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Transient Analysis of $M^{[X1]}$,$M^{[X2]}$/G_1 ,G_2/1 Retrial Queueing System with Priority Services, Working Breakdown, Start Up/Close Down Time, Bernoulli Vacation, Reneging and Balking

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Abstract

This paper considers $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ general retrial queueing system with priority services. Two types of customers from different classes arrive at the system in different independent compound Poisson processes. The server follows the pre-emptive priority rule subject to working breakdown, startup/closedown time and Bernoulli vacation with general (arbitrary) vacation periods. After completing the service, if there are no priority customers present in the system the server may go for a vacation or close down the system. On completion of the close down, the server needs some time to set up the system. The priority customers who find the server busy are queued in the system. A low-priority customer who find the server busy are routed to a retrial (orbit) queue that attempts to get the service. The system may breakdown at any point of time when it is in operation. However, when the system fails, instead of stopping service completely, the service is continued only to the high priority customers at a slower rate. We consider balking to occur to the low priority customer while the server is busy or idle, and reneging to occur at the high priority customers during server's vacation, start up/close down time. Using the supplementary variable technique, we derive the joint distribution of the server state and the number of customers in the system. Finally, some performance measures and numerical examples are presented.

Key Words: Retrial; Working Breakdown; Startup/Close down time; Balking; Reneging

Mathematical Subject Classification: $60K25, 68M30, 90B22$

1. Introduction

Several authors studied about retrial queue, where the customers join the orbit and retry for service. Notably, Atencia et.al (2005) described the retrial queue with general retrial times where the single server provides the service. The fact is that perfectly reliable servers are virtually nonexistent has been often overlooked by many researchers. In fact, the servers may well be subject to lengthy and unpredictable breakdowns while serving a customer. For example, in manufacturing systems the machine may breakdown due to technical or job related problems. This results in a period of unavailable time until the servers are repaired. Such a system with repairable server has been studied as a queueing model and reliability model by many authors. Further while studying queueing models with server breakdown, it is generally assumed that the server stops service when he breaks down. In most of the models considered so far in queueing systems with server breakdowns, the underlying assumption has been that a server breakdown disrupts the service completely.

In computer, the presence of a virus in the system may slow down the performance of the computer system. The computer may still be able to perform various tasks but at a considerably slower rate. Here the failure of the computer system does not stop the work completely. Motivated by this factor, we have therefore considered in this paper a new class of queueing systems with working breakdown policy with various parameters.

Kalidass et.al (2012) introduced the working breakdown policy, in which the server works at a lower service rate rather than stopping service during the breakdown period. During working breakdown time the server can receive complaints from the customers who should wait for the server to be repaired and reduce the cost of waiting customers. Therefore, the working breakdown service is a more reasonable repair policy for unreliable queueing systems. Tao Li et.al (2017) and Zaiming Liu et.al (2014) described more about working breakdowns. Recently, Kim B.K. et.al (2016) studied the M/G/1 queueing system with disasters and working breakdowns. Cheng-Dar Liou (2013) applied the matrix-geometric method to examine an infinite capacity Markovian queue with an unreliable server subject to working breakdowns and impatient customers. Tao Li et.al (2018) investigated An M/G/1 retrial queue with balking customers and Bernoulli working vacation interruption.Pavai Madheswari et.al (2019) studied about Analysis of M/G/1 retrial queues with second optional service and customer balking under two types of Bernoulli vacation schedule.

Reflecting more practical situations, the server may turns off his service by a close down time and a start up time is required before starting service. Dimitriou et.al (2009) studied a queueing model with startup/closedown time and retrial customers. Servi L.D et.al (2012) deal with working vacation with single server Markovian queue. However, a working breakdown can occur at any time point at which the server is busy with the service of a customer in the system. Sridharan et.al (1996) have considered a finite capacity queueing system with both partial and total failure of the server. However, they have assumed that the breaksdown occurs even when the server is idle. In our paper, but we have assumed that the server fails only in the operational state. Customer impatience can be viewed as a potential loss of customers. Subha rao (1967) considered balking and reneging in M/G/1 systems. Recenty, Yang et.al (2017) studied the analysis of a finite capacity system with working breakdown and retention of impatient customers. Ammar et.al (2019) explained about Performance Analysis of Pre-emptive Priority Retrial Queueing System with Disaster under Working Breakdown Services.

The suggested model can be implemented in computer networks. The server is interconnected with two or more networks in a packet-switched network and it is used to transmit the packets within a network for transmission. A batch of packets arrives at the server according to a Poisson process. When packets arrive at the server, one of the packets is picked for service and other packets will be kept in the buffer (orbit). In the buffer, each packet waits for some time and demands the service again. After transmission of each packet, some preservation activities, such as routing information backup (vacations) can be programmed regularly or may transmission to another packet. After transmission of each packet or information backup or repair the server serves the packets from the buffer (retrial) or waiting to transmit (idle state) to a new packet.

This paper considers $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ general retrial queueing system with priority services. Two different sorts of customers arrive at the system in two independent compound Poisson processes. Under the pre-emptive priority rule, the server providing general service to the arriving customers is subject to breakdown and Bernoulli vacation with general vacation time. We propose a retrial queueing model with the additional characteristics of server's working breakdown, repair, startup/closedown times, balking and reneging. Models with this type of a working breakdown can be used to analyse computer networks with virus affection and breakdowns due to a reset order. Arriving high priority customers who find the server busy with high(low) priority customer are queued (pre-empts the low priority service) and then are served in accordance with FCFS discipline. The arriving low-priority customers on finding the server busy cannot be queued. They leave the service area and join the orbit as retrial customer. After completing service, if there is no high priority customer present in the system, the server may go for a vacation or close down the system. After completing the close down time the server need some time to set up the system. After completing vacation, repair, setup, if there is no high priority customer present in the system then the server becomes idle. We consider reneging to occur for high priority customers during vacation, startup/closedown time and balking to occur for low priority customers during server's busy or idle period.

The summary of the paper is as follows. Section 1 is an introduction to priority retrial queueing discipline and comprises of literature review. Section 2 deals with model description, notations used, mathematical formulation and governing equations of the model. Section 3 elucidates the steady state solutions of the system. Section 4 demonstrates the performance measures of the model. In Section 5 the numerical results and graphs are computed following which a conclusion is given.

2. Model Description

The basic operation of the model can be described as follows.

Arrival and Retrial Process: Two class of customers arrive at the system in two independent compound Poisson processes with arrival rates λ_1 and λ_2 respectively. Let $\lambda_1 c_{1,i}\, dt$ and $\lambda_2 c_{2,i}\, dt$ $(i = 1,2,3,\dots)$ be the first order probability that a batch of 'i' customers arrive at the system during a short interval of time $(t, t + dt)$, where $0 \leq c_{1,i} \leq 1, \sum_{i=1}^{\infty} c_{1,i} = 1$, $0 \leq c_{2,i} \leq 1, \sum_{i=1}^{\infty} c_{2,i} = 1$ and $\lambda_1 > 0, \lambda_2 > 0$. The high priority customer who finds the server busy is queued and then served using FCFS rule. The arriving low-priority customers on finding the server busy, are routed to a retrial queue and they follow classical retrial policy that attempts to get the service. The retrial time is generally distributed with distribution function $I(s)$ and density function $i(s)$. Let $\eta(x)dx$ be the conditional probability of completion of retrial during the interval $(x, x + dx]$ given that elapsed retrial time is x.

