Estimation Accuracy of Exponential Distribution Parameters

Muhammad Zahid Rashid
College of Statistical and Actuarial Sciences
University of the Punjab, Q.A. Campus
Lahore, Pakistan
zahidcsas@gmail.com

Ahmad Saeed Akhter
College of Statistical and Actuarial Sciences
University of the Punjab, Q.A. Campus
Lahore, Pakistan
akhtar@stat.pu.edu.pk

Abstract

The exponential distribution is commonly used to model the behavior of units that have a constant failure rate. The two-parameter exponential distribution provides a simple but nevertheless useful model for the analysis of lifetimes, especially when investigating reliability of technical equipment.

This paper is concerned with estimation of parameters of the two parameter (location and scale) exponential distribution. We used the least squares method (LSM), relative least squares method (RELS), ridge regression method (RR), moment estimators (ME), modified moment estimators (MME), maximum likelihood estimators (MLE) and modified maximum likelihood estimators (MMLE). We used the mean square error MSE, and total deviation TD, as measurement for the comparison between these methods. We determined the best method for estimation using different values for the parameters and different sample sizes.

Keywords: Exponential distribution, Parameter estimation, Least squares method, Relative least squares, Ridge regression, Moment estimators, Modified moment estimators, Maximum likelihood estimators, Modified maximum likelihood estimators, Total deviation, Mean square error.

Introduction

Exponential distribution plays an important part in the field of reliability engineering and life testing. The exponential distribution would be an appropriate choice for an experimenter, if failure rate appears to be more or less constant. “This is a distribution of the time to an event when the probability of the event occurring in the next small time interval does not vary through time. It is used in the theory of waiting lines or queues, which are found in many situations: from the gates at the entrance to toll roads through the time taken for an answer to a telephone enquiry, to the time taken for an ambulance to arrive at the scene of an accident” Evans et al; (2000). Exponential distribution is a special case of the Weibull distribution where $\beta=1$.

The exponential distribution can be found with one or two parameters; scale and location parameters. The values of these parameters must be more than zero except for the location parameter which can be greater than or equal to zero.

The probability density function of two parameter exponential distribution is given by

$$f(t_i,\gamma,\beta)=\frac{1}{\beta}\exp[-\frac{(t_i-\gamma)}{\beta}]t\geq\gamma;\beta>0,\gamma<t<\infty \quad (1)$$

The parameters γ and β are interpreted as measure of guarantee and failure rate respectively.

Maguire, Pearson and Wynn (1952) studied mine accidents and showed that time intervals between industrial accidents follow exponential distribution. Cohen and Helm (1973) used (BLUE), (MLE), (ME), (MVUE) and MME to estimate the parameters of the exponential distribution. Peter (1974) used robust M-estimation method for the scale parameter, with application to the exponential distribution. Cohen and Whitten (1982) used the moment and modified moment estimators for the Weibull distribution. Samia and Mohamed (1993) used five modifications of moments to estimate the parameters of the Pareto distribution. Lalitha and Anand (1996) used modified maximum likelihood to estimate the scale parameter of the Rayleigh distribution. Kang and Young (1997) estimated the parameters of a Pareto distribution by jackknife and bootstrap methods. Razali et al. (2009) studied the estimation accuracy of three parameter Weibull distribution.

In this paper, we use the least squares method, relative least squares, ridge regression, moment and modified moment estimators, maximum and modified maximum likelihood estimators to estimate the two parameter of the exponential distribution. The present paper introduces third modified moment estimators and first modified maximum likelihood estimators. Also, we compared between these methods using two parameter exponential distributions to find the most accurate method.

Methodology

1. Least squares method (LSM)

For the estimation of probability distribution parameters, the least squares method (LSM) is extensively used in reliability engineering and mathematics problems.

