}}{m_1},$$
$$\Delta_{1i}^u = \delta_i\alpha_{1i}^u,\ \ \Delta_{2i}^u = (1-\delta_i)\alpha_{2i}^u,\ \ \Delta_i^u = \Delta_{1i}^u - \Delta_{2i}^u,\ \ \Delta_1 = \sum_{i=1}^n \Delta_i^u,$$
$$\gamma_{1j}^v = \delta_j\beta_{1j}^v,\ \gamma_{2j}^v = \left(1-\delta_j\right)\beta_{2j}^v,\ \gamma_j^v = \gamma_{1j}^v + \gamma_{2j}^v,\ m_2 = \sum_{j=1}^m \gamma_j^v,\ K_2 = \frac{\sum_{j=1}^m \gamma_j^v v_{(j)}}{m_2},$$
$$\Delta_{1j}^v = \delta_j\alpha_{1j}^v,\ \ \Delta_{2j}^v = \left(1-\delta_j\right)\alpha_{2j}^v,\ \ \Delta_j^v = \Delta_{1j}^v - \Delta_{2j}^v,\ \ \Delta_2 = \sum_{j=1}^m \Delta_j^v,$$
$$A = \sum_{i=1}^n \delta_i + \sum_{j=1}^m \delta_j = n + m - (r + r'),$$
$$B = \sum_{i=1}^n \Delta_i^u\left(u_{(i)} - K_1\right) + \sum_{j=1}^m \Delta_j^v\left(v_{(j)} - K_2\right),$$
$$C = \sum_{i=1}^n \gamma_i^u \left(u_{(i)} - K_1\right)^2 + \sum_{j=1}^m \gamma_j^v \left(v_{(j)} - K_2\right)^2. \tag{16}$$

By the inverse transformations defined in (11), we obtain the MML estimators of the Weibull parameters σ_1, σ_2 and p as

$$\hat{\sigma}_{1_{MML,SRS}} = e^{\hat{p}_{MML,SRS}\hat{\mu}_{1_{MML,SRS}}}, \quad \hat{\sigma}_{2_{MML,SRS}} = e^{\hat{p}_{MML,SRS}\hat{\mu}_{2_{MML,SRS}}} \text{ and}$$
$$\hat{p}_{MML,SRS} = \frac{1}{\hat{\eta}_{MML,SRS}}. \tag{17}$$

These estimators have closed form expressions. They are the functions of the sample observations and are easy to compute. Asymptotically, they are fully efficient under some mild regularity conditions. It should be noted that the fully efficient estimators are unbiased and their variances are equal to the Rao-Cramer lower bound. They are asymptotically equivalent to ML estimators, see Vaughan and Tiku (2000).

It should be noted that since the MML estimators can provide the explicit form of the parameter estimators, they are used as initial value for the iterative methods for the ML estimators of the unknown parameters.

By incorporating MML estimators of the scale parameters into the equation (3), the MML estimator of R is obtained as follows

$$\hat{R}_{MML,SRS} = \frac{\hat{\sigma}_{2_{MML,SRS}}}{\hat{\sigma}_{1_{MML,SRS}} + \hat{\sigma}_{2_{MML,SRS}}}. \tag{18}$$

3. Estimators of *R* based on Type-II Right Censored RSS Data

In this section, we derive the ML and the MML estimators of R based on type-II right censored RSS data.

Let's first describe how to obtain the right censored RSS data. Traditionally, in complete RSS, m_x-dimensional m_x sets are selected via SRS. Without doing certain measurements, the sampling units are ranked with respect to virtual comparisons, expert opinion or auxiliary variables. Then, in the first set the smallest ranked unit, in the next set the second smallest ranked unit and finally in the last set the largest ranked unit are selected for actual measurements. In this way, we obtain m_x-measured units. This complete procedure is called a cycle and repeated r_x times until the sample size $n = m_x r_x$ is obtained. See the following table to better understanding the RSS procedure:

$$\begin{array}{lllll} \text{Cycle 1} & X_{(1)1} & X_{(2)1} & \cdots & X_{(m_x)1} \\ \text{Cycle 2} & X_{(1)2} & X_{(2)2} & \cdots & X_{(m_x)2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \text{Cycle } r_x & X_{(1)r_x} & X_{(2)r_x} & \cdots & X_{(m_x)r_x}. \end{array}$$

Here, $X_{(i)c}$, $(i = 1, \dots, m_x;\ c = 1, \dots, r_x)$ is the ith smallest observation in the ith set and the cth cycle. In censoring procedure, similar to SRS, the largest r observations in each cycle are censored. In other words, if the observation $x_{(i)c} \le x_{(m_x-r)c}$, then we take it as it is, otherwise we replace it with the value of $(m_x - r)$th observation in each cycle.

3.1 ML Estimator of *R*

At the beginning of this subsection, we will give some abbreviations for better understanding the rest of the paper. They are given as follows;

m_x and m_y: the set sizes for X and Y, respectively,

r_x and r_y: the number of cycles for X and Y, respectively,

r and r': the number of censored observation(s) in each cycle for X and Y, respectively.

Here, $X_{(i)c}$ $(i = 1, \dots, m_x;\ c = 1, \dots, r_x)$ and $Y_{(j)l}$ $\left(j = 1, \dots, m_y;\ l = 1, \dots, r_y\right)$ are the RSS data for the random variables $X \sim Weibull(p, \sigma_1)$ and $Y \sim Weibull(p, \sigma_2)$, respectively. We use the following representations for obtaining the type-II right censored data in the context of RSS

$$x_{(i)c} = \begin{cases} x_{(i)c}, & if\ x_{(i)c} \le x_{(m_x-r)c} \\ x_{(m_x-r)c}, & if\ x_{(i)c} > x_{(m_x-r)c} \end{cases} \text{ and } y_{(j)l} = \begin{cases} y_{(j)l}, & if\ y_{(j)l} \le y_{(m_y-r')l} \\ y_{(m_y-r')l}, & if\ y_{(j)l} > y_{(m_y-r')l} \end{cases}.$$

Let $\delta_{(i)c}$ and $\delta_{(j)l}$ be the censoring indicator and taking the values of 1 or 0. They return the value 0 when the censoring occurs, otherwise they return 1.

Then the likelihood function is given by

$$L = \prod_{c=1}^{r_x} \prod_{i=1}^{m_x} \left[f_i\left(x_{(i)c}\right)\right]^{\delta_{(i)c}} \left[1 - F_i\left(x_{(i)c}\right)\right]^{1-\delta_{(i)c}} \prod_{l=1}^{r_y} \prod_{j=1}^{m_y} \left[f_j\left(y_{(j)l}\right)\right]^{\delta_{(j)l}} \left[1 - F_j\left(y_{(j)l}\right)\right]^{1-\delta_{(j)l}} \tag{19}$$

where $f_i(x)$ and $F_i(x)$ are the pdf and the cdf of the ith order statistic, respectively. They are given below

$$f_i(x) = \frac{m_x!}{(i-1)!\,(m_x-i)!} [F(x)]^{i-1} [1 - F(x)]^{m_x - i} f(x), \tag{20}$$

$$F_i(x) = \frac{m_x!}{(i-1)!\,(m_x-i)!} \int_0^{F(x)} u^{i-1} (1-u)^{m_x - i} du. \tag{21}$$

It is clear from the equation (21) that $\mathrm{F_i(x)}$ is incomplete Beta function. It can be shown as below

$$\mathrm{F_i(x)} = \frac{\mathrm{m_x!}}{\mathrm{(i-1)!\,(m_x-i)!}} \int_0^{1-\mathrm{e}^{-\mathrm{x^p}/\sigma_1}} \mathrm{u^{i-1}(1-u)^{m_x-i}du}$$
$$= \frac{1}{B(i, m_x - i + 1)} \int_0^{1-e^{-x^p/\sigma_1}} u^{i-1}(1-u)^{m_x-i}du = I_{1-e^{-x^p/\sigma_1}}(i, m_x - i + 1).$$

$\mathrm{f_j(y)}$ and $\mathrm{F_j(y)}$ in (19) are defined similar to $\mathrm{f_i(x)}$ and $\mathrm{F_i(x)}$, respectively. Therefore, we do not reproduce them for brevity.