Service Process: If a high priority customer arrives in a batch and finds a low priority customer in service, they pre-empt the low priority customer who is undergoing service and the service of the pre-empted low priority customer begins only after the completion of service of all high priority customers present in the system. The service times for the high priority and low priority customers are generally (arbitrary) distributed with distribution functions $B_i(s)$ and density functions $b_i(s)$, i = 1,2 respectively. Let $\mu_i(x)dx$ be the conditional probability of completion of high priority and low priority customers service during the interval $(x, x + dx]$, given that the elapsed service time is x.

Bernoulli Vacation: After every service completion to the high/low priority customer the server may take a vacation with probability θ or continue service to the next customer with probability $1 - \theta$. Vacation time is generally distributed with distribution function $V(s)$ and density function $v(s)$. Let $\beta(x)dx$ be the conditional probability of completion of vacation during the interval $(x, x + dx]$ given that the elapsed vacation time is x.

Working Breakdown State: The server may become inactive during busy period. At the time of breakdown the high priority customer who is in service will get service continuosly at a lower service rate $\mu_3$which follows exponential distribution. But, the low priority customer who is in service will be sent to the orbit.

Repair Process: The broken down server is sent for repair immediately so as to regain its functionality with exponential repair rate γ. Immediately after returning from repair, the server starts to serve high priority/low priority customers, if any present in the system.

Close Down Time: After completing a vacation or repair or service, if there are no high priority customers present in the system the server will close down the system. Close down time is generally distributed with distribution function $C(s)$ and density function $c(s)$. Let $\varphi(x)dx$ be the conditional probability of a completion of close down time during the interval $(x, x + dx]$ given that the elapsed close down time is x.

Startup Time: On completion of close down time, the server takes some time to start up the system to increase the efficiency of the service. Startup time is generally distributed with distribution function $M(s)$ and density function $m(s)$. Let $\delta(x)dx$ be the conditional probability of completion of a startup time during the interval $(x, x + dx]$ given that the elapsed startup time is x.

Balking and Reneging: If the server is unavailable in the system, the high priority customer may renege the system with an exponential rate ξ. If the server is busy or unavailable in the system, an arriving low-priority customer either joins the orbit with probability b or balks with probability (1 - b).

Idle State: After completing a startup or vacation or repair if there is any high priority customer waiting in the system the server starts providing the service. Otherwise the server will be idly present in the system for the customers to arrive.

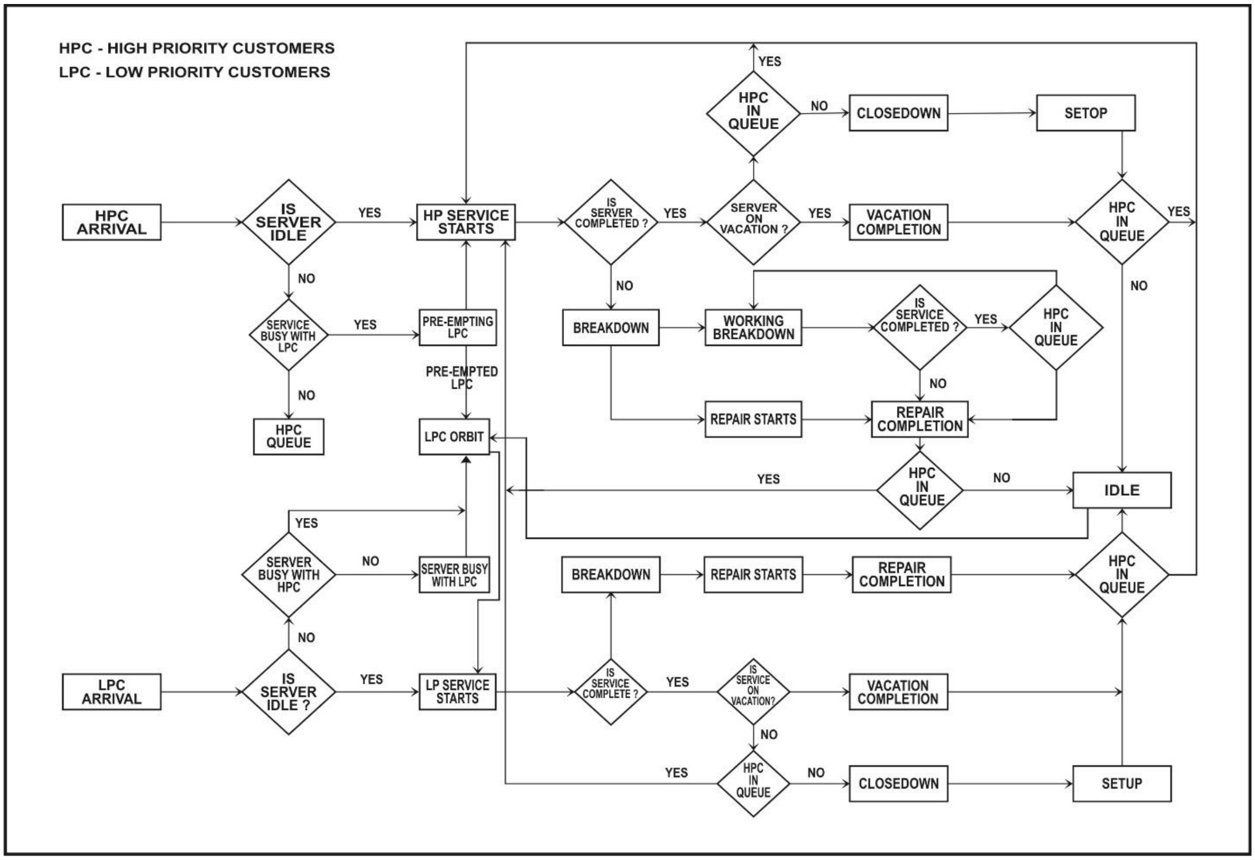

Figure 1: Flow Chart Of The Model Description

2.1 Definitions and Notations

The state of the system at time t is defined by the Markov process is $\{S(t); t\geq 0\} = \{Y(t), N_1(t), N_2(t), t\geq 0\}$, where $N_1(t)$ and $N_2(t)$ denotes the quantity of customers in the high priority queue and the low priority orbit respectively.The state of the server at time t is given by,

$$Y(t) = \begin{cases} 0, & if \ the \ server \ is \ in \ idle \ state; \\ 1, & if \ the \ server \ is \ busy \ with \ high \ priority \ customer; \\ 2, & if \ the \ server \ is \ busy \ with \ low \ priority \ customer; \\ 3, & if \ the \ server \ is \ in \ vacation; \\ 4, & if \ the \ server \ is \ working \ breakown \ state; \\ 5, & if \ the \ server \ is \ in \ repair \ process; \\ 6, & if \ the \ server \ is \ in \ closedown \ state; \\ 7, & if \ the \ server \ is \ in \ startup \ state. \end{cases}$$