The cumulative distribution function of (1) is given by

$$F(t_i)=1-\exp[-\frac{(t_i-\gamma)}{\beta}]$$

$$\Rightarrow \exp[-\frac{(t_i-\gamma)}{\beta}]=1-F(t_i)$$

To get a linear relation between the two parameters taking the natural logarithm of above equation as follows

$$\frac{(t_i-\gamma)}{\beta}=\ln[1-F(t_i)$$

After simplification, we get

$$t_i=\gamma+\beta[-\ln(1-F(t_i)]$$

The last equation can be represented by

$$y_i=a+bx_i \qquad (2)$$

Where $y_i=t_i$, $a=\gamma$, $b=\beta$, $x_i=[-\ln(1-F(t_i))]$ $i=1,2,............n$ and *n* is the sample size.

Let $t_1,t_2,t_3,.......n$ be a random sample of t_i and $F(t_i)$ is estimated and replaced by the median rank method as follows:

$$F(t_i)=\frac{(i-0.3)}{(n+0.4)} \qquad t_i, i=1,2,......,n \quad (t_1<t_2<t_3...<t_n)$$

Because $F(t_i)$ of the mean rank method

$$F(t_i)=\frac{i}{(n+1)}$$

May be a larger value for smaller i and a smaller value for larger i.

Thus, equation (2) is a linear equation and is expressed as

$$y_i=a+bx_i$$

To compute a and b by simple linear regression we proceed as follows.

Let

$$S(a,b)=\sum_{i=1}^{n}(Y_i-a-bX_i)^2$$

Differentiating S w.r.t a and b then equate to zero, we obtain the following two normal equations

$$\sum_{i=1}^{n}Y_i=na+b\sum_{i=1}^{n}X_i$$

$$\sum_{i=1}^{n}X_iY_i=a\sum_{i=1}^{n}X_i+b\sum_{i=1}^{n}X_i^2$$

Solving the above two equations for a and b, we obtain the least square estimates (LSE) of a and b as:

$$\hat{a} = \frac{\sum_{i=1}^{n}[-\ln(1-F(t_i))]\sum_{i=1}^{n}t_i[-\ln(1-F(t_i))] - \sum_{i=1}^{n}t_i\sum_{i=1}^{n}[-\ln(1-F(t_i))]^2}{[\sum_{i=1}^{n}(-\ln(1-F(t_i)))]^2 - n\sum_{i=1}^{n}[-\ln(1-F(t_i))]^2} \quad (3)$$

$$\hat{b} = \frac{\sum_{i=1}^{n}t_i[-\ln(1-F(t_i))] - n\sum_{i=1}^{n}t_i[-\ln(1-F(t_i))]}{[\sum_{i=1}^{n}(-\ln(1-F(t_i)))]^2 - n\sum_{i=1}^{n}[-\ln(1-F(t_i))]^2} \quad (4)$$

2. Relative Least squares Method (RELS)

The relative least squares estimators of a and b can be obtained by minimizing the sum of squares of the relative residuals, Pablo and Bruce (1992), w.r.t. a and b as follows

$$S = \sum_{i=1}^{n}\left(\frac{Y_i - a - bX_i}{Y_i}\right)^2$$

$$S = \sum_{i=1}^{n}\left(1 - aw_i - bz_i\right)^2 \quad (5)$$

Differentiating w.r.t, a and b then equate to zero

$$\sum_{i=1}^{n} w_i = a\sum_{i=1}^{n} w_i^2 + b\sum_{i=1}^{n} w_i z_i$$

$$\sum_{i=1}^{n} z_i = a\sum_{i=1}^{n} w_i z_i + b\sum_{i=1}^{n} z_i^2$$

After simplification, we get

$$\hat{a} = \frac{\sum_{i=1}^{n} w_i z_i \sum_{i=1}^{n} z_i - \sum_{i=1}^{n} w_i \sum_{i=1}^{n} z_i^2}{\left(\sum_{i=1}^{n} w_i z_i\right)^2 - \sum_{i=1}^{n} w_i^2 \sum_{i=1}^{n} z_i^2} \quad (6)$$

$$\hat{b} = \frac{\sum_{i=1}^{n} w_i z_i \sum_{i=1}^{n} w_i - \sum_{i=1}^{n} w_i^2 \sum_{i=1}^{n} z_i^2}{\left(\sum_{i=1}^{n} w_i z_i\right)^2 - \sum_{i=1}^{n} w_i^2 \sum_{i=1}^{n} z_i^2} \quad (7)$$

Where $w_i = \frac{1}{t_i}$ and $z_i = \frac{\left[-\ln\left(1-F\left(t_i\right)\right)\right]}{t_i}$

3. Ridge regression method (RR):

The ridge regression (RR) estimates of A and B can be obtained by minimizing the error sum of squares for the model

$$y_i = a + bx_i$$

Subject to the single constraint that $a^2 + b^2 = \phi$ where ϕ is a finite positive constant.

