A Modified DDF-Based Super-Efficiency Modelhandling Negative Data

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Abstract

In conventional DEA models, inputs and outputs are assumed to be non-negative while negative data may occur in some DEA application such as the performance analysis of socially responsible and mutual funds; and the macroeconomic performance where “rate of growth of GDP per capita” can be either negative or positive. To handle the negative data and provide a measure of efficiency for all units, many researches have been studied. In this paper, the radial super-efficiency model based on Directional Distance Function (DDF) is modified to provide a complete ranking order of the DMUs (including efficient and inefficient ones). The proposed model shows more reliability on differentiating efficient DMUs from inefficient ones via a new super-efficiency measure. The properties of proposed model include feasibility, monotonicity and unit invariance. Moreover, the model can produce positive outputs when data are non-negative. Apart from numerical examples, an empirical study in bank sector demonstrates the superiority of the proposed model.

Keywords: Data envelopment analysis; Super-efficiency; Negative data; DDF model.

1. Introduction

Data Envelopment Analysis (DEA) is a powerful tool in the context of production management for performance measurement. The purpose of DEA is to measure the relative efficiency of a set of decision making units (DMUs) where multiple inputs convert into multiple outputs (Charnes*et al.* (1978)). Conventional DEA models assume non-negative values for inputs and outputs. However, there are many applications in which one or more inputs and/or outputs are necessarily negative such as the performance analysis of socially responsible and mutual funds (Basso & Funari (2014)), and the macroeconomic performance where “rate of growth of GDP per capita” can be either negative or positive (Lovell(1995)).In DEA literature, there have been various approaches for dealing with unrestricted in sign variables.

Pastor (1996) approached negative data using a translation invariance classification, for the first time. That is, in light of the translation in variance property in basic DEA models such as the additive model, the original negative data can be equivalently converted to positive data by adding a constant number. However, many DEA models such as CCR may not have this property to be applied as a treatment of negative data (Ali & Seiford, 1990). A number of significant contributions have been developed in the DEA literature to address the occurrence of negative data (e.g., Kerstens & Van de Woestyne (2011); Silva Portela et al. (2004))

Silva Portelaet al. (2004) proposed range directional measure (RDM) model using some variations of the DDF. Sharp et al. (2007) extended a modified slack-based measure for negative data inspired by the Silva's RDM model. Emrouznejadet al. (2010) proposed a Semi-Oriented Radial Measure (SORM). While Kerstens and Van de Woestyne (2011) modified the traditional proportional distance function, Cheng et al. (2013) suggested variant of the traditional input- or output-oriented radial efficiency measure to handle negative inputs and outputs. Kerstens and Van de Woestyne (2014) highlighted some shortcomings in Cheng's method using a more general case of the DDF proposed by Kerstens and Van de Woestyne (2011). An overview of the many DEA modeling approaches can be found in Pastor and Aparicio (2015).

The super-efficiency procedure presents the possible capability of an efficient DMU in expanding its inputs and/or reducing its outputs without becoming inefficient (Chen, Du, & Hoa (2013)). Banker and Chang (2006)exploited the super-efficiency model to detect and remove the outliers. Further, the super-efficiency DEA approach can be viewed as atool for sensitivity analysis where a DMU under evaluation is excluded from reference set (see, e.g., Rousseau & Semple (1995); and Zhu(2001)).

Whereas, in the absence of negative data, the classical super-efficiency model under constant returns to scale (CRS) does not suffer from the infeasibility problem[1],the super-efficiency model based upon the variable returns to scale (VRS) may be infeasible for a given DMU under evaluation (see, e.g., Chen & Liang(2011), Lee et al.(2011) and Lee & Zhu (2012)).Many modified VRS radial super-efficiency DEA models were proposed to address the infeasibility issue (see, e.g., Cook et al.(2009), Leeet al.(2011)). On the other hand, Ray (2008) suggested the VRS Nerlove-Luenberger super-efficiency DEA model, based on the DDF model and showed that apart from two exceptions the model is feasible. By choosing proper directions, Chen et al. (2013) proposed a DDF-based VRS super-efficiency DEA model to address the infeasibility issue in the two exceptions. Lin and Chen (2015) considered the model in Chen et al. (2013) when zero data exist in outputs. All these modified super-efficiency DEA models are proposed for the non-negative data and the infeasibility issue when there are negative inputs or outputs still exists. In 2013, for the first time, Hadi-Vencheh and Esmaeilzadeh (2013) provided a super-efficiency model based on the RDM model (VE model) for ranking DMUs in the presence of negative data. However, Pourmahmoud et al. (2016) highlighted some shortcomings in VE model and proved the model suffers from the common infeasibility and unboundedness problems. Recently, Lin and Chen (2017) proposed a novel DDF-based VRS radial super-efficiency DEA model which is feasible and is able to handle negative data. They claimed that their proposed model can provide a measure of efficiency for all DMUs in the presence of negative data. This paper highlights some cases that their model is not responding for ranking of DMUs for example when DMUs consume the same inputs. Apart from Hadi-Vencheh and Esmaeilzadeh (2013), and Lin and Chen (2017), super-efficiency models with negative data have received no attention in the literature. The contribution of this paper is fivefold:

[1]Note that zero data can make the super-efficiency model under CCR infeasible (Lee and Zhu (2012))

1. A modified DDF based super-efficiency model interacting with negative data is proposed.
2. The proposed model is always feasible and conveys good properties such as unit invariance, monotonicity, and providing positive outputs when data are non-negative.
3. The proposed model can provide a ranking order for all DMUs via a new super-efficiency measure and produce improved targets for inefficient units.
4. By using different changing rates for inputs and outputs in the proposed model, DMU reaches the frontier with maximum potential in inputs and outputs.
5. This study shows that in distinguishing DMUs to efficient and inefficient ones, proposed model shows higher reliability than the other super-efficiency model compared in this study.

The rest of the paper is outlined as follows. Section 2 briefly presents the concept of DDF,DDF-based super-efficiency model and the model proposed by Lin and Chen (2017). In Section 3, a modified DDF-based super-efficiency model handling negative data is introduced. In section 4, the proposed model is applied to a numerical example. The penultimate section is devoted to an illustration application and finally Section 6 concludes this study.

2. Preliminaries

2.1. DDF model

Consider a set of n observed DMUs, $\{DMU_j\ (j = 1,2, \dots, n)\}$ where each observation transforms m inputs,$x_{ij}\ (i = 1,2, \dots, m)$, into outputs, $y_{rj}\ (r = 1,2, \dots, s)$. Consider an input-output bundle of $DMU_o(x_o, y_o)$ and a reference input-output bundle(g^x, g^y). Furthermore, assume that all data are non-negative. Production possibility set $T_o(x, y)$ from the observed input-output for n DMUs can be defined as follows:

$$T_o(x,y) = \left\{(x,y): x \geq \sum_{j=1}^{n} \lambda_j x_j;\ y \leq \sum_{j=1}^{n} \lambda_j y_j; \sum_{j=1}^{n} \lambda_j = 1\,; \lambda_j \geq 0; (j = 1,2,\dots,n)\right\}$$

which is constructed assuming convexity, free disposibility of inputs and outputs, and VRS.

Based on T_o, the DDF regarding $T_o(x, y)$ can be expressed as follows (Chambers *et al.*(1996)):

$$D(x_o, y_o; g^x, g^y) = \max \beta : (x_o - g^x, y_o + g^y) \in T_o. \tag{1}$$

The reference bundle (g^x, g^y) can be chosen in an arbitrary way and this makes the DDF varies with reference to the evaluated DMU. The VRS DEA formulation for model (1) is as follows:

$$max \quad \beta \tag{2}$$

$$s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io} - \beta g^x, \quad \mathrm{i} = 1, 2, \dots, \mathrm{m},$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} + \beta g^y, \quad r = 1, 2, \ldots, s,$$

$$\sum_{j=1}^{n} \lambda_j = 1,$$

$$\lambda_j \geq 0, \quad j = 1, 2, \ldots, n..$$

Model (2) combines the features of both an input- and output-oriented models in which each input and output of the unit under assessment are decreased and increased respectively, at the same time by the same portion β.The factor β^*as the optimal value of βin model (2) is the Nerlove–Luenberger (N–L) measure of technical inefficiency for the evaluated DMU. By implication, its efficiency equals $1 - \beta^*$ (Ray (2008)).