2.2 Queue Size Distribution

Since service time, vacation time, closedown time and startup time are not exponential, the process $\{Y(t), N_1(t), N_2(t)\}$ is not Marovian. In such case we introduce supplementary variables corresponing to elapsed times to make it Markovian (1955). Joint distributions of the server state and queue size are defined as,

$$\overline{I}_{0,n}(s,t) = Pr\{Y(t)=0, N_1(t)=0, N_2(t)=n\}, n \geq 1$$

$$\overline{P}^{(1)}_{m,n}(x,s,t)dx = Pr\{Y(t)=1, x<\pi(t)\leq x+dx, N_1(t)=m, N_2(t)=n\}, m\geq 0, n\geq 0$$

$$\overline{P}^{(2)}_{0,n}(x,s,t)dx = Pr\{Y(t)=2, x<\pi(t)\leq x+dx, N_1(t)=0, N_2(t)=n\}, n\geq 0$$

$$\overline{V}_{m,n}(x,s,t)dx = Pr\{Y(t)=3, x<\pi(t)\leq x+dx, N_1(t)=m, N_2(t)=n\}, m\geq 0, n\geq 0$$

$$\overline{Q}_{m,n}(s,t)dx = Pr\{Y(t)=4, N_1(t)=m, N_2(t)=n\}, m\geq 0, n\geq 0$$

$$\overline{R}_{0,n}(s,t)dx = Pr\{Y(t)=5, N_1(t)=0, N_2(t)=n\}, n\geq 0$$

$$\overline{C}_{m,n}(x,s,t)dx = Pr\{Y(t)=6, x<\pi(t)\leq x+dx, N_1(t)=m, N_2(t)=n\}, m\geq 0, n\geq 0$$

$$\overline{M}_{m,n}(x,s,t)dx = Pr\{Y(t)=7, x<\pi(t)\leq x+dx, N_1(t)=m, N_2(t)=n\}), m\geq 0, n\geq 0$$

2.3 Equations Governing the System

The Kolmogorov forward equations which govern the model:

$$\frac{\partial}{\partial t}P^{(1)}_{m,n}(x,t) + \frac{\partial}{\partial x}P^{(1)}_{m,n}(x,t) = -(\lambda_1+\lambda_2+\alpha+\mu_1(x))P^{(1)}_{m,n}(x,t)$$
$$+(1-\delta_{m0})\lambda_1\sum_{i=1}^{m} c_{1,i}P^{(1)}_{m-i,n}(x,t) + (1-\delta_{0n})\lambda_2 b\sum_{i=1}^{n} c_{2,i}P^{(1)}_{m,n-i}(x,t)$$
$$+\lambda_2(1-b)P^{(1)}_{m,n}(x,t) + \gamma Q_{m,n}(t);\ m\geq 0, n\geq 0, \tag{1}$$

$$\frac{\partial}{\partial t}V_{m,n}(x,t) + \frac{\partial}{\partial x}V_{m,n}(x,t) = -(\lambda_1+\lambda_2+\xi+\beta(x))V_{m,n}(x,t)$$
$$+(1-\delta_{m0})\lambda_1\sum_{i=1}^{m} c_{1,i}V_{m-i,n}(x,t) + (1-\delta_{0n})\lambda_2 b\sum_{i=1}^{n} c_{2,i}V_{m,n-i}(x,t)$$
$$+\lambda_2(1-b)V_{m,n}(x,t) + \xi V_{m+1,n}(x,t) m\geq 0, n\geq 0, \tag{2}$$

$$\frac{\partial}{\partial t}P^{(2)}_{0,n}(x,t) + \frac{\partial}{\partial x}P^{(2)}_{0,n}(x,t) = -(\lambda_1+\lambda_2+\alpha+\mu_2(x))P^{(2)}_{0,n}(x,t)$$
$$+(1-\delta_{0n})\lambda_2 b\sum_{i=1}^{n} c_{2,i}P^{(2)}_{0,n-i}(x,t) + \lambda_2(1-b)P^{(2)}_{0,n}(x,t);\ n\geq 0, \tag{3}$$

$$\frac{\partial}{\partial t}C_{m,n}(x,t) + \frac{\partial}{\partial x}C_{m,n}(x,t) = -(\lambda_1+\lambda_2+\xi+\varphi(x))C_{m,n}(x,t)$$
$$+(1-\delta_{m0})\lambda_1\sum_{i=1}^{m} c_{1,i}C_{m-i,n}(x,t) + (1-\delta_{0n})\lambda_2 b\sum_{i=1}^{n} c_{2,i}C_{m,n-i}(x,t)$$
$$+\lambda_2(1-b)C_{m,n}(x,t) + \xi C_{m+1,n}(x,t); m\geq 0, n\geq 0, \tag{4}$$

$$\frac{\partial}{\partial t}M_{m,n}(x,t) + \frac{\partial}{\partial x}M_{m,n}(x,t) = -(\lambda_1+\lambda_2+\xi+\delta(x))M_{m,n}(x,t)$$
$$+(1-\delta_{m0})\lambda_1\sum_{i=1}^{m} c_{1,i}M_{m-i,n}(x,t) + (1-\delta_{0n})\lambda_2 b\sum_{i=1}^{n} c_{2,i}M_{m,n-i}(x,t)$$
$$+\lambda_2(1-b)M_{m,n}(x,t) + \xi M_{m+1,n}(x,t); m\geq 0, n\geq 0, \tag{5}$$

$$\frac{d}{dt}Q_{m,n}(t) = -(\lambda_1+\lambda_2+\gamma+\mu_3)Q_{m,n}(t) + (1-\delta_{m0})\lambda_1\sum_{i=1}^{m} c_{1,i}Q_{m-i,n}(t)$$
$$+(1-\delta_{0n})\lambda_2 b\sum_{i=1}^{n} c_{2,i}Q_{m,n-i}(t) + \lambda_2(1-b)Q_{m,n}(t) + \mu_3 Q_{m+1,n}(t)$$

$$+\alpha \int_0^\infty P_{m,n}^{(1)}(x,t)dx; m \geq 0, n \geq 0, \tag{6}$$

$$\frac{d}{dt}R_{0,n}(t) = -(\lambda_1 + \lambda_2 + \gamma)R_{0,n}(t) + (1 - \delta_{0n})\lambda_2 b \sum_{i=1}^{n} c_{2,i}R_{0,n-i}(t) + \lambda_2(1-b)R_{0,n}(t)$$

$$+\alpha \int_0^\infty P_{0,n-1}^{(2)}(x,t)dx;\ m \geq 0, n \geq 1, \tag{7}$$

$$\frac{d}{dt}I_{0,0}(t) = -(\lambda_1 + \lambda_2)I_{0,0}(t) + \int_0^\infty M_{0,0}(x,t)\delta(x)dx + \int_0^\infty V_{0,0}(x,t)\beta(x)dx + \gamma R_{0,0}(t) \tag{8}$$

$$\frac{\partial}{\partial t}I_{0,n}(x,t) + \frac{\partial}{\partial x}I_{0,n}(x,t) = -(\lambda_1 + \lambda_2 + \eta(x))I_{0,n}(x,t);\ n \geq 1. \tag{9}$$

