A New Multivariate Weibull Distribution

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Abstract

We have proposed a new multivariate Weibull distribution as a compound distribution of univariate Weibull distributions. We have studied some properties of the proposed distribution. The multivariate distribution of records for proposed distribution has also been studied. Estimation of the parameters has been done alongside application to the real data set.

Keywords: Moments of residual life, Goodness-of-fit, Order Statistics, Maximum Likelihood Estimation.

Introduction

Weibull distribution, introduced by Weibull (1951), has been a popular model for modelling the lifetime data. The density and distribution function of Weibull distribution can be presented in various ways, see for example Johnson et al. (1995). A relatively simple form of the density function is

$$f\left(x;\sim,\ \right)=\sim\ \ x^{\ -1}\exp\left(-\sim x\ \ \right);\ x,\sim,\ \ >0. \qquad (1)$$

The distribution function corresponding to (1) is

$$F\left(x;\sim,\ \right)=1-\exp\left(-\sim x\ \ \right);\ x,\sim,\ \ >0. \qquad (2)$$

Since its emergence the distribution has attracted so many scholars. Various authors have used the distribution to model lifetime data and in developing new distributions. Madholkar and Srivastava (1993) have introduced the exponentiated Weibull distribution as an extension of Weibull distribution. Famoye et al. (2005) have used the Weibull distribution in context of Beta-G distribution of Eugene et al. (2002) to propose the Beta-

Weibull distribution. Cordeiro et al. (2010) have proposed the Kumaraswamy-Weibull distribution as an alternative to Weibull distribution with much wider applicability.

Chandler (1952) introduced record statistics as a model for ordered random variables. The records are defined in a sequence of independent and identically distributed random variables as under.

Suppose a sequence of independently and identically distributed random variables $X_1, X_2, \ldots, X_n$ is available with same distribution function $F(x)$. Suppose further that $Y_n = \max\{X_1, X_2, \ldots, X_n\}$ for $n \geq 1$, then X_j is an Upper Record Value of the sequence $\{X_n, n \geq 1\}$ if $Y_j > Y_{j-1}$. From this definition it is clear that X_1 is an upper record value. We also associate the indices to each record value with which they occur. These indices are called the record time $\{U(n), n > 0\}$ where

$$U(n) = \min\left\{j \mid j > U(n-1), X_j > X_{U(n-1)}, n > 1\right\}.$$

We can readily see that $U(1) = 1$. The upper record values are denoted by $X(n)$. The density function of *n*th upper record statistics is given by Chandler (1952) as

$$f_{X(n)}(x) = \frac{1}{\Gamma(n)}\left[R(x)\right]^{n-1} f(x)\,;\ -\infty < x < \infty. \tag{3}$$

The joint density function of *m*th and *n*th upper records is given by Chandler (1952) as

$$f_{X(m),X(n)}(x_1, x_2) = \frac{1}{\Gamma(m)\Gamma(n-m)} r(x_1)\, f(x_2)\left[R(x_1)\right]^{m-1} \times\left[R(x_2) - R(x_1)\right]^{n-m-1}\,;\ -\infty < x_1 < x_2 < \infty \tag{4}$$

where $R(x) = -\ln\{1 - F(x)\}$ and $r(x) = R'(x) = f(x)/\{1 - F(x)\}$.

Record statistics has been studied by several authors for various probability distributions. Ahsanullah (1992) has provided general results for distribution of records for continuous probability distributions. A comprehensive review of record statistics can be found in Ahsanullah (1995) and Nevzorov (2001). Often we have a sample from some bivariate distribution and the sample is arranged with respect to one of the variable. The automatically shuffled other variable is called concomitant of ordered variable. The concomitant of records appear when the bivariate sample is arranged with respect to records. The density function of nth concomitant of record is given by Ahsanullah (1995) as

$$f_{[Y(n)]}(y) = \int_{-\infty}^{\infty} f(y \mid x)\, f_{X(n)}(x)\, dx \tag{5}$$

where $f_{X(n)}(x)$ is defined in (3). The joint distribution of two concomitants of record values is given by Ahsanullah (1995) as

$$f_{[Y(m),Y(n)]}(y_1, y_2) = \int_{-\infty}^{\infty}\int_{x_1}^{\infty} f(y_1 \mid x_1)\, f(y_2 \mid x_2)\, f_{X(m),X(n)}(x_1, x_2)\, dx_2 dx_1, \tag{6}$$

where $f_{X(m),X(n)}(x_1,x_2)$ is given in (4). Distribution of concomitants of records for various distributions has been studied by many authors. Shahbaz et al. (2011) have studied concomitants of record statistics for a new bivariate Weibull distribution.

The distribution of concomitants has been extended to bivariate case by Shahbaz et al. (2012). The density function of nth bivariate record statistics is given as

$$f_{[Y(n),Z(n)]}(y,z)=\int_{-\infty}^{\infty} f(y,z\,|\,x)\, f_{X(n)}(x)\,dx\,, \tag{7}$$

where $f(y,z\,|\,x)$ is the conditional distribution of *Y* and *Z* given *X* = *x*. Shahbaz et al. (2014) and Shahbaz et al. (2012) have studied bivariate concomitants of records for trivariate Weibull and trivariate Inverse Weibull distribution.