The method of Lagrange's multiplier requires the differentiation of

$$L = \sum_{i=1}^{n} (y_i - a - bx_i)^2 + \lambda(a^2 + b^2 - \phi)$$

W.r.t a and b. when these derivatives are equated to zero, we obtain the following two equations

$$\sum_{i=1}^{n} y_i = (n+\lambda)a + b\sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b(\lambda + \sum_{i=1}^{n} x^2{}_i)$$

Solving above two equations for a and b we get

$$\hat{a} = \frac{(\sum_{i=1}^{n} x_i)(\sum_{i=1}^{n} x_i y_i) - \sum_{i=1}^{n} y_i(\lambda + \sum_{i=1}^{n} x^2{}_i)}{(\sum_{i=1}^{n} x_i)^2 - (n+\lambda)(\lambda + \sum_{i=1}^{n} x^2{}_i)}$$

$$\hat{b} = \frac{(\sum_{i=1}^{n} x_i)(\sum_{i=1}^{n} y_i) - (n+\lambda)\sum_{i=1}^{n} x_i y_i}{(\sum_{i=1}^{n} x_i)^2 - (n+\lambda)(\lambda + \sum_{i=1}^{n} x^2{}_i)}$$

For two parameter exponential distribution we know that $y_i = t_i$, $a = \gamma$, $b = \beta$
$x_i = [-\ln(1 - F(t_i))]$ $\qquad i = 1,2,\ldots\ldots\ldots\ldots n$

$$\hat{\gamma} = \frac{\sum_{i=1}^{n}[-\ln(1-F(t_i))]\sum_{i=1}^{n} t_i[-\ln(1-F(t_i))] - \sum_{i=1}^{n} t_i(\lambda + \sum_{i=1}^{n}[-\ln(1-F(t_i))]^2)}{[\sum_{i=1}^{n}(-\ln(1-F(t_i)))]^2 - (n+\lambda)(\lambda + \sum_{i=1}^{n}[-\ln(1-F(t_i))]^2)} \qquad (8)$$

$$\hat{\beta} = \frac{\sum_{i=1}^{n} t_i \sum_{i=1}^{n}(-\ln(1-F(t_i))) - (n+\lambda)\sum_{i=1}^{n} t_i(-\ln(1-F(t_i)))}{[\sum_{i=1}^{n}(-\ln(1-F(t_i)))]^2 - (n+\lambda)(\lambda + \sum_{i=1}^{n}[-\ln(1-F(t_i))]^2} \qquad (9)$$

Where $0 < \lambda < 1$ the ridge coefficient is the readers may see Ronald and Raymond (1978) if $\lambda = 0$ we obtain the least square estimates.

4. Moment Estimators (ME)

For the two parameter exponential distribution, the first two moments of exponential distribution are given by

$$\mu_1 = E(x) = \gamma + \beta \tag{10}$$

$$\mu_2' = E(x^2) = \gamma^2 + 2\beta^2 + 2\gamma\beta \tag{11}$$

The variance of x is given by

$$\mathrm{var}(x) = \mu_2' - \mu^2{}_1$$

$$= \gamma^2 + 2\beta^2 + 2\gamma\beta - (\gamma + \beta)$$

$$= \beta^2 \tag{12}$$

By equating the first two population moments with sample moments

$$\bar{x} = \frac{\left(\sum_{i=1}^{n} x_i\right)}{n} \tag{13}$$

$$\Rightarrow \qquad m_1 = \bar{x}$$

and $$s^2 = \frac{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}{n-1} \tag{14}$$

$$m_1 = \gamma + \beta \tag{15}$$

$$s^2 = \beta^2 \tag{16}$$

From above equation the estimate of β is given by $\hat{\beta} = s$.

