Estimation of $P(Y < X)$ Using Record Values from the Generalized Inverted Exponential Distribution

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Abstract

This article deals with the estimation of $R = P(Y < X)$ when X and Y are distributed as two independent generalized inverted exponential with common scale parameter and different shape parameters. The maximum likelihood and Bayesian estimators of R are obtained on the basis of upper record values and upper record ranked set samples. The Bayesian estimator cannot be obtained in explicit form, and therefore it has been achieved using Lindley approximation. Simulation study is performed to compare the reliability estimators in each record sampling scheme with respect to biases and mean square errors.

Keywords: Reliability; upper record ranked set sample; maximum likelihood estimator; Bayesian estimator; Lindley approximation.

1. Introduction

Abouammoh and Alshingiti (2009) introduced the *generalized inverted exponential distribution* (*GIED*) as a generalization of the inverted exponential distribution. They declared that the *GIED* can be better than the inverted exponential distribution for real data set based on the likelihood ratio test and the Kolmogorov-Smirnov statistic. The *GIED* has been widely used in varied fields such as, accelerated life testing, horse racing, queues, sea currents and wind speeds.

The *probability density function* (pdf) of the *GIED* with the shape parameter α and the scale parameter λ takes the following form

$$f(x;\alpha,\lambda) = \frac{\alpha\lambda}{x^2}\left(1 - e^{\frac{-\lambda}{x}}\right)^{\alpha-1} e^{\frac{-\lambda}{x}} \qquad ; x > 0, \alpha, \lambda > 0. \tag{1}$$

The corresponding *cumulative distribution function* (cdf) is as follows

$$F(x;\alpha,\lambda) = 1 - \left(1 - e^{\frac{-\lambda}{x}}\right)^{\alpha} \qquad ; x > 0, \alpha, \lambda > 0. \tag{2}$$

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Record data are very important in the situations that the observations are difficult to obtain or are destroyed in experimental tests. Record data arise in many real life applications such as industrial stress testing, meteorology, sports, hydrology and economics. A record value of some phenomenon is the largest (smallest) observation any one has ever made. The mathematical theory and statistical study of record values started by Chandler (1952) is now spread in many directions.The prominent theoretical contributions and inference issues may refer to Nagaraja (1988), Arnold *et al.* (1998) and Ahsanullah and Nevzorov (2015).

According to Arnold *et al.* (1998), record values can be classified into lower and *upper record value* (URV). For a sequence of *independent and identically distributed* (iid) random variables, an observation $X_j, j \geq 1$ is called an URV if its value exceed that all of previous observations $\left(i\,e., X_j > X_i, for\,every\; i < j\right)$. While an observation is called a lower record if its value is less than all previous observations $\left(i\,e., X_i < X_j, for\,every\; i > j\right)$.

Suppose that the first *m* URV $U_1 = u_1, U_2 = u_2, \ldots, U_m = u_m$ from cdf $F(.;\theta)$ and pdf $f(.;\theta)$ of the sampling population respectively, then the joint distribution of the first *m* URV is defined by Arnold *et al.* (1998) as

$$f(\underline{u},\theta) = f(u_m,\theta) \prod_{i=1}^{m-1} \frac{f(u_i,\theta)}{1-F(u_i,\theta)} \qquad ;-\infty < u_1 < \ldots < u_m < \infty, \ \theta \in \Theta \qquad (3)$$

where $\underline{u} = (u_1,\ldots,u_m)$, Θ is the parameter space and $\theta \in \Theta$ may be a vector.

Ranked set sampling was first proposed by McIntyre in 1952 as a more efficient technique than simple random sample for estimating the population parameters of interest. Salehi and Ahmadi (2014) proposed a new sampling scheme for generating record data called record ranked set sampling to help scientists in situations where the only observations that are going to be used are the last record data such as athletic data, weather data and Olympic data. Since the new sampling scheme is based generally on ranked set sampling, so it is called record ranked set sampling.

As described by Salehi and Ahmadi (2014) for record ranked set sampling, suppose that there exist *n* independent sequential sequences of continuous random variables, the *i*th sequence sampling is terminated when the *i*th record value is observed. The only observations that are used for analysis are the last record value in each sequence. The last record value of the *i*th sequence in this plane is denoted by $U_{i,i}$, then the available observations are $U = (U_{1,1}, U_{2,2}, \ldots, U_{n,n})^T$, i.e.

$$
\begin{array}{llllllll}
1: & U_{(1)1} & U_{(2)1} & U_{(3)1} & \ldots & U_{(m)1} & \rightarrow & U_{(1)1} = U_{1,1} \\
2: & U_{(1)2} & U_{(2)2} & U_{(3)2} & \ldots & U_{(m)2} & \rightarrow & U_{(2)2} = U_{2,2} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ldots & \vdots \\
n: & U_{(1)n} & U_{(2)n} & U_{(3)n} & \ldots & U_{(m)n} & \rightarrow & U_{(m)n} = U_{n,n}
\end{array}
$$

where, , $U_{(i)j}$ is the *i*th record in the *j*th sequence. It is recognized that unlike the record values; here $U_{i,i}'s$ are independent random variables but not ordered. Let $\underline{U}_{i,i} = (U_{1,1}, U_{2,2}, \ldots, U_{n,n})^T$ be the *upper record ranked set sample* (URRSS), then the joint density function of $\underline{U}_{i,i}$, which is denoted by $f_{\underline{U}_{i,i}}(\underline{u}_{i,i}, \theta)$, is obtained using the marginal density of URV as follows

$$f_{\underline{U}_{i,i}}(\underline{u}_{i,i}, \theta) = \prod_{i=1}^{n} \frac{\left[-\ln\left(1 - F\ (u_{i,i}; \theta)\right)\right]^{i-1}}{(i-1)!} f\ (u_{i,i}; \theta) \qquad ; \theta \in \Theta \qquad (4)$$

where, $\underline{u}_{i,i} = \left(u_{1,1}, u_{2,2}, \ldots, u_{n,n}\right)^T$ is the observed values of $\underline{U}_{i,i}$, θ is real valued parameter and Θ is the parameter space.