2.2. Super-efficiency model based on DDF

The idea behind the super-efficiency method is that a DMU under analysis is excluded from the reference set so that the efficientDMUs can receive scores greater than or equal to the unity while the score for the inefficient DMUs do not change. In so doing, the super-efficiency version of model (1) is obtained when DMU_ounder evaluation is removed from the reference set.$T_o^s(x, y)$ of super-efficiency for *n*DMUs can be defined as follows:

$$T_o^s(x, y) = \left\{ (x, y): x \geq \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j x_j;\ y \leq \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j y_j; \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j = 1;\ \lambda_j \geq 0; (j = 1, 2, \ldots, n; j \neq o) \right\}$$

The super-efficiency based on DDF model (Model (1)) is as follows:

$D(x_o, y_o; g^x, g^y) = \max \beta : (x_o - g^x, y_o + g^y) \in T_o^s$.

DDF-based super-efficiency DEA model can be established as follows:

$$\max \quad \beta \tag{3}$$

$$\text{s.t.} \quad \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j x_{ij} \leq x_{io} - \beta g^x, \quad i = 1, 2, \ldots, m,$$

$$\sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j y_{rj} \geq y_{ro} + \beta g^y, \quad r = 1, 2, \ldots, s,$$

$$\sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j = 1,$$

$$\lambda_j \geq 0, \quad j = 1, 2, \ldots, n;\ j \neq o.$$

Ray (2008) defined the super-efficiency score of the evaluated DMU_o equals$1-\beta_o^*$, where β_o^* is the optimum value of model (3). The smaller the value of β_o^*, more efficient the DMU_o is. For any efficient DMU_o, $1-\beta_o^*$ is no less than 1.

The direction vector (g^x, g^y) should be non-negative and non-zero, and can be chosen in anarbitrary way (Chen *et al.* (2013), Ray (2008)). I fall input and output data are non-negative, the standard DDF for the DMU_o is adopted by choosing (x_o, y_o) as (g^x, g^y) (Chambers *et al.* (1998)) and the N-L super-efficiency model (NLS model) is obtained. The NLS model is very often feasible for non-negative data, but it fails in two cases (Ray (2008)). To address these infeasibility issues, Chen *et al.* (2013) selected a new reference input–output bundle for the DDF and propose a modified DDF-based VRS super-efficiency model. However Lin and Chen (2015) showed that the model proposed by Chen *et al.* (2013) does not fully eliminate the infeasibility issue in Ray (2008). In this regards, Lin and Chen (2015) proposed a modified DDF-based super-efficiency DEA model (LCS model) by choosing$\left(x_{io}+\max_{j\neq o}\{x_{ij}\}, y_{ro}\right)$ as (g^x, g^y). The LCS model successfully addresses the infeasibility issue in conventional VRS radial super-efficiency DEA models and the NLS model under non-negative data.

2.3. Proposed model by Lin and Chen(2017)

Lin and Chen (2017) showed that in the presence of negative data, both the NLS and LCS models might be infeasible. This is because their related direction vectors might be negative, which could result in the DMU_o to be further away from the super-efficiency frontier and thus lead to infeasibility. Accordingly, they choose a new direction vector which is always non-negative and non-zero, independent of inputs and outputs being non-negative or not. Their proposed model is as follows:

$$\max \quad \beta \tag{4}$$

$$\text{s.t.} \quad \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j x_{ij} \leq (1-\beta)x_{io} - a_i\beta, \quad i=1,2,\dots,m,$$

$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j y_{rj} \geq (1+\beta)y_{ro} - b_r\beta, \quad r=1,2,\dots,s,$$

$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j = 1,$$

$$\lambda_j \geq 0, \quad j=1,2,\dots,n;\ j\neq o$$

Where$a_i = k * \max_{j=1,2,\dots,n}\{|x_{ij}|\},\ i=1,2,\dots,m$ and $b_r = \min_{j=1,2,\dots,n}\{y_{rj}\},\ r=1,2,\dots,s$; k is a constant, satisfying $k \geq 3$.

To exemplify the Lin and Chen's proposed model (4), let us consider the numerical example presented in Table1 where there are eight DMUs with one positive input(x), and two free in sign-valued outputs$(y_1\ and\ y_2)$.

Table 1: Numerical example

DMUs	x	y_1	y_2
A	1	-6	5
B	1	-6	3
C	1	-5	-2
D	1	-2	-5
E	1	2	-6
F	1	-3.5	3.5
G	1	6.5	-3
H	1	5	2

The results of applying model (4) to the DMUs in Table 1 are presented in Table 2. The optimal values of $1-\beta^*$ besides the optimal slack values (s^*;t_1^*,t_2^*)are shown in columns two-five. The input and outputs projections($x^*; y_1^*, y_2^*$) are represented in the columns six-eight. Projection points are computed by inserting the optimal value in the right-hand side of the input and output inequalities in model (4).

Table 2. The results of numerical example

DMUs	$1-\beta^*$	s^*	t_1^*	t_2^*	x^*	y_1^*	y_2^*
A	1.1364	0.5455	2.5000	0.0000	1.5455	-6.0000	3.5000
B	1.0000	0.0000	7.3333	0.0000	1.0000	-6.0000	3.0000
C	1.0000	0.0000	10.0000	4.0000	1.0000	-5.0000	-2.0000
D	1.0000	0.0000	7.0000	7.0000	1.0000	-2.0000	-5.0000
E	1.0000	0.0000	3.0000	8.0000	1.0000	2.0000	-6.0000
F	1.0000	0.0000	3.0000	0.0000	1.0000	-3.5000	3.5000
G	1.1200	0.4800	0.0000	5.3600	1.4800	5.0000	-3.3600
H	1.2657	1.0627	0.0000	0.0000	2.0627	2.0776	-0.1254

Table 2 reports that$\beta_B^* = \beta_C^* = \beta_D^* = \beta_E^* = \beta_F^* = 0$,$\beta_A^* = -0.1364$,$\beta_G^* = -0.1200$ and$\beta_H^* = -0.2657$. DMUs A, G and H are Pareto-efficient, while DMUs B, C, D, E and F are inefficient due to the optimal slack-values. Table 1 shows that all the DMUs are on the frontier in their input components meaning that input level is efficient; but due to illogical results for DMUs A, G and H the input projections are not on the efficient frontier, as represented in the sixth column of Table 2. This is because$x_{io} + a_i > 0,\ i = 1, 2, \dots, m$ for each $o \in \{1, 2, \dots, n\}$ and model (4) uses a unified changing rate β for both inputs and outputs. Thus, when DMUs consume the same inputs, our expectation is $\beta^* = 0$ and $x^* = 1$ for all DMUs whether efficient or inefficient. This demonstrates that the optimal values of $\beta^*$and the projection points for DMUs A, G and H are illogical results. Consequently, using the $1-\beta^*$ as the super-efficiency measure, model (4) is unable to provide a complete ranking order for all DMUs. Note that this expectation is not true, when DMUs produce the same outputs; because in this case $y_{ro} - b_r = 0,\ r = 1, 2, \dots, s$ for each $o \in \{1, 2, \dots, n\}$and the output constraints in model (4) is disappeared due to convexity constraint.