Recently Arnold et al. (2009) have introduced ordering of vectors by using the distribution of concomitants. The distribution of multivariate concomitants is given by Arnold et al. (2009) as

$$f_{[\mathbf{y}(n)]}(\mathbf{y})=\int_{-\infty}^{\infty} f(\mathbf{y}\,|\,x)\, f_{X(n)}(x)\,dx \tag{8}$$

where $f(\mathbf{y}\,|\,x)$ is the conditional distribution of vector **y** given *X* and $f_{X(n)}(x)$ is defined in (3).

In this paper we have proposed a new multivariate Weibull distribution and have studied its properties. The new multivariate distribution is proposed in section 2 with some graphical study. In section 3 some common properties of the proposed distribution has been studied. Multivariate concomitants of the distribution has been studied in section 4. Application of proposed distribution is given in section 5.

The New Multivariate Weibull Distribution

Shahbaz et al. (2012) has proposed a trivariate Weibull distribution by using compounding of distributions. The density function of proposed trivariate Weibull distribution is

$$\begin{aligned} f(x,z_1,z_2) &= \alpha_1\alpha_2\alpha_3 ¤_1(x)¤_2(x,z_1)\, x^{\alpha_1-1} z_1^{\alpha_2-1} z_2^{\alpha_3-1} \\ &\quad\times \exp\left[-\left\{x^{\alpha_1}+¤_1(x)\,z_1^{\alpha_2}+¤_2(x,z_1)\,z_2^{\alpha_3}\right\}\right], \\ &\quad ¤_1(x)>0\,;\ ¤_2(x,z_1)>0\,;\ x,z_1,z_2,\alpha_1,\alpha_2,\alpha_3>0, \end{aligned} \tag{9}$$

where $¤_1(x)$ is some function of *X* and $¤_2(x,z_1)$ is some function of *X* and Z_1. Shahbaz et al. (2012) have used $¤_1(x)=x^{\alpha_1}$ and $¤_2(x,z_1)=x^{\alpha_1}z_1^{\alpha_2}$ to propose following specific density function

$$f(x,z_1,z_2)=\alpha_1\alpha_2\alpha_3 x^{3\alpha_1-1} z_1^{2\alpha_2-1} z_2^{\alpha_3-1}\exp\left[-x^{\alpha_1}\left\{1+z_1^{\alpha_2}+z_1^{\alpha_2}z_2^{\alpha_3}\right\}\right], \tag{10}$$

with $x,z_1,z_2,\alpha_1,\alpha_2,\alpha_3>0$. Using the idea given by Shahbaz et al. (2012), we have proposed the multivariate Weibull distribution in the following.

Suppose X has a Weibull distribution with density function

$$f\left(x;\alpha\right)=\alpha x^{\alpha-1}\exp\left(-x^{\alpha}\right);\ x,\alpha>0. \tag{11}$$

We write X has $W\left(1,\alpha\right)$. Now suppose that $Y_1|x$ has $W\left(\theta_1\left(x\right),\beta_1\right)$, $Y_2|\left(x,y_1\right)$ has $W\left(\theta_2\left(x,y_1\right),\beta_2\right)$, $Y_3|\left(x,y_1,y_2\right)$ has $W\left(\theta_3\left(x,y_1,y_2\right),\beta_3\right)$ and so on, such that $Y_p|\left(x,y_1,y_2,\ldots,y_{p-1}\right)$ has $W\left(\theta_p\left(x,y_1,y_2,\ldots y_{p-1}\right),\beta_p\right)$ with densities

$$f\left(y_1|x\right)=\beta_1\theta_1\left(x\right)y_1^{\beta_1-1}\exp\left\{-\theta_1\left(x\right)y_1^{\beta_1}\right\}$$

$$f\left(y_2|x,y_1\right)=\beta_2\theta_2\left(x,y_1\right)y_2^{\beta_2-1}\exp\left\{-\theta_2\left(x,y_1\right)y_2^{\beta_2}\right\}$$

$$f\left(y_3|x,y_1,y_2\right)=\beta_3\theta_3\left(x,y_1,y_2\right)y_3^{\beta_3-1}\exp\left\{-\theta_3\left(x,y_1,y_2\right)y_3^{\beta_3}\right\}$$

and

$$f\left(y_p|x,y_1,\ldots y_{p-1}\right)=\beta_p\theta_p\left(x,y_1,\ldots,y_{p-1}\right)y_3^{\beta_3-1}\exp\left\{-\theta_p\left(x,y_1,\ldots,y_{p-1}\right)y_p^{\beta_p}\right\},$$

where $\beta_i>0$ and $\theta_i\left(x,.\right)>0;i=1,2,\ldots,p$. For simplicity we use

$$f\left(y_i|x,\mathbf{y}_{i-1}\right)=\beta_i\theta_i\left(x,\mathbf{y}_{i-1}\right)\exp\left\{-\theta_i\left(x,\mathbf{y}_{i-1}\right)y_i\right\};\ i=1,2,\ldots,p,$$