In reliability context, the parameter $R = P(Y < X)$ is called stress-strength reliability. The stress-strength reliability arises naturally when we consider a random stress *Y* applied to a certain device with strength *X*. If the stress exceeds the strength, i.e. $Y > X$, the component will fail. Thus, the reliability is defined as the probability of not failing or $P(Y < X)$. The reliability function *R* represents the relation between the stress and strength of the component, and it can be considered as a measure of a component performance.

The reliability function $R = P(Y < X)$ has attracted the attention of many authors and become popular in many fields besides life testing, psychology, reliability and medical sciences. Lately, estimation of reliability function associated with record values have been raised in many fields such as industrial tests.

In recent years there has been a growing interest in the study of inference problems associated with stress-strength model and record values. The estimation problem of $R = P(Y < X)$ based on record values was firstly considered by Baklizi (2008) who estimated the reliability function based on URV for one and two parameters exponential distribution. Subsequent papers extended this work assuming various lifetime distributions for stress and strength random variables, for instance Baklizi (2012) estimated *R* based on URV for the Weibull distribution. Wang and Zhang (2013) estimated *R* for a class of distributions. Latterly, Salehi and Ahmadi (2015) considered the estimation of *R* based on URRSS from one-parameter exponential distribution and studied its performance.

This article aims to estimate the reliability function $R = P(Y < X)$ when the strength and the stress are two independent variables of *GIED* based on URV and URRSS. Assuming that the scale parameter is common and known, maximum likelihood and Bayesian estimators of *R* based on independent gamma priors for the unknown parameters are obtained under squared error loss function. The procedures of this study are encapsulated by analyzing a simulated data.

The rest of the paper is organized as follows. In the next section, maximum likelihood estimator of *R* based on URV and URRSS are discussed. In Section (3), Bayesian estimator of *R* using squared error loss function is discussed for both record schemes. A

simulation study and numerical results are given in Section (4). Finally, conclusions appear in Section (5).

2. Maximum Likelihood Method of Estimation

This section provides the *maximum likelihood* (ML) estimator of $R = P(Y < X)$ depending on the two record schemes, namely, URV and URRSS.

2.1 ML Estimator of R Based on URV

Let the random variable X represents the strength of the component available to overcome the stress Y applied on that component. Let X and Y be two independent random variables from the *GIED* with common scale parameter λ and different shape parameters α and β respectively, then the reliability of a system is obtained as follows

$$R = \int_0^\infty \frac{\alpha\lambda}{x^2}\left(1-e^{\frac{-\lambda}{x}}\right)^{\alpha-1} e^{\frac{-\lambda}{x}}\left[1-\left(1-e^{\frac{-\lambda}{x}}\right)^{\beta}\right]dx = \frac{\beta}{\alpha+\beta}. \tag{5}$$

To obtain the ML estimator of R; suppose $u_1,\ldots,u_m$ be URV observed from $X \sim GIED(\alpha,\lambda)$. Let, also $s_1,\ldots,s_n$ is URV observed from $Y \sim GIED(\beta,\lambda)$. Therefore, the likelihood functions of α,λ and β for the observed samples $\underline{u}$ and $\underline{s}$ variables based on URV denoted by $L(\alpha\mid\underline{u})$ and $L(\beta\mid\underline{s})$ are given by

$$L(\alpha\mid\underline{u}) = \left(1-e^{\frac{-\lambda}{u_m}}\right)^{\alpha} \prod_{i=1}^{m}\frac{\alpha\lambda}{u_i^2}e^{\frac{-\lambda}{u_i}}\left(1-e^{\frac{-\lambda}{u_i}}\right)^{-1}, \tag{6}$$

and,

$$L(\beta\mid\underline{s}) = \left(1-e^{\frac{-\lambda}{s_n}}\right)^{\beta} \prod_{j=1}^{n}\frac{\beta\lambda}{s_j^2}e^{\frac{-\lambda}{s_j}}\left(1-e^{\frac{-\lambda}{s_j}}\right)^{-1}. \tag{7}$$

The joint likelihood function of the observed record values $\underline{u}$ and $\underline{s}$ is given by

$$L(\alpha,\beta\mid\underline{u},\underline{s}) = \left(1-e^{\frac{-\lambda}{u_m}}\right)^{\alpha}\left(1-e^{\frac{-\lambda}{s_n}}\right)^{\beta} \prod_{i=1}^{m}\frac{\alpha\lambda}{u_i^2}e^{\frac{-\lambda}{u_i}}\left(1-e^{\frac{-\lambda}{u_i}}\right)^{-1} \prod_{j=1}^{n}\frac{\beta\lambda}{s_j^2}e^{\frac{-\lambda}{s_j}}\left(1-e^{\frac{-\lambda}{s_j}}\right)^{-1}.$$

The joint log-likelihood function of the observed record values $\underline{u}$ and $\underline{s}$, denoted by l, is obtained as

$$l = \ln L(\alpha,\beta\mid\underline{u},\underline{s}) = \alpha\ln\left(1-e^{\frac{-\lambda}{u_m}}\right)+\beta\ln\left(1-e^{\frac{-\lambda}{s_n}}\right)+\sum_{i=1}^{m}\left(\ln\alpha+\ln\lambda-2\ln u_i-\frac{\lambda}{u_i}-\ln\left(1-e^{\frac{-\lambda}{u_i}}\right)\right)$$

$$+\sum_{j=1}^{n}\left(\ln\beta+\ln\lambda-2\ln s_j-\frac{\lambda}{s_j}-\ln\left(1-e^{\frac{-\lambda}{s_j}}\right)\right).$$

Assuming that the scale parameter λ is known, the ML estimators of α and β, say $\hat{\alpha}$ and $\hat{\beta}$ based on the observed URV can be obtained as the solutions of

$$\frac{\partial l}{\partial \alpha} = \ln\left(1 - e^{\frac{-\lambda}{u_m}}\right) + \frac{m}{\alpha}, \quad \text{and} \quad \frac{\partial l}{\partial \beta} = \ln\left(1 - e^{\frac{-\lambda}{s_n}}\right) + \frac{n}{\beta}. \tag{8}$$

From (8), we have

$$\hat{\alpha} = \frac{-m}{\ln\left(1 - e^{\frac{-\lambda}{u_m}}\right)}, \quad \text{and} \quad \hat{\beta} = \frac{-n}{\ln\left(1 - e^{\frac{-\lambda}{s_n}}\right)}. \tag{9}$$