3. Proposed super-efficiency model

The single input and both outputs cannot be moved at the same rate to the frontier due to the fact that input level is already efficient, as shown in Table 1. The proposed model by Lin and Chen (2017) is unable to provide a complete ranking order for all the DMUs when DMUs consume the same inputs. Different rates β_x and β_y should be used for inputs and outputs, respectively. To this end, the proposed model is as follows:

$$\max \quad \beta_x + \beta_y \tag{5}$$

$$s.t. \quad \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j x_{ij} \le x_{io} - (x_{io} + a_i)\beta_x, \quad i = 1, 2, \dots, m,$$

$$\sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j y_{rj} \ge y_{ro} + (y_{ro} - b_r)\beta_y, \quad r = 1, 2, \dots, s,$$

$$\beta_x . \beta_y \ge 0$$

$$\sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j = 1,$$

$$\lambda_j \ge 0, \;\; \mathrm{j} = 1, 2, \dots, \mathrm{n}; \; \mathrm{j} \neq \mathrm{o}$$

In order to the evaluated DMU reach to the super-efficiency frontier, following conditions are required. If DMU_o is efficient, inputs should be increased and outputs should be contracted, which means $\beta_x \le 0$and$\beta_y \le 0$. And if DMU_o is inefficient, inputs should be contracted and outputs should be increased, which means$\beta_x \ge 0$and$\beta_y \ge 0$. These conditions are incorporated by enforcing $\beta_x . \beta_y \ge 0$ into the model (5). Due to the constraint of $\beta_x . \beta_y \ge 0$, model (5) is a non-linear programming problem. This non-linear programming problem can be transformed to a linear programming problem using the following procedure. Two binary variables, w and z are introduced and the non-linear constraint $\beta_x . \beta_y \ge 0$is transformed into a set of linear constraints as follows:

$$-M(1-w) \le \beta_x \le Mw$$
$$-Mz \le \beta_y \le M(1-z)$$
$$z + w = 1, z \in \{0,1\}, \quad w \in \{0,1\}$$

where M is a sufficiently large number. Obviously, w=1 and z=0 signify $\beta_x \ge 0$and$\beta_y \ge 0$, respectively; and w=0 and z=1 signify $\beta_x \le 0$ and $\beta_y \le 0$, respectively. By substitution of this set of the linear constraints for $\beta_x . \beta_y \ge 0$, model (5) becomes a mixed integer linear programming problem. When the evaluated DMU moves simultaneously to the frontier through direction of $\beta_x$and β_y, the non-zero slacks may be survived. To verify the existence of non-zero slack(s), a non-Archimedean infinitesimal with sum of slacks is incorporated into the objective function of model (5) to reflect the optimal slack calculation process in the standard DEA model.

Proposition 1. Model (5) is always feasible and the following inequalities are hold:

a) $0 \le \beta_x^* < 1$and$0 \le \beta_y^*$for $(x_{io}, y_{ro}) \in T_o^s$;and also

b) $-1 \le \beta_x^* < 0$and$-1 \le \beta_y^* < 0$ for$(x_{io}, y_{ro}) \notin T_o^s$.

Proof. The proof is given in Appendix A.

Corollary 1. Let β_x^* and β_y^* be the optimal solutions of model (5), then$\frac{1-\beta_x^*}{1+\beta_y^*} \geq 0$.

According to corollary 1, the measure of super-efficiency can be defined as following:

Definition 1. Let$\rho^* = \frac{1-\beta_x^*2}{1+\beta_y^*}$, then

(a) If$\rho^* > 1$, then DMU_o is an extreme efficient unit.

(b) If $\rho^* = 1$andthe optimal value of the slacks generated by model (5) are zero, then DMU_o is a non-extreme efficient unit.

(c) If $\rho^* = 1$and the optimal value of the slacks produce by model (5) are non-zero, then DMU_o is a weak efficient unit.

(d) If $\rho^* < 1$, then DMU_o is an inefficient unit.

From model (5) the output-projections for DMU_o are

$$y_{ro}^* = (1+\beta_y^*)\mathrm{y_{ro}} - b_r\beta_y^* = y_{ro} + (y_{ro} - b_r)\beta_y^*, \quad \mathrm{r} = 1, 2, \dots, \mathrm{s}$$

whereβ_y^* is the optimal value of model (5).According to Proposition 1,

$$y_{ro}^* = y_{ro} + (y_{ro} - b_r)\beta_y^* \geq y_{ro}, \text{when} (x_{io}, y_{ro}) \in T_o^s \quad \text{and}$$

$$y_{ro}^* = y_{ro} + (y_{ro} - b_r)\beta_y^* \geq y_{ro} - (y_{ro} - b_r) = b_r, \text{when } (x_{io}, y_{ro}) \notin T_o^s. \blacksquare$$

Therefore, the following Lemma is hold.

Lemma 2. For the data set with non-negative outputs, $y_{ro}^* \geq 0$satisfies for any DMU_o $(o \in \{1,2, \dots, n\})$.

Corollary 2. If$\beta_x = \beta_y = \beta$ in model (5), the proposed model is equivalent with the model (4), consequently the conceptual problem described in Ray (2008) does not occur.

Proposition 2. Model (5) is unit invariant.

Proof. The proof is given in Appendix B.

Proposition 3. If inputs (outputs) of the DMU_o are reduced (increased), the optimal value of model (5) does not increase.

Proof. The proof is given in Appendix C.

Further examination of the proposed method is made by applying DMUs in Table 1. Table 3 reports the results when proposed model is applied to the numerical example in Table1. The optimal solutions of the proposed model i.e. the optimal values of β_x^* andβ_y^* are shown in the second and third columns of Table 3, respectively; and the super-efficiency measure $(\rho^*)$is presented in the fourth column. The columns five-seven of Table 3 show the projection point for a DMU under evaluation.

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[2]Note that When$\beta_y^* = -1$, ρ^* diverges to infinity, hence the super-efficiency measure is assumed to be infinity.

Table 3: The results of applying proposed model for data set in Table 1 (M=100)

DMUs	β_x^*	β_y^*	ρ^*	x^*	y_1^*	y_2^*	ranking order
A	0.0000	-0.1364	1.1579	1.0000	-6.0000	3.5000	2
B	0.0000	0.2222	0.8182	1.0000	-6.0000	5.0000	5
C	0.0000	1.5745	0.3884	1.0000	-3.4255	4.2979	7
D	0.0000	2.1163	0.3209	1.0000	6.4651	-2.8837	8
E	0.0000	0.5625	0.6400	1.0000	6.5000	-6.0000	6
F	0.0000	0.0804	0.9256	1.0000	-3.2991	4.2634	4
G	0.0000	-0.1200	1.1364	1.0000	5.0000	-3.3600	3
H	0.0000	-0.2657	1.3618	1.0000	2.0776	-0.1254	1

The results show that DMUs A, G and H are efficient; since their supper-efficiency measures are greater than one. However, DMUs B, C, D, E and F are inefficient, since their supper-efficiency measures are less than one. Using different changing rates for input and outputs, proposed model provides $\beta_x^* = 0$ for all DMUs, unlike the model (4). Column five shows that$x^* = 1$for all the DMUs, and this logical outcome was expected. The proposed model provided ranking order for all the DMUs, shown in column eight: $H \succ A \succ G \succ F \succ B \succ E \succ C \succ D$.

4. Numerical example

In this section, data set of "the notional effluent processing system" from Sharp *et al.* (2007) is used to show the applicability and merits of the proposed model.

The data set is presented in Table 4. There are 13 DMUs, with two inputs {x_1, x_2} and three outputs {y_1, y_2, y_3}:one positive input (cost), one non-positive input (effluent), one positive output (saleable output), and two non-positive outputs (methane and CO2).