where $\theta_1\left(x,\mathbf{y}_0\right)=\theta_1\left(x\right),\theta_2\left(x,\mathbf{y}_1\right)=\theta_2\left(x,y_1\right),\theta_3\left(x,\mathbf{y}_2\right)=\theta_3\left(x,y_1,y_2\right)$ and so on. Now the density function of $\left(p+1\right)-$variate distribution is

$$f\left(x,\mathbf{y}\right)=f\left(x\right)\times\prod_{i=1}^{p}f\left(y_i|x,\mathbf{y}_{i-1}\right),$$

where $\mathbf{y}=\left[y_1\quad y_2\quad\cdots\cdots\quad y_p\right]$. Using above densities, the density function of $\left(p+1\right)-$variate Weibull distribution is

$$f\left(x,\mathbf{y}\right)=\alpha x^{\alpha-1}\exp\left(-x^{\alpha}\right)\times\prod_{i=1}^{p}\beta_i\theta_i\left(x,\mathbf{y}_{i-1}\right)\exp\left\{-\theta_i\left(x,\mathbf{y}_{i-1}\right)y_i\right\}, \tag{12}$$

where $x,y_i>0$. The distribution (12) can be studied for various choices of $\theta_i\left(x,\mathbf{y}_{i-1}\right)$. In this paper we have studied the distribution (12) for following choices of $\theta_i\left(x,\mathbf{y}_{i-1}\right)$

$\theta_1\left(x\right)=x^{\alpha}$, $\theta_2\left(x,y_1\right)=x^{\alpha}y_1^{\beta_1}$, $\theta_3\left(x,y_1,y_2\right)=x^{\alpha}y_1^{\beta_1}y_2^{\beta_2}$ and

$\theta_p\left(x,y_1,\ldots,y_{p-1}\right)=x^{\alpha}y_1^{\beta_1}\cdots y_{p-1}^{\beta_{p-1}}$. Using these in (12), the density function of $\left(p+1\right)-$variate Weibull distribution is

$$\begin{aligned}f\left(x,\mathbf{y}\right)=&\ \alpha x^{\alpha-1}\exp\left(-x^{\alpha}\right)\times\beta_1x^{\alpha}y_1^{\beta_1-1}\exp\left(-x^{\alpha}y_1^{\beta_1}\right)\\&\times\beta_2x^{\alpha}y_1^{\beta_1}y_2^{\beta_2-1}\exp\left(-x^{\alpha}y_1^{\beta_1}y_2^{\beta_2}\right)\times\beta_3x^{\alpha}y_1^{\beta_1}y_2^{\beta_2}y_3^{\beta_3-1}\exp\left(-x^{\alpha}y_1^{\beta_1}y_2^{\beta_2}y_3^{\beta_3}\right)\times\cdots\\&\times\beta_3\left(x^{\alpha}y_1^{\beta_1}y_2^{\beta_2}\cdots y_3^{\beta_3-1}\right)\exp\left(-x^{\alpha}y_1^{\beta_1}y_2^{\beta_2}y_3^{\beta_3}\cdots y_p^{\beta_p}\right)\end{aligned}$$

or

$$\begin{aligned}f\left(x,\mathbf{y}\right)=&\ \alpha\left(\prod_{i=1}^{p}\beta_i\right)\times x^{\left(p+1\right)\alpha-1}\times\left(\prod_{i=1}^{p}y_i^{\left(p+1-i\right)\beta_i-1}\right)\\&\times\exp\left[-x^{\alpha}\left(1+\sum_{i=1}^{p}\prod_{j=1}^{i}y_j^{\beta_j}\right)\right],\end{aligned} \tag{13}$$

with $x > 0$ and $\mathbf{y} > 0$. The joint marginal distribution of **y** is readily obtained as

$$f(\mathbf{y}) = \int_0^\infty f(x,\mathbf{y})\,dx = \int_0^\infty \left(\prod_{i=1}^{p} \sim_i\right) \times x^{(p+1)\ \ -1} \times \left(\prod_{i=1}^{p} y_i^{(p+1-i)\sim_i -1}\right)$$

$$\times\times \exp\left[-x\ \left(1+\sum_{i=1}^{p}\prod_{j=1}^{i} y_j^{\sim_j}\right)\right] dx,$$

which after simplification becomes

$$f(\mathbf{y}) = \frac{\Gamma(p)\left(\prod_{i=1}^{p}\sim_i\right)\left(\prod_{i=1}^{p} y_i^{(p+1-i)\sim_i -1}\right)}{\left(1+\sum_{i=1}^{p}\prod_{j=1}^{i} y_j^{\sim_j}\right)^{p+1}}\ ;\ \mathbf{y} > 0\ ,\ \sim_i > 0. \tag{14}$$