Hence, by using the invariance property of ML estimation, then ML estimator of R, denoted by $\hat{R}$, becomes

$$\hat{R} = \frac{\hat{\beta}}{\hat{\alpha} + \hat{\beta}} = \left[1 + \frac{m \ln\left(1 - e^{\frac{-\lambda}{s_n}}\right)}{n \ln\left(1 - e^{\frac{-\lambda}{u_m}}\right)}\right]^{-1}. \tag{10}$$

2.2 ML Estimator of *R* Based on URRSS

Let $u_{1,1}, \ldots, u_{m,m}$ and $s_{1,1}, \ldots, s_{n,n}$ be two independent URRSS from the strength and stress random variables with parameters (α, λ) and (β, λ) respectively. Then the joint likelihood function of the observed URRSS $\underline{u}_{i,i}$ and $\underline{s}_{j,j}$ is obtained as follows

$$\ell(\alpha, \beta \mid \underline{u}_{i,i}, \underline{s}_{j,j}) = \prod_{i=1}^{m} \frac{[-\alpha \ln(1 - e^{\frac{-\lambda}{u_{i,i}}})]^{i-1}}{(i-1)!} \frac{\alpha \lambda}{u_{i,i}^2} (1 - e^{\frac{-\lambda}{u_{i,i}}})^{\alpha-1} e^{\frac{-\lambda}{u_{i,i}}}$$

$$\prod_{j=1}^{n} \frac{[-\beta \ln(1 - e^{\frac{-\lambda}{s_{j,j}}})]^{j-1}}{(j-1)!} \frac{\beta \lambda}{s_{j,j}^2} (1 - e^{\frac{-\lambda}{s_{j,j}}})^{\beta-1} e^{\frac{-\lambda}{s_{j,j}}}.$$

The joint log-likelihood function of the observed URRSS $\underline{u}_{i,i}$ and $\underline{s}_{j,j}$, denoted by l, is obtained as

$$l = \sum_{i=1}^{m} \left\{ (i-1)\ln[-\alpha \ln(1 - e^{\frac{-\lambda}{u_{i,i}}})] - \ln\{(i-1)!\} + \ln \alpha + \ln \lambda - 2\ln(u_{i,i}) + (\alpha-1)\ln(1 - e^{\frac{-\lambda}{u_{i,i}}}) - \frac{\lambda}{u_{i,i}} \right\}$$

$$+ \sum_{j=1}^{n} \left\{ (j-1)\ln[-\beta \ln(1 - e^{\frac{-\lambda}{s_{j,j}}})] - \ln\{(j-1)!\} + \ln \beta + \ln \lambda - 2\ln(s_{j,j}) + (\beta-1)\ln(1 - e^{\frac{-\lambda}{s_{j,j}}}) - \frac{\lambda}{s_{j,j}} \right\}.$$

The ML estimators of α and β, based on the observed URRSS are obtained, respectively, by equating the following equations with zero and solving them numerically

$$\frac{\partial l}{\partial \alpha} = \frac{m(m+1)}{2\alpha} + \sum_{i=1}^{m} \ln\left(1 - e^{\frac{-\lambda}{u_{i,i}}}\right), \quad \text{and} \quad \frac{\partial l}{\partial \beta} = \frac{n(n+1)}{2\beta} + \sum_{j=1}^{n} \ln\left(1 - e^{\frac{-\lambda}{s_{j,j}}}\right).$$

Assuming that the scale parameter λ is known, the ML estimators of α and β, denoted by $\hat{\hat{\alpha}}$ and $\hat{\hat{\beta}}$ are obtained as follows

$$\hat{\hat{\alpha}} = \frac{-m(m+1)}{2\sum_{i=1}^{m} \ln\left(1 - e^{\frac{-\lambda}{u_{i,i}}}\right)}, \quad \text{and} \quad \hat{\hat{\beta}} = \frac{-n(n+1)}{2\sum_{j=1}^{n} \ln\left(1 - e^{\frac{-\lambda}{s_{j,j}}}\right)}. \tag{11}$$

Hence, the ML estimator of R based on URRSS, denoted by $\hat{\hat{R}}$, is obtained by substituting $\hat{\hat{\alpha}}$ and $\hat{\hat{\beta}}$ in (5) as follows

$$\hat{\hat{R}} = \frac{\hat{\hat{\beta}}}{\hat{\hat{\alpha}} + \hat{\hat{\beta}}} = \left[1 + \frac{m(m+1)\sum_{j=1}^{n} \ln\left(1 - e^{\frac{-\lambda}{s_{j,j}}}\right)}{n(n+1)\sum_{i=1}^{m} \ln\left(1 - e^{\frac{-\lambda}{u_{i,i}}}\right)}\right]^{-1}. \tag{12}$$

3. Bayesian Estimation

In this section, the Bayesian estimator of R is obtained under the assumption of independent gamma priors for the shape parameters α and β. Bayes estimator of R is derived based on URV and URRSS using Lindley approximation.

3.1 Bayesian Estimator of R Based on URV

According to Dey and Dey (2014), the gamma priors for the shape parameters α and β, and a *squared error loss function* (SELF) are used. Therefore, the prior densities of α and β are

$$\pi_1(\alpha) \propto \alpha^{a-1} e^{-b\alpha}, \tag{13}$$

and,

$$\pi_2(\beta) \propto \beta^{c-1} e^{-d\beta}, \tag{14}$$

where a, b, c and d are the parameters of the prior distributions. Through combining the prior density (13) and the likelihood function (6), the posterior density of α can be obtained as follows

$$\pi_1^*(\alpha) \propto \alpha^{m+a-1} \left[\left(1 - e^{\frac{-\lambda}{u_m}}\right) e^{-b}\right]^{\alpha}. \tag{15}$$

Similarly, through combining the prior density (14) and the likelihood function (7). The posterior density of β is given by

$$\pi_2^*(\beta) \propto \beta^{n+c-1} \left[\left(1 - e^{\frac{-\lambda}{s_n}} \right) e^{-d} \right]^\beta. \tag{16}$$