The above set of equations are to be solved under the following boundary conditions at x=0.

$$I_{0,n}(0,t) = \int_0^\infty M_{0,n}(x,t)\delta(x)dx + \int_0^\infty V_{0,n}(x,t)\beta(x)dx + \gamma R_{0,n}(t);\ n \geq 1. \tag{10}$$

$$P_{m,n}^{(1)}(0,t) = \lambda_1 c_{1,m+1} I_{0,n}(t) + (1-\theta)\int_0^\infty P_{m+1,n}^{(1)}(x,t)\mu_1(x)dx + (1-\delta_{0n})\lambda_1 c_{1,m+1}$$

$$\int_0^\infty P_{0,n-1}^{(2)}(x,t)dx + \int_0^\infty M_{m+1,n}(x,t)\delta(x)dx + \int_0^\infty V_{m+1,n}(x,t)\beta(x)dx;$$

$$m \geq 0, n \geq 1, \tag{11}$$

$$P_{0,0}^{(1)}(0,t) = \lambda_1 c_{1,1} I_{0,0}(t) + (1-\theta)\int_0^\infty P_{1,0}^{(1)}(x,t)\mu_1(x)dx + \int_0^\infty M_{1,n}(x,t)\delta(x)dx$$

$$+\int_0^\infty V_{1,n}(x,t)\beta(x)dx; n = 0, \tag{12}$$

$$P_{0,n}^{(2)}(0,t) = \int_0^\infty I_{0,n+1}(x,t)\eta(x)dx + \lambda_2 b c_{2,n+1} I_{0,0}(t) + \lambda_2 b \sum_{i=1}^{n} c_{2,i}\int_0^\infty I_{0,n+1-i}(x,t)dx; n \geq 0, \tag{13}$$

$$V_{m,n}(0,t) = \theta \int_0^\infty P_{m,n}^{(1)}(x,t)\mu_1(x)dx;\ m \geq 0, n \geq 0, \tag{14}$$

$$V_{0,n}(0,t) = \theta \int_0^\infty P_{0,n}^{(1)}(x,t)\mu_1(x)dx + \theta \int_0^\infty P_{0,n}^{(2)}(x,t)\mu_2(x)dx;\ m = 0, n \geq 0, \tag{15}$$

$$C_{0,n}(0,t) = (1-\theta)\int_0^\infty P_{0,n}^{(1)}(x,t)\mu_1(x)dx + (1-\theta)\int_0^\infty P_{0,n}^{(2)}(x,t)\mu_2(x)dx;\ n \geq 0, \tag{16}$$

$$M_{m,n}(0,t) = \int_0^\infty C_{m,n}(x,t)\varphi(x)dx;\ m \geq 0,\ n \geq 0. \tag{17}$$

We assume that initially there are no customers in the system and the server is idle. Then the initial conditions become,

$$P_{m,n}^{(1)}(0) = P_{0,n}^{(2)}(0) = V_{m,n}(0) = C_{m,n}(0) = M_{m,n}(0) = R_0(0) = Q_0(0) = 0; m \geq 0, n \geq 0$$

$$I_{0,n}(0) = 0 \text{ and } I_{0,0}(0) = 1. \tag{18}$$

The Probability Generating Function(PGF) of this model:

$$I(x,z_2,t) = \sum_{n=1}^{\infty} z_2^n I_{0,n}(x,t), P_0^{(2)}(x,z_2,t) = \sum_{n=0}^{\infty} z_2^n P_{0,n}^{(2)}(x,t), R_0(z_2,t) = \sum_{n=0}^{\infty} z_2^n R_{0,n}(t),$$

$$Q(z_1,z_2,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} z_1^m z_2^n Q_{m,n}(t), B(x,z_1,z_2,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} z_1^m z_2^n B_{m,n}(x,t)$$

where $B = P^{(1)}, V, C, M$.

By taking Laplace transform from (1) to (17) and solving the equations, we get,

$$\bar{I}_0(x,s,z_2) = \bar{I}_0(0,s,z_2)[1-\bar{I}(f(a,s))]e^{-f(a,s)x}, \tag{19}$$

$$\bar{P}^{(1)}(x,s,z_1,z_2) = \bar{P}^{(1)}(0,s,z_1,z_2)[1-\bar{B}_1(f_1(s,z_1,z_2))]e^{-f_1(s,z_1,z_2)x}, \tag{20}$$

$$\bar{P}_0^{(2)}(x,s,z_2) = \bar{P}_0^{(2)}(0,s,z_2)[1-\bar{B}_2(f_2(s,z_2))]e^{-f_2(s,z_2)x}, \tag{21}$$

$$\bar{V}(x,s,z_1,z_2) = \bar{V}(0,s,z_1,z_2)[1-\bar{V}(f_5(s,z_1,z_2))]e^{-f_5(s,z_1,z_2)x}, \tag{22}$$

$$\bar{Q}(s,z_1,z_2) = \frac{\alpha\bar{P}^{(1)}(0,s,z_1,z_2)[1-\bar{B}_1(f_1(s,z_1,z_2))]e^{-f_1(s,z_1,z_2)x}}{f_3(s,z_1,z_2)}, \tag{23}$$

$$\bar{R}_0(s,z_2) = \frac{\alpha z_2\bar{P}_0^{(2)}(0,s,z_2)[1-\bar{B}_2(f_2(s,z_2))]e^{-f_2(s,z_2)x}}{f_4(s,z_2)}, \tag{24}$$

$$\bar{M}(x,s,z_1,z_2) = \bar{M}(0,s,z_1,z_2)[1-\bar{M}(f_5(s,z_1,z_2))]e^{-f_5(s,z_1,z_2)x}, \tag{25}$$

$$\bar{C}(x,s,z_1,z_2) = \bar{C}(0,s,z_1,z_2)[1-\bar{C}(f_5(s,z_1,z_2))]e^{-f_5(s,z_1,z_2)x}. \tag{26}$$

where,

$$f(a,s) = s+\lambda_1+\lambda_2,$$

$$f_1(s,z_1,z_2) = s+\lambda_1[1-C_1(z_1)]+\lambda_2 b[1-C_2(z_2)]+\alpha[1-\frac{\gamma}{f_3(s,z_1,z_2)}],$$

$$f_2(s,z_2) = s+\lambda_1+\lambda_2 b[1-C_2(z_2)]+\alpha,$$

$$f_3(s,z_1,z_2) = s+\lambda_1[1-C_1(z_1)]+\lambda_2 b[1-C_2(z_2)]+\mu_3[1-\frac{1}{z_2}]+\gamma,$$

$$f_4(s,z_2) = s+\lambda_1+\lambda_2 b[1-C_2(z_2)]+\gamma,$$

$$f_5(s,z_1,z_2) = s+\lambda_1[1-C_1(z_1)]+\lambda_2 b[1-C_2(z_2)]+\xi[1-\frac{1}{z_1}].$$