The marginal distribution of *i*th component of **y** is

$$f(y_i) = \int_0^\infty \cdots \int_0^\infty \int_0^\infty \cdots \int_0^\infty f(\mathbf{y})\,dy_1 \cdots dy_{i-1} dy_{p+1} dy_p$$

$$= \int_0^\infty \cdots \int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{\Gamma(p)\left(\prod_{i=1}^{p}\sim_i\right)\left(\prod_{i=1}^{p} y_i^{(p+1-i)\sim_i -1}\right)}{\left(1+\sum_{i=1}^{p}\prod_{j=1}^{i} y_j^{\sim_j}\right)^{p+1}} dy_1 \cdots dy_{i-1} dy_{i+1} dy_p$$

which after simplification become

$$f(y_i) = \frac{\sim_i y_i^{\sim_i -1}}{\left(1+y_i^{\sim_i}\right)^2}\ ;\ y_i, \sim_i > 0. \tag{15}$$

The joint marginal distribution of any pair (y_i, y_k) is given as

$$f(y_i) = \int_0^\infty \cdots \int_0^\infty \int_0^\infty \cdots \int_0^\infty \int_0^\infty \cdots \int_0^\infty f(\mathbf{y})\,dy_1 \cdots dy_{i-1} dy_{i+1} \cdots dy_{k-1} dy_{k+1} \cdots dy_p$$

$$= \int_0^\infty \cdots \int_0^\infty \int_0^\infty \cdots \int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{\Gamma(p)\left(\prod_{i=1}^{p}\sim_i\right)\left(\prod_{i=1}^{p} y_i^{(p+1-i)\sim_i -1}\right)}{\left(1+\sum_{i=1}^{p}\prod_{j=1}^{i} y_j^{\sim_j}\right)^{p+1}}$$

$$\times dy_1 \cdots dy_{i-1} dy_{i+1} \cdots dy_{k-1} dy_{k+1} \cdots dy_p,$$

which after simplification becomes

$$f(y_i, y_k) = \frac{2\sim_i \sim_k y_i^{2\sim_i -1} y_k^{\sim_k -1}}{\left(1+y_i^{\sim_i} + y_i^{\sim_i} y_k^{\sim_k}\right)^3}\ ;\ y_i, y_k, \sim_i, \sim_k > 0. \tag{16}$$

We now obtain the entries of mean vector and covariance matrix of **y** as under.

The *q*th moment of Y_i is

$$E\left(Y_i^q\right) = \int_0^\infty y_i^q f(y_i)\,dy_i = \int_0^\infty y_i^q \frac{\sim_i y_i^{\sim_i -1}}{\left(1+y_i^{\sim_i}\right)^2} dy_i$$

$$= \frac{q}{\sim_i}\Gamma\left(\frac{q}{\sim_i}\right)\Gamma\left(1-\frac{q}{\sim_i}\right);\ \sim_i > q.$$

Using $q = 1$, the *i*th entry of mean vector of Y_i is

$$E(Y_i) = \frac{1}{\sim_i}\Gamma\left(\frac{1}{\sim_i}\right)\Gamma\left(1-\frac{1}{\sim_i}\right);\ \sim_i > 1. \tag{17}$$

Further, using q = 2, we have

$$E\left(Y_i^2\right)=\frac{2}{\sim_i}\Gamma\left(\frac{2}{\sim_i}\right)\Gamma\left(1-\frac{2}{\sim_i}\right);\sim_i>2. \qquad (18)$$

The variance can be easily obtained from (17) and (18). Further

$$E\left(Y_iY_k\right)=\int_0^\infty\int_0^\infty y_iy_k\,f\left(y_i,y_k\right)dy_idy_k=\int_0^\infty\int_0^\infty y_iy_k\,\frac{2\sim_i\sim_k y_i^{2\sim_i-1}y_k^{\sim_k-1}}{\left(1+y_i^{\sim_i}+y_i^{\sim_i}y_k^{\sim_k}\right)^3}dy_idy_k,$$

which after simplification becomes

$$E\left(Y_iY_k\right)=\Gamma\left(1-\frac{1}{\sim_i}\right)\Gamma\left(1+\frac{1}{\sim_k}\right)\Gamma\left(1+\frac{1}{\sim_i}-\frac{1}{\sim_k}\right);\sim_i>1,\ \sim_k>1. \qquad (19)$$

The covariance can be obtained from (17) and (19). Following table contains means, variances, covariances and correlations between Y_i and Y_k for selected values of $\sim_i$ and $\sim_k$. From the table we can see that that the means, variances, covariances and correlations decrease with increase in $\sim_i$ and $\sim_k$.

Further, the joint conditional distribution of **y** given $X = x$ is obtained from (11) and (13) and is given as

$$f\left(\mathbf{y}\,|\,x\right)=\frac{f\left(x,\mathbf{y}\right)}{f\left(x\right)}=\left(\prod_{i=1}^p\sim_i\right)\times x^p\ \times\left(\prod_{i=1}^p y_i^{(p+1-i)\sim_i-1}\right)$$
$$\times\exp\left[-x\ \left(1+\sum_{i=1}^p\prod_{j=1}^i y_j^{\sim_j}\right)\right]. \qquad (20)$$

The conditional moments of order q for distribution (20) are given as

$$E\left(Y_1^q\,|\,x\right)=x^{-qp/\sim_1}\Gamma\left(1+\frac{q}{\sim_1}\right). \qquad (21)$$

and $$E\left(Y_2^q\,|\,x\right)=\frac{p}{\sim_i}\Gamma\left(\frac{q}{\sim_i}\right)\Gamma\left(1-\frac{q}{\sim_1}\right);\sim_1>q\,;\,i=2,3,\ldots,p. \qquad (22)$$