Assume that α and β are independent, and let $T_1 = \left(1 - e^{\frac{-\lambda}{u_m}} \right)$ and $T_2 = \left(1 - e^{\frac{-\lambda}{s_n}} \right)$, the

joint bivariate posterior density of α and β will be

$$\pi^*(\alpha, \beta \mid \underline{u}, \underline{s}) \propto \alpha^{m+a-1} \beta^{n+c-1} \left[T_1 e^{-b} \right]^\alpha \left[T_2 e^{-d} \right]^\beta. \tag{17}$$

By applying the transformation technique, let

$$r_B = \frac{\beta}{\alpha + \beta} \quad \text{and } q = \alpha + \beta, \tag{18}$$

then $\beta = r_B q$ and $\alpha = q(1 - r_B)$, $q > 0$, $0 < r_B < 1$. Therefore, the joint bivariate posterior density function (17) can be written as

$$\pi^*(r_B, q \mid \underline{u}, \underline{s}) \propto q^{n+m+a+c-2} (1 - r_B)^{m+a-1} r_B^{n+c-1} \left[T_1 e^{-b} \right]^{q[1-r_B]} \left[T_2 e^{-d} \right]^{r_B q}.$$

Consequently, the posterior pdf of r_B can be formulated by integrating with respect to q as follows

$$\pi^*(r_B \mid \underline{u}, \underline{s}) = \frac{r_B^{n+c-1}(1 - r_B)^{m+a-1} \int_0^\infty q^{n+m+a+c-2} [T_1 e^{-b}]^{q[1-r_B]} [T_2 e^{-d}]^{r_B q} dq}{\int_0^1 \int_0^\infty r_B^{n+c-1}(1 - r_B)^{m+a-1} q^{n+m+a+c-2} [T_1 e^{-b}]^{q[1-r_B]} [T_2 e^{-d}]^{r_B q} dq \, dr_B}. \tag{19}$$

SELF implies the cost obtained by replacing the actual value of the parameter with the parameter estimate. Under SELF, the Bayes risk of R, is the expected value of r_B. This expected value contains an integral which is usually not obtained in a simple closed form

$$E(r_B) = \frac{\int_0^1 \int_0^\infty r_B^{n+c} (1 - r_B)^{m+a-1} q^{n+m+a+c-2} \left[T_1 e^{-b} \right]^{q[1-r_B]} \left[T_2 e^{-d} \right]^{r_B q} dq dr_B}{\int_0^1 \int_0^\infty r_B^{n+c-1} (1 - r_B)^{m+a} q^{n+m+a+c-2} \left[T_1 e^{-b} \right]^{q[1-r_B]} \left[T_2 e^{-d} \right]^{r_B q} dq dr_B}.$$

Lindley's approximation is a procedure for the evaluation of the ratio of two integrals and it can be used to obtain the approximate Bayesian estimator. See, for example, (Ahmad *et al.* (1997); Nadar and Papadopoulos (2011)). Alternatively, Raqab and Kundu (2005) used the approximation of Lindley (1980) following the approach of Ahmad *et al.* (1997) to compute the Bayes estimate of *R*. They obtained the approximate Bayes estimate of

R, under SELF through Lindley's approximation form expanding about the posterior mode.

The Bayes estimator of R using SELF under URV cannot be computed analytically, therefore the Lindley's approximation is applied in the following subsection.

3.2 Lindley's Approximation of *R* Based on URV

It is known that the Bayes risk under SELF is the mean of the posterior distribution. Since the Bayes risk of R using SELF under URV cannot be computed analytically. Hence, the numerical calculations are needed by using Lindley's approximation for expanding the posterior mode. Consequently, Bayesian estimator of R say; R^* is obtained. In the two parameters case (α, β), Lindley's approximation leads to

$$\hat{U}_B = E[U(\alpha, \beta)] = U(\alpha, \beta) + \frac{1}{2}\left[B + Q_{30}B_{12} + Q_{21}C_{12} + Q_{12}C_{21} + Q_{03}B_{21}\right]. \tag{20}$$

where, $B = \sum_{i=1}^{2}\sum_{j=1}^{2}U_{ij}\tau_{ij}$, $Q_{\eta\xi} = \dfrac{\partial^{\eta+\xi}Q}{\partial^{\eta}\alpha\,\partial^{\xi}\beta}$; $\eta,\ \xi = 0,1,2,3$, $\eta+\xi = 3$, for $i, j = 1,2$, and for $i \neq j$,

$B_{ij} = (U_i\tau_{ii} + U_j\tau_{ij})\tau_{ii}$, $C_{ij} = 3U_i\tau_{ii}\tau_{ij} + U_j(\tau_{ii}\tau_{jj} + 2\tau_{ij}^2)$, τ_{ij} is the $(i, j)th$ element in the inverse of the matrix $Q^* = (-Q_{ij}^*)$; $i, j = 1,2$ such that $Q^* = \dfrac{\partial^2 Q}{\partial\alpha\,\partial\beta}$.

In our case $U(\alpha, \beta) \equiv R$. The reliability function (5) is to be evaluated at $(\tilde{\alpha}, \tilde{\beta})$ which are the mode of the posterior density of URV. To show that the posterior density function is unimodal, it suffices to show that the function $Q = L(\alpha, \beta) + \rho(\alpha, \beta)$ has the unique mode. Note that $L(\alpha, \beta)$ is a logarithm of the likelihood function and $\rho(\alpha, \beta)$ is a logarithm of the joint prior distribution. Therefore, Q can be rewritten as

$$Q = \alpha\,ln\,(1 - e^{\frac{-\lambda}{u_m}}) + \beta\,ln(1 - e^{\frac{-\lambda}{s_n}}) + m\,ln\,\alpha - 2\sum_{i=1}^{m}ln\,u_i - \lambda\sum_{i=1}^{m}\frac{1}{u_i} - \sum_{i=1}^{m}ln\,(1 - e^{\frac{-\lambda}{u_i}}) + n\,ln\,\beta$$

$$+ (m+n)\,ln\,\lambda - 2\sum_{j=1}^{n}ln\,s_j - \lambda\sum_{j=1}^{n}\frac{1}{s_j} - \sum_{j=1}^{n}ln\,(1 - e^{\frac{-\lambda}{s_j}}) + (a-1)ln\,\alpha - b\alpha + (c-1)ln\,\beta - d\beta.$$

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The joint posterior mode, denoted by $(\tilde{\alpha}, \tilde{\beta})$ is obtained as

$$\tilde{\alpha} = \frac{m + a - 1}{b - K_1} \quad \text{and} \quad \tilde{\beta} = \frac{n + c - 1}{d - K_2} \tag{21}$$

where, $K_1 = ln\left(1 - e^{\frac{-\lambda}{u_m}}\right)$ and $K_2 = ln\left(1 - e^{\frac{-\lambda}{s_n}}\right)$.