By substitute about $\hat{\beta}$ into (15) the estimate of γ is given by $\hat{\gamma} = m_1 - s$.

5. First Modified Moment Estimators (MME-I)

In this modification of the moment estimators, the second moment of two parameter exponential distribution is replaced by

$$E[F(t_{(1)})] = F(t_{(1)})$$

Where $t_{(1)}$ is the first order statistic

$$\frac{1}{n+1} = 1 - \exp[-\frac{(t_{(1)} - \gamma)}{\beta}]$$

$$-\frac{(t_{(1)} - \gamma)}{\beta} = \ln(\frac{n}{n+1}) \tag{17}$$

We know that the first sample standard moment

$$m_1 = \gamma + \beta$$

$$\gamma = m_1 - \beta \tag{18}$$

The solution of (18) and (17) is given the estimate of β as

$$\hat{\beta} = \frac{m_1 - t_{(1)}}{1 + \ln(\frac{n}{n+1})}$$

$$\hat{\beta} = Z(m_1 - t_{(1)}) \tag{19}$$

Where

$$Z = \frac{1}{1 + \ln(\frac{n}{n+1})}$$

From (18) and (19) we can obtain the estimate of γ as

$$\hat{\gamma} = m_1 - Z(m_1 - t_{(1)})$$

$$\hat{\gamma} = (1 - Z)m_1 + Z(t_{(1)}) \tag{20}$$

6. Second Modified Moment Estimators (MME-II)

In this case, the second standard moment of two parameter exponential distribution is replaced by

$$E(t_{(1)}) = t_{(1)}$$

That is

$$\gamma + \frac{\beta}{n} = t_{(1)} \tag{21}$$

Solution of (18) and (21) is given the estimate of

$$\hat{\beta} = \frac{n(m_1 - t_{(1)})}{(n-1)} \tag{22}$$

From equations (21) and (22) we can obtain the estimate of γ as

$$\hat{\gamma} = \frac{n(t_{(1)} - m_1)}{(n-1)} \tag{23}$$

7. Third Modified Moment estimators (MME-III)

In this modification the second moment is replaced by $M_{et} = t_{me}$, where Me_t is the population median and t_{me} is the sample median.

The median of two parameter exponential distribution is given by

$$\frac{1}{\beta}\int_0^a \exp[-\frac{(t_i-\gamma)}{\beta}]dt = \frac{1}{2}$$

$$a = \gamma + \beta(\ln 2)$$

Thus, we have

$$\gamma + \beta(\ln 2) = t_{me} \qquad (24)$$

From equation (18) and (24) we obtain

$$\hat{\beta} = \frac{(m_1 - t_{me})}{(1-\ln 2)} \qquad (25)$$

From equation (24) and (25), we obtain

$$\hat{\gamma} = \frac{(t_{me} - m_1(\ln 2))}{(1-\ln 2)} \qquad (26)$$

8. Maximum likelihood estimator (MLE)

The likelihood function can be shown as follows,

$$L(\gamma,\beta) = \prod\nolimits^{n}{}_{i=1} f(t_{i,}\gamma,\beta) \qquad (27)$$

The maximum likelihood estimator MLE of the parameter is the value of the parameter that maximizes L and MLE for 2-parameter exponential distribution can be obtained by solving the equations resulting from setting the two partial derivatives of L(γ,β) to zero;

$$L(\gamma,\beta) = \prod_{i=1}^{n} \frac{1}{\beta}\exp[-\frac{(t_i-\gamma)}{\beta}] \qquad (28)$$

$$\frac{\partial \ln L}{\partial \gamma} = \frac{n}{\beta} = 0 \qquad (29)$$

It can be seen that $L(\gamma, t_m)$ is monotonically increasing with t_m i.e. the greater the value of t_m the greater the value of likelihood function.