Similarly for the boundary conditions we can get,

$$\begin{aligned} z_1\bar{P}^{(1)}(0,s,z_1,z_2) &= \lambda_1 C_1(z_1)\bar{I}_0(x,s,z_2)+(1-\theta)\int_0^\infty \bar{P}^{(1)}(x,s,z_1,z_2)\mu_1(x)dx \\ &+\lambda_1 C_1(z_1)z_2\int_0^\infty \bar{P}_0^{(2)}(x,s,z_2)dx-(1-\theta)\int_0^\infty \bar{P}_0^{(1)}(x,s,z_2)\mu_1(x)dx \\ &+\int_0^\infty \bar{M}(x,s,z_1,z_2)\delta(x)dx-\int_0^\infty \bar{M}_0(x,s,z_2)\delta(x)dx. \\ &+\int_0^\infty \bar{V}(x,s,z_1,z_2)\beta(x)dx-\int_0^\infty \bar{V}_0(x,s,z_2)\beta(x)dx. \end{aligned} \tag{27}$$

$$\begin{aligned} \bar{I}_0(0,s,z_2) &= 1-(s+\lambda_1+\lambda_2)\bar{I}_{0,0}(s)+\int_0^\infty \bar{M}_0(x,s,z_2)\delta(x)dx \\ &+\int_0^\infty \bar{V}_0(x,s,z_2)\beta(x)dx+\gamma\bar{R}_0(s,z_2) \end{aligned} \tag{28}$$

$$z_2\bar{P}_0^{(2)}(0,s,z_2) = \bar{I}_0(0,s,z_2)[\bar{I}(f(a,s))+\lambda_2 bC_2(z_2)[\frac{1-\bar{I}(f(a,s))}{f(a,s)}]]+\lambda_2 bC_2(z_2)\bar{I}_{0,0}(s). \tag{29}$$

By applying Rouche's theorem on (27), we get,

$$\overline{P}_0^{(1)}(0,s,z_2)\overline{B}_1(f_1(s,z_2)) = \frac{\lambda_1 C_1(g(z_2))\overline{I}_0(0,s,z_2)[\frac{1-\overline{I}(f(a,s))}{f(a,s)}]+\overline{P}_0^{(2)}(0,s,z_2)\overline{G}_3(s,z_2)}{\overline{G}_4(s,z_2)}. \tag{30}$$

By substituting (30) in required equations we get,

$$\overline{P}_0^{(1)}(0,s,z_1,z_2) = \frac{\left[\begin{array}{l}\overline{I}_0(x,s,z_2)\{\lambda_1 C_1(z_1)\overline{G}_4(s,z_2)-\lambda_1 C_1(g(z_2))\overline{G}_1(s,z_1,z_2)\} \\ + \quad \overline{P}_0^{(2)}(0,s,z_2)\{\overline{G}_2(s,z_1,z_2)\overline{G}_4(s,z_2)-\overline{G}_1(s,z_1,z_2)\overline{G}_3(s,z_2)\}\end{array}\right]}{[\overline{G}_4(s,z_2)\{z_1-(1-\theta+\theta\overline{V}(f_5(s,z_1,z_2)))\overline{B_1}(f_1(s,z_1,z_2))\}]}, \tag{31}$$

$$\overline{P}_0^{(2)}(0,s,z_2) = \frac{\left[\begin{array}{l}(1-(s+\lambda_1+\lambda_2)\overline{I}_{0,0}(s))\overline{G}_4(s,z_2)\{\overline{I}+\lambda_2 bC_2(z_2)[\frac{1-\overline{I}}{f(a,s)}]\} \\ + \quad \lambda_2 bC_2(z_2)\overline{I}_{0,0}(s)\{\overline{G}_4(s,z_2)-\lambda_1 C_1(g(z_2))[\frac{1-\overline{I}(f(a,s))}{f(a,s)}]\overline{G}_5(s,z_2)\}\end{array}\right]}{\left[\begin{array}{l}z_2\{\overline{G}_4(s,z_2)-\lambda_1 C_1(g(z_2))[\frac{1-\overline{I}(f(a,s))}{f(a,s)}]\overline{G}_5(s,z_2)\} \\ - \quad \{\overline{G}_3(s,z_2)\overline{G}_5(s,z_2)+\overline{G}_4(s,z_2)\overline{G}_6(s,z_2)\}\{\overline{I}+\lambda_2 bC_2(z_2)[\frac{1-\overline{I}(f(a,s))}{f(a,s)}]\}\end{array}\right]} \tag{32}$$

$$\overline{I}_0(0,s,z_2) = \frac{\left[\begin{array}{l}z_2(1-(s+\lambda_1+\lambda_2)\overline{I}_{0,0}(s))\overline{G}_4(s,z_2) \\ + \quad \lambda_2 bC_2(z_2)\overline{I}_{0,0}(s)\{\overline{G}_3(s,z_2)\overline{G}_5(s,z_2)+\overline{G}_4(s,z_2)\overline{G}_6(s,z_2)\}\end{array}\right]}{\left[\begin{array}{l}z_2\{\overline{G}_4(s,z_2)-\lambda_1 C_1(g(z_2))[\frac{1-\overline{I}(f(a,s))}{f(a,s)}]\overline{G}_5(s,z_2)\} \\ - \quad \{\overline{G}_3(s,z_2)\overline{G}_5(s,z_2)+\overline{G}_4(s,z_2)\overline{G}_6(s,z_2)\}\{\overline{I}+\lambda_2 bC_2(z_2)[\frac{1-\overline{I}(f(a,s))}{f(a,s)}]\}\end{array}\right]} \tag{33}$$

$$\overline{V}(0,s,z_1,z_2) = \theta\overline{P}^{(1)}(0,s,z_1,z_2)\overline{B_1}(f_1(s,z_1,z_2)) + \theta\overline{P}_0^{(2)}(0,s,z_2)\overline{B_2}(f_2(s,z_2)), \tag{34}$$

$$\overline{C}(0,s,z_2) = (1-\theta)\overline{P}_0^{(1)}(0,s,z_2)\overline{B_1}(f_1(s,z_2)) + (1-\theta)\overline{P}_0^{(2)}(0,s,z_2)\overline{B_2}(f_2(s,z_2)),$$