Therefore, the Bayes estimator of R based on URV can be obtained from (20) as follows

$$R^* = \tilde{R}\left[1 + \left(\frac{\tilde{R}^2\tilde{\alpha}^2}{\tilde{\beta}^2(m + a - 1)} - \frac{\tilde{R}^2\tilde{\alpha}}{\tilde{\beta}(n + c - 1)} - \frac{\tilde{R}\tilde{\alpha}}{\tilde{\beta}(m + a - 1)} + \frac{\tilde{R}\tilde{\alpha}}{\tilde{\beta}(n + c - 1)}\right)\right],$$

where, $\tilde{R}$ is the reliability function evaluated at $(\tilde{\alpha}, \tilde{\beta})$.

3.3 Bayesian Estimator of R Based on URRSS

The conjugate priors of both α and β are assumed to be independent gamma distributions as defined in (14) and (15). Hence, the posterior density of α and β is obtained, respectively, as follows

$$\pi_1^{**}(\alpha) \propto (\alpha)^{\sum_{i=1}^{m} i + a - 1} e^{-b\alpha} \prod_{i=1}^{m} \left(1 - e^{\frac{-\lambda}{u_{i,i}}} \right)^{\alpha},$$

and,

$$\pi_2^{**}(\beta) \propto (\beta)^{\sum_{j=1}^{n} j + c - 1} e^{-d\beta} \prod_{j=1}^{n} \left(1 - e^{\frac{-\lambda}{s_{j,j}}} \right)^{\beta}.$$

Then, the joint bivariate posterior density of α and β will be

$$\pi^{**}(\alpha, \beta | \underline{u}_{i,i}, \underline{s}_{j,j}) \propto (\alpha)^{(\sum_{i=1}^{m} i) + a - 1} (\beta)^{(\sum_{j=1}^{n} j) + c - 1} e^{-[b\alpha + d\beta]} \prod_{i=1}^{m} (1 - e^{\frac{-\lambda}{u_{i,i}}})^{\alpha} \prod_{j=1}^{n} (1 - e^{\frac{-\lambda}{s_{j,j}}})^{\beta}.$$

By applying the transformation technique defined in Equation (18), therefore, the joint bivariate posterior density function can be written as

$$\pi^{**}(r_B, q | \underline{u}_{i,i}, \underline{s}_{j,j}) \propto (1 - r_B)^{(\sum_{i=1}^{m} i) + a - 1} (r_B)^{(\sum_{j=1}^{n} j) + c - 1} D,$$

where,

$$D = (q)^{\left(\sum_{i=1}^{m} i + \sum_{j=1}^{n} j\right) + a + c - 2} e^{-\left[b(1-r_B) + dr_B\right]q} \prod_{i=1}^{m} \left(1 - e^{\frac{-\lambda}{u_{i,i}}} \right)^{q(1-r_B)} \prod_{j=1}^{n} \left(1 - e^{\frac{-\lambda}{s_{j,j}}} \right)^{q r_B}.$$

Consequently, the posterior pdf of r_B is as follows

$$\pi^{**}(r_B / \underline{u}_{i,i}, \underline{s}_{j,j}) = \frac{(1 - r_B)^{\left(\sum_{i=1}^{m} i\right) + a - 1} (r_B)^{\left(\sum_{j=1}^{n} j\right) + c - 1} \int_0^{\infty} D dq}{\int_0^1 \int_0^{\infty} (1 - r_B)^{\left(\sum_{i=1}^{m} i\right) + a - 1} (r_B)^{\left(\sum_{j=1}^{n} j\right) + c - 1} D dq dr_B}.$$

Under SELF, the Bayes risk of R, is the expected value of r_B. This expected value contains an integral which is usually not obtained in a simple closed form

$$E(r_B) = \frac{\int_0^1 \int_0^{\infty} (1 - r_B)^{\left(\sum_{i=1}^{m} i\right) + a - 1} (r_B)^{\left(\sum_{j=1}^{n} j\right) + c} D dq dr_B}{\int_0^1 \int_0^{\infty} (1 - r_B)^{\left(\sum_{i=1}^{m} i\right) + a - 1} (r_B)^{\left(\sum_{j=1}^{n} j\right) + c - 1} D dq dr_B},$$

so the Lindley's approximation is employed.

3.4 Lindley's Approximation of *R* Based on URRSS

Here, $U(\alpha, \beta) \equiv R$, the reliability function (5) is evaluated at $\left(\tilde{\tilde{\alpha}}, \tilde{\tilde{\beta}}\right)$ which is the mode of the posterior density; based on URRSS. In this case Q can be rewritten as

$$Q = \frac{m(m+1)}{2}\ln\alpha + \sum_{i=1}^{m}(i-1)\ln\left[-\ln(1-e^{\frac{-\lambda}{u_{i,i}}}) - \sum_{i=1}^{m}\ln\{(i-1)! - 2\sum_{i=1}^{m}\ln(u_{i,i}) +\right.$$

$$(\alpha-1)\sum_{i=1}^{m}\ln(1-e^{\frac{-\lambda}{u_{i,i}}}) - \lambda\sum_{i=1}^{m}\frac{1}{u_{i,i}} + \frac{n(n+1)}{2}\ln\beta + \sum_{j=1}^{n}(j-1)\ln\left[-\ln(1-e^{\frac{-\lambda}{s_{j,j}}}) - \sum_{j=1}^{n}\ln\{(j-1)!\right.$$

$$+(m+n)\ln\lambda - 2\sum_{j=1}^{n}\ln(s_{j,j}) + (\beta-1)\sum_{j=1}^{n}\ln(1-e^{\frac{-\lambda}{s_{j,j}}}) - \lambda\sum_{j=1}^{n}\frac{1}{s_{j,j}} + (a-1)\ln\alpha - b\,\alpha$$

$$+(c-1)\ln\beta - d\,\beta.$$

Again, the joint posterior mode based on URRSS, denoted by $\left(\tilde{\tilde{\alpha}}, \tilde{\tilde{\beta}}\right)$ is obtained as

$$\tilde{\tilde{\alpha}} = \frac{\sum_{i=1}^{m}i + a - 1}{b - W_1} \quad \text{and} \quad \tilde{\tilde{\beta}} = \frac{\sum_{j=1}^{n}j + c - 1}{d - W_2}, \tag{22}$$

where, $W_1 = \sum_{i=1}^{m}\ln\left(1-e^{\frac{-\lambda}{u_{i,i}}}\right)$ and $W_2 = \sum_{j=1}^{n}\ln\left(1-e^{\frac{-\lambda}{s_{j,j}}}\right)$.