Hence since t ≥ t_m we conclude

$$\hat{\gamma} = \min(t_i) = y_1 \qquad (30)$$

$$\frac{\partial \ln L}{\partial \gamma} = -\frac{n}{\beta} + \frac{\sum_{i=1}^{n}(t_i-\gamma)}{\beta^2}$$

$$\beta = \frac{\sum_{i=1}^{n}(t_i - \gamma)}{n} = \bar{t} - \gamma$$

$$\hat{\beta} = \bar{t} - y_1 \tag{31}$$

9. First Modified Maximum Likelihood estimator

The first equation of MLM $\hat{y} = y_1$ is replaced by

$$E(Y_{(1)}) = Y_{(1)}$$

$$\gamma + \frac{\beta}{n} = y_{(1)} \tag{32}$$

By solving equation (32) and (31) simultaneously, we obtain the parameter estimates

$$\hat{\gamma} = \frac{(n+1)y_{(1)} - \bar{t}}{n} \tag{33}$$

Goodness of Fit Analysis

Some methods of goodness of fit analysis are employed here. Mean square error MSE and total deviation TD are two measurements that give an indication of the accuracy of parameter estimation. AL-Fawzan (2000) referred to the use of the procedure of MSE and TD.

a. Mean Square Errors (MSE)

The MSE can be calculated as below

$$MSE = \sum_{i=1}^{n}\{\hat{F}(t_i) - F(t_i)\}^2$$

Where $\hat{F}(t_i) = 1 - \exp[-\frac{(t_i - \hat{\gamma})}{\hat{\beta}}]$

is the value of the cumulative distribution function of the two parameter exponential distribution using the estimated parameters, and

$$F(t_i) = 1 - \exp[-\frac{(t_i - \gamma)}{\beta}]$$

b. Total Derivation (TD)

The total derivation TD, calculated for each method is as follows

$$TD = \left|\frac{\hat{\gamma} - \gamma}{\gamma}\right| + \left|\frac{\hat{\beta} - \beta}{\beta}\right|$$

Where γ and β are the known parameters, and $\hat{\gamma}$ $and \hat{\beta}$ are the estimated parameters by any method.

Application

A simulation study is used in order to compare the performance of the proposed estimation methods. According to limitations of the computer time, we carry out this comparison taking the sample sizes as n = 20, 40, 60, 80 and 100 with pairs of (γ,β) = (1,1), (1,2), (2,3) we generated random samples of different sizes by observing that if U is uniform (0,1), then

$$T=\gamma+\beta[-\ln(1-u)] \text{ is exponential of } (\gamma,\beta)$$

Such generated data have been used to obtain estimates of the unknown parameters. The results obtained from parameter estimation of the 2-parameter exponential distribution using different sample sizes and different values of parameters with mean square error MSE and total deviation TD, are given in tables (1), (2), (3), (4) and (5).

Table 1: The estimates for n=20

Method	True Values		Estimated Values		MSE	TD
	γ	β	$\hat{\gamma}$	$\hat{\beta}$		
LSM	1	1	0.999997763	0.9999864	3.83633×10^{-11}	0.0000158
	1	2	1.00002381	1.99998	6.45067×10^{-10}	0.0000333
	2	3	2.00000623	3.00000825	6.0299×10^{-11}	0.0000059
RELS	1	1	0.95797	1.0000018	0.0114253	0.0420285
	1	2	0.9999	2.18104	0.0097119	0.0905306
	2	3	1.99999	2.999998	1.64×10^{-12}	0.000001
RR	1	1	0.973875	1.00162	0.00432129	0.027745
	1	2	0.999202	1.97792	0.0001753	0.011838
	2	3	1.97308	2.97954	0.0008347	0.02028
ME	1	1	1.06899	1.02620	0.0408251	0.09519
	1	2	1.13799	2.0524	0.0408306	0.16419
	2	3	2.206977	3.0786	0.0408276	0.129689
(MME-I)	1	1	0.944557	1.15063	0.0150322	0.0206073
	1	2	0.88912	2.30126	0.0150319	0.26151
	2	3	1.83367	3.45190	0.0150327	0.233798
(MME-II)	1	1	0.943095	1.15209	0.0153931	0.208995
	1	2	0.88618	2.3042	0.0153951	0.26592
	2	3	1.82927	3.45630	0.0153951	0.237465
(MME-III)	1	1	1.00664	1.08855	0.0596031	0.09519
	1	2	1.1771021	2.0132779	0.0596043	0.183741
	2	3	2.26565	3.01993	0.0596040	0.139468
MLE	1	1	1.00070	1.09449	0.0107806	0.09519
	1	2	1.00139	2.18899	0.0107800	0.095885
	2	3	2.00209	3.283485	0.0107805	0.09554
MMLE-I	1	1	0.9459755	1.09449	0.0098298	0.148515
	1	2	0.8919405	2.18899	0.0098315	0.202555
	2	3	1.837915	3.283485	0.0098310	0.175538

