Concomitants of Generalized Order Statistics for a Bivariate Weibull Distribution

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Abstract

In this paper we have studied the distribution of *r*–th concomitant and joint distribution of *r*–th and *s*–th concomitant of generalized order statistics for a bivariate Weibull distribution. We have derived the expression for single and product moments. Numerical study has also been conducted to see the behavior of mean of concomitants for selected values of the parameters.

Keywords: Concomitants, Generalized Order Statistics, Bivariate Weibull distribution.

1. Introduction

A bivariate Weibull distribution is defined by Hanif Shahbaz and Ahmad (2009) and by Ahsanullah et al. (2010) as a compound distribution of two Weibull random variables. The density function of bivariate Weibull distribution defined by Hanif Shahbaz and Ahmad (2009) is

$$f(x,y)=\beta\phi(x)\alpha_1\alpha_2x^{\alpha_1-1}y^{\alpha_2-1}exp\left[-\left\{\beta x^{\alpha_1}+\phi(x)y^{\alpha_2}\right\}\right];$$
$$\alpha_1>0\ ,\ \alpha_2>0\ ,\ \beta>0\ ,\ \phi(x)>0\ ,\ x>0\ ,\ y>0, \qquad (1.1)$$

where $\phi(x)$ is any positive real function of *X*. Ahsanullah et al. (2010) have studied the distribution (1.1) for $\phi(x)=x^{\alpha_1}$. The distribution in that case is given as

$$f(x,y)=\beta\alpha_1\alpha_2x^{2\alpha_1-1}y^{\alpha_2-1}exp\left[-x^{\alpha_1}\left(\beta+y^{\alpha_2}\right)\right];\ \beta>0,\ \alpha_1>0,\alpha_2>0,x>0,y>0. \qquad (1.2)$$

Distribution (1.1) can be studied for other choices of $\phi(x)$. The distribution (1.2) has been study in context of order statistics and record values by Ahsanullah et al. (2010) and in context of order statistics by Hanif Shahbaz et al. (2011).

The generalized order statistics (*gos*) has been defined by Kamps (1995) as a unified model for ordered random variables. Kamps (1995) has argued that the quantities $X_{r:n,m,k}$ are called *gos* if their joint distribution is given as

$$f_{1,\ldots,n:n,m,k}(x_1,x_2,\ldots,x_n)=k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left\{1-F(x_n)\right\}^{k-1}f(x_n)$$
$$\times\left[\prod_{j=1}^{n-1}\left\{1-F(x_i)^m\right\}f(x_i)\right]; \qquad (1.3)$$

where *n* is sample size, *m* and *k* are parameters of the model and quantities γ_j are given as $\gamma_j = k + (n-r)(m+1)$. The density function of a single *gos* is given by Kamps (1995) as

$$f_{r:n,m,k}(x) = \frac{C_{r-1}}{(r-1)!} f(x)\{1-F(x)\}^{\gamma_r - 1} g_m^{r-1}\left[F(x)\right], \tag{1.4}$$

where $C_{r-1} = \prod_{j=1}^{r} \gamma_j ; r = 1, 2, \ldots, n$, and

$$g_m(x) = h_m(x) - h_m(0) = \begin{cases} \left[1-(1-x)^{m+1}\right] \big/ (m+1); & m \neq -1 \\ -\ln(1-x) & m = -1. \end{cases}$$

We also have

$$h_m(x) = \begin{cases} -(1-x)^{m+1} \big/ (m+1); & m \neq -1 \\ -\ln(1-x) & m = -1. \end{cases}$$

Kamps (1995) has further shown that the joint density function of two GOS $X_{r:n,m,k}$ and $X_{s:n,m,k}$ for $r < s$ is given as

$$f_{r,s:n,m,k}(x_1, x_2) = \frac{C_{s-1}}{(r-1)!(s-r-1)!} f(x_1) f(x_2)\{1-F(x_1)\}^m g_m^{r-1}\{F(x_1)\}$$
$$\times \{1-F(x_2)\}^{\gamma_s - 1}\left[h_m\{F(x_2)\} - h_m\{F(x_1)\}\right]^{s-r-1}; -\infty < x_1 < x_2 < \infty. \tag{1.5}$$

The density functions of GOS given in (1.4) and (1.5) provide several models of ordered random variables as special case. Specifically for $m = 0$ and $k = 1$ the model reduces to *Ordinary Order Statistics* as given by David and Nagaraja (2003). Also for $m = -1$ we obtain *k*th upper record values introduced by Chandler (1952). Other models like fractional order statistics given by Stigler (1977), sequential order statistics etc. can also be obtained for various values of the parameters involved. Other special cases of *gos* can be seen in Shahbaz et al. (2017).

Sometime it happen that a sample is available from a bivariate distribution, say $F(x, y)$, and sample is arranged with respect to one of the variable, say *X*. The other variable, *Y*, is shuffled alongside the variable *X* and is called the concomitant of *X*. When sample is arranged using order statistics then we have concomitants of order statistics and is discussed in David and Nagaraja (2003). Ahsanullah (1995) has discussed concomitants of record values. The concomitants of *gos* has been discussed by Ahsanullah and Nevzorov (2001) and by Shahbaz et al. (2017). The density function of *r*th concomitant of *gos* is given as

$$f_{[r:n,m,k]}(y) = \int_{-\infty}^{\infty} f(y \mid x) f_{r:n,m,k}(x) dx, \tag{1.6}$$

where $f(y \mid x)$ is conditional distribution of *Y* given *X* = *x* and $f_{r:n,m,k}(x)$ is defined in (1.4). The joint distribution of two concomitants is given as

$$f_{[r,s:n,m,k]}(y_1, y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_2} f(y_1 \mid x_1) f(y_2 \mid x_2) f_{r,s:n,m,k}(x_1, x_2) dx_1 dx_2, \tag{1.7}$$

where $f_{r,s:n,m,k}(x_1, x_2)$ is given in (1.5).

Various authors have studied