The Bayesian estimator of *R* denoted by R^{**}, based on URRSS can be obtained from (20) as follows

$$R^{**} = \tilde{\tilde{R}}\left[1 + \left(\frac{\tilde{\tilde{R}}^2\,\tilde{\tilde{\alpha}}^2}{\tilde{\tilde{\beta}}^2\left(\frac{m(m+1)}{2}+a-1\right)} - \frac{\tilde{\tilde{R}}^2\,\tilde{\tilde{\alpha}}}{\tilde{\tilde{\beta}}\left(\frac{n(n+1)}{2}+c-1\right)} - \frac{\tilde{\tilde{R}}\,\tilde{\tilde{\alpha}}}{\tilde{\tilde{\beta}}\left(\frac{m(m+1)}{2}+a-1\right)} + \frac{\tilde{\tilde{R}}\,\tilde{\tilde{\alpha}}}{\tilde{\tilde{\beta}}\left(\frac{n(n+1)}{2}+c-1\right)}\right)\right],$$

where $\tilde{\tilde{R}}$ is the reliability function evaluated at $\left(\tilde{\tilde{\alpha}}, \tilde{\tilde{\beta}}\right)$.

4. Simulation Study and Discussion

In this section, a simulation study is designed to compare the different reliability estimators in stress-strength model based on URV and URRSS. The exact values of reliability are selected as $R = 0.25, 0.5, 0.7$ and 0.9. Different reliability estimates are compared using the *mean square error* (MSE) and biases for both ML and Bayesian techniques. The efficiency of the estimates is obtained through using the following relations
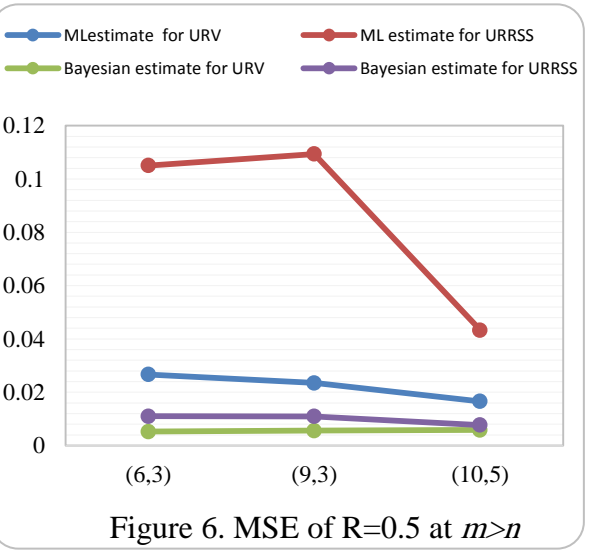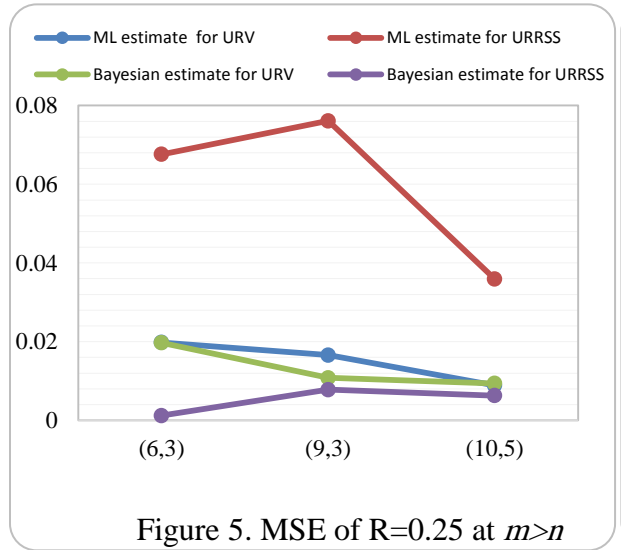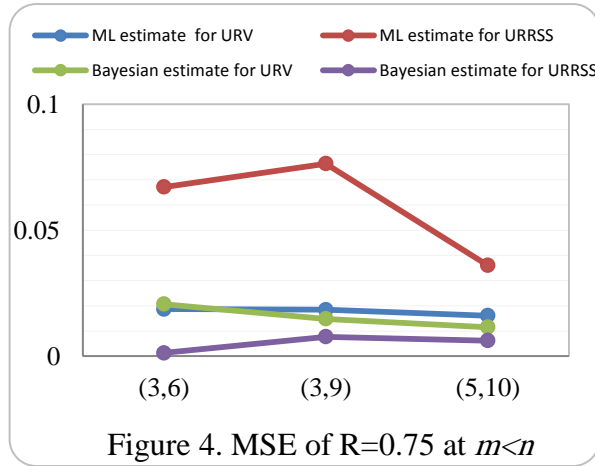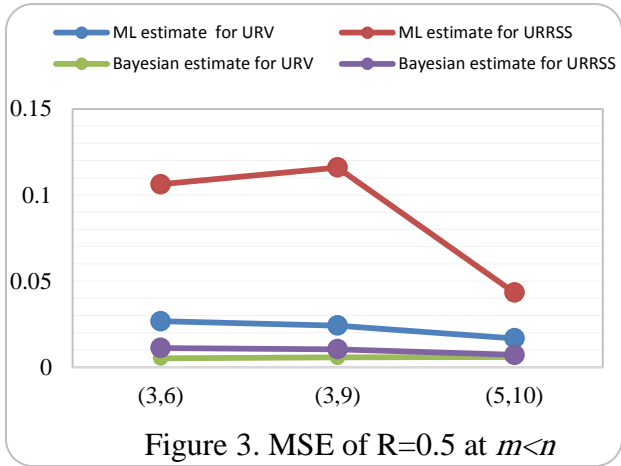
$$eff\left(\hat{\hat{R}},\hat{R}\right)=\frac{MSE\left(\hat{\hat{R}}\right)}{MSE\left(\hat{R}\right)}, \quad and \quad eff\left(R^{**},R^{*}\right)=\frac{MSE\left(R^{**}\right)}{MSE\left(R^{*}\right)}.$$