(35)$\overline{M}(0,s,z_1,z_2) = \{(1-\theta)\overline{P}_0^{(1)}(0,s,z_2)\overline{B_1}(f_1(s,z_2)) + (1-\theta)\overline{P}_0^{(2)}(0,s,z_2)\overline{B_2}(f_2(s,z_2))\}$

$$\overline{C}(f_5(s,z_2)) \tag{36}$$

Where,

$$\overline{G}_1(s,z_1,z_2) = \{1-\theta+\theta\overline{V}(f_5(s,z_2))\} + (1-\theta)\{\overline{C}(f_5(s,z_2))\overline{M}(f_5(s,z_2))$$

$$-\overline{C}(f_5(s,z_1,z_2))\overline{M}(f_5(s,z_1,z_2))\},\quad \overline{G}_2(s,z_1,z_2) = \theta\overline{B}_2(f_2(s,z_2))[\overline{V}(f_5(s,z_1,z_2)) - \overline{V}(f_5(s,z_2))] + (1-\theta)\overline{B}_2(f_2(s,z_2))$$

$$\{\overline{C}(f_5(s,z_1,z_2))\overline{M}(f_5(s,z_1,z_2)) - \overline{C}(f_5(s,z_2))\overline{M}(f_5(s,z_2))\}$$

$$+\lambda_1 C_1(z_1)z_2[\frac{1-\overline{B}_2(f_2(s,z_2))}{f_2(s,z_2)}],$$

$$\overline{G}_3(s,z_2) = \theta\overline{B}_2(f_2(s,z_2))[\overline{V}(f_5(s,g(z_2))) - \overline{V}(f_5(s,z_2))] + (1-\theta)\overline{B}_2(f_2(s,z_2))$$

$$\{\overline{C}(f_5(s,g(z_2)))\overline{M}(f_5(s,g(z_2))) - \overline{C}(f_5(s,z_2))\overline{M}(f_5(s,z_2))\}$$

$$+\lambda_1 C_1(g(z_2))z_2[\frac{1-\overline{B}_2(f_2(s,z_2))}{f_2(s,z_2)}],$$

$$\overline{G}_4(s,z_2) = (1-\theta)\{\overline{C}(f_5(s,z_2))\overline{M}(f_5(s,z_2)) - \overline{C}(f_5(s,g(z_2)))\overline{M}(f_5(s,g(z_2)))\} \qquad [1-\theta+\theta\overline{V}(f_5(s,z_2))]$$

$$\overline{G}_5(s,z_2) = (1-\theta)\overline{C}(f_5(s,z_2))\overline{M}(f_5(s,z_2)) + \theta\overline{V}(f_5(s,z_2))$$

$$\overline{G}_6(s,z_2) = (1-\theta)\overline{B}_2(f_2(s,z_2))\overline{C}(f_5(s,z_2))\overline{M}(f_5(s,z_2))$$

$$+\theta\overline{B}_2(f_2(s,z_2))\overline{V}(f_5(s,z_2)) + \frac{\alpha\gamma z_2[1-\overline{B}_2(f_2(s,z_2))]}{f_2(s,z_2)f_4(s,z_2)}.$$

Theorem 1 . The inequality

$$P^{(1)}(1,1) + P^{(2)}(0,1) + Q(1,1) = \rho < 1,$$

is a necessary and sufficient condition for the system to be stable, under this condition the marginal PGF of the server's state, queue size and orbit size distributions are given by,

$$\bar{I}_{(0)}(s,z_2) = \bar{I}_0(0,s,z_2)[\tfrac{1-\bar{I}(f(a,s))}{f(a,s)}], \tag{37}$$

$$\bar{P}^{(1)}(s,z_1,z_2) = \bar{P}^{(1)}(0,s,z_1,z_2)[\tfrac{1-\overline{B_1}(f_1(s,z_1,z_2))}{f_1(s,z_1,z_2)}], \tag{38}$$

$$\bar{V}(s,z_1,z_2) = \{\theta\bar{P}^{(1)}(0,s,z_1,z_2)\overline{B_1}(f_1(s,z_1,z_2)) + \theta\bar{P}_0^{(2)}(0,s,z_2)\overline{B_2}(f_1(s,z_2))\}$$
$$[\tfrac{1-\bar{V}(f_5(s,z_1,z_2))}{f_5(s,z_1,z_2)}], \tag{39}$$

$$\bar{P}_0^{(2)}(s,z_2) = \bar{P}_0^{(2)}(0,s,z_2)[\tfrac{1-\overline{B_2}(f_2(s,z_2))}{f_2(s,z_2)}], \tag{40}$$

$$\bar{C}(s,z_1,z_2) = \{(1-\theta)\bar{P}_0^{(1)}(0,s,z_2)[\tfrac{1-\overline{B_1}(f_1(s,z_2))}{f_1(s,z_2)}] + (1-\theta)\,\bar{P}_0^{(2)}(0,s,z_2)$$
$$\times[\tfrac{1-\overline{B_2}(f_2(s,z_2))}{f_2(s,z_2)}]\}[\tfrac{1-\bar{C}(f_5(s,z_1,z_2))}{f_5(s,z_1,z_2)}], \tag{41}$$

$$\bar{Q}(s,z_1,z_2) = \frac{\alpha\bar{P}^{(1)}(0,s,z_1,z_2)[1-\bar{B}_1(f_1(s,z_1,z_2))]}{f_1(s,z_1,z_2)f_3(s,z_1,z_2)}, \tag{42}$$

$$\bar{R}(s,z_2) = \frac{\alpha z_2\bar{P}_0^{(2)}(0,s,z_2)[1-\bar{B}_2(f_2(s,z_2))]}{f_2(s,z_2)f_4(s,z_2)}, \tag{43}$$

$$\bar{M}(s,z_1,z_2) = \{\{(1-\theta)\bar{P}_0^{(1)}(0,s,z_2)[\tfrac{1-\overline{B_1}(f_1(s,z_2))}{f_1(s,z_2)}] + (1-\theta)\,\bar{P}_0^{(2)}(0,s,z_2)$$
$$\times[\tfrac{1-\overline{B_2}(f_2(s,z_2))}{f_2(s,z_2)}]\}\bar{C}(f_5(s,z_1,z_2))\} \times [\tfrac{1-\bar{M}(f_5(s,z_1,z_2))}{f_5(s,z_1,z_2)}]. \tag{44}$$