Table 4: Data sets extracted from Sharp

DMUs	x_1	x_2	y_1	y_2	y_3
A	1.03	-0.05	0.56	-0.09	-0.44
B	1.75	-0.17	0.74	-0.24	-0.31
C	1.44	-0.56	1.37	-0.35	-0.21
D	10.8	-0.22	5.61	-0.98	-3.79
E	1.3	-0.07	0.49	-1.08	-0.34
F	1.98	-0.1	1.61	-0.44	-0.34
G	0.97	-0.17	0.82	-0.08	-0.43
H	9.82	-2.32	5.61	-1.42	-1.94
I	1.59	0	0.52	0.00	-0.37
J	5.96	-0.15	2.14	-0.52	-0.18
K	1.29	-0.11	0.57	0.00	-0.24
L	2.38	-0.25	0.57	-0.67	-0.43
M	10.3	-0.16	9.56	-0.58	0.00

Table 5, shows the results of applying model (4) and model (5) on data sets used in Table 4. The optimal values ofβ_x^*, β_y^* and the super-efficiency measure ρ^* are represented in the columns four-sixin Table 5, respectively. DMUs C, G, H, K and M are efficient, since their super-efficiency measures are greater than 1. Other DMUs are inefficient, since their supper-efficiency measures are less than 1. The ranking order for DMU M is superior to other DMUs, as shown in the seventh column in Table 5.

Table 5: Applying the proposed model for data set in Table 4(M=100)

DMUs	$1-\beta^*$	Ranking Order	β_x^*	β_y^*	ρ^*	Ranking Order
A	0.9982	7	0.0000	0.0136	0.9866	7
B	0.9863	10	0.0108	0.0253	0.9648	9
C	1.0412	3	-0.0629	0.0000	1.0629	3
D	0.9192	13	0.0000	0.7501	0.5714	13
E	0.9955	8	0.0000	0.0296	0.9713	8
F	0.9921	9	0.0068	0.0398	0.9552	10
G	1.0108	5	-0.0120	0.0000	1.0120	5
H	1.4023	2	-0.4239	0.0000	1.4239	2
I	1.0000	6	0.0088	0.0000	0.9912	6
J	0.9829	11	0.0518	0.0000	0.9482	11
K	1.0292	4	-0.0083	-0.0310	1.0406	4
L	0.9694	12	0.0270	0.0655	0.9132	12
M	1.5402	1	0.0000	-0.5402	2.1747	1

As it is shown in the second and the sixth columns in Table 5, both models are feasible for all DMUs and they can differentiate the performance of both efficient and inefficient DMUs for used data set. The ranking orders of both models are close; however their super-efficiency measures are different. This is due to the fact that, in proposed model i.e., model (5) different rates, β_x and β_y for inputs and outputs respectively are used, while the same rates are used for both inputs and outputs in the model (4). The super-efficiency measure provided by Model (4) for DMU I is 1.0000 however, the measure of ρ^* as the measure of super-efficiency yielded by model (5) is 0.9912. This shows that model (5) is more responsive than model (4) and it can differentiate the DMUs more discretely. Table 6 shows the target input-output values of inefficient DMUs, determined by the model (5). The proposed model demonstrates that in each inefficient DMU, the inputs and the outputs should be reduced and expanded, respectively, in order to tend to the super-efficiency frontier. Hence, the proposed model can provide improved target inputs and outputs for all the inefficient DMUs.

Table 6: Improved targets for inefficient DMUs

DMUs	x_1^*	x_2^*	y_1^*	y_2^*	y_3^*
A	1.0300	-0.0500	0.5610	-0.0719	-0.3944
B	1.3801	-0.2436	0.7463	-0.2102	-0.2220
D	10.8000	-0.2200	9.4503	-0.6500	-3.7900
E	1.3000	-0.0700	0.4900	-1.0699	-0.2380
F	1.7479	-0.1463	1.6546	-0.4010	-0.2027
I	1.2900	-0.0614	0.5200	0.0000	-0.3700
J	3.9724	-0.5029	2.1400	-0.5200	-0.1800
L	1.4400	-0.4314	0.5752	-0.6209	-0.2100

Lin and Chen (2017) calculated the improved targets for inefficient DMUs. By comparison of their results and the results obtained using proposed model, shown in Table 6, it can be concluded that for some of the DMUs the targets obtained using model (4) is more improved (in some components) than the one obtained using model (5). In other DMUs the proposed model provided more improved targets (in some components) than the model (4). These variations are due to having different directions in their movements to reach the super-efficiency frontier.

5. An empirical application

In this section the proposed model is illustrated by applying it to a real world data of the 61 banks in the GCC[3] countries. In this evaluation, the input variables are total assets, capital and deposits. The output variables are loans and equity in each branch. Note that the last output could take both positive and negative values among the banks. For full definitions of variables see Emrouznejad and Anouze (2010). Table 7 below shows the descriptive statistics of the variables.

Table 7: Descriptive statistics of the banks data

Variables (million $)	Min	Max	Mean	Median	St. Dev
Inputs					
Assets	252.49	29313	5569.16	2390.31	6667.20
Equity	50.19	2381.04	627.15	398.84	615.02
Deposit	26.05	25251.31	4495.24	2006.6	5560.15
Outputs					
Loan	120.97	15379	2777.32	1427.89	3222.04
Profit	-51	647.7	93.11	41.59	128.45

The outcomes after applying assumed data set in Model (4) and in model (5) are reported in Table 8.

[3]The Gulf Cooperation Council (GCC), is a trade bloc involving the six Arab states of thePersian Gulf with many economic and social objectives (for full details see www.gcc-sg.org).

Table 8: Outcomes after applying the assume data on the model (4) and the model (5) (M=100)