The computations are performed based on 5000 replications using several combinations of the sample sizes (m,n) = (3,3), (3,6), (3,9), (5,5), (5,10), (6,3), (7,7), (9,3), (10,5),(10,10) and (12,12). Generate random samples from the $GIED\left(\alpha,\lambda\right)$ and $GIED\left(\beta,\lambda\right)$ using the transformation technique. The URV and URRSS are obtained for each of the strength and stress random samples. The ML estimate of R is computed based on URV and URRSS. Also, for a given values of the prior distributions parameters $a=c=4, b=d=2,$ the Bayes estimates of R under SELF are obtained using Lindley approximation.

Tables 1 and 2 report the simulation results and represented for selected values of R through Figures (1-6). From these tables and figures the following results can be observed

1. The ML estimate of R based on URV is more efficient than the corresponding based on URRSS for different sample sizes and different exact values of R (see Table 1). Also, the Bayesian estimator of R based on URRSS is more efficient than the corresponding based on URV for different sample sizes and different exact values of R, expect few cases (see Table 2).

2. The MSE for the Bayesian estimates of R based on URV is less than the Bayesian estimate of R based on URRSS at $R = 0. 5$ for all $m = n$, $m > n$ and $m < n$ (see Figures 2, 3 and 6).

3. The MSE for the Bayesian estimate of R based on URRSS has the smallest values for $R = 0.75$ when $m < n$ (see Figure 4). Also, it has the smallest values for $R = 0.25$ when $m > n$ (see Figure 5).

4. As seen from Figure 4, the MSEs for the ML estimate of R based on URV are less than the ML estimate of R based on URRSS at $R = 0.75$ for $m < n$.

5. The MSE for the ML estimate of R based on URRSS has the largest MSE (see Figures; 1,3,4,5 and 6). The MSEs for ML estimate of R based on URRSS are less than the corresponding Bayesian estimate for most values of (m, n) at $R = 0.9$.

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Figures

Figure 1. MSE of R=0.25 at *m=n*

Figure 2. MSE of R=0.5 at *m=n*

Figure 3. MSE of R=0.5 at *m<n*

Figure 4. MSE of R=0.75 at *m<n*

Figure 5. MSE of R=0.25 at *m>n*

Figure 6. MSE of R=0.5 at *m>n*

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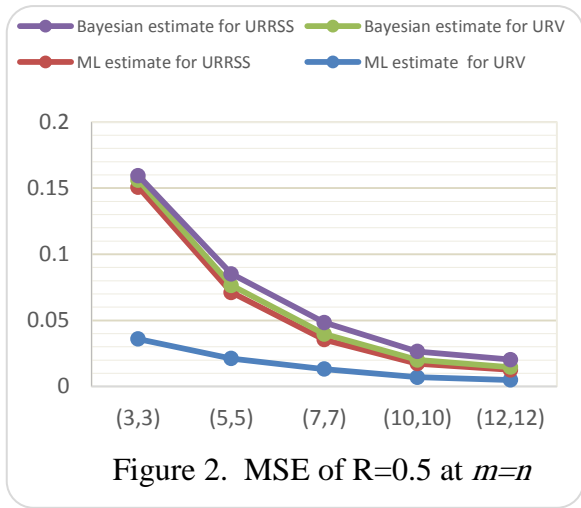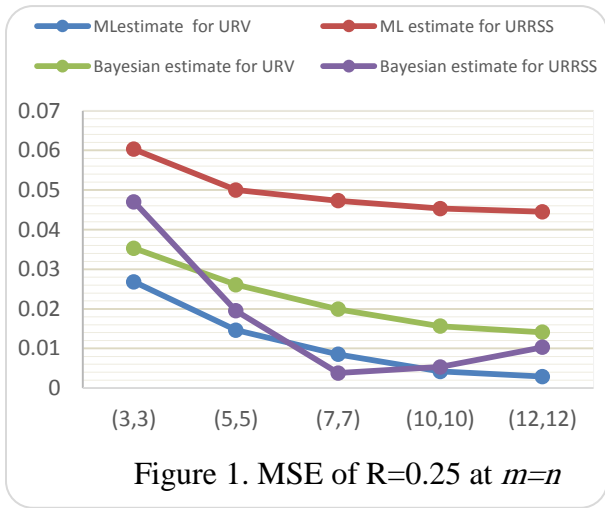
Table 1: MSE and bias for the ML estimates of *R* for different sample sizes based on URV and URRSS