3. Steady State Analysis: Limiting Behaviour

By applying the well-known Tauberian property,

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t),$$

to the above equations, we obtain the steady- state solutions of this model. In order to determine $I_{0,0}$, we use the normalizing condition

$$P^{(1)}(1,1) + V(1,1) + P^{(2)}(1) + Q(1,1) + R(1) + M(1,1) + C(1,1) + I_0(0,1) + I_{0,0} = 1.$$

For this, let $P_q(z_1,z_2)$ be the probability generating function of the queue size irrespective of the state of the system. Then adding the above equation, we obtain,

$$P_q(z_1,z_2) = P^{(1)}(z_1,z_2) + V(z_1,z_2) + P_0^{(2)}(z_2) + M(z_1,z_2) + C(z_1,z_2) + Q(z_1,z_2) + R(z_2), \tag{45}$$

$$P_q(z_1,z_2) = \frac{N_1(z_1,z_2)}{D_1(z_1,z_2)} + \frac{N_2(z_1,z_2)}{D_2(z_1,z_2)} + \frac{N_3(z_1,z_2)}{D_3(z_1,z_2)},$$

where

$$N_1(z_1,z_2) = I_0(0,z_2)[\tfrac{1-\bar{I}(f(a))}{f(a)}]\{G_4(z_2)f_5(z_1,z_2) + \lambda_1 C_1(g(z_2))\{(1-\theta)$$

$$[1-\overline{C}(f_5(z_1,z_2))\overline{M}(f_2(z_1,z_2))]\},$$

$$N_2(z_1,z_2)=P_0^{(2)}(0,z_2)\{G_4(z_2)\{[\frac{1-\overline{B_2}(f_2(z_2))}{f_1(z_2)}](f_4(z_2)+\alpha z_2)f_5(z_1,z_2)$$

$$+\overline{B}_2(f_2(z_2))f_4(z_2)[1-(1-\theta)\overline{C}(f_5(z_1,z_2))\overline{M}(f_5(z_1,z_2))+\theta\overline{V}(f_5(z_1,z_2))]\}$$

$$+f_2(z_2)f_4(z_2)(1-\theta)[1-\overline{C}(f_5(z_1,z_2))\overline{M}(f_5(z_1,z_2))]G_3(z_2)\}\{q\theta\overline{B_1}(f_1(z_2))$$

$$[1-\overline{V}(f_2(z_1,z_2))]\overline{M}(f_2(z_1,z_2)))\}\}$$

$$N_3(z_1,z_2)=P^{(1)}(0,z_1,z_2)\{(1-\overline{B}_1(f_1(z_1,z_2)))f_5(z_1,z_2)[\alpha+f_3(z_1,z_2)]+$$

$$\theta\overline{B}_1(f_1(z_1,z_2))(1-\overline{V}(f_5(z_1,z_2)))f_1(z_1,z_2)f_3(z_1,z_2)\},$$

$$D_1(z_1,z_2)=G_4(z_2)f_5(z_1,z_2),$$

$$D_2(z_1,z_2)=f_2(z_1,z_2)f_4(z_2)f_5(z_1,z_2)G_4(z_2),$$

$$D_3(z_1,z_2)=f_1(z_1,z_2)f_3(z_1,z_2)f_5(z_1,z_2).$$

In order to obtain the probability of idle time $I_{0,0}$, we use the normalizing condition

$$P_q^{(1)}(1,1)+I_{0,0}=1.$$

From which we can have,

$$I_{0,0}=\frac{(\lambda_1+\alpha)(\lambda_1+\gamma)\gamma\, G_4(1)}{Dr},$$

$$Dr=(\lambda_1+\alpha)(\lambda_1+\gamma)\gamma\, G_4(1)+\gamma P_0^{(2)}(0,1)\{G_4(1)+\{\overline{B}_2(\lambda_1+\alpha)(\lambda_1+\gamma)[(1-\theta)$$

$$[E(K)+E(M)]+\theta E(V)]+(1-\overline{B}_2)(\lambda_1+\alpha+\gamma)\}$$

$$+(\lambda_1+\alpha)(\lambda_1+\gamma)(1-\theta)[E(K)+E(M)]G_3(1)\}$$

$$+I_0(0,1)[\frac{1-\overline{I}(f(a))}{f(a)}]\{G_4(1)+\lambda_1[E(K)+E(M)]\}(\lambda_1+\alpha)(\lambda_1+\gamma)\gamma$$

$$+P^{(1)}(0,1,1)G_4(1)\{E(B_1)(\alpha+\gamma)+\theta E(V)\gamma\}(\lambda_1+\alpha)(\lambda_1+\gamma).$$

3.1 The Average Queue Length

The Mean number of customers in the queue and in the orbit under the steady state condition are,

$$L_{q_1}=\frac{d}{dz_1}P_{q_1}(z_1,1)|_{z_1=1},\quad L_{q_2}=\frac{d}{dz_2}P_{q_2}(1,z_2)|_{z_2=1}. \tag{46}$$

Then,

$$L_{q_1}=\frac{D_1{}'(1,1)N_1{}''(1,1)-D_1{}''(1,1)N_1{}'(1,1)}{2(D_1{}'(1,1))^2}+\frac{D_2{}'(1,1)N_2{}''(1,1)-D_2{}''(1,1)N_2{}'(1,1)}{2(D_2{}'(1,1))^2}$$

$$+\frac{D_3{}''(1,1)N_3{}'}{3(D_3{}''(1,1))^2},\quad L_{q_2}=\frac{d_1{}'(1,1)n_1{}''(1,1)-d_1{}''(1,1)n_1{}'(1,1)}{2(d_1{}'(1,1))^2}+\frac{d_2{}'(1,1)n_2{}''(1,1)-d_2{}''(1,1)n_2{}'(1,1)}{2(d_2{}'(1,1))^2}$$

$$+\frac{d_3{}''(1,1)n_3{}'}{3(d_3{}''(1,1))^2}.$$

3.2 The Average Waiting Time in the Queue and Orbit

Average waiting time of a customer in the high priority queue and the low priority orbit is

$$W_{q_1} = \frac{L_{q_1}}{\lambda_1}, \quad W_{q_2} = \frac{L_{q_2}}{\lambda_2}. \tag{47}$$

where L_{q_1} and L_{q_2} are given above.

4. Particular Cases

Case: 1 $M^X/G/1$ Queueing model:

If there are no priority arrival, no vacation, no working breakdown, no balking, no retrial, no closedown/startup time and batch arrival The model under study becomes classical $M^X/G/1$ queueing system. In this case, the PGF of the busy state is given as,

$$P(z) = \frac{-(1-\overline{B}(\lambda-\lambda C(z)))I_{0,0}}{z-\overline{B}(\lambda-\lambda C(z))}$$

Case: 2 $M/G/1$ Queueing model:

If there are no priority arrival, no vacation, no working breakdown, no balking, no retrial, no closedown/startup time and single arrival. The model under study becomes classical $M/G/1$ queueing system. In this case, the PGF of the busy state is given as,

$$P(z) = \frac{-(1-\overline{B}(\lambda-\lambda z))I_{0,0}}{z-\overline{B}(\lambda-\lambda z)}$$

The above two results are coincide with the results of Gross.D and Harris.M (1985).

5. Numerical Results

In order to see the effect of different parameters espeacially the high priority arrival and closedown rate on the different states of the server, the utilization factor and proportion of idle time, we compute some numerical results. We consider the service time, vacation time and repair time to be exponentially distributed to numerically illustrate the feasibility of our results. Giving the suitable values which satisfy the stability condition, we compute the following table values.

The values of the parameters are: $\mu = 14$, $\mu_3=2$, $\alpha = 1.0$, $\beta = 1.0$, $\gamma = 5.0$, $\delta = 1.0$, $\eta = 3.0$, $\theta = 0.3$, $\xi = 0.5$ and $\phi = 1.0$.