The conditional means and conditional variances can be computed from (21) and (22). Further, the joint conditional moments for distribution (20) are

$$E\left(Y_1Y_2\,|\,x\right)=\frac{1}{\sim_2}x^{-\ /\sim_1}\Gamma\left(1+\frac{1}{\sim_1}-\frac{1}{\sim_2}\right)\Gamma\left(\frac{1}{\sim_2}\right)$$

$$E\left(Y_1Y_3\,|\,x\right)=\frac{1}{\sim_1\sim_3}x^{-\ /\sim_1}\Gamma\left(\frac{1}{\sim_1}\right)\Gamma\left(\frac{1}{\sim_3}\right)\Gamma\left(1-\frac{1}{\sim_2}\right)$$

$$E\left(Y_iY_k\,|\,x\right)=\Gamma\left(1-\frac{1}{\sim_i}\right)\Gamma\left(1+\frac{1}{\sim_k}\right)\Gamma\left(1+\frac{1}{\sim_i}-\frac{1}{\sim_k}\right);\,i>1,\,k>3.$$

The conditional mean vector and covariance matrix can be computed for specific values of $\sim_i$, $\sim_k$,   and x.

We now obtain the distribution of multivariate concomitants for the new multivariate Weibull distribution.

Table 1: Means, Variances and Covariance

α_i	α_k	μ_{Yi}	μ_{Yi}	$\sigma^2_{Y_i}$	$\sigma^2_{Y_i}$	$\sigma_{Y_iY_k}$	$\rho_{Y_iY_k}$
3.0	3.0	1.209	1.209	0.956	0.956	-0.253	-0.265
3.0	3.5	1.209	1.148	0.956	0.523	-0.201	-0.284
3.0	4.0	1.209	1.111	0.956	0.337	-0.167	-0.294
3.0	4.5	1.209	1.086	0.956	0.238	-0.143	-0.300
3.0	5.0	1.209	1.069	0.956	0.179	-0.125	-0.304
3.5	3.0	1.148	1.209	0.523	0.956	-0.215	-0.304
3.5	3.5	1.148	1.148	0.523	0.523	-0.170	-0.325
3.5	4.0	1.148	1.111	0.523	0.337	-0.141	-0.336
3.5	4.5	1.148	1.086	0.523	0.238	-0.121	-0.342
3.5	5.0	1.148	1.069	0.523	0.179	-0.106	-0.346
4.0	3.0	1.111	1.209	0.337	0.956	-0.188	-0.331
4.0	3.5	1.111	1.148	0.337	0.523	-0.148	-0.353
4.0	4.0	1.111	1.111	0.337	0.337	-0.123	-0.365
4.0	4.5	1.111	1.086	0.337	0.238	-0.105	-0.371
4.0	5.0	1.111	1.069	0.337	0.179	-0.092	-0.375
4.5	3.0	1.086	1.209	0.238	0.956	-0.168	-0.352
4.5	3.5	1.086	1.148	0.238	0.523	-0.132	-0.375
4.5	4.0	1.086	1.111	0.238	0.337	-0.109	-0.386
4.5	4.5	1.086	1.086	0.238	0.238	-0.094	-0.393
4.5	5.0	1.086	1.069	0.238	0.179	-0.082	-0.396
5.0	3.0	1.069	1.209	0.179	0.956	-0.152	-0.368
5.0	3.5	1.069	1.148	0.179	0.523	-0.120	-0.391
5.0	4.0	1.069	1.111	0.179	0.337	-0.099	-0.403
5.0	4.5	1.069	1.086	0.179	0.238	-0.084	-0.409
5.0	5.0	1.069	1.069	0.179	0.179	-0.074	-0.413

Distribution of Multivariate Concomitants

The concomitants of record values have been studied by several authors. Shahbaz et al. (2012) have studied the bivariate concomitants of record values for a trivariate Weibull distribution. Arnold et al. (2009) have provided the distribution of multivariate concomitants of record values which is given in (8) as

$$f_{[\mathbf{y}(n)]}(\mathbf{y}) = \int_{-\infty}^{\infty} f(\mathbf{y} \mid x)\, f_{X(n)}(x)\,dx,$$

where

$$f_{X(n)}(x) = \frac{1}{\Gamma(n)}\left[R(x)\right]^{n-1} f(x)\,;\ -\infty < x < \infty.$$

