A Note on Blest's Measure of Kurtosis
(With reference to Weibull Distribution)

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Abstract

A number of measures of peakedness, shoulder heaviness and tailedness have been proposed in literature. A Weibull family of distributions has peakedness as well as other characteristics with varying degrees depending on the shape parameter of the distribution. In this paper we apply Blest's approach for measuring kurtosis of Weibull distributions and make some remarks on its comparison with Pearson's measure of kurtosis.

Keywords: Negatively Skewed Distribution (NSD), Positively Skewed Distribution (PSD)

1. Introduction

Pearson's measure of kurtosis has been often criticized as it does not focus adequately on the central part of a distribution. Its other limitation is that it distorts its real purpose for skewed distributions. Now there exists in literature a number of other measures of kurtosis such as proposed by Horn (1983), Hosking (1992), Blest (2003), and Zenga (2005). Some consistency is reported to be prevailing among these measures for selected distributions. This paper aims at applying Blest's approach in measuring kurtosis of the Weibull family as it includes both skewed and symmetrical distributions. Some remarks are also made on its relation with Pearson's measure of kurtosis.

2. Pearson's and Blest's measures of Kurtosis

Pearson's measures of skewness and kurtosis are:

$$\alpha_3 = \frac{\mu_3}{\mu_2^{3/2}} \qquad , \qquad \alpha_4 = \frac{\mu_4}{\mu_2^2}$$

Where μ_2, μ_3, μ_4 are the second, third and fourth standardized moments of a densityfunction. The following graph provides these measures for Weibull distribution when its parameter varies; the lines for zero skewness and the normal kurtosis are shown drawn for comparison.

Fig.1: Kurtosis and Skewness for the Weibull distribution
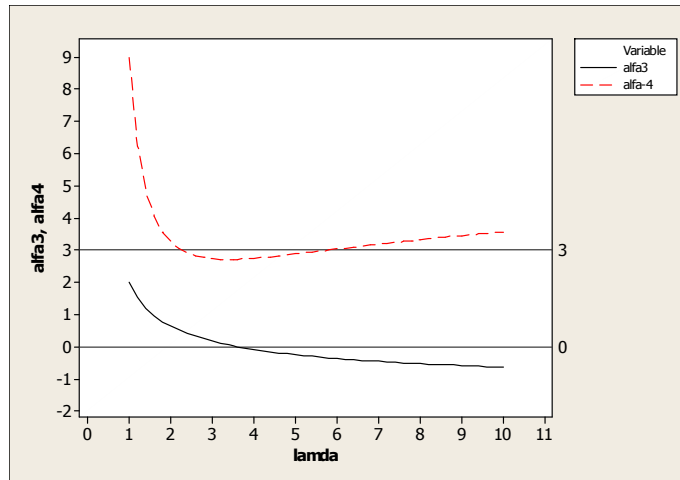
From the fig we see that the coefficient of Kurtosis has a value 3 at the two points $\lambda = 2.2, \lambda = 5.6$. For $\lambda = 2.2$ the distribution is positively skewed ($\alpha_3 = 0.508696$), and for $\lambda = 5.6$ the corresponding distribution is negatively skewed ($\alpha_3 = -0.329871$), the distribution with $\lambda = 3.6$ is symmetrical and its coefficient of skewness α_3 has a value >0, =0, <0 depending on Weibull's parameter λ. The effect of increase in lamda causes slow skewness in a distribution and a slow increase in Kurtosis. But this effect is too rapid in skewness and kurtosis when the parameter lamda decreases. We classify these distributions as:

Class A	$\lambda \leq 3.4$	1.1
Class B	$3.4 < \lambda < 3.8$	
Class C	$\lambda \geq 3.8$	

2.2 Blest's measure of Kurtosis

Dayson (1943), Kaplansky (1945), Finucan (1964) and some others consider the standardized fourth central moment α_4 as a poor indicator of a distribution's shape. Blest also asserts that since α_4 includes the effect of skewness, it does not reveal a correct picture of the peakedness and tailedness of a distribution. Blest's approach separates the effect of skewness in Pearson's kurtosis measure α_4 to improve his measure of kurtosis α_4^*. He proposes a measure of Kurtosis adjusted for skewness introducing standardized meson about which the third moment vanishes. The quantity $\zeta = \mu + f\sigma$ is the standardized Meson and the third moment

$$E((X - \zeta)^3) = 0 \Rightarrow f^3 + 3f - \alpha_3 = 0 \qquad 2.1$$

$$f = (\sqrt{.25\alpha_3^2 + 1} + .5\alpha_3)^{1/3} - (\sqrt{.25\alpha_3^2 + 1} - .5\alpha_3)^{1/3}$$

is the standardized value of Meson. f is the effect of asymmetry that is based on the third moment about the mean. f Involves only the coefficient of skewness that may be > 0, =0, < 0 for the positively, symmetrical, and negatively skewed

distribution so the value of $f > 0$, $=0$, < 0. The following graph shows the value of α_3 and f against the value of Weibull's parameter.