Table 1: Effect of λ_1 with varying value of λ_2 on various queue characteristics

λ_1	λ_2	$I_{0,0}$	Lq_1	Lq_2	Wq_1	Wq_2
	1.45	0.0705	1.3397	1.7501	0.7443	1.2069
1.8	1.55	0.0703	1.3609	23.6856	0.7561	15.2810
	1.65	0.0702	1.3812	29.7515	0.7561	18.0312
	1.45	0.0641	2.3041	8.3007	1.2127	5.7246
1.9	1.55	0.0639	2.3244	31.5489	1.2234	20.3541
	1.65	0.0638	2.3440	37.4151	1.2337	22.6758
	1.45	0.0584	3.4326	15.4959	1.7163	10.6868
2.0	1.55	0.0583	3.4510	40.1845	1.7255	25.9255
	1.65	0.0582	3.4689	45.8188	1.7345	27.7689

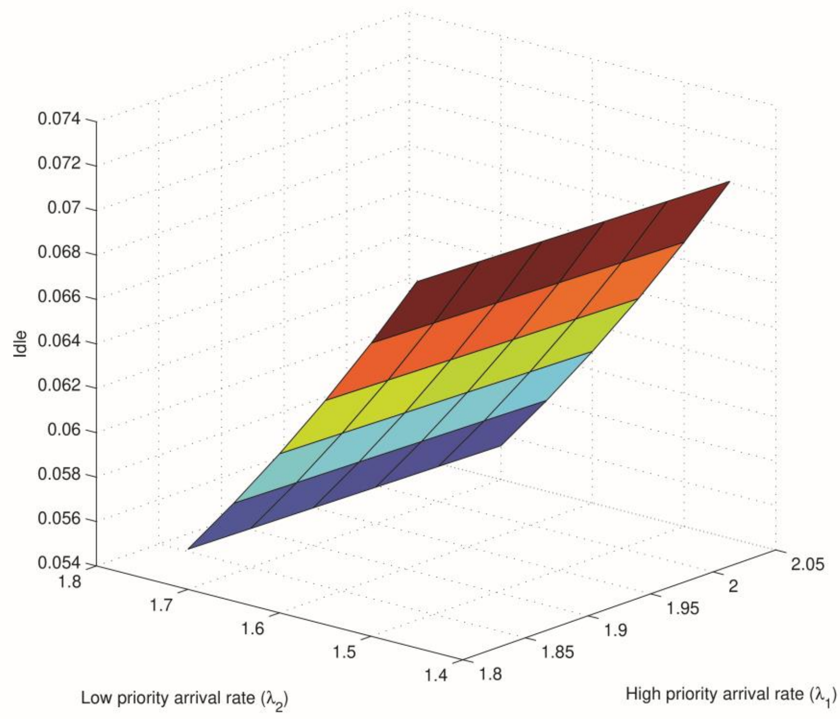

Figure 2: Idle Vs λ_1 and λ_2

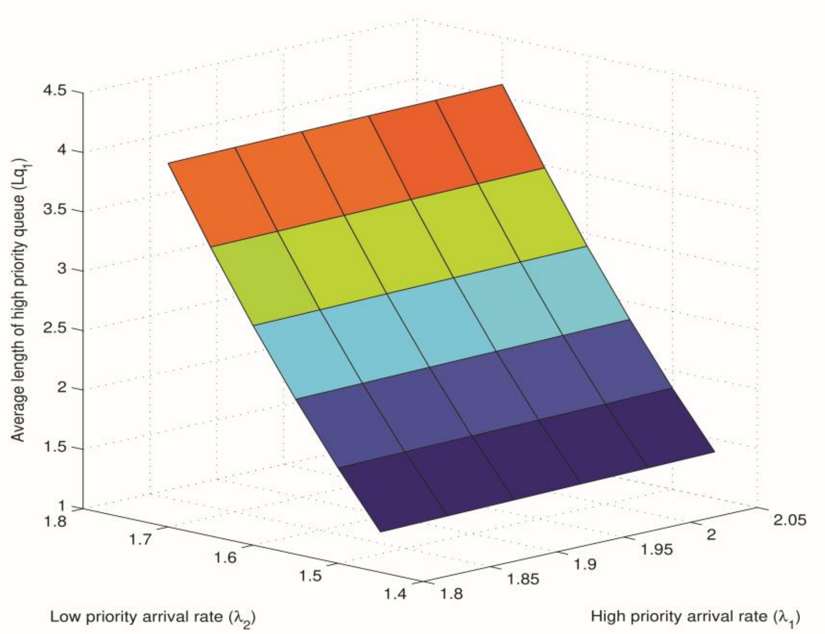

Figure 3: Lq_1 Vs λ_1 and λ_2

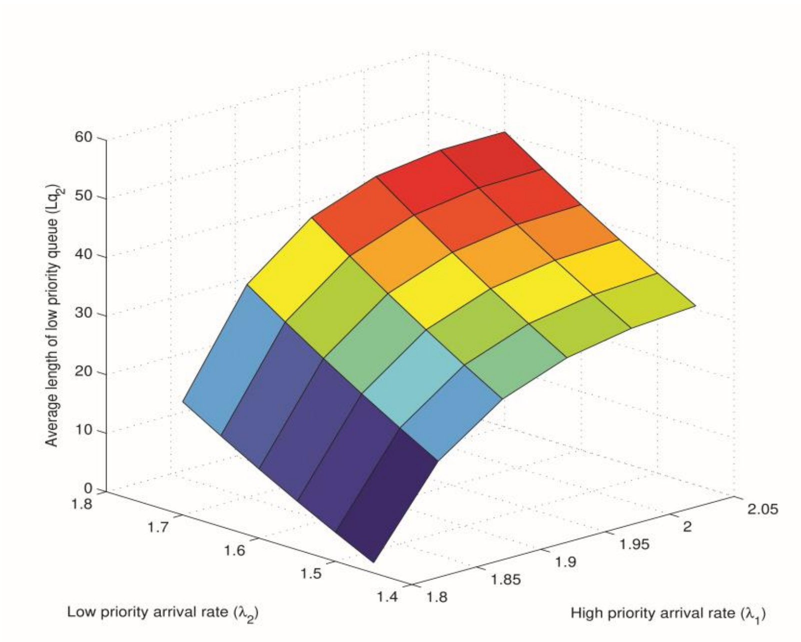

Figure 4: Lq_2 Vs λ_1 and λ_2

Table 1 clearly shows that as long as the arrival rate of high priority customers (λ_1) and low priority customers (λ_2) increases the server's idle time ($I_{0,0}$) decreases. Simultaneously the average queue length for both high priority (Lq_1) and low priority customers (Lq_2)are increases.

6. Conclusion

In this paper we have analysed a $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ retrial queue with priority service under Bernoulli vacation subject to working breakdown, close down/start up time also investigated. In addition, the effect of impatient behaviour of the customer on a service system is studied. The joint distribution of the number of customers in the queue and the number of customers in the orbit are derived. Numerical examples have been carried out to observe the trend of the mean number of customers in the system for varying parametric values.

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