The ML estimators of the unknown parameters p, σ_1 and σ_2 are the solutions of the likelihood equations given in below

$$\frac{\partial \ln L}{\partial p} = \frac{\sum_{c=1}^{r_x}\sum_{i=1}^{m_x}\delta_{(i)c} + \sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\delta_{(j)l}}{p} + \sum_{c=1}^{r_x}\sum_{i=1}^{m_x}\delta_{(i)c}\ln x_{(i)c}$$
$$+\frac{1}{\sigma_1}\sum_{c=1}^{r_x}\sum_{i=1}^{m_x}\delta_{(i)c}(i-1)\frac{x_{(i)c}^p \ln x_{(i)c}}{e^{x_{(i)c}^p/\sigma_1}-1} - \frac{1}{\sigma_1}\sum_{c=1}^{r_x}\sum_{i=1}^{m_x}\delta_{(i)c}(m_x-i+1)x_{(i)c}^p \ln x_{(i)c}$$
$$-\frac{1}{p}\sum_{c=1}^{r_x}\sum_{i=1}^{m_x}\left(1-\delta_{(i)c}\right)\frac{\ln x_{(i)c}\, f_i\left(x_{(i)c}\right)}{1-I_{1-e^{-x_{(i)c}{}^p/\sigma_1}}(i, m_x-i+1)} + \sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\delta_{(j)l}\ln y_{(j)l}$$
$$+\frac{1}{\sigma_2}\sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\delta_{(j)l}(j-1)\frac{y_{(j)l}^p \ln y_{(j)l}}{e^{y_{(j)l}^p/\sigma_2}-1} - \frac{1}{\sigma_2}\sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\delta_{(j)l}\left(m_y-j+1\right)y_{(j)l}^p \ln y_{(j)l}$$
$$-\frac{1}{p}\sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\left(1-\delta_{(j)l}\right)\frac{\ln y_{(j)l}\, f_j\left(y_{(j)l}\right)}{1-I_{1-e^{-\frac{y_{(j)l}{}^p}{\sigma_2}}}\left(j, m_y-j+1\right)} = 0, \qquad (22)$$

$$\frac{\partial \ln L}{\partial \sigma_1} = -\frac{\sum_{c=1}^{r_x}\sum_{i=1}^{m_x}\delta_{(i)c}}{\sigma_1} - \frac{1}{\sigma_1^2}\sum_{c=1}^{r_x}\sum_{i=1}^{m_x}\delta_{(i)c}\left[(i-1)\frac{x_{(i)c}^p}{e^{x_{(i)c}^p/\sigma_1}-1} + (m_x-i+1)x_{(i)c}^p\right]$$
$$-\frac{1}{p\sigma_1}\sum_{c=1}^{r_x}\sum_{i=1}^{m_x}\left(1-\delta_{(i)c}\right)\frac{f_i\left(x_{(i)c}\right)}{1-I_{1-e^{-x_{(i)c}{}^p/\sigma_1}}(i, m_x-i+1)} = 0, \qquad (23)$$

$$\frac{\partial \ln L}{\partial \sigma_2} = -\frac{\sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\delta_{(j)l}}{\sigma_2} - \frac{1}{\sigma_2^2}\sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\delta_{(j)l}\left[(j-1)\frac{y_{(j)l}^p}{e^{y_{(j)l}^p/\sigma_2}-1} + \left(m_y-j+1\right)y_{(j)l}^p\right]$$
$$-\frac{1}{p\sigma_2}\sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\left(1-\delta_{(j)l}\right)\frac{f_j\left(y_{(j)l}\right)}{1-I_{1-e^{-y_{(j)l}{}^p/\sigma_2}}\left(j, m_y-j+1\right)} = 0. \qquad (24)$$

Because of the similar reasons mentioned in subsection 2.1, the ML estimators of the parameters cannot be obtained explicitly. For this reason, we resort to iterative methods for solving the likelihood equations. Solutions of these equations are the ML estimators of the parameters p, σ_1 and σ_2. They are represented by $\hat{p}_{ML,RSS}$, $\hat{\sigma}_{1_{ML,RSS}}$ and $\hat{\sigma}_{2_{ML,RSS}}$, respectively. The ML estimator of R is obtained by replacing the parameters σ_1 and σ_2 with the corresponding ML estimators of them in equation (3), similar as in equation (9).

3.2 MML Estimator of *R*

The MML estimators of the parameters of the Weibull distribution are derived by using the relationship between the Weibull and EV distribution as shown in subsection 2.2.

Let $U_{(i)c}$ $(i = 1, \dots, m_x;\ c = 1, \dots, r_x)$ and $V_{(j)l}$ $(j = 1, \dots, m_y;\ l = 1, \dots, r_y)$ denote the ith and the jth order statistics in the cth and lth cycles, respectively. Here, the distribution of the random variables U and V are EV with parameters (μ_1, η) and (μ_2, η), respectively, as mentioned earlier.

To obtain the MML estimators of μ_1, μ_2 and η based on type-II right censored RSS data, the likelihood function can be written as follows

$$L = C\prod_{c=1}^{r_x}\prod_{i=1}^{m_x}\left[\frac{1}{\eta}f(z_{(i)c})F(z_{(i)c})^{i-1}\left(1-F(z_{(i)c})\right)^{m_x-i}\right]^{\delta_{(i)c}}\left[1-F_i(z_{(i)c})\right]^{1-\delta_{(i)c}}$$
$$\prod_{l=1}^{r_y}\prod_{j=1}^{m_y}\left[\frac{1}{\eta}f(w_{(j)l})F(w_{(j)l})^{j-1}\left(1-F(w_{(j)l})\right)^{m_y-j}\right]^{\delta_{(j)l}}\left[1-F_j(w_{(j)l})\right]^{1-\delta_{(j)l}} \quad (25)$$

where, $z_{(i)c} = (u_{(i)c} - \mu_1)/\eta$, $i = 1, \dots, m_x$, $c = 1, \dots, r_x$ and $w_{(j)l} = (v_{(j)l} - \mu_2)/\eta$, $j = 1, \dots, m_y, l = 1, \dots, r_y$ are the standardized ordered statistics. Then, the likelihood equations are obtained by taking derivation of log-likelihood function with respect to unknown parameters. Likelihood equations have no explicit solutions because of the following awkward functions

$$g_1(z) = \frac{f'(z)}{f(z)},\quad g_2(z) = \frac{f(z)}{F(z)},\quad g_3(z) = \frac{f(z)}{1-F(z)},\quad g_4(z) = \frac{f_i(z)}{1-F_i(z)}. \quad (26)$$

Also, $g_1(w)$, $g_2(w)$, $g_3(w)$ and $g_4(w)$ are defined similarly as in (26). To apply the MML estimation procedure, we first linearize the functions in (26) by using the first two terms of Taylor series expansion around $t^u_{(i)c} = E(z_{(i)c})$ and $t^v_{(j)l} = E(w_{(j)l})$. The linearized functions are given below

$$g_1(z_{(i)c}) \cong \alpha^u_{1ic} - \beta^u_{1ic}z_{(i)c},\ g_2(z_{(i)c}) \cong \alpha^u_{2ic} - \beta^u_{2ic}z_{(i)c},$$
$$g_3(z_{(i)c}) \cong \alpha^u_{3ic} + \beta^u_{3ic}z_{(i)c}, g_4(z_{(i)c}) \cong \alpha^u_{4ic} + \beta^u_{4ic}z_{(i)c}, i = 1, \dots, m_x, c = 1, \dots, r_x,$$

where

$$\alpha^u_{1ic} = 1 - e^{t^u_{(i)c}} + t^u_{(i)c}e^{t^u_{(i)c}},\quad \beta^u_{1ic} = e^{t^u_{(i)c}},$$
$$\alpha^u_{2ic} = \frac{f(t^u_{(i)c})}{F(t^u_{(i)c})} + t^u_{(i)c}\beta^u_{2ic},\quad \beta^u_{2ic} = \frac{\left(e^{t^u_{(i)c}}-1\right)f(t^u_{(i)c})F(t^u_{(i)c})+f^2(t^u_{(i)c})}{F^2(t^u_{(i)c})},$$
$$\alpha^u_{3ic} = e^{t^u_{(i)c}} - t^u_{(i)c}e^{t^u_{(i)c}},\quad \beta^u_{3ic} = e^{t^u_{(i)c}},$$
$$\alpha^u_{4ic} = \frac{f_i(t^u_{(i)c})}{1-F_i(t^u_{(i)c})} - t^u_{(i)c}\beta^u_{4ic},\quad \beta^u_{4ic} = \frac{f_i^2(t^u_{(i)c})+f_i'(t^u_{(i)c})\left(1-F_i(t^u_{(i)c})\right)}{\left(1-F_i(t^u_{(i)c})\right)^2},$$

$$t_{(i)}^{u}=\begin{cases}\ln\left(-\ln\left(1-\frac{i}{m_x+1}\right)\right), & \text{if } x_{(i)c}\le x_{(m_x-r)c}\\ \ln\left(-\ln\left(1-\frac{m_x-r}{m_x+1}\right)\right), & \text{if } x_{(i)c}> x_{(m_x-r)c}\end{cases}. \quad (27)$$

$g_1(w_{(j)l})$, $g_2(w_{(j)l})$, $g_3(w_{(j)l})$ and $g_4(w_{(j)l})$ linearized functions, the coefficients α and β, i.e., $(\alpha_{1jl}^{v},\beta_{1jl}^{v})$, $(\alpha_{2jl}^{v},\beta_{2jl}^{v})$, $(\alpha_{3jl}^{v},\beta_{3jl}^{v})$ and $(\alpha_{4jl}^{v},\beta_{4jl}^{v})$ for the sample V are exactly the same as in (27), except that $t_{(i)c}^{u}$ is replaced by $t_{(j)l}^{v}$.