Banks	VE model	$1-\beta^*$	Ranking Order	β_x^*	β_y^*	ρ^*	Ranking Order
1	**Infeasible**	1.0052	6	-0.0052	0.0000	1.0052	6
2	0.9639	0.9904	37	0.0000	0.1116	0.8996	36
3	1.0077	1.0016	11	-0.0016	0.0000	1.0016	11
4	0.9327	0.9796	46	0.0000	0.405	0.7117	55
5	1.0121	1.0030	8	-0.0031	0.0000	1.0031	8
6	0.9038	0.9743	49	0.0000	0.6089	0.6215	56
7	1.523	1.0946	3	-0.1352	0.0000	1.1352	3
8	0.9442	0.9869	41	0.0000	0.6786	0.5957	57
9	**Infeasible**	1.1699	2	-0.0067	-0.1699	1.2127	2
10	0.9009	0.9673	52	0.0000	0.3600	0.7353	53
11	0.997	0.9994	19	0.0000	0.1358	0.8805	40
12	0.9879	0.9977	27	0.0000	0.6824	0.5944	58
13	0.9677	0.9910	36	0.0000	0.3466	0.7426	51
14	0.9991	0.9998	16	0.0000	0.0116	0.9885	19
15	0.9920	0.9983	25	0.0000	0.0401	0.9615	23
16	0.9602	0.9873	40	0.0000	0.2658	0.7900	49
17	0.9014	0.9752	48	0.0000	0.8026	0.5548	59
18	0.9951	0.9989	22	0.0000	0.0617	0.9419	28
19	0.975	0.9936	33	0.0000	0.0513	0.9512	26
20	0.8804	0.9689	50	0.0000	0.8955	0.5276	60
21	0.9969	0.9992	21	0.0000	0.0409	0.9607	24
22	1.0051	1.0015	12	-0.0015	0.0000	1.0015	12
23	0.9899	0.9971	28	0.0000	0.0359	0.9653	22
24	0.9952	0.9985	23	0.0000	0.0344	0.9668	21
25	0.973	0.9916	35	0.0000	0.2267	0.8152	47
26	1.0585	1.0130	4	-0.0140	0.0000	1.0140	4
27	0.9962	0.9993	20	0.0000	0.0105	0.9896	18
28	0.9802	0.9946	31	0.0000	0.2103	0.8262	45
29	0.9825	0.9952	30	0.0000	0.0558	0.9471	27
30	0.7877	0.9194	53	0.0000	1.2558	0.4433	61
31	1.0004	1.0001	15	-0.0001	0.0000	1.0001	16
32	0.9986	0.9996	18	0.0000	0.0140	0.9862	20
33	0.9991	0.9997	17	0.0000	0.0024	0.9977	17
34	0.9754	0.9946	31	0.0000	0.2291	0.8136	48
35	0.9643	0.9901	38	0.0000	0.1251	0.8888	38
36	0.9984	0.9996	18	0.0000	0.0139	0.9363	31
37	0.9489	0.9850	43	0.0000	0.1324	0.8831	39
38	0.9563	0.9867	42	0.0000	0.3006	0.7689	50
39	0.8865	0.9675	51	0.0000	0.3934	0.7177	54
40	0.9949	0.9985	23	0.0000	0.0469	0.9552	25
41	0.9179	0.9794	47	0.0000	0.1405	0.8768	42
42	0.9837	0.9967	29	0.0000	0.1512	0.8686	43
43	0.9935	0.9982	26	0.0000	0.0690	0.9354	32
44	0.9979	0.9996	18	0.0000	0.0657	0.9384	29
45	1.0017	1.0003	14	-0.0003	0.0000	1.0004	14
46	**Infeasible**	1.2310	1	0.0000	-0.2310	1.3004	1
47	0.994	0.9984	24	0.0000	0.0674	0.9368	30
48	1.0249	1.0036	7	-0.0037	0.0000	1.0037	7
49	1.0025	1.0008	13	-0.0008	0.0000	1.0008	13
50	1.0206	1.0028	9	-0.0029	0.0000	1.0029	9
51	1.0059	1.0020	10	-0.0021	0.0000	1.0021	10
52	0.9301	0.9823	44	0.0000	0.2250	0.8163	46
53	0.9157	0.9799	45	0.0000	0.1384	0.8784	41
54	1.028	1.0058	5	-0.0071	0.0000	1.0071	5
55	0.9562	0.9884	39	0.0000	0.0972	0.9114	35
56	0.9794	0.9939	32	0.0000	0.0717	0.9331	33
57	0.956	0.9867	42	0.0000	0.2026	0.8315	44
58	0.9938	0.9997	17	0.0000	0.0780	0.9277	34
59	0.9755	0.9926	34	0.0000	0.1245	0.8893	37
60	1.0009	1.0003	14	-0.0003	0.0000	1.0003	15
61	0.9681	0.9939	32	0.0000	0.3468	0.7425	52

From the second, third and the sixth columns in Table 8, VE model is infeasible for DMUs 1, 9 and 46. Both models (4) and (5) are feasible for all DMUs; however their super-efficiency measures are different. 16 DMUs are found efficient by both models. The super-efficiency measure provided by Model (4) for DMUs 14, 32, 33, 36, 44 and 58 is almost 1.0000 (this is the case when the values are rounded with 3 decimal digits); however the measure of ρ^* as the measure of super-efficiency yielded by proposed model is 0.9885, 0.9862, 0.9977, 0.9863, 0.9384 and 0.9277. The result shows that model (5) is more precise and responsive than model (4) in discriminating the DMUs. From Table 8,all the super-efficiency scores yielded by model (5) for inefficient(efficient) DMUs are less than or equal (bigger than or equal)to those generated by model (4) as shown in Figure 1.Thus, super-efficiency scores vary from 0.9194 to 1.2310 under the Lin and Chen's model, whereas they vary from 0.4433 to 1.3004 under our proposed model. As can be seen,in general, the super-efficiency scores obtained from model (4) is around 1.0000 for inefficient DMUs, whereas these scores yielded from model (5) have bigger changing ranges for inefficient ones. From Table 8, DMUs 46 and 30 have the best and the worst performance, respectively under both models. Column seven presents a complete ranking order for all DMUs (both efficient and inefficient ones) using proposed model. However, Lin and Chen's model cannot put discriminations between some inefficient DMUs: between DMUs 45 and 60,DMUs 33 and 58,DMUs 32, 36 and 44,DMUs 24 and 40,DMUs 28 and 34,DMUs 56 and 61,and also DMUs 38 and 57.

Figure 1: Comparison of efficiency score from Lin and Chen's model and our proposed model.

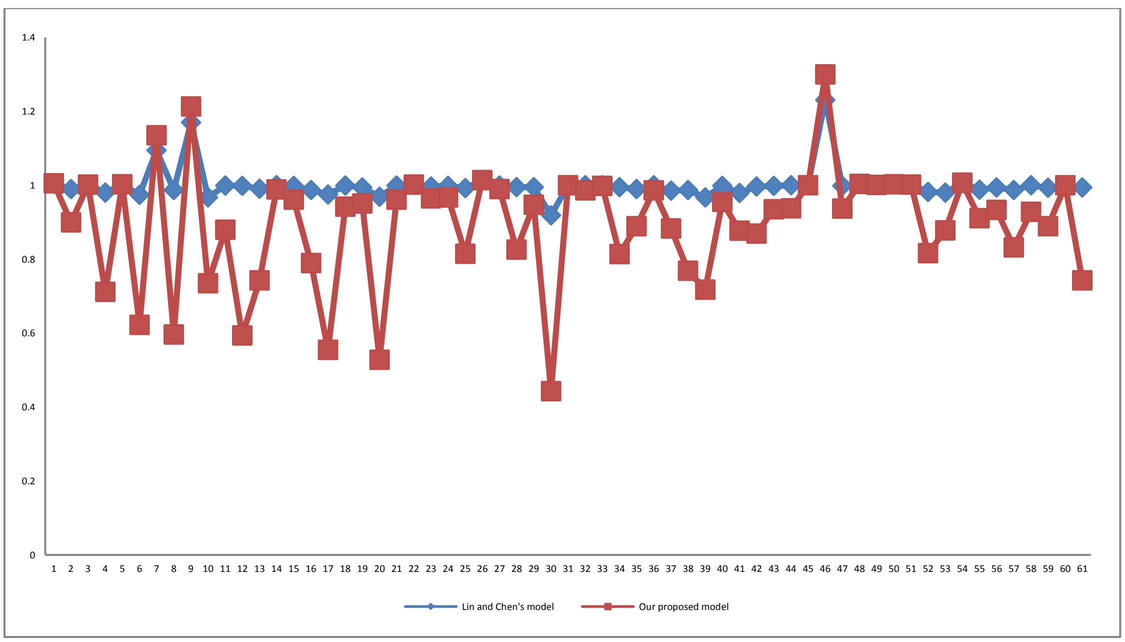

Table 9 shows the target input-output values of inefficient DMUs, determined by both models.