We now obtain the distribution of multivariate concomitants for the new multivariate Weibull distribution given in previous section. The density function of new multivariate Weibull distribution is given in (13). The marginal distribution of X is given in (11) and the conditional distribution of y given x is given in (20). Now for distribution (11) we have

$$F(x) = 1 - \exp(-x) \; ; \; x > 0 \, ,$$

so

$$R(x) = -\ln\left[1 - F(x)\right] = x \; ,$$

and

$$f_{X(n)}(x) = \frac{}{\Gamma(n)} x^{n \; -1} \exp(-x). \tag{23}$$

Now using (20) and (23) in (8), the distribution of multivariate concomitant of order statistics for new multivariate Weibull distribution is obtained as under

$$f_{[\mathbf{y}(n)]}(\mathbf{y}) = \frac{}{\Gamma(n)} \int_0^\infty \left(\prod_{i=1}^{p} \sim_i\right) \times x^{(p+n) \; -1} \times \left(\prod_{i=1}^{p} y_i^{(p+1-i)\sim_i - 1}\right)$$
$$\times \exp\left[-x \; \left(1 + \sum_{i=1}^{p} \prod_{j=1}^{i} y_j^{\sim_j}\right)\right] dx$$

or

$$f_{[\mathbf{y}(n)]}(\mathbf{y}) = \frac{}{\Gamma(n)} \left(\prod_{i=1}^{p} \sim_i\right) \left(\prod_{i=1}^{p} y_i^{(p+1-i)\sim_i - 1}\right) \int_0^\infty x^{(p+n) \; -1} \exp\left[-x \; \left(1 + \sum_{i=1}^{p} \prod_{j=1}^{i} y_j^{\sim_j}\right)\right] dx$$

which after simplification becomes

$$f_{[\mathbf{y}(n)]}(\mathbf{y}) = \frac{\Gamma(n+p)}{\Gamma(n)} \frac{\left(\prod_{i=1}^{p} \sim_i\right)\left(\prod_{i=1}^{p} y_i^{(p+1-i)\sim_i - 1}\right)}{\left\{\left(1 + \sum_{i=1}^{p} \prod_{j=1}^{i} y_j^{\sim_j}\right)\right\}^{n+p}} \; ; \; y_i > 0 \, . \tag{24}$$

The marginal distribution of *i*th concomitant is readily written from (24) as

$$f_{[Y_i(n)]}(y_i) = \frac{n \sim_i y_i^{\sim_i - 1}}{\left(1 + y_i^{\sim_i}\right)^{n+p}} \; ; \; y_i > 0 \, .$$

The joint marginal distribution of two concomitants is obtained from (24) as

$$f_{[Y_i(n), Y_k(n)]}(y_i, y_k) = \frac{n(n+1) \sim_i \sim_k y_i^{2\sim_i - 1} y_k^{\sim_k - 1}}{\left(1 + y_i^{\sim_i} + y_i^{\sim_i} y_k^{\sim_k}\right)^{n+2}} \; ; \; y_i, y_k > 0 \, . \tag{25}$$

The *q*th moment of *i*th concomitant is readily obtained from (24) as

$$E\left(Y_{i(n)}^{q}\right) = \frac{1}{\Gamma(n)} \Gamma\left(1 + \frac{q}{\sim_i}\right) \Gamma\left(n - \frac{q}{\sim_i}\right) ; \sim_i > \frac{q}{n} \cdot \tag{26}$$

The mean and variance can be easily obtained from (26). Further, the product moment between *i*th and *k*th concomitant is

$$E\left(Y_{i(n)} Y_{k(n)}\right) = \frac{1}{\Gamma(n)} \Gamma\left(n - \frac{1}{\sim_i}\right) \Gamma\left(1 + \frac{1}{\sim_k}\right) \Gamma\left(1 + \frac{1}{\sim_i} - \frac{1}{\sim_k}\right). \tag{27}$$

The covariance con be computed by using (26) and (27). The mean, variance, covariance and correlation coefficient can be computed for various choices of n, $\sim_i$ and $\sim_k$.

Estimation

In this section we have discussed maximum likelihood estimation for parameters of new multivariate Weibull distribution. For this consider the density function of new multivariate Weibull distribution given in (13) as

$$f(x,\mathbf{y}) = \left(\prod_{i=1}^{p}\sim_i\right)\times x^{(p+1)\quad -1}\times\left(\prod_{i=1}^{p} y_i^{(p+1-i)\sim_i -1}\right)$$
$$\times\exp\left[-x\quad\left(1+\sum_{i=1}^{p}\prod_{j=1}^{i} y_j^{\sim_j}\right)\right].$$

The log of density function is given as

$$\ln\{f(x,\mathbf{y})\} = \ln\quad + \sum_{i=1}^{p}\ln\sim_i + \{(p+1)\quad -1\}\ln x + \sum_{i=1}^{p}\{(p+1-i)\sim_i - 1\}\ln y_i$$
$$-x\quad\left(1+\sum_{i=1}^{p}\prod_{j=1}^{i} y_j^{\sim_j}\right).$$

The log of likelihood function is immediately written as

$$\ln\{L(\sim_i,\quad)\} = n\ln\quad + n\sum_{i=1}^{p}\ln\sim_i + \{(p+1)\quad -1\}\sum_{h=1}^{n}\ln x_h$$
$$+\sum_{i=1}^{p}\{(p+1-i)\sim_i - 1\}\sum_{h=1}^{n}\ln y_{ih} - \sum_{h=1}^{n} x_h\quad\left(1+\sum_{i=1}^{p}\prod_{j=1}^{i} y_{jh}^{\sim_j}\right).$$