By incorporating linearized functions into likelihood equations, we obtain the modified likelihood equations $\partial\ln L^*/\partial\mu_1=0$, $\partial\ln L^*/\partial\mu_2=0$ and $\partial\ln L^*/\partial\eta=0$. See Appendix for the details of the modified likelihood equations. Algebraic solutions of these equations are the following closed form estimators called as MML

$$\hat{\mu}_{1_{MML,RSS}}=K_1-\frac{\Delta_1}{m_1}\hat{\eta}_{MML,RSS},\quad \hat{\mu}_2=K_2-\frac{\Delta_2}{m_2}\hat{\eta}_{MML,RSS} \text{ and}$$
$$\hat{\eta}_{MML,RSS}=\frac{-B+\sqrt{B^2+4AC}}{2A}, \quad (28)$$

where

$$\gamma_{1ic}^{u}=\delta_{(i)c}(\beta_{1ic}^{u}+(i-1)\beta_{2ic}^{u}+(m_x-i)\beta_{3ic}^{u}),\quad \gamma_{2ic}^{u}=\left(1-\delta_{(i)c}\right)\beta_{4ic}^{u},$$
$$\gamma_{ic}^{u}=\gamma_{1ic}^{u}+\gamma_{2ic}^{u},\quad m_1=\sum_{c=1}^{r_x}\sum_{i=1}^{m_x}\gamma_{ic}^{u},\quad K_1=\frac{\sum_{c=1}^{r_x}\sum_{i=1}^{m_x}\gamma_{ic}^{u}u_{(i)c}}{m_1}$$
$$\Delta_{1ic}^{u}=\delta_{(i)c}(\alpha_{1ic}^{u}+(i-1)\alpha_{2ic}^{u}-(m_x-i)\alpha_{3ic}^{u}),\quad \Delta_{2ic}^{u}=\left(1-\delta_{(i)c}\right)\alpha_{4ic}^{u},$$
$$\Delta_{ic}^{u}=\Delta_{1ic}^{u}-\Delta_{2ic}^{u},\quad \Delta_1=\sum_{c=1}^{r_x}\sum_{i=1}^{m_x}\Delta_{ic}^{u},$$
$$\gamma_{1jl}^{v}=\delta_{(j)l}\left(\beta_{1jl}^{v}+(j-1)\beta_{2jl}^{v}+\left(m_y-j\right)\beta_{3jl}\right),\quad \gamma_{2jl}^{v}=\left(1-\delta_{(j)l}\right)\beta_{4jl}^{v},$$
$$\gamma_{jl}^{v}=\gamma_{1jl}^{v}+\gamma_{2jl}^{v},\quad m_2=\sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\gamma_{jl}^{v},\quad K_2=\frac{\sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\gamma_{jl}^{v}v_{(j)l}}{m_2}$$
$$\Delta_{1jl}^{v}=\delta_{(j)l}\left(\alpha_{1jl}^{v}+(j-1)\alpha_{2jl}^{v}-\left(m_y-j\right)\alpha_{3jl}^{v}\right),\quad \Delta_{2jl}^{v}=\left(1-\delta_{(j)l}\right)\alpha_{4jl}^{v},$$
$$\Delta_{jl}^{v}=\Delta_{1jl}^{v}-\Delta_{2jl}^{v},\quad \Delta_2=\sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\Delta_{jl}^{v},$$
$$A=\sum_{c=1}^{r_x}\sum_{i=1}^{m_x}\delta_{(i)c}+\sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\delta_{(j)l},$$
$$B=\sum_{c=1}^{r_x}\sum_{i=1}^{m_x}\Delta_{ic}^{u}\left(u_{(i)c}-K_1\right)+\sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\Delta_{jl}^{v}\left(v_{(j)l}-K_2\right),$$
$$C=\sum_{c=1}^{r_x}\sum_{i=1}^{m_x}\gamma_{ic}^{u}\left(u_{(i)c}-K_1\right)^2+\sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\gamma_{jl}^{v}\left(v_{(j)l}-K_2\right)^2. \quad (29)$$

Similar to subsection 2.2, we use the inverse transformations defined in equation (11) to obtain the MML estimators of the Weibull parameters σ_1, σ_2 and p denoted by $\hat{\sigma}_{1_{MML,RSS}}$, $\hat{\sigma}_{2_{MML,RSS}}$ and $\hat{p}_{MML,RSS}$, respectively. They are given below

$$\hat{\sigma}_{1_{MML,RSS}}=e^{\hat{p}_{MML,RSS}\hat{\mu}_{1_{MML,RSS}}},\quad \hat{\sigma}_{2_{MML,RSS}}=e^{\hat{p}_{MML,RSS}\hat{\mu}_{2_{MML,RSS}}} \quad \text{and}$$
$$\hat{p}_{MML,RSS}=\frac{1}{\hat{\eta}_{MML,RSS}}. \quad (30)$$

Then, the MML estimators of R based on type-II right censored RSS data is obtained as in equation (18).

4. Simulation Study

In this section, we perform Monte-Carlo simulation study to compare the performances of the proposed estimators of the system reliability R based on SRS and RSS data. In the comparisons, we use the bias and the mean square error (MSE) criteria defined below

$$\mathrm{Bias}(\hat{\mathrm{R}}) = \mathrm{E}(\hat{\mathrm{R}} - \mathrm{R}) \quad \text{and} \quad \mathrm{MSE}(\hat{\mathrm{R}}) = \mathrm{E}(\hat{\mathrm{R}} - \mathrm{R})^2 \tag{31}$$

respectively.

We just reproduce the bias and the relative efficiency (RE) values of the estimators for the sake of brevity, see Table 1. RE of the estimator $\hat{R}$ with respect to the estimator $\hat{R}^*$ is defined as shown below

$$\mathrm{RE} = \frac{\mathrm{MSE}(\hat{\mathrm{R}}^*)}{\mathrm{MSE}(\hat{\mathrm{R}})}. \tag{32}$$

It is known that $\hat{R}$ is more efficient than $\hat{R}^*$ if $RE > 1$ and vice versa. In this study, we calculate the values of the following REs

$$\mathrm{RE}_1 = \frac{\mathrm{MSE}(\hat{\mathrm{R}}_{\mathrm{ML,SRS}})}{\mathrm{MSE}(\hat{\mathrm{R}}_{\mathrm{ML,RSS}})}, \quad \mathrm{RE}_2 = \frac{\mathrm{MSE}(\hat{\mathrm{R}}_{\mathrm{MML,SRS}})}{\mathrm{MSE}(\hat{\mathrm{R}}_{\mathrm{MML,RSS}})}, \quad \mathrm{RE}_3 = \frac{\mathrm{MSE}(\hat{\mathrm{R}}_{\mathrm{ML,SRS}})}{\mathrm{MSE}(\hat{\mathrm{R}}_{\mathrm{MML,SRS}})},$$
$$\mathrm{RE}_4 = \frac{\mathrm{MSE}(\hat{\mathrm{R}}_{\mathrm{ML,RSS}})}{\mathrm{MSE}(\hat{\mathrm{R}}_{\mathrm{MML,RSS}})} \quad \text{and} \quad \mathrm{RE}_5 = \frac{\mathrm{MSE}(\hat{\mathrm{R}}_{\mathrm{ML,SRS}})}{\mathrm{MSE}(\hat{\mathrm{R}}_{\mathrm{MML,RSS}})}. \tag{33}$$

All the simulations are performed in Matlab R2013a. In simulation setup, the set sizes and the number of cycles are taken to be as $m_x = m_y = m =$6, 8, 10 and $r_x = r_y = r =$1, 5, respectively. It is obvious that the sample sizes for the stress X and the strength Y become $n = m_x r_x$ and $m = m_y r_y$ in the context of RSS. It should also be realized that the sample sizes are taken to be n and m in SRS throughout the simulation study. We use different values of the shape parameter p such as 0.5, 1.5 and 2.5. We also use the following parameter settings for the scale parameters σ_1 and σ_2; (σ_1, σ_2) =(1,1), (1,2), (1,3). Therefore, the true values of R equal to 0.5, 0.67 and 0.75, respectively.