Table 9: Improved targets for inefficient DMUs provided by two models

Banks	Input targets (million $)						Output targets (million $)			
	Proposed model			Lin & Chen's model			Proposed model		Lin & Chen's model	
	ASST	EQTY	DEPO	ASST	EQTY	DEPO	LOAN	PROF	LOAN	PROF
2	7538.18	1197.54	5907.17	6617.272	1117.092	5119.522	5253.485	183.8281	4782.684	162.2876
4	4380.82	565.94	3731.92	2496.008	408.551	2109.128	2680.895	110.8868	1980.148	66.5723
6	5135.9	830.62	3819.89	2747.486	626.0042	1777.93	3414.922	126.6405	2220.816	62.2433
8	1578.94	343.73	1196.72	403.3107	245.406	186.1356	568.6948	47.9013	391.2029	8.6938
10	11827.75	947.4	10201.28	8565.898	682.8822	7390.993	5771.99	226.6537	4412.044	159.835
11	557.8217	101.8401	448.9115	506.7941	97.6627	404.9726	481.3384	12.7962	438.4429	5.2024
12	578.9017	73.8301	490.7114	373.3876	57.0744	313.6925	414.831	11.7863	296.0455	-13.5934
13	1519.85	200.55	1295.13	718.7452	134.7874	605.155	809.5766	41.4435	636.9193	18.2648
14	894.68	123.3	739.03	875.712	121.7485	722.697	731.5667	21.9483	724.6789	21.1254
15	4011.57	398.84	3350.71	3859.332	386.3531	3219.74	3310.991	64.0195	3193.218	59.7731
16	3230.48	348.62	2618.47	2071.636	253.3934	1622.288	1775.329	78.4453	1444.502	52.5598
17	4737.69	508.77	4146.69	2437.568	318.8595	2163.655	3568.177	108.4197	2080.813	39.635
18	604.9817	114.3901	484.5915	508.2422	106.4609	401.2964	376.2554	19.7516	361.6827	15.7128
19	11938.44	1268.61	10014.88	11299.11	1214.765	9465.86	5898.156	282.2494	5651.285	268.009
20	5847.25	608.25	4873.6	2933.961	367.4686	2369.061	4474.92	126.5676	2489.353	45.59
21	790.47	107.73	673.22	722.5455	102.1793	614.7132	591.107	20.1165	572.9657	17.3723
23	6162.47	704.78	4205.81	5892.144	682.2353	3976.109	3335.32	136.4975	3232.844	130.52
24	1686.95	216.09	1313.85	1552.081	205.0158	1197.878	891.8053	44.7726	867.3114	41.7293
25	2036.58	224.86	1771.65	1280.482	162.9441	1120.174	1056.687	51.5756	890.1801	33.3227
27	7410.83	1227.33	5138.18	7347.426	1221.764	5084.39	5663.239	154.2958	5609.497	152.3051
28	1357.11	182.06	1144.07	872.186	142.2805	726.4742	891.0535	36.5915	760.6852	21.763
29	6666.75	852.58	5191.39	6209.315	813.9194	4800.005	3730.952	163.2356	3556.642	152.8911
30	16236.5	1128.2	10846.5	7838.055	461.3808	3864.926	7192.661	256.4717	3508.525	96.2883
32	749.66	153.97	586.46	712.0383	150.8746	554.0765	538.5225	24.3397	532.9347	23.3315
33	8316.03	852.87	6435.87	8290.827	850.7763	6414.35	6043.059	178.3379	6030.717	177.8599
34	2034.35	351.27	1408.3	1551.683	311.0659	994.3586	1623.848	55.9112	1350.229	36.4466
35	6448.93	920.3	5239.81	5517.134	840.698	4440.241	3839.523	161.0573	3458.708	139.3407
36	547.0617	209.3101	329.2715	509.4482	206.1848	296.9301	151.0723	21.493	150.6726	20.5304
37	10632.27	1049.15	9271.54	9154.37	926.3216	7996.735	7638.406	207.6757	6858.962	180.8549
38	2937.26	406.95	2454.04	1725.153	306.2472	1410.903	1558.978	77.9793	1241.377	49.4927
39	9575.26	1495.09	8005.01	6401.541	1213.949	5278.975	3458.211	234.7276	2593.98	160.7339
40	731.25	168.62	549.36	596.649	157.5208	433.532	451.9965	25.0855	437.66	21.7903
41	17607.16	1784.31	15033.36	15434.78	1600.563	13164.75	8140.124	411.2825	7297.072	362.6828
42	2390.31	249.19	2006.6	2094.683	224.9967	1752.108	2026.527	6.9297	1781.627	-0.5153
43	797.94	157.57	630.01	638.8151	144.4783	493.0367	565.3628	25.4458	537.4154	20.6382
44	653.38	181.7	431.15	615.9367	178.6042	398.9506	612.4373	19.3466	582.3349	15.0379
47	942.39	121.52	806.86	801.8951	110.0368	685.8403	662.1367	24.4553	628.7614	19.8017
52	8531.06	1368.26	7038.11	6821.406	1217.42	5570.859	6687.406	205.4391	5576.113	162.0398
53	17944.87	2239.12	14832.36	15814.53	2050.354	13009.8	7232.414	426.3225	6493.494	376.7259
55	12342.48	1143.57	10589.42	11174.81	1047.08	9584.043	5952.318	285.4342	5497.787	259.2105
56	7182.1	614.89	6201.61	6603.553	567.7041	5703.139	3455.216	159.9511	3250.942	147.0272
57	5328.43	576.72	4536.16	4083.508	473.6765	3464.458	2830.325	129.6702	2403.911	101.2353
58	1047.94	254.01	49.1	1023.128	251.9474	27.9636	364.9945	18.9507	347.4031	13.9081
59	4006.94	462.38	3495.95	3322.783	405.7885	2906.263	2643.642	97.9223	2381.133	82.4255
61	939.38	210.85	495.51	393.6935	165.6988	27.3614	120.97	30.5334	120.97	9.9117

The proposed model demonstrates that in each inefficient DMU, the inputs and the outputs should be reduced and expanded, respectively, in order to tend to the super-efficiency frontier.

From the theoretical analyses it is concluded that, the same as Lin and Chen's model, the proposed model can deal with the data set with free in sign values and can provide improved targets for inefficient DMUs.

6. Conclusion

Conventional DEA models are introduced to evaluate DMUs with non-negative data, while in practice there are important DMUs with negative data and they need to be evaluated. Recently, Lin and Chen (2017) proposed a novel DDF-based VRS radial super-efficiency DEA model which is feasible and is able to handle negative data. They claimed that their proposed model can provide a measure of efficiency for all DMUs. In this study, it is shown that although their proposed model can overcome the common infeasibility problem, the model has failing in some cases. It is unable to provide a complete ranking order and logical results in such a case that all DMUs consume the same inputs. This is because(i) in this model a unified changing rate for both inputs and outputs is used and (ii) the input improvement direction is strictly positive. In this study, a modified radial DDF-based super-efficiency model is proposed to provide a complete ranking order for all the DMUs via a new super-efficiency measure.

References

1. Ali, A.I., & Seiford, L.M., (1990).Translation invariance in data envelopment analysis, Operations Research Letters, 9(6), 403-405.
2. Andersen, P., & Petersen, N.C., (1993).A procedure for ranking efficient units in data envelopment analysis. Management Science, 39 (10), 1261–1294.
3. Banker, R.D., &Chang, H., (2006).The super-efficiency procedure for outlier identification, not for ranking efficient units. European Journal of Operational Research 175,1311–1320.
4. Basso, A., & Funari, S., (2014).Constant and variable returns to scale DEA models for socially responsible investment funds. European Journal of Operational Research, 235(3), 775-783.
5. Chambers, R., Chung, Y., & Färe, R., (1998). Profit, directional distance functions, and Nerlovian efficiency. Journal of Optimization Theory and Applications, 98: 351–364.
6. Chambers, R. G., Chung, Y., & Färe, R., (1996). Benefit and distance functions. Journal of Economic Theory, 70, 407–419
7. Charnes A., Cooper W. W., & Rhodes E., (1978).Measuring the efficiency of decision making units. European Journal of Operational Research, 2(6): 429-444.
8. Chen, Y., Du, J., & Hoa, J., (2013). Super-efficiency based on a modified directional distance function. Omega, 41, 621-625.
9. Chen, Y., & Liang, L., (2011). Super-efficiency DEA in the presence of infeasibility: One model approach. European Journal of Operational Research, 213, 359–360.

10. Cheng, G., Zervopoulos, P., & Qian, Z., (2013).A variant of radial measure capable of dealing with negative inputs and outputs in data envelopment analysis. European Journal of Operational Research, 225(1), 100–105.