The derivatives of log of likelihood function *wrt* the parameter is

$$\frac{\partial\ln\{L(\sim_i,\quad)\}}{\partial\quad} = \frac{n}{\quad} + (p+1)\sum_{h=1}^{n}\ln x_h - \sum_{h=1}^{n} x_h\quad\ln x_h$$
$$\times\left(1+\sum_{i=1}^{p}\prod_{j=1}^{i} y_{jh}^{\sim_j}\right). \qquad (28)$$

and

$$\frac{\partial\ln\{L(\sim_i,\quad)\}}{\partial\sim_i} = \frac{n}{\sim_i} + (p+1-i)\sum_{h=1}^{n}\ln y_{ih} - \sum_{h=1}^{n} x_h\quad\ln y_{ih}\left(\sum_{k=i}^{p}\prod_{j=1}^{k} y_{jh}^{\sim_j}\right). \qquad (29)$$

The maximum likelihood estimators can be obtained by solving (p + 1)–equations in (28) and (29). The solution is obviously done by using some numerical method.
The entries of observed Fisher information matrix for parameter are given as

$$\frac{\partial^2\ln\{L(\sim_i,\quad)\}}{\partial\quad^2} = -\frac{n}{\quad^2} - \sum_{h=1}^{n} x_h\quad(\ln x_h)^2\left(1+\sum_{i=1}^{p}\prod_{j=1}^{i} y_{jh}^{\sim_j}\right),$$

and

$$\frac{\partial^2\ln\{L(\sim_i,\quad)\}}{\partial\quad\partial\sim_m} = \sum_{h=1}^{n} x_h\quad(\ln y_{ih}y_{mh})^2\left(\sum_{k=m}^{p}\prod_{j=1}^{k} y_{jh}^{\sim_j}\right).$$

The entries of Fisher information matrix *wrt* the parameters $\sim 's$ are

$$\frac{\partial^2 \ln\left\{L\left(\sim_i, \quad\right)\right\}}{\partial \sim_i^2} = -\frac{n}{\sim_i^2} - \sum_{h=1}^{n} x_h \left(\ln y_{ih}\right)^2 \left(\sum_{k=i}^{p} \prod_{j=1}^{k} y_{jh}^{\sim_j}\right),$$

and

$$\frac{\partial^2 \ln\left\{L\left(\sim_i, \quad\right)\right\}}{\partial \sim_i \partial \sim_m} = -\sum_{h=1}^{n} x_h \left(\ln y_{ih} \ln y_{mh}\right)\left(\sum_{k=m}^{p} \prod_{j=1}^{k} y_{jh}^{\sim_j}\right).$$

The observed Fisher Information matrix can be computed for a given data.

Simulation Study and Application

In this section we have given simulation study and real data application of the proposed multivariate Weibull distribution. The simulation study has been conducted to see the performance of maximum likelihood estimates and the real data application has been conducted to see suitability of proposed multivariate distribution.

Simulation Study

In this sub-section we have presented the simulation results to see the performance of maximum likelihood estimates of proposed multivariate Weibull distribution. The simulation study has been carried out by using trivariate Weibull distribution. The algorithm for simulation is given below

1. Draw sample of size n, for n = 50; 100; 300 and 500, from Weibull distribution with shape parameter and scale parameter 1. Denote this sample with variable *X*.
2. For each observation of X, draw sample of size 1 from Weibull distribution with shape parameter $\sim_1$ and scale parameter x . Repeat this procedure for all observations of *X*. Denote this sample with variable Y_1.
3. For each observation of X and Y_1, draw sample of size 1 from Weibull distribution with shape parameter $\sim_2$ and scale parameter $x\ y_1^{\sim_1}$. Repeat this procedure for all observations of *X* and Y_1. Denote this sample with variable Y_2.
4. Using the trivariate sample $\left(x, y_1, y_2\right)$ obtain maximum likelihood estimates of $, \sim_1$ and $\sim_2$.
5. Repeat steps 1–4 for 5000 times to get 5000 estimates of $, \sim_1$ and $\sim_2$.
6. Compute mean and standard deviation of 5000 estimates obtained in step 5 to get the results.

The results of simulation study are given in Table–2 below.

Table 2: Results of Simulation Study

True Values	Sample Size	Estimates			Standard Errors		
		$\hat{}$	$\hat{\sim}_1$	$\hat{\sim}_2$	$SE(\hat{})$	$SE(\hat{\sim}_1)$	$SE(\hat{\sim}_2)$
$= 0.5$, $\sim_1 = 2.0$, $\sim_2 = 2.5$	50	0.4915	1.9823	2.4960	0.0098	0.3965	0.2995
	100	0.5039	2.0042	2.5022	0.0101	0.1804	0.2252
	300	0.5020	2.0028	2.5011	0.0033	0.0668	0.0917
	500	0.4980	1.9992	2.4984	0.0010	0.0320	0.0550
$= 1.5$, $\sim_1 = 2.0$, $\sim_2 = 3.5$	50	1.5092	2.0075	3.5105	0.0204	0.2007	0.4017
	100	1.4982	1.9911	3.4971	0.0100	0.1792	0.2497
	300	1.4992	1.9972	3.4997	0.0017	0.0399	0.0917
	500	1.5020	2.0009	3.5012	0.0020	0.0400	0.0300
$= 1.5$, $\sim_1 = 3.0$, $\sim_2 = 3.5$	50	1.5023	3.0067	3.5040	0.0100	0.3612	0.2504
	100	1.5038	3.0043	3.5099	0.0050	0.2004	0.1255
	300	1.5007	3.0020	3.5003	0.0017	0.0400	0.0917
	500	1.5019	3.0016	3.5012	0.0015	0.0240	0.0450

From the results we can see that the performance of maximum likelihood estimators increases with increase in the sample size.