Under the assumption of type-II right censored SRS data, the largest $[\![\mathrm{qn} + 0.5]\!]$ and $[\![\mathrm{qm} + 0.5]\!]$ observations are censored for both the samples corresponding to the stress and the strength (i.e., X and Y), respectively. Similarly, in the presence of type-II right censored RSS data, the largest observations $[\![\mathrm{qm_x} + 0.5]\!]$ and $[\![\mathrm{qm_y} + 0.5]\!]$ are censored in each of the cycles corresponding to the samples X and Y, respectively. Here, q is the proportion of censoring and it is taken as 10%, 20% and 30%. Also, $[\![.]\!]$ represents the greatest integer value. Simulations are done based on $[\![100{,}000/\min(\mathrm{n},\mathrm{m})]\!]$ Monte-Carlo runs. Type-II right censored samples are generated from $\mathrm{Weibull}(\mathrm{p}, \sigma_1)$ and $\mathrm{Weibull}(\mathrm{p}, \sigma_2)$ for the stress X and the strength Y, respectively. Simulation results are reported in Tables 1-3.

<u>Comparisons with respect to the bias:</u> It is observed from Tables 1-3 that all the estimators have negligible biases regardless of the values of the shape parameter p. However, the amount of the bias increases when the scale parameters σ_1 and σ_2 are not equal.

Table 1. Biases of the ML and the MML estimators of R based on SRS and RSS and the RE values when p =0.5.

			Bias				*Relative Efficiency*				
r	m	q	$\hat{R}_{ML.SRS}$	$\hat{R}_{MML.SRS}$	$\hat{R}_{ML.RSS}$	$\hat{R}_{MML.RSS}$	RE_1	RE_2	RE_3	RE_4	RE_5
			$\sigma_1 = 1, \sigma_2 = 1;\ R = 0.5$								
1	6	10	0.0001	0.0012	-0.0006	-0.0003	3.49	3.40	1.06	1.03	3.63
		20	-0.0025	-0.0031	-0.0012	-0.0024	3.18	3.16	1.06	1.05	3.36
		30	-0.0029	-0.0010	0.0013	0.0031	2.89	2.85	1.07	1.06	3.07
	8	10	-0.0004	-0.0003	0.0012	0.0012	4.18	3.92	1.05	0.98	4.14
		20	-0.0036	-0.0029	0.0008	0.0012	4.07	3.81	1.06	0.99	4.06
		30	-0.0085	-0.0078	-0.0016	-0.0012	3.91	3.60	1.06	0.97	3.82
	10	10	-0.0022	-0.0021	0.0004	0.0003	5.27	4.89	1.04	0.97	5.11
		20	-0.0007	-0.0008	0.0003	0.0003	5.06	4.72	1.05	0.98	4.97
		30	-0.0006	-0.0007	0.0005	0.0004	4.89	4.38	1.05	0.94	4.63
5	6	10	0.0018	0.0018	0.0002	0.0003	2.88	2.86	1.01	1.01	2.91
		20	0.0012	0.0011	0.0005	0.0007	3.23	3.17	1.01	0.99	3.22
		30	-0.0001	-0.0001	0.0012	0.0008	2.87	2.92	1.01	1.03	2.97
	8	10	0.0021	0.0021	-0.0002	-0.0003	3.85	3.78	1.01	0.99	3.83
		20	0.0017	0.0017	-0.0010	-0.0009	3.67	3.60	1.01	0.99	3.65
		30	0.0012	0.0011	0.0003	0.0002	4.13	4.03	1.01	0.99	4.09
	10	10	-0.0005	-0.0005	-0.0006	-0.0006	4.86	4.65	1.00	0.96	4.68
		20	0.0001	0.0001	0.0000	-0.0001	5.29	5.16	1.00	0.98	5.21
		30	-0.0004	-0.0003	-0.0003	-0.0003	4.99	4.68	1.01	0.94	4.73
			$\sigma_1 = 1, \sigma_2 = 2;\ R = 0.67$								
1	6	10	0.0116	0.0045	0.0141	0.0082	2.83	2.70	1.05	1.00	2.83
		20	0.0206	0.0137	0.0178	0.0142	2.51	2.34	1.06	0.98	2.48
		30	0.0227	0.0157	0.0049	-0.0010	2.48	2.55	1.06	1.10	2.73
	8	10	0.0105	0.0058	0.0086	0.0065	3.80	3.54	1.04	0.97	3.69
		20	0.0176	0.012	0.0132	0.0096	3.74	3.47	1.05	0.98	3.67
		30	0.0163	0.0115	0.0163	0.0129	3.60	3.33	1.05	0.97	3.50
	10	10	0.0067	0.0032	0.0059	0.0054	4.74	4.33	1.03	0.94	4.47
		20	0.0102	0.0060	0.0094	0.0085	4.59	4.15	1.04	0.94	4.33
		30	0.0115	0.0068	0.0098	0.0099	4.49	3.95	1.05	0.92	4.16
5	6	10	0.0024	0.0012	0.0024	-0.0041	2.78	2.62	1.01	0.95	2.64
		20	0.0057	0.0042	0.0030	-0.0034	3.14	2.88	1.01	0.93	2.92
		30	0.0045	0.0029	0.0042	-0.0061	2.64	2.48	1.01	0.95	2.52
	8	10	0.0030	0.0020	0.0026	-0.0007	3.77	3.47	1.00	0.92	3.50
		20	0.0025	0.0014	0.0026	-0.0021	3.49	3.09	1.01	0.89	3.13
		30	0.0035	0.0023	0.0020	-0.0026	3.88	3.62	1.01	0.94	3.66
	10	10	0.0011	0.0003	0.0019	0.0007	4.49	4.17	1.00	0.93	4.20
		20	0.0045	0.0037	0.0013	-0.0003	4.44	4.00	1.01	0.91	4.05
		30	0.0039	0.0030	0.0025	0.0010	4.10	3.66	1.01	0.90	3.70

Table 1. (continued).

			Bias				*Relative Efficiency*				
r	m	q	$\hat{R}_{ML.SRS}$	$\hat{R}_{MML.SRS}$	$\hat{R}_{ML.RSS}$	$\hat{R}_{MML.RSS}$	*RE₁*	*RE₂*	*RE₃*	*RE₄*	*RE₅*
					$\sigma_1 = 1, \sigma_2 = 3;\ R = 0.75$						
1	6	10	0.0242	0.0170	0.0145	0.0077	2.86	2.57	1.03	0.93	2.67
		20	0.0182	0.0112	0.0214	0.0140	2.73	2.66	1.04	1.01	2.77
		30	0.0210	0.0121	0.0154	0.0089	2.34	2.21	1.05	0.99	2.32
	8	10	0.0122	0.0063	0.0101	0.0069	3.78	3.36	1.02	0.91	3.46
		20	0.0194	0.0119	0.0186	0.0142	3.37	3.01	1.04	0.93	3.15
		30	0.0216	0.0141	0.0177	0.0139	3.09	2.84	1.04	0.95	2.96
	10	10	0.0096	0.0047	0.0070	0.0062	4.54	4.06	1.02	0.91	4.14
		20	0.0127	0.0071	0.0101	0.0088	4.00	3.61	1.02	0.92	3.70
		30	0.0172	0.0110	0.0135	0.0132	3.85	3.37	1.03	0.90	3.48
5	6	10	0.0027	0.0011	0.0039	-0.0057	2.61	2.22	1.00	0.85	2.24
		20	0.0070	0.0051	0.0044	-0.0054	2.85	2.48	1.00	0.87	2.50
		30	0.0076	0.0054	0.0057	-0.008	2.60	2.29	1.01	0.89	2.32
	8	10	0.0031	0.0019	0.0031	-0.0019	3.16	2.72	1.00	0.86	2.73
		20	0.0021	0.0007	0.0038	-0.0024	3.22	2.78	1.00	0.86	2.80
		30	0.0041	0.0025	0.0030	-0.0037	3.64	3.17	1.01	0.87	3.20
	10	10	0.0024	0.0014	0.0006	-0.0007	4.07	3.55	1.00	0.87	3.57
		20	0.0012	0.0001	0.0009	-0.0018	3.89	3.40	1.00	0.87	3.41
		30	0.0016	0.0004	0.0035	0.0016	3.87	3.29	1.00	0.85	3.31

Comparisons with respect to the sampling methods: In columns corresponding to RE_1 and RE_2, we compare the performances of the ML and the MML estimators of R based on SRS to the corresponding estimators of R based on RSS. It is clear from the values of RE_1 and RE_2 that estimators based on RSS are much more efficient than the estimators based on SRS. In other words, RE_1 and RE_2 values are much greater than 1 in all cases. It can also be seen that the efficiencies of the ML and the MML estimators of R based on RSS increase as the set sizes m_x and m_y increase. However, the efficiencies of the corresponding estimators of R are more or the less the same for different values of the cycles (r_x, r_y) and the shape parameter p.