Fig.2: Meson and Pearson's Skewness.
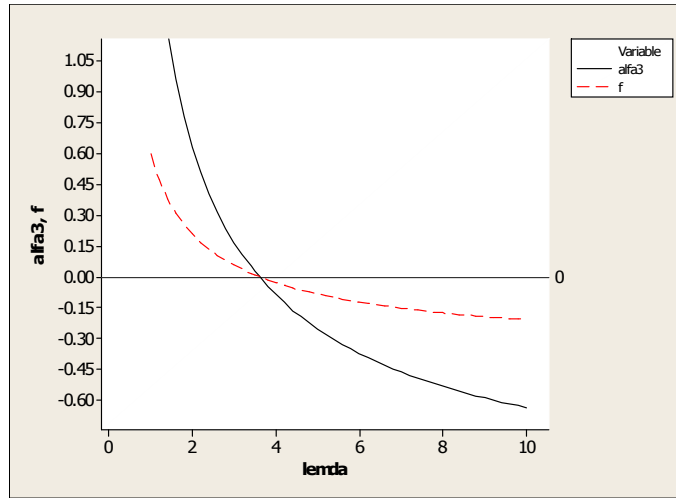
This graph presents the behavior of f and α_3 for -1 to 1 as lamda increases. Both indicate a downward trend as lamda increases, and assume a value zero at λ=3.4 and different patterns in Classes A, C of Weibull distributions. In the following graph the effect of f^3 nearly vanishes and equation 2.1 reduces to $\alpha_3 = 3f$.

Fig.3: Contributions of $3f$ and f^3
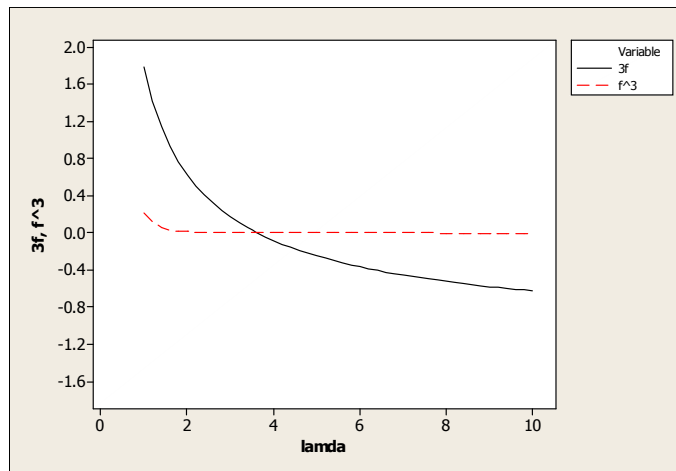
Blest concludes that for distributions with the coefficient of skewness α₃ between -1 to 1 the following holds.

i)   f = (1/3) α₃.
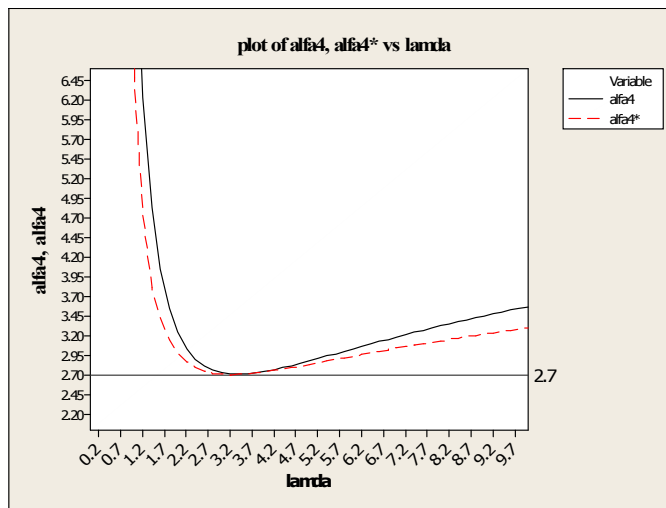
ii) $\alpha_4{}^* = \alpha_4 - 3((1 + f^2)^2 - 1)$

For Weibull distribution the above can be readily verified. As for the second remark, the impact of f is that if $3((1+f^2)^2 - 1) > 0$

Then $\alpha_4^* < \alpha_4$ and

If $3((1+f^2)^2 - 1) = 0$, then $\alpha_4^* = \alpha_4$.