R	(*m*, *n*)	URRSS		URV		*eff* $(\hat{\hat{R}}, \hat{R})$
		MSE$(\hat{\hat{R}})$	bias	MSE$(\hat{R})$	bias	
0.25	(3,3)	0.060	-0.180	0.027	0.029	2.250
	(3,6)	0.059	-0.240	0.017	0.013	3.427
	(3,9)	0.060	-0.243	0.016	0.033	3.696
	(5,5)	0.050	-0.214	0.015	0.019	3.425
	(5,10)	0.055	-0.234	0.012	0.047	4.516
	(6,3)	0.068	-0.066	0.020	0.025	3.414
	(7,7)	0.047	-0.215	0.009	0.009	5.565
	(9,3)	0.076	0.000	0.017	0.005	4.584
	(10,5)	0.036	-0.167	0.009	-0.021	4.034
	(10,10)	0.045	-0.212	0.004	0.006	10.786
	(12,12)	0.045	-0.210	0.003	0.003	15.345
0.5	(3,3)	0.115	-0.006	0.036	0.000	3.198
	(3,6)	0.106	-0.222	0.027	-0.008	3.980
	(3,9)	0.116	-0.272	0.024	0.019	4.810
	(5,5)	0.050	-0.005	0.021	0.000	2.359
	(5,10)	0.043	-0.129	0.017	0.046	2.587
	(6,3)	0.105	0.219	0.027	0.008	3.933
	(7,7)	0.022	0.000	0.013	0.000	1.697
	(9,3)	0.109	0.257	0.024	-0.023	4.649
	(10,5)	0.043	0.124	0.017	-0.045	2.607
	(10,10)	0.010	0.000	0.007	0.001	1.464
	(12,12)	0.008	-0.040	0.005	-0.001	1.512
0.75	(3,3)	0.062	0.171	0.022	-0.027	2.768
	(3,6)	0.067	0.062	0.019	-0.031	3.595
	(3,9)	0.076	-0.005	0.018	-0.046	4.151
	(5,5)	0.063	-0.241	0.014	-0.017	4.455
	(5,10)	0.036	0.165	0.016	-0.030	2.238
	(6,3)	0.058	0.239	0.015	0.003	3.826
	(7,7)	0.063	-0.241	0.008	-0.011	7.494
	(9,3)	0.059	0.243	0.013	-0.001	4.600
	(10,5)	0.055	0.233	0.012	-0.046	4.586
	(10,10)	0.046	0.212	0.004	-0.004	10.484
	(12,12)	0.062	-0.250	0.003	-0.003	19.904
0.9	(3,3)	0.018	0.077	0.011	-0.033	1.656
	(3,6)	0.018	0.076	0.009	-0.029	2.016
	(3,9)	0.018	0.075	0.039	-0.142	0.463
	(5,5)	0.010	0.100	0.005	-0.014	2.205
	(5,10)	0.010	0.100	0.003	-0.001	3.948
	(6,3)	0.010	0.100	0.005	-0.013	1.934
	(7,7)	0.010	0.100	0.003	-0.009	3.607
	(9,3)	0.010	0.100	0.005	-0.017	2.178
	(10,5)	0.010	0.100	0.003	-0.019	3.068
	(10,10)	0.010	0.100	0.001	-0.005	7.258
	(12,12)	0.010	0.100	0.002	-0.017	6.010

Table 2: MSE and bias for the Bayesian estimates of R for different sample sizes based on URV and URRSS

R	(m, n)	URRSS		URV		$eff\ (R^*, R^{**})$
		$MSE(R^{**})$	bias	$MSE\ (R^*)$	bias	
0.25	(3,3)	0.047	0.213	0.035	0.179	1.331
	(3,6)	0.086	0.286	0.039	0.189	2.225
	(3,9)	0.088	0.290	0.048	0.215	1.817
	(5,5)	0.020	0.125	0.026	0.151	0.747
	(5,10)	0.021	0.136	0.038	0.190	0.559
	(6,3)	0.001	0.014	0.020	0.129	0.061
	(7,7)	0.004	0.029	0.020	0.132	0.191
	(9,3)	0.008	-0.085	0.011	0.090	0.722
	(10,5)	0.006	-0.068	0.009	0.082	0.670
	(10,10)	0.005	-0.062	0.016	0.118	0.340
	(12,12)	0.010	-0.097	0.014	0.113	0.730
0.5	(3,3)	0.003	-0.001	0.005	0.000	0.644
	(3,6)	0.011	0.072	0.005	0.012	2.161
	(3,9)	0.010	0.074	0.006	0.037	1.837
	(5,5)	0.009	-0.003	0.005	0.000	1.603
	(5,10)	0.007	0.010	0.006	0.043	1.215
	(6,3)	0.011	-0.072	0.005	-0.012	2.111
	(7,7)	0.009	-0.001	0.004	0.000	2.074
	(9,3)	0.011	-0.079	0.006	-0.039	1.947
	(10,5)	0.008	-0.014	0.006	-0.043	1.322
	(10,10)	0.007	0.000	0.003	0.001	2.418
	(12,12)	0.006	-0.034	0.002	-0.001	2.799
0.75	(3,3)	0.048	-0.215	0.036	-0.184	1.325
	(3,6)	0.001	0.015	0.021	-0.134	0.063
	(3,9)	0.008	0.084	0.015	-0.111	0.519
	(5,5)	0.063	-0.250	0.026	-0.149	2.441
	(5,10)	0.006	0.067	0.012	0.012	0.539
	(6,3)	0.087	-0.288	0.036	-0.182	2.414
	(7,7)	0.063	-0.250	0.020	-0.133	3.103
	(9,3)	0.088	-0.291	0.040	-0.194	2.201
	(10,5)	0.022	-0.137	0.038	-0.189	0.576
	(10,10)	0.005	0.063	0.016	-0.118	0.350
	(12,12)	0.063	-0.250	0.014	-0.113	4.428
0.9	(3,3)	0.134	-0.364	0.094	-0.302	1.431
	(3,6)	0.026	-0.158	0.057	-0.234	0.456
	(3,9)	0.003	-0.051	0.057	-0.236	0.052
	(5,5)	0.079	-0.274	0.067	-0.253	1.176
	(5,10)	0.005	-0.062	0.032	-0.174	0.159
	(6,3)	0.194	-0.436	0.095	-0.304	2.052
	(7,7)	0.032	-0.170	0.051	-0.222	0.617
	(9,3)	0.196	-0.438	0.103	-0.318	1.892
	(10,5)	0.082	-0.282	0.078	-0.276	1.058
	(10,10)	0.005	-0.066	0.039	-0.194	0.137
	(12,12)	0.001	-0.018	0.050	-0.222	0.016

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5. Conclusion

In this paper, the problem of estimating $R = P(Y < X)$ for the *GIED* is addressed using both URV and URRSS. ML and Bayesian estimators of R are compared via the MSE. Also, the efficiency of estimators based on URRSS with respect to the corresponding estimators based on URV is computed. When comparing the performance of the estimated R, it is observed that the ML estimate based on the URV is, in general, better than the corresponding based on URRSS relative to their biases and MSEs. MSE of the Bayes estimate based on URRSS approach is less than the MSE of the corresponding ML estimate. The simulation study indicates that in order to estimate the reliability function in stress-strength model for *GIED* using ML method of estimation, URV scheme is preferable than URRSS scheme.

Acknowledgement

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