Comparisons with respect to the estimation methods: Columns corresponding to RE_3 and RE_4 are used to compare the efficiencies of the ML and the MML estimators under SRS and RSS, respectively. In case of SRS, it is clear from the column corresponding to RE_3 that the efficiencies of the ML and the MML estimators of R are almost equal for all values of the sample sizes (n, m) and the shape parameter p. In case of RSS, when the number of cycles are equal to 1, i.e. $r_x = r_y = 1$, and the scale parameters are equal each other, i.e. $\sigma_1 = \sigma_2 = 1$, the *MSE*s of the ML and the MML estimators of R are very close to each other. However, when the number of cycles increase, i.e. $r_x = r_y = 5$, and the scale parameters are differ each other, i.e. $\sigma_1 = 1, \sigma_2 = 2$ and $\sigma_1 = 1, \sigma_2 = 3$, the ML estimator of R is more efficient than the corresponding MML estimator, see the column corresponding to RE_4. These situations do not show difference with respect to the values of the different shape parameters.

Table 2. Biases of the ML and the MML estimators of R based on SRS and RSS and the RE values when p =1.5.

			Bias				*Relative Efficiency*				
r	m	q	$\hat{R}_{ML.SRS}$	$\hat{R}_{MML.SRS}$	$\hat{R}_{ML.RSS}$	$\hat{R}_{MML.RSS}$	RE_1	RE_2	RE_3	RE_4	RE_5
				$\sigma_1 = 1, \sigma_2 = 1;\ \ R = 0.5$							
		10	0.0003	0.0010	-0.0017	-0.0013	3.47	3.34	1.06	1.02	3.55
	6	20	0.0037	0.0028	-0.0000	0.0002	3.10	2.98	1.07	1.03	3.20
		30	-0.0018	-0.0008	0.0006	-0.0032	2.87	2.74	1.08	1.04	2.99
		10	0.0012	0.0013	-0.0001	0.0001	4.30	4.03	1.05	0.98	4.24
1	8	20	0.0015	0.0014	-0.0004	-0.0005	4.10	3.82	1.06	0.99	4.06
		30	-0.0014	-0.0013	0.0004	0.0006	4.07	3.84	1.06	1.00	4.09
		10	0.0003	0.0004	0.0002	0.0003	5.30	4.95	1.04	0.97	5.16
	10	20	-0.0006	-0.0004	0.0005	0.0005	5.20	4.82	1.05	0.97	5.08
		30	-0.0007	-0.0008	-0.0011	-0.0010	4.82	4.36	1.05	0.95	4.61
		10	0.0001	0.0001	-0.0003	-0.0001	2.88	2.84	1.01	1.00	2.88
	6	20	0.0014	0.0014	-0.0004	-0.0007	3.18	3.14	1.01	1.00	3.19
		30	0.0006	0.0007	0.0003	0.0001	2.80	2.88	1.01	1.05	2.94
		10	0.0009	0.0009	0.0000	0.0000	4.30	4.16	1.01	0.97	4.21
5	8	20	0.0017	0.0016	-0.0003	-0.0003	3.62	3.51	1.01	0.98	3.55
		30	0.0010	0.0010	0.0001	0.0002	4.29	4.22	1.01	0.99	4.27
		10	0.0004	0.0004	0.0001	0.0002	5.07	4.89	1.00	0.97	4.93
	10	20	0.0038	0.0038	0.0016	0.0014	4.61	4.35	1.01	0.95	4.40
		30	0.0016	0.0016	-0.0001	0.0001	4.83	4.52	1.01	0.94	4.57
				$\sigma_1 = 1, \sigma_2 = 2;\ \ R = 0.67$							
		10	-0.0022	-0.0081	0.0143	0.0079	3.30	3.11	1.04	0.99	3.27
	6	20	0.0210	0.0158	0.0151	0.0051	3.27	3.16	1.07	1.03	3.39
		30	0.0160	0.0094	0.0304	0.0227	2.65	2.50	1.06	1.00	2.66
		10	0.0109	0.0062	0.0095	0.0076	3.92	3.57	1.04	0.95	3.73
1	8	20	0.0128	0.0075	0.0119	0.0087	3.74	3.46	1.05	0.97	3.64
		30	0.0109	0.0059	0.0152	0.0128	3.74	3.38	1.05	0.95	3.56
		10	0.0081	0.0048	0.0060	0.0057	4.80	4.36	1.03	0.94	4.52
	10	20	0.0110	0.0069	0.0081	0.0072	4.51	4.08	1.04	0.94	4.25
		30	0.0131	0.0086	0.0123	0.0124	4.40	3.90	1.04	0.92	4.08
		10	0.0031	0.0020	0.0025	-0.0044	2.89	2.69	1.00	0.94	2.71
	6	20	0.0031	0.0017	0.0022	-0.0046	3.05	2.80	1.01	0.93	2.84
		30	0.0052	0.0037	0.0042	-0.0066	2.60	2.44	1.01	0.95	2.48
		10	0.0019	0.0011	0.0012	-0.0024	3.56	3.29	1.00	0.93	3.32
5	8	20	0.0020	0.0010	0.0011	-0.0035	3.52	3.20	1.00	0.91	3.23
		30	0.0036	0.0023	0.0021	-0.0026	3.97	3.77	1.01	0.95	3.82
		10	0.0014	0.0007	0.0007	-0.0005	4.79	4.48	1.00	0.93	4.51
	10	20	0.0040	0.0031	0.0012	-0.0007	4.31	3.96	1.00	0.92	3.98
		30	0.0034	0.0024	0.0026	0.0013	4.10	3.64	1.01	0.89	3.68

Table 2. (continued).

			Bias				*Relative Efficiency*				
r	m	q	$\hat{R}_{ML.SRS}$	$\hat{R}_{MML.SRS}$	$\hat{R}_{ML.RSS}$	$\hat{R}_{MML.RSS}$	RE_1	RE_2	RE_3	RE_4	RE_5
$\sigma_1 = 1, \sigma_2 = 3;\ R = 0.75$											
		10	0.0013	-0.0055	0.0270	0.0198	3.40	3.19	1.02	0.96	3.28
	6	20	0.0196	0.0109	0.0213	0.0156	3.09	2.95	1.03	0.99	3.07
		30	0.0326	0.0234	0.0271	0.0152	2.84	2.46	1.05	0.90	2.59
		10	0.0159	0.0099	0.0128	0.0098	3.45	3.07	1.02	0.91	3.16
1	8	20	0.0154	0.0084	0.0124	0.0084	3.42	3.10	1.04	0.94	3.23
		30	0.0188	0.0120	0.0161	0.0129	3.22	2.94	1.04	0.95	3.07
		10	0.0115	0.0067	0.0075	0.0067	4.32	3.81	1.02	0.90	3.90
	10	20	0.0129	0.0073	0.0099	0.0085	4.01	3.55	1.03	0.91	3.66
		30	0.0156	0.0095	0.0136	0.0133	3.85	3.38	1.03	0.90	3.50
		10	0.0046	0.0030	0.0037	-0.0058	2.58	2.17	1.00	0.84	2.19
	6	20	0.0037	0.0018	0.0036	-0.0060	3.01	2.53	1.00	0.84	2.55
		30	0.0055	0.0034	0.0063	-0.0084	2.44	2.09	1.01	0.86	2.11
		10	0.0029	0.0017	0.0023	-0.0020	3.20	2.73	1.00	0.86	2.75
5	8	20	0.0030	0.0016	0.0036	-0.0032	3.16	2.76	1.00	0.88	2.78
		30	0.0056	0.0040	0.0039	-0.0026	3.66	3.18	1.00	0.87	3.21
		10	0.0035	0.0025	0.0014	-0.0007	4.12	3.59	1.00	0.87	3.61
	10	20	0.0031	0.0020	0.0016	-0.0014	4.10	3.49	1.00	0.85	3.51
		30	0.0033	0.0021	0.0029	0.0005	3.95	3.22	1.00	0.82	3.25

In column corresponding to RE_5, the performances of the MML estimator of R based RSS, i.e. $\hat{R}_{MML,RSS}$, and the ML estimator of R based on SRS, i.e. $\hat{R}_{ML,SRS}$, are compared. The reason of why we make this comparison is that to determine the more efficient estimator. It is clear that $\hat{R}_{MML,RSS}$ provides explicit solution for the system reliability R. However, $\hat{R}_{ML,SRS}$ is obtained iteratively. It can be seen that $\hat{R}_{MML,RSS}$ is significantly more efficient than the $\hat{R}_{ML,SRS}$ in all cases.