Table 2: The estimates for n=40

Method	True Values		Estimated Values		MSE	TD
	γ	β	$\hat{\gamma}$	$\hat{\beta}$		
LSM	1	1	0.999996	0.999924	2.041×10^{-8}	0.0000789
	1	2	1.004568	1.99696	0.0000482	0.0060837
	2	3	2.00000185	2.99999	4.1×10^{-12}	0.0000012
RELS	1	1	0.99999	0.99999746	4.0544×10^{-11}	0.0000032
	1	2	0.99992	2.0227	0.0003941	0.0113576
	2	3	2.0001945	2.999844	3.567×10^{-8}	0.0001493
RR	1	1	0.992288	0.994180	0.0013807	0.013532
	1	2	1.01501	1.96564	0.0004904	0.03219
	2	3	2.00730	2.95982	0.0003316	0.0170433
ME	1	1	1.15174	0.654301	0.414524	0.497439
	1	2	1.30347	1.30860	0.414503	0.64917
	2	3	2.45521	1.962901	0.414511	0.573305
(MME-I)	1	1	0.982780	0.823257	0.157346	0.193965
	1	2	0.965560	1.646513	0.157347	0.211184
	2	3	1.94834	2.46977	0.157347	0.202573
(MME-II)	1	1	0.982520	0.823516	0.157587	0.193964
	1	2	0.965040	1.647032	0.157587	0.211444
	2	3	1.947560	2.470548	0.157857	0.202704
(MME-III)	1	1	1.00950	0.796538	0.138321	0.212962
	1	2	1.01900	1.59308	0.138317	0.22246
	2	3	2.02849	2.38962	0.138324	0.217705
MLE	1	1	1.00311	0.802928	0.141786	0.200182
	1	2	1.00622	1.60586	0.141783	0.20329
	2	3	2.00932	2.40878	0.141796	0.201733
MMLE-I	1	1	0.983035	0.802928	0.193612	0.214037
	1	2	0.966070	1.60586	0.193608	0.231
	2	3	1.94911	2.40878	0.193610	0.222518

Table 3: The estimates for n=60

Method	True Values		Estimated Values		MSE	TD
	γ	β	$\hat{\gamma}$	$\hat{\beta}$		
LSM	1	1	0.9999982	0.9999975	6.5×10^{-10}	0.0000094
	1	2	1.00005263	2.0004348	0.0000003	0.00027
	2	3	2.0000054	2.999996	3.66×10^{-11}	1.0000016
RELS	1	1	1.0000094	1.0000038	2.1×10^{-9}	0.0000132
	1	2	1.0000174	1.999886	9.68×10^{-9}	0.0000741
	2	3	1.999963	2.99995	6.4×10^{-9}	0.000034
RR	1	1	0.990869	1.00078	0.0013624	0.009911
	1	2	0.998391	1.99403	0.0000859	0.004594
	2	3	1.98926	2.99481	0.0003236	0.0071
ME	1	1	1.01216	1.11234	0.0719734	0.1245
	1	2	1.02432	2.22468	0.0719734	0.13666
	2	3	2.03648	3.337022	0.0719741	0.130581
(MME-I)	1	1	0.996175	1.12833	0.0623616	0.132155
	1	2	0.992349	2.25665	0.0623561	0.135976
	2	3	1.988523	3.384975	0.0623559	0.134063
(MME-II)	1	1	0.996017	1.12848	0.0622806	0.132463
	1	2	0.992034	2.25697	0.0622854	0.136451
	2	3	1.988051	3.385448	0.0622831	0.134457
(MME-III)	1	1	1.04236	1.08214	0.103420	0.1245
	1	2	1.08471	2.16429	0.103414	0.166855
	2	3	2.12707	3.24643	0.103416	0.145678
MLE	1	1	1.01483	1.10967	0.0740230	0.1245
	1	2	1.02965	2.219349	0.0740185	0.139324
	2	3	2.04448	3.3290	0.0740204	0.131913
MMLE-I	1	1	0.996330	1.10967	0.0458705	0.11334
	1	2	0.992660	2.219349	0.0458743	0.117014
	2	3	1.98899	3.32902	0.0458733	0.115178