The following graph shows that Blest's kurtosis always has a value less than that of Pearson's kurtosis except when lamda in Class B. The gap increases in Class C.

**Fig. 4: Relationship between α_4 and α_4^***

2.3 Relation between median, mean and meson

In the neighborhood of λ =3.4 the Weibull distribution is nearly symmetrical, with its Mean, Median and Meson about equal. In general, the following holds depending on λ.

❊ Meson > Mean > median

❊ Meson =Mean = median

❊ Meson < Mean < median

A Weibull distribution may be positively skewed, symmetrical or negatively skewed. In the first situation λ ≤ 3.4, in the second, is 3.4<λ<3.8., and in the third, λ≥3.8. Blest considers only the case of positively skewed distribution but in this paper we examine his proposed measure of kurtosis for the negatively skewed distribution. For this purpose we construct the following graphs.

Fig.5: Mean, Median and Meson Weibull Distributions
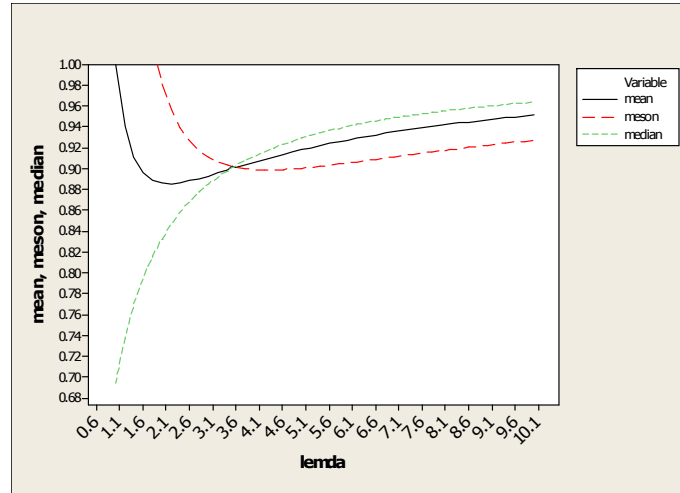
The mean, median amd meson, Mean, median are close in the neighborhood of λ=3.4. For class A the gaps among them increases rapidly unlike that in Class C. The order of these measures is opposite in Classes A and C; for Class A the order is Meson > Mean > median.*The gap between median and meson is another way measuring the degree of kurtosis. Distributions in Class C are far flatter than distributions in Class A.*

2.4 Additional aspects of Blest' approach

Prob (a)

For the positively skewed distribution (PSD) in Class A, the mass between Median to Mean (Prob a) decreases as the value of λ increases. For a symmetrical distribution, Prob(a) is small and approaches zero at $\lambda = 3.6$. For the NSD in Class C the Prob(a) C increases as the value of λ increases. Prob (a) is calculated as follows for PSD and NSD respectively.

$$\Pr ob(a) = \Pr o(Mx < X < \mu) = \int_{Mx}^{\mu} f(X)dx$$

$$\Pr ob(a) = Pro(\zeta < X < \mu) = \int_{\zeta}^{\mu} f(X)dx$$

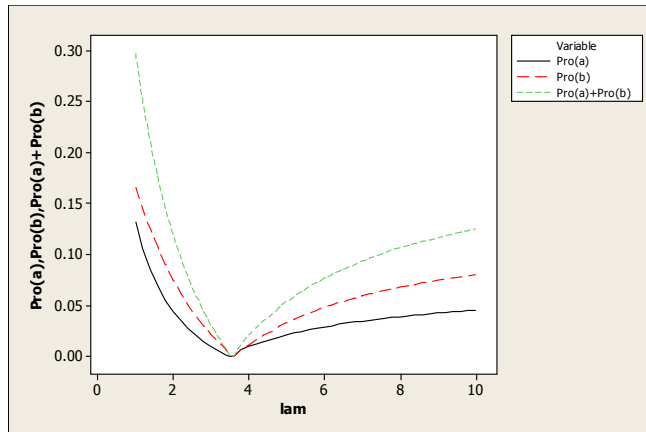Blest proposed Prob (a) only for the PSD.

Prob (b)

For PSD the mass between Mean to Meson Prob.(b) in class Adecreases as the value λ increases. In class B the Prob(b) is zero. For the NSD, Prob(b) increases as the value of λ increases in class C.