Comparisons with respect to proportion of censoring: From the simulation study, we show that when the proportion of censoring q increases the estimators of R lose their efficiencies for $r_x = r_y = 1$ and in all set sizes as expected. Also, it should be noted that the MSEs of the estimators increase as the proportion of censoring q increases when the number of cycles change but the set sizes stay the same. However, we don't reproduce the values of MSEs in Tables 1-3 for the sake of brevity.

Table 3. Table 2. Biases of the ML and the MML estimators of R based on SRS and RSS and the RE values when p =2.5.

			Bias				*Relative Efficiency*				
r	m	q	$\hat{R}_{ML.SRS}$	$\hat{R}_{MML.SRS}$	$\hat{R}_{ML.RSS}$	$\hat{R}_{MML.RSS}$	RE_1	RE_2	RE_3	RE_4	RE_5
					$\sigma_1 = 1, \sigma_2 = 1;\ R = 0.5$						
1	6	10	-0.0084	-0.0085	-0.0002	-0.0004	3.32	3.18	1.07	1.03	3.44
		20	-0.0255	-0.0235	-0.0046	-0.0031	3.23	3.13	1.08	1.05	3.40
		30	0.0060	0.0056	0.0026	0.0020	3.08	2.96	1.08	1.04	3.21
	8	10	-0.0007	-0.0010	-0.0001	-0.0001	4.24	3.99	1.05	0.99	4.21
		20	-0.0003	-0.0001	0.0009	0.0011	4.15	3.94	1.06	1.00	4.18
		30	0.0026	0.0023	0.0006	0.0007	3.95	3.71	1.06	1.00	3.96
	10	10	0.0019	0.0020	-0.0003	-0.0005	5.22	4.91	1.04	0.98	5.12
		20	0.0003	0.0002	0.0000	-0.0001	5.11	4.74	1.05	0.97	4.98
		30	-0.0017	-0.0016	0.0005	0.0005	4.79	4.31	1.05	0.95	4.57
5	6	10	0.0017	0.0017	0.0006	0.0007	2.91	2.85	1.01	0.99	2.89
		20	-0.0007	-0.0007	-0.0015	-0.0012	3.46	3.42	1.01	1.00	3.48
		30	-0.0013	-0.0013	-0.0003	-0.0001	2.79	2.85	1.02	1.04	2.90
	8	10	-0.0017	-0.0018	-0.0002	-0.0003	4.08	3.96	1.01	0.97	4.00
		20	0.0018	0.0018	-0.0011	-0.0010	3.53	3.46	1.01	0.99	3.51
		30	0.0001	0.0002	-0.0002	0.0001	4.41	4.36	1.01	1.00	4.43
	10	10	-0.0013	-0.0014	-0.0004	-0.0003	4.87	4.69	1.01	0.97	4.75
		20	-0.0013	-0.0013	-0.0000	0.0000	4.98	4.83	1.01	0.97	4.88
		30	-0.0020	-0.0020	0.0001	-0.0001	4.40	4.05	1.01	0.93	4.10
					$\sigma_1 = 1, \sigma_2 = 2;\ R = 0.67$						
1	6	10	0.0077	0.0021	0.0172	0.0131	3.09	2.89	1.05	0.98	3.05
		20	0.0113	0.0055	0.0154	0.0105	3.02	2.80	1.06	0.98	2.98
		30	0.0091	0.0026	0.0334	0.0244	2.91	2.85	1.07	1.05	3.06
	8	10	0.0103	0.0058	0.0096	0.0073	3.96	3.67	1.04	0.96	3.83
		20	0.0115	0.0064	0.0125	0.0096	3.88	3.55	1.05	0.96	3.75
		30	0.0165	0.0111	0.0145	0.0110	3.77	3.40	1.05	0.95	3.58
	10	10	0.0096	0.0061	0.0063	0.0056	4.69	4.27	1.03	0.94	4.42
		20	0.0129	0.0085	0.0076	0.0067	4.61	4.16	1.04	0.94	4.34
		30	0.0119	0.0072	0.0114	0.0117	4.48	3.95	1.04	0.92	4.14
5	6	10	0.0021	0.0010	0.0034	-0.0037	2.68	2.52	1.01	0.95	2.55
		20	0.0039	0.0024	0.0026	-0.0042	3.08	2.82	1.01	0.92	2.86
		30	0.0040	0.0024	0.0054	-0.0051	2.80	2.68	1.01	0.97	2.72
	8	10	0.0037	0.0028	0.0024	-0.0009	4.01	3.67	1.01	0.92	3.71
		20	0.0024	0.0014	0.0027	-0.0017	3.75	3.50	1.00	0.94	3.54
		30	0.0050	0.0039	0.0023	-0.0025	4.18	3.82	1.01	0.92	3.87
	10	10	0.0043	0.0035	0.0012	0.0002	4.14	3.85	1.00	0.93	3.88
		20	0.0007	-0.0000	0.0026	0.0005	4.82	4.41	1.00	0.92	4.44
		30	0.0022	0.0012	0.0023	0.0006	4.45	3.89	1.00	0.88	3.93

Table 3. (continued).

			Bias				*Relative Efficiency*				
r	m	q	$\hat{R}_{ML.SRS}$	$\hat{R}_{MML.SRS}$	$\hat{R}_{ML.RSS}$	$\hat{R}_{MML.RSS}$	*RE₁*	*RE₂*	*RE₃*	*RE₄*	*RE₅*
			$\sigma_1 = 1, \sigma_2 = 3;\ R = 0.75$								
1	6	10	0.0133	0.0069	0.0213	0.0139	2.90	2.72	1.05	0.98	2.87
		20	0.0168	0.0097	0.0160	0.0075	2.85	2.73	1.02	0.98	2.80
		30	0.0043	-0.0037	0.0397	0.0248	2.54	2.53	1.04	1.04	2.65
	8	10	0.0125	0.0066	0.0112	0.0078	3.60	3.26	1.02	0.93	3.35
		20	0.0134	0.0067	0.0161	0.0110	3.54	3.22	1.03	0.93	3.33
		30	0.0106	0.0038	0.0149	0.0100	3.29	3.07	1.04	0.97	3.20
	10	10	0.0099	0.0053	0.0078	0.0071	4.43	3.93	1.02	0.90	4.01
		20	0.0118	0.0063	0.0106	0.0093	4.06	3.65	1.02	0.92	3.74
		30	0.0153	0.0092	0.0136	0.0134	3.93	3.41	1.03	0.89	3.51
5	6	10	0.0025	0.0010	0.0030	-0.0062	2.66	2.26	1.00	0.85	2.27
		20	0.0053	0.0035	0.0035	-0.0063	2.84	2.45	1.01	0.87	2.48
		30	0.0064	0.0044	0.0053	-0.0093	2.39	2.00	1.01	0.84	2.03
	8	10	0.0023	0.0012	0.0020	-0.0028	3.39	2.91	1.00	0.86	2.92
		20	0.0032	0.0018	0.0046	-0.0022	3.03	2.60	1.00	0.86	2.62
		30	0.0045	0.0030	0.0031	-0.0035	3.60	3.12	1.01	0.87	3.15
	10	10	0.0012	0.0003	0.0014	-0.0006	4.42	3.77	1.00	0.85	3.79
		20	0.0022	0.0011	0.0025	-0.0004	3.81	3.39	1.00	0.89	3.41
		30	0.0059	0.0047	0.0033	0.0013	3.88	3.22	1.00	0.83	3.25

5. Real Data Applications

In this section, we analyze two different data sets. First one is the strength data taken from the literature and the other one is the wind speed data obtained from the Turkish State Meteorological Service. The first data set is widely used in the engineering literature in the context of reliability studies and the second data set is very popular among the people working in the area of renewable energy.

5.1 Strength data

Here, we reanalyze the widely used strength data taken from the literature (Badar and Priest, 1982 and Ghitany et al., 2015) by using the methodologies presented in this study. Strength data is about the strength measured in GPA for single carbon fibers, and impregnated 1000 carbon fiber tows.

Single fibers were tested under tension at gauge lengths of 20 mm (Data Set 1) and 50 mm (Data Set 2) for the sample sizes 69 and 65, respectively.

To identify the distribution of the strength data, we use the Q-Q plot technique. Q-Q plots indicate that the Weibull distribution beautifully models the both data sets, see Figure 1.

We first consider the strength data (for both 20 mm and 50 mm) mentioned above as populations of interest. Then, we randomly draw samples from these populations via the SRS and the RSS techniques.

SRS Technique: We randomly select 21 observations from each of the data sets corresponding to the gauge lengths of 20 mm (X) and 50 mm (Y).