Table 4: The estimates for n=80

Method	True values		Estimated Values		MSE	TD
	γ	β	$\hat{\gamma}$	$\hat{\beta}$		
LSM	1	1	0.999998	0.9999977	1.84312×10^{-10}	0.0000039
	1	2	0.99999802	1.999998	4.0589×10^{-11}	0.0000025
	2	3	1.9999955	2.999996	1.14738×10^{-10}	0.0000036
RELS	1	1	1.0000176	1.0000132	0.0000001	0.0000898
	1	2	1.0000082	2.0000144	1.355×10^{-9}	0.0000153
	2	3	1.999994	2.99997	7.9446×10^{-10}	0.0000109
RR	1	1	0.995201	0.998303	0.0008376	0.006496
	1	2	1.00363	1.98818	0.0001032	0.00954
	2	3	1.99883	2.98649	0.0001617	0.0050883
ME	1	1	1.053605	0.783958	0.193896	0.269647
	1	2	1.0721	1.56792	0.223028	0.28814
	2	3	2.16082	2.35187	0.193897	0.296453
(MME-I)	1	1	0.989691	0.847872	0.205316	0.162437
	1	2	0.979383	1.69574	0.205309	0.172742
	2	3	1.96907	2.54362	0.205317	0.167592
(MME-II)	1	1	0.989625	0.847938	0.205412	0.162437
	1	2	0.979250	1.69588	0.2054069	0.17281
	2	3	1.96887	2.54382	0.205414	0.167625
(MME-III)	1	1	0.959586	0.877977	0.262005	0.162437
	1	2	0.919172	1.75595	0.262011	0.202583
	2	3	1.87876	2.63393	0.2620024	0.182643
MLE	1	1	1.00022	0.837339	0.191808	0.162881
	1	2	1.00045	1.67468	0.191787	0.16311
	2	3	2.00067	2.51202	0.191792	0.162995
MMLE-I	1	1	0.989757	0.837339	0.234187	0.172904
	1	2	0.979514	1.67468	0.234184	0.183146
	2	3	1.969271	2.51202	0.234184	0.178024

Table 5: The estimates for n=100

Method	True Values		Estimated Values		MSE	TD
	γ	β	$\hat{\gamma}$	$\hat{\beta}$		
LSM	1	1	1.00000202	1.0000017	2.58×10^{-10}	0.0000037
	1	2	1.00000213	2.0000022	7.884×10^{-11}	0.0000032
	2	3	2.00000312	2.999967	6.105×10^{-10}	0.0000127
RELS	1	1	1.00000113	1.00000111	8.72×10^{-11}	1.000002
	1	2	0.9999394	2.0000044	3.52×10^{-8}	0.000067
	2	3	2.0000157	2.999997	9.81×10^{-10}	0.0000089
RR	1	1	0.995823	0.999050	0.0007896	0.005127
	1	2	1.00085	1.99307	0.0000546	0.004315
	2	3	1.99667	2.992125	0.0001602	0.00429
ME	1	1	0.930781	0.914596	0.394559	0.154623
	1	2	0.861562	1.82919	0.394562	0.223843
	2	3	1.79234	2.74379	0.394564	0.189233
(MME-I)	1	1	0.995422	0.849955	0.196354	0.15458
	1	2	0.990844	1.69991	0.196344	0.159201
	2	3	1.98627	2.54987	0.196344	0.156908
(MME-II)	1	1	0.995380	0.849998	0.196423	0.154622
	1	2	0.990760	1.699995	0.196424	0.159243
	2	3	1.98614	2.54999	0.196426	0.156933
(MME-III)	1	1	0.6126	1.23278	2.18395	0.62018
	1	2	0.225200	2.46555	2.18398	1.00758
	2	3	0.837800	3.69833	2.18397	0.813877
MLE	1	1	1.00388	0.841498	0.184035	0.162382
	1	2	1.00776	1.68300	0.184029	0.16626
	2	3	2.01164	2.52449	0.184038	0.164323
MMLE-I	1	1	0.995465	0.841498	0.219908	0.163037
	1	2	0.990930	1.68300	0.219903	0.16757
	2	3	1.98639	2.52449	0.219921	0.165308