Application

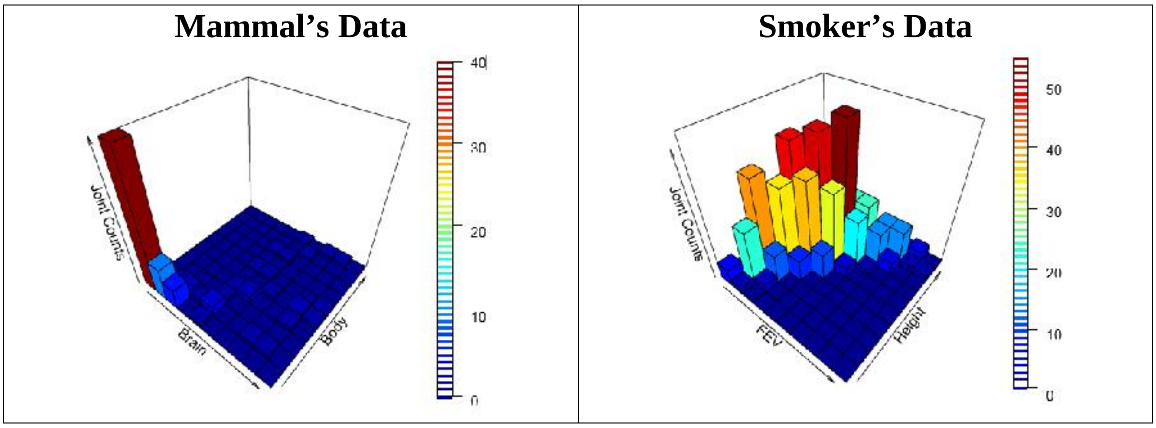
In this subsection we have applied the proposed multivariate Weibull distribution on two real data sets. The data set 1 contains information about body weight and brain weight of 84 mammals as reported by Ramsay and Schafer (1997). The second data set that we have used contains information about height and forced expiratory volume (FEV) of 655 smokers as reported by Rosner (1999). The bivariate Weibull distribution has been _tted on the data. The results are shown in the Table–3 below.

Table 3: Results of Fitted Distributions

Data Set	$\hat{\sim}$	$\hat{}$	*LL*	*AIC*
1	0.1783	0.1060	-996.338	1996.676
2	0.5786	0.0877	-770.671	1545.342

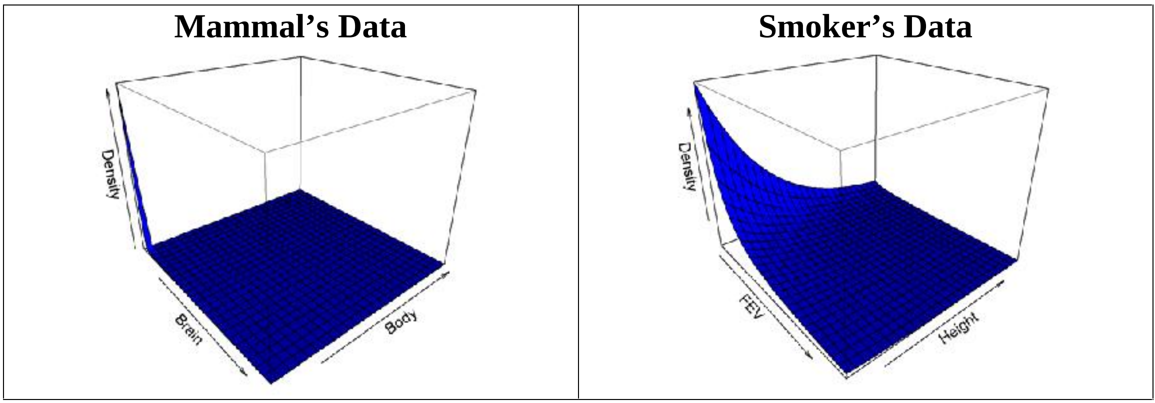
The bivariate histogram for two datasets given below

Figure 1: Bivariate Histograms of Two Data Sets



We have further constructed the bivariate surface plots for two fits and are given in figure 2 below.

Figure 2: Surface Plots of Fitted Distributions



We can see that the surface plot is close to bivariate histograms. Hence the bivariate Weibull distribution fits data reasonably well.

Conclusions and Recommendation

In this paper we have proposed a multivariate Weibull distribution and have studied some of its common properties. The distribution of multivariate concomitants of records for the proposed distribution has also been obtained. The distribution can be used to obtain the distribution of concomitants for any number of variables.

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