RSS Technique: To estimate the system reliability R, we select RSS sample with the set sizes $m_x = m_y$ =3 and the number of cycles $r_x = r_y$ =7. In this technique, we need $m_x^2 r_x = m_y^2 r_y$ =63 observations to obtain samples. However, we only use $m_x r_x = m_y r_y$ =21 of them for both the data sets corresponding to the gauge lengths of 20 mm (X) and 50 mm (Y).

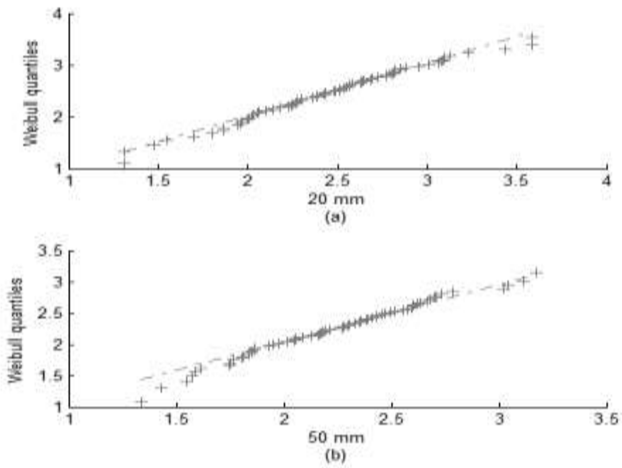

Figure 1. Weibull Q-Q plots of gauge lengths of 20 mm (a) and 50 mm (b) data

Therefore, the sample sizes for the SRS and the RSS techniques become 21 for each of the data sets, i.e. $n = m$ =21. They are given in Table 4 and 5, respectively.

Table 4. Gauge lengths of 20 mm and 50 mm based on SRS, $n = m$ =21

Data set 1 (Gauge lengths of 20 mm)			Data set 2 (Gauge lengths of 50 mm)		
1.479	1.552	1.803	1.339	1.549	1.589
1.966	1.997	2.006	1.613	1.746	1.807
2.098	2.24	2.27	1.852	2.019	2.055
2.272	2.426	2.566	2.058	2.162	2.171
2.642	2.773	2.818	2.335	2.386	2.471
2.821	2.88	2.954*	2.558	2.633*	2.67*
3.012*	3.067*	3.233*	2.699*	2.785*	3.116*

*: Censored observations

In the context of censoring, we assume that the sampling units which are greater than 2.90 and 2.61 are censored for X and Y samples, respectively. It is clear from Table 4 and 5 that censored observations are equal to 19% of the samples represented by X and Y for both the SRS and the RSS samples.

Table 5. Gauge lengths of 20 mm and 50 mm based on RSS, $m_x = m_y$ =3 and $r_x = r_y$ =7

Data set 1 (Gauge lengths of 20 mm)				Data set 2 (Gauge lengths of 50 mm)			
	Set				Set		
Cycle	1	2	3	Cycle	1	2	3
1	1.865	2.642	3.433*	1	1.807	2.051	2.299
2	1.7	2.848	2.301	2	1.549	1.852	2.62*
3	2.098	2.478	3.128*	3	2.055	2.431	3.174*
4	2.27	2.684	3.233*	4	1.974	3.02*	2.67*
5	1.479	2.809	2.554	5	2.171	2.601	2.514
6	1.314	2.24	3.585*	6	2.577	2.272	2.125
7	2.027	2.773	2.586	7	1.812	1.764	2.604

*: Censored observations

We then compute the ML and the MML estimates of the system reliability R based on SRS and RSS techniques. By using bootstrap method, we compare the efficiencies of ML and MML estimators of *R* based on SRS and RSS. In view of SRS, we use the methodology proposed by Efron (1982). Moreover, in the context of RSS, we use the bootstrap RSS by rows method originated by Modarres et al. (2006) for the bootstrap standard error (BSE) and bootstrap confidence interval (BCI) of R. Here, let B be the number of bootstrap replications, R^* be the bootstrap estimates of R and $\overline{\mathrm{R}}^* = (1/\mathrm{B})\sum_{\mathrm{i}=1}^{\mathrm{B}} \mathrm{R}^*$, then the BSE is calculated as shown below

$$\mathrm{BSE} = \left\{\frac{1}{\mathrm{B}-1}\sum_{\mathrm{i}=1}^{\mathrm{B}}\left(\widehat{\mathrm{R}}^* - \overline{\mathrm{R}}^*\right)^2\right\}^{1/2}. \quad (34)$$

After ranking $\widehat{\mathrm{R}}_1^*, \ldots, \widehat{\mathrm{R}}_{\mathrm{B}}^*$ from the smallest to the largest, i.e., $\left(\widehat{\mathrm{R}}_{(1)}^*, \ldots, \widehat{\mathrm{R}}_{(\mathrm{B})}^*\right)$, we construct approximate $100(1-\alpha)\%$ BCI of R as given

$$\left(\widehat{\mathrm{R}}_{((\alpha/2)\mathrm{B})}^*, \widehat{\mathrm{R}}_{((1-\alpha/2)\mathrm{B})}^*\right). \quad (35)$$

The ML and the MML estimates of *R* with the BSEs and the corresponding 95% BCIs of *R* based on SRS and RSS are given in Table 6.

Table 6. The ML and the MML estimates of *R* for right censored strength data

$\hat{R}_{ML,SRS}$	$\hat{R}_{MML,SRS}$	$\hat{R}_{ML,RSS}$	$\hat{R}_{MML,RSS}$
0.3738	0.3739	0.3849	0.3839
(0.0872)*	(0.0747)*	(0.0532)*	(0.0587)*
(0.2322,0.5659)**	(0.2457,0.5377)**	(0.2873,0.4966)**	(0.2795,0.5092)**

*: BSE
**: BCI

It is clear that the BSE values based on RSS are smaller than the BSE values based on SRS. Also, the length of the BCIs based on RSS are shorter than the corresponding BCIs based on SRS. Therefore, the estimates based on RSS are more reliable than the estimates based on SRS as expected from the simulation results given earlier. The estimate of $P(X < Y)$, based on right censored data, is approximately .38, i.e., $P(X > Y)$ is .62. It indicates that the single carbon fibres with length 20 mm are stronger than the single carbon fibres with length 50 mm.

5.2 Wind speed data

In this real life application, we use hourly wind speed data (m/s) obtained from Bursa and Eskisehir, Turkey during the spring of 2009 to make an implementation of proposed methods. To do this, 1933 observations were taken for each of the wind speed data (i.e., Bursa and Eskisehir) at the heights of 10m. These data sets were obtained from the Turkish State Meteorological Service, see also Arslan et al. (2017).

Basically, the stress-strength model bases on the idea of the probability of X less than Y. Here, our aim is to estimate the probability of the wind speed of Bursa (X) is less than the wind speed of Eskisehir (Y), in other words, $P(X < Y)$. Before starting to analyze the data set, we first ensure that Weibull distribution provides good fit for the wind speed data obtained from Bursa and Eskisehir (i.e., X and Y). Under the assumption of equal shape parameter, p is estimated to be 1.6771. Based on this value of p, the scale parameters corresponding to the data sets X and Y are obtained to be $\sigma_1 = 4.3463$ and $\sigma_2 = 7.3858$, respectively.

Similar to first application, we treat the wind speed data of Bursa and Eskisehir as the populations of interests. Then, we select SRS and RSS samples from these data sets. In the context of RSS, we select samples with the set sizes $m_x = m_y$ =8 and the number of cycles $r_x = r_y$ =5 then we obtain the sample sizes as $n = m$ =40. Sample sizes of SRS samples are also $n = m$ =40. Then, we draw the Q-Q plots of the wind speed data for both Bursa (X) and Eskisehir (Y) and also we draw the plot of the empirical cdf against the fitted cdf, see Figure 2 and 3.