Conclusion

The results are listed in the tables (1), (2), (3), (4) and (5). From the computations, we note that, the estimates of parameters from the LSM are too close to the true values, and the values of MSE and TD are very small. The parameter estimates from RELS, RR method are close to the true values but not as LS estimates, because the values of MSE and TD are greater than the corresponding values from LSM.

In this article, our proposed third modified moment estimators (MME-III) is given better accuracy as compared to previously defined traditional method of moments (ME) and first modified moment estimators (MME-I) for small samples of size n=20, 40. For the samples of large size n=100, the values of MSE and TD obtained from (MME-111) are too large.

Another modification which we made in this article is the first modified maximum likelihood estimators (MMLE-I). First modified maximum likelihood estimators (MMLE-I) is given better accurate estimates as compared to previously defined traditional method of moments (ME) and maximum likelihood method for the samples of size n=20, 60, because they gave the least value for the mean square error.

References

1. Cohen, A.C., and Helm, F.R. (1973). Estimation in Exponential Distribution. *Technometrics* 15, 415-418.
2. Cohen, A.C. and Whitten, B.J. (1982b). Modified Maximum Likelihood and Moment Estimation of three Parameters Weibull Distribution. *Commun. Statist. Theory and Method* 11(23), 2631-2656.
3. Evans, M., Hasting, N.A.J. and Peacocks, J.B. (2000). *Statistical Distribution* (3^{rd} ed.). Wiley Series in Probability and Statistic, New York.
4. Kang, S.B. and Young, C.S. (1997). Estimation of the Parameters in a Pareto Distribution by Jackknife and Bootstrap Methods. *Journal of Information and Optimization Sci*. 18, 289-300.
5. Keceioglu, D. (1991). *Reliability Engineering Handbook*. Prentice Hall, Eaglewood Cliffs, NJ.
6. Lalitha, S. and Anand, M. (1996). Modified Maximum Likelihood Estimators for Rayleigh Distribution. *Commun. Statist*., *Theory and Method* 25 (2), 389-401.
7. Maguire, B.A., Pearson, E.S. and Wynn, A.H.A. (1952). The time intervals between industrial accidents. *Biometrika* 39, 168-180.
8. Mohammed A. AL-Fawzan (2000). Algorithms for Estimating the Parameters of the Weibull Distribution, *King Abdul-Aziz City for Science and Technology, Riyadh, Saudi Arabia*.

9. Pablo, B.S. and Bruce, E.R. (1992). Model Parameter Estimation using Least Squares. *Water Res.* 26(6), 789-796.

10. Peter, F. Thall, (1974). Huber-Sense Robust M-Estimation of a Scale Parameter, with Application to the Exponential Distribution. *JASA*74, 147-152.

11. Ronald, E.W. and Raymond, H.M. (1978). *Probability and Statistics for Engineers and Scientists* (2nd ed). McMillan Publishing Co., Inc., New York.

12. Samia, A.S. and Mohammed M.M. (1993). Modified Moment Estimators for three parameters Pareto distribution. *ISSR*, Cairo University 28 (2).

13. Sidney, Y. and Ferenc, S. (1989). *An Introduction to Numerical Computations* (2nd ed.). Macmillan Publishing Company, New York.

14. Yidirim, F. (1990). Least Squares Estimation for the Weibull Parameters, *Pakistan Journal of Statistics*, 6(1) A, 93-104.