11. Cook, W.D., Liang, L., Zha, Y., & Zhu, J., (2009). A modified super-efficiency DEA model for infeasibility. Journal of Operational Research Society, 60, 276–281.

12. Emrouznejad, A., Anouze, A. L., & Thanassoulis, E., (2010).A semi-oriented radial measure for measuring the efficiency of decision making units with negative data, using DEA. European Journal of Operational Research 200(1), 297-304.,

13. Emrouznejad, A. & Anouze, A. L., (2010).Data envelopment analysis with classification and regression tree - A case of banking efficiency. Expert Systems with Applications, 27(4): 231-246.

14. Hadi-Vencheh, A., & Esmaeilzadeh, A., (2013).A new super-efficiency model in the presence of negative data. Journal of the Operational Research Society, 64(3), 396-401.

15. Kerstens, K., & Van de Woestyne, I., (2011). Negative data in DEA: a simple proportional distance function approach. Journal of the Operational Research Society, 62(7), 1413-1419.

16. Kerstens, K., & Vande Woestyne, I., (2014).A note on a variant of radial measure capable of dealing with negative inputs and outputs in DEA. European Journal of Operational Research, 234(1), 341-342.

17. Lee, H. S., & Zhu, J., (2012).Super-efficiency infeasibility and zero data in DEA. European Journal of Operational Research, 216, 429–433.

18. Lee, H. S., Chu, C. W., & Zhu, J., (2011).Super-efficiency DEA in the presence of infeasibility. European Journal of Operational Research 212, 141–147.

19. Lin, R., & Chen, Z., (2015). Super-efficiency measurement under variable return to scale: an approach based on a new directional distance function. Journal of the Operational Research Society 66, 1506–1510.

20. Lin, R., & Chen, Z., (2017). A directional distance based super-efficiency DEA model handling negative data. Journal of the Operational Research Society 68(11), 1312-1322.

21. Lovell, C.A.K., (1995), Measuring the Macroeconomic Performance of the Taiwanese Economy. International Journal of Production Economics, 39, 165-178.

22. Pastor, J.T., & Aparicio, J., (2015).Translation Invariance in Data Envelopment Analysis (pp. 245-268).Springer US.

23. Pastor, J.T., (1996). Translation invariance in data envelopment analysis: A generalization. Annals of Operations Research, 66, 93-102.

24. Pourmahmoud, J., Hatami-Marbini, A., & Babazadeh, E., (2016).A comment on a new super-efficiency model in the presence of negative data. Journal of the Operational Research Society 67(3), 530-534.

25. Ray, S. C., (2008). The directional distance function and measurement of super-efficiency: an application to airlines data. Journal of the Operational Research Society 59(6), 788-797.

26. Rousseau, J.J., Semple, J.H., (1995). Radii of classification preservation in data envelopment analysis: A case study of 'Program Follow-Through'. Journal of the Operational Research Society 943–957.

27. Silva Portela, M.C.A., Thanassoulis, E., & Simpson, G., (2004). A directional distance approach to deal with negative data in DEA: An application to bank branches. Journal of the Operational Research Society, 55 (10): 1111-1121.

28. Sharp, J. A., Meng, W., & Liu, W., (2007).A modified slacks-based measure model for data envelopment analysis with 'natural' negative outputs and inputs. Journal of the Operational Research Society 58(12), 1672-1677.

29. Zhu, J., (2001). Super-efficiency and DEA sensitivity analysis. European Journal of Operational Research 129 (2), 443–455.

Appendix A

Proposition 1. Model (5) is always feasible and the following inequalities are hold:

a) $0 \le \beta_x^* < 1$ and $0 \le \beta_y^*$ for $(x_{io}, y_{ro}) \in T_o^s$; and also

b) $-1 \le \beta_x^* < 0$ and $-1 \le \beta_y^* < 0$ for $(x_{io}, y_{ro}) \notin T_o^s$.

Proof. Since $x_{io} + a_i > 0$, we have

$$\beta_x \le \frac{x_{io} - \sum_{j=1, j\neq o}^{n} \lambda_j x_{ij}}{x_{io} + a_i}, \quad i = 1,2,\ldots,m. \tag{6}$$

Following the notations used by Lin and Chen (2017), let $J_o = \{r \mid y_{ro} - b_r > 0, r = 1,2,\ldots,s\}$ and $O_o = \{r \mid y_{ro} - b_r = 0, r = 1,2,\ldots,s\}$ for each $o \in \{1,2,\ldots,n\}$. Thus, $y_{ro} - b_r \ge 0$ implies that $J_o \cup O_o = \{r = 1,2,\ldots,s\}$. Due to convexity constraint i.e. $\sum_{j=1, j\neq o}^{n} \lambda_j = 1$, we have

$$\sum_{j=1, j\neq o}^{n} \lambda_j y_{rj} \ge \min_{j\neq o}\{y_{rj}\} \ge \min_{j}\{y_{rj}\} = b_r = y_{ro}, \ r \in O_o.$$

This shows that the output constraints in model (5) satisfy for all $r \in O_o$. Hence, the output constraints in model (5) are equivalent to

$$\beta_y \le \frac{\sum_{j=1, j\neq o}^{n} \lambda_j y_{rj} - y_{ro}}{y_{ro} - b_r}, \ r \in J_o. \tag{7}$$

There are two cases as follows:

Case (I) when $(x_{io}, y_{ro}) \in T_o^s$:

We have $\sum_{j=1, j\neq o}^{n} \lambda_j x_{ij} \le x_{io},\ i = 1,2,\ldots,m$ and $\sum_{j=1, j\neq o}^{n} \lambda_j y_{rj} \ge y_{ro},\ r = 1,2,\ldots,s$. So,

$$\frac{x_{io} - \sum_{j=1, j\neq o}^{n} \lambda_j x_{ij}}{x_{io} + a_i} \ge 0, \qquad i = 1,2,\ldots,m, \tag{8}$$

$$\frac{\sum_{j=1, j\neq o}^{n} \lambda_j y_{rj} - y_{ro}}{y_{ro} - b_r} \ge 0, \ r \in J_o. \tag{9}$$

Inequalities of (6)-(9) result that $\beta_x = \beta_y = 0$ is a feasible solution of model (5), and consequently $\beta_x^* \ge 0$, and $\beta_y^* \ge 0$ always hold for $o \in \{1,2,\ldots,n\}$. In addition

$$\beta_x \le \frac{x_{io} - \sum_{j=1, j\neq o}^{n} \lambda_j x_{ij}}{x_{io} + a_i} \le \frac{x_{io} + \max_{j\neq o}\{|x_{ij}|\}}{x_{io} + a_i} \le \frac{x_{io} + \max_{j=1,2,\ldots,n}\{|x_{ij}|\}}{x_{io} + a_i} < 1.$$

Thus, $0 \le \beta_x^* < 1$ and $0 \le \beta_y^*$ for $(x_{io}, y_{ro}) \in T_o^s$.

Case (II) when $(x_{io}, y_{ro}) \notin T_o^s$:

In this case $\exists i$: $\sum_{j=1, j\neq o}^{n} \lambda_j x_{ij} > x_{io}$ and/or $\exists r$: $\sum_{j=1, j\neq o}^{n} \lambda_j y_{rj} < y_{ro}$ which implies that $x_{io} - \sum_{j=1, j\neq o}^{n} \lambda_j x_{ij} < 0$ and/or $\sum_{j=1, j\neq o}^{n} \lambda_j y_{rj} - y_{ro} < 0$. Due to (6), (7) and the constraint $\beta_x . \beta_y \ge 0$, model (5) is still feasible; and $\beta_x^* \le 0$ and/or $\beta_y^* \le 0$ are the optimal solutions. In general,

(a) if $\exists i$: $\sum_{j=1, j\neq o}^{n} \lambda_j x_{ij} > x_{io}$, then $\beta_x^* < 0$ and $\beta_y^* = 0$,

(b) if $\exists r$: $\sum_{j=1, j\neq o}^{n} \lambda_j y_{rj} < y_{ro}$, then $\beta_x^* = 0$ and $\beta_y^* < 0$,

(c) if$\exists i$: $\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j x_{ij} > x_{io}$ and $\exists r$: $\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j y_{rj} < y_{ro}$, then $\beta_x^* < 0$ and $\beta_y^* < 0$.