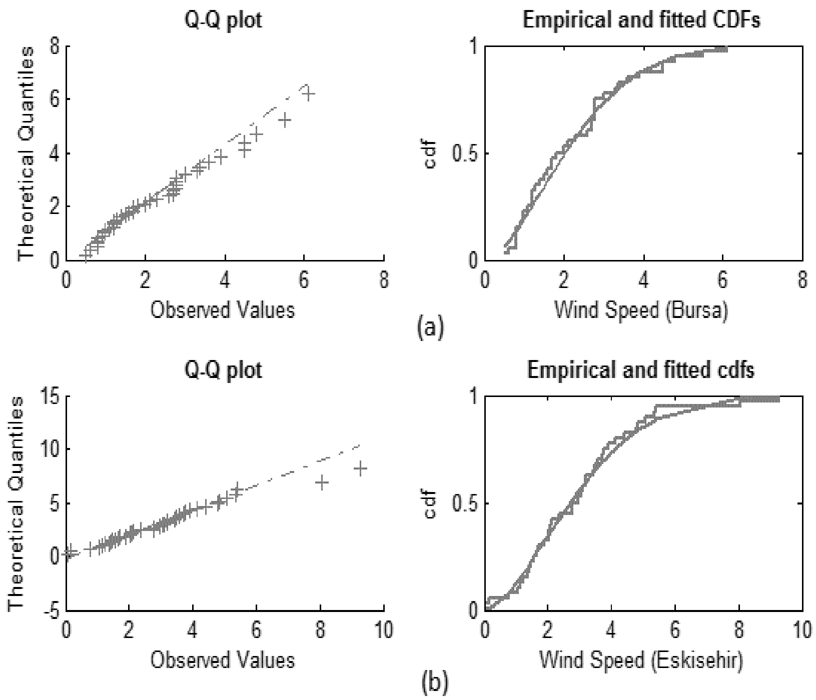

Figure 2a. Diagnostic plots for the wind speed data obtained from Bursa based on SRS
Figure 2b**.** Diagnostic plots for the wind speed data obtained from Eskisehir based on SRS

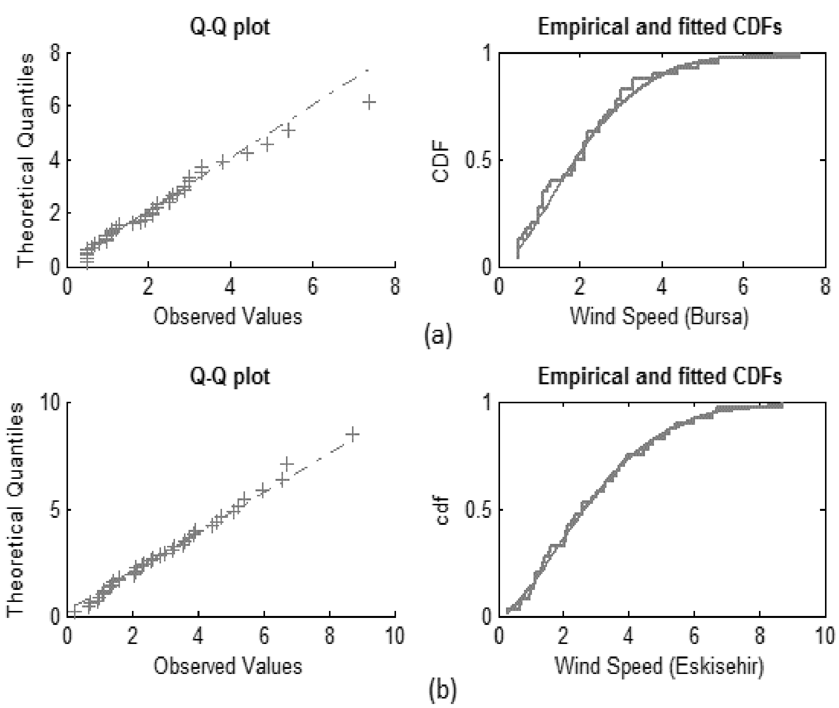

Figure 3a. Diagnostic plots for the wind speed data obtained from Bursa based on RSS
Figure 3b. Diagnostic plots for the wind speed data obtained from Eskisehir based on RSS

Under the assumption of right censoring, we suppose that the observations which are greater than 4 and 5 are censored for the samples X and Y, respectively. Censored observations are almost 12.5% of the complete samples. Similar to subsection 5.1, the resulting ML and MML estimates of R based on SRS and RSS are given in Table 7.

Table 7. The ML and the MML estimates of *R* for right censored wind speed data

$\hat{R}_{ML,SRS}$	$\hat{R}_{MML,SRS}$	$\hat{R}_{ML,RSS}$	$\hat{R}_{MML,RSS}$
0.6068	0.6052	0.6460	0.6453
(0.0569)*	(0.0526)*	(0.0210)*	(0.0204)*
(0.4878,0.7133)**	(0.4965,0.7044)**	(0.6056,0.6877)**	(0.6068,0.6855)**

*: BSE
**: BCI

It is clear form Table 7, BSEs of the ML and the MML estimators of R based on RSS is less than the corresponding estimators of R based on SRS. The length of BCIs based on RSS are much more smaller than the length of BCIs based on SRS. These results are in agreement with the results of first application. The estimate of $P(X < Y)$, based on right censored data, is greater than .60. It implies that the wind speed for Bursa is less than the wind speed for Eskisehir during the spring of 2009.

6. Conclusions

Based on type-II right censored SRS and RSS data, we derive the estimators of $R = P(X < Y)$ when the distributions of both the stress and the strength are Weibull with the different scale and the same shape parameters. In the estimation procedure, we use the ML and the MML methodologies. An extensive Monte-Carlo simulation study and empirical studies using two real data sets have been done to compare the efficiencies of the estimators of system reliability R. Simulation results show that the most efficient estimator of R is the ML estimator based on RSS as expected. It is followed by the MML estimator based on RSS. We see that the estimators based on SRS are the least efficient among the all estimators. Therefore, the ML estimator of R based on RSS can be used when our interest is efficiency especially when the scale parameters are not equal, i.e., $\sigma_1 \neq \sigma_2$. On the other hand, if our focus is to obtain the explicit and the efficient estimator of R, we suggest to use the MML estimator based on RSS when the scale parameters $\sigma_1 = \sigma_2 = 1$ and the set sizes m_x and m_y are small or moderate (i.e., 6 or 8).

Appendix: Modified likelihood equations based on RSS

$$\frac{\partial \ln L^*}{\partial \mu_1} = -\frac{1}{\eta}\sum_{c=1}^{r_x}\sum_{i=1}^{m_x}\delta_{(i)c}\Big[\big(\alpha_{1ic}^u - \beta_{1ic}^u z_{(i)c}\big) + (i-1)\big(\alpha_{2ic}^u - \beta_{2ic}^u z_{(i)c}\big) - (m_x - i)\big(\alpha_{3ic}^u + \beta_{3ic}^u z_{(i)c}\big)\Big] + \frac{1}{\eta}\sum_{c=1}^{r_x}\sum_{i=1}^{m_x}\big(1-\delta_{(i)c}\big)\big(\alpha_{4ic}^u + \beta_{4ic}^u z_{(i)c}\big) = 0,$$

$$\frac{\partial \ln L^*}{\partial \mu_2} = -\frac{1}{\eta}\sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\delta_{(j)l}\Big[\big(\alpha_{1jl}^v - \beta_{1jl}^v w_{(j)l}\big) + (j-1)\big(\alpha_{2jl}^v - \beta_{2jl}^v w_{(j)l}\big) - \big(m_y - j\big)\big(\alpha_{3jl}^v + \beta_{3jl}^v w_{(j)l}\big)\Big] + \frac{1}{\eta}\sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\big(1-\delta_{(j)l}\big)\big(\alpha_{4jl}^v + \beta_{4jl}^v w_{(j)l}\big) = 0,$$

$$\begin{aligned}\frac{\partial \ln L^*}{\partial \eta} = &-\frac{\sum_{c=1}^{r_x}\sum_{i=1}^{m_x}\delta_{(i)c} + \sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\delta_{(j)l}}{\eta} \\ &-\frac{1}{\eta}\sum_{c=1}^{r_x}\sum_{i=1}^{m_x}\delta_{(i)c}\Big[\big(\alpha_{1ic}^u - \beta_{1ic}^u z_{(i)c}\big) + (i-1)\big(\alpha_{2ic}^u - \beta_{2ic}^u z_{(i)c}\big) - (m_x - i)\big(\alpha_{3ic}^u + \beta_{3ic}^u z_{(i)c}\big)\Big] z_{(i)c} \\ &+\frac{1}{\eta}\sum_{c=1}^{r_x}\sum_{i=1}^{m_x}\big(1-\delta_{(i)c}\big)\big(\alpha_{4ic}^u + \beta_{4ic}^u z_{(i)c}\big)z_{(i)c} \\ &-\frac{1}{\eta}\sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\delta_{(j)l}\Big[\big(\alpha_{1jl}^v - \beta_{1jl}^v w_{(j)l}\big) + (j-1)\big(\alpha_{2jl}^v - \beta_{2jl}^v w_{(j)l}\big) - \big(m_y - j\big)\big(\alpha_{3jl}^v + \beta_{3jl}^v w_{(j)l}\big)\Big] w_{(j)l} \\ &+\frac{1}{\eta}\sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\big(1-\delta_{(j)l}\big)\big(\alpha_{4jl}^v + \beta_{4jl}^v w_{(j)l}\big)w_{(j)l} = 0.\end{aligned}$$

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