Prob(b) is calculated in the following manner.

$$\Pr ob(b) = \Pr ob(\mu < X < \zeta) = \int_{\mu}^{\zeta} f(X)dx$$

$$\Pr ob(b) = \Pr o(\mu < X < Mx) = \int_{\mu}^{Mx} f(X)dx$$

Fig.6: Probabilitiesof a, b, a+b

The above graph shows probability of a, b, and a+b (median to meson for positively skewed distributions) against lamda. Blest claims that for the same degree of skewness the probability of a+b remains constant. It is true for all the distributions he considered in his paper. To verify this point we applied this concept to the Weibull family and found it holding for as well. The following graph shows the relation between Pr (meson < X < median) for Class A and Pr (median < X < meson) for Class C. For Class B this probability is obviously zero.

Fig.7: {Pro(a)+Pro(b)} and α_3
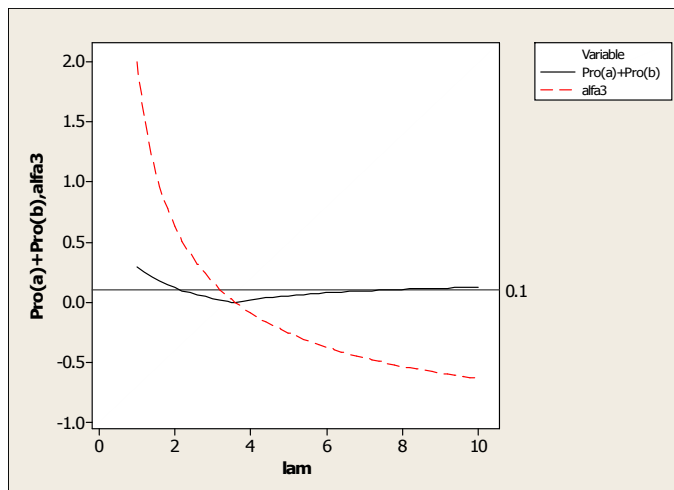
As lamda increases there is a somewhat sharp fall in the probability (a+b) till it has a value 3.5 and then there is a slow rise to 0.1 as it further increases. ***This probability also reflects the condition of kurtosis suggesting that positively skewed Weibull distributions have narrower peaks than negatively skewed Weibull distributions. This fact correlates well with both Blest's and Pearson's measures of kurtosis in Fig 4.***

2.5 General conclusions

Work on kurtosis in literature has followed two paths; one setting up orderings among them criteria to compare distribution functions, and the other in defining kurtosis measures. This paper follows the second approach to compare Blest's measure with Pearson's measure of kurtosis.

Applying these two measures to the Weibull family of distributions we find that they produce different information on kurtosis. Certain similarities do exist for some members of this family.

For nearly symmetrical Weibull distributions, Pearson's and Blest's measures almost coincide but as the amount of positive or negative skewness increases Blest's picture of the existing peakedness of a distribution becomes clearer as it removes the effect of asymmetry. Pearson's measure in this regard is found to be over pronouncingthe peakedness. The deviation between the two measures increases when a Weibull distribution is more skewed to the right.

The gap between median (meson) and meson (median) is another way measuring the degree of kurtosis for a negatively (positively) skewed distribution. Positively skewed distributions of Weibull family cover higher areas between meson and median and so they are more peaked than its negatively skewed distributions.

Blest does not prove that the probability between meson and mean remains constant for a given coefficient of skewness, but taking a number of distributions he supports this claim. We find that this assertion also holds true for Weibull distributions.

References

1. Blest, D. C. (2003). A New Measure of Kurtosis Adjusted for Skewness. *Australian & New Zealand Journal of Statistics,* 45(2), 175-179.

2. Balanda, K. P. &MacGillivray, H.L. (1988). A Critical Review. *The American Statistical Association*, 42 (2), 111-119.

3. Balanda, K. P. &MacGillivray, H.L. (1990). Kurtosis and Spread. *Canadian Journal of Statistics* 18, 17-30.

4. Dyson, F.J. (1943). A Note on Kurtosis. *Journal of the Royal Statistical Society*. 106 (4), 360-361.

5. Finucan, H. M. (1964). A Note on Kurtosis. *Journal of the Royal Statistical Society*, 26 (1), 111-112.

6. Fiori, A. M. &Zenga, M. (2005). The Meaning of Kurtosis: TheInfluence Function and an Early Intuition by L. Faleschini. *StatisticaAnno,* 65 (2), 131-140.

7. Hosking, J. R. M (1992). Moments or L moments? An example comparing two measures of distributional shape. *The American Statistician Association*, 46 (3), 186-189.

8. Horn, P.S. (1983). A Measure for Peakedness. *The American Statistician*, 37 (1), 55- 56.

9. Kaplansky, Irving (1945). A common error concerning Kurtosis. *The American Statistician Association*, 46 (230), 259.

10. Pearson K. (1905). The fault law and its generalizations by Fechner and Pearson. *Biometrika*, 4(1/2), 169-212.