In addition, it is obvious that $\max_{j=1,2,\dots,n}\{|x_{ij}|\} \leq 2x_{io} + a_i$due to$k \geq 3$. Consequently,

$$\frac{x_{io} - \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j x_{ij}}{x_{io} + a_i} \geq \frac{x_{io} - \max_{j=1,2,\dots,n}\{|x_{ij}|\}}{x_{io} + a_i} \geq \frac{x_{io} - 2x_{io} - a_i}{x_{io} + a_i} = -1, \quad i = 1, 2, \dots, m. \tag{10}$$

On the other hand,

$$\frac{\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j y_{rj} - y_{ro}}{y_{ro} - b_r} \geq \frac{\min_j\{y_{rj}\} - y_{ro}}{y_{ro} - b_r} = -1, \quad r \in J_o. \tag{11}$$

According to the objective function of model (5), inequalities of (10) and (11) implies that $-1 \leq \beta_x^* < 0$ and $-1 \leq \beta_y^* < 0$ for $(x_{io}, y_{ro}) \notin T_o^S$, ■

Appendix B

Proposition 2. Model (5) is unit invariant.

Proof. To show the units invariance of model (5), assume that the inputs x_{ij} and outputs y_{rj} are multiplied by the positive $\alpha_i$and μ_r, respectively. Let $\tilde{x}_{ij} = \alpha_i x_{ij}$ $(i = 1, 2, \dots, m;\ j = 1,2, \dots, n)$, $\tilde{y}_{rj} = \mu_r y_{rj}$ $(r = 1, 2, \dots, s;\ j = 1, 2, \dots, n)$, $\tilde{a}_i = k * \max_{j=1,2,\dots,n}\{|\tilde{x}_{ij}|\}$ $(i = 1,2, \dots, m)$ and $\tilde{b}_r = \min_{j=1,2,\dots,n}\{\tilde{y}_{rj}\}$ $(r = 1, 2, \dots, s)$.

Hence, the model (5) using the transformed date is written as following:

$$\max \quad \beta_x + \beta_y$$

$$s.t. \quad \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j \tilde{x}_{ij} \leq \tilde{x}_{io} - (\tilde{x}_{io} + \tilde{a}_i)\beta_x, \quad i = 1, 2, \dots, m,$$

$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j \tilde{y}_{rj} \geq \tilde{y}_{ro} + (\tilde{y}_{ro} - \tilde{b}_r)\beta_y, \quad r = 1, 2, \dots, s,$$

$$\beta_x . \beta_y \geq 0$$

$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j = 1,$$

$$\lambda_j \geq 0, \ \ j = 1, 2, \dots, n;\ j \neq o$$

This model is transformed to the model (5), in terms of the untransformed data, after substitution of $\alpha_i x_{ij}$ for $\tilde{x}_{ij}$ in the input constraints and $\mu_r y_{rj}$ for $\tilde{y}_{rj}$ in the output constraints, and cancellation of the common factors from both sides of the inequalities.

Appendix C

Model (5) has also monotonicity property. Suppose that the inputs and the outputs of DMU$_o$ are reduced by Δx_{io} and increased by Δy_{ro}, respectively; and let $x_{io} \geq 0$, i = 1, 2, ..., m, and $y_{ro} \geq 0$, r = 1, 2, ..., s. Since the input and output data of DMU$_o$ are changed, the constants a_i and $b_r$should be adjusted correspondingly. According to model (5), a_i and b_r should be redefined by

$$a_i = k * \max_{j=1,2,\dots,n}\{|x_{ij}|, \forall j, |x_{io} - \Delta x_{io}|\}, \qquad i = 1,2,\dots,m \tag{12}$$

$$b_r = \min_{j=1,2,\dots,n}\{y_{rj}, \forall j, y_{ro} + \Delta y_{ro}\},\ r = 1,2,\dots,s. \tag{13}$$

Following conclusion is made after redefining $a_i$and b_r.

Proposition 3. If inputs (outputs) of the DMU_o are reduced (increased), the optimal value of model (5) does not increase for a_i and b_r defined in (12) and (13).

Proof. If specified input reduction and output expansion happens, the direction vector is $(x_{io} - \Delta x_{io} + a_i, y_{ro} + \Delta y_{ro} - b_r)$. The following statement is made by having the definitions of (12) and (13). $x_{io} - \Delta x_{io} + a_i > 0$, i = 1, 2, . . . ,m, and $y_{ro} + \Delta y_{ro} - b_r > 0$, r = 1, 2, . . . , s. Consequently the corresponding model (5) for the DMU_o is rewritten as

$$\max \quad \beta_x + \beta_y \tag{14}$$

$$\text{s.t.} \quad \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j x_{ij} \le (1-\beta_x)(x_{io} - \Delta x_{io}) - a_i\beta_x, \quad i = 1,2,\dots,m,$$

$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j y_{rj} \ge (1+\beta_y)(y_{ro} + \Delta y_{ro}) - b_r\beta_y, \quad r = 1,2,\dots,s,$$

$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j = 1,$$

$$\beta_x.\beta_y \ge 0$$

$$\lambda_j \ge 0,\ \ j = 1,2,\dots,n, j \neq o$$

Assume the optimal solution of model (14) as $(\lambda_j', \beta_x', \beta_y')$. A similar derivation made in (10) is applied for input constraint of the model (14) using (12) as following:

$$\beta_x' \le \frac{x_{io} - \Delta x_{io} - \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j' x_{ij}}{x_{io} - \Delta x_{io} + a_i} \le \frac{x_{io} - \Delta x_{io} + \max_{j\neq o}\{|x_{ij}|\}}{x_{io} - \Delta x_{io} + a_i} < 1, \ \ i = 1,2,\dots,m \tag{15}$$

Thus, $\beta_x' < 1$. A similar derivation made in (11) is applied for output constraint of the model (14) using (13) as following.

$$\beta_y' \ge \frac{\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j y_{rj} - y_{ro} - \Delta y_{ro}}{y_{ro} + \Delta y_{ro} - b_r} \ge \frac{\min_{j=1,2,\dots,n}\{y_{rj}\} - y_{ro} - \Delta y_{ro}}{y_{ro} + \Delta y_{ro} - b_r} = -1, \ \ r \in J_o' \tag{16}$$

where $J_o' = \{r | y_{ro} + \Delta y_{ro} - b_r > 0, r = 1,2,\dots,s\}$. Since we maximize β_x and β_y in model (14), $\beta_y' \ge -1$. Following statements is made using (15) and (16).

$1 - \beta_x'$and$1 + \beta_y'$ are non-negative,

$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j' x_{ij} \leq (1-\beta_x')(x_{io}-\Delta x_{io}) - a_i\beta_x' \leq (1-\beta_x')x_{io} - a_i\beta_x', \quad i = 1, 2, \dots, m, \tag{17}$$

$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j' y_{rj} \geq \left(1+\beta_y'\right)(y_{ro}+\Delta y_{ro}) - b_r\beta_y' \geq \left(1+\beta_y'\right)y_{ro} - b_r\beta_y', \quad r = 1, 2, \dots, s. \tag{18}$$

Therefore, $(\lambda_j', \beta_x', \beta_y')$ is a feasible solution for model (14).
Maximizing of β_x and β_y is aimed in model (5), hence $\beta_x^* \geq \beta_x'$and $\beta_y^* \geq \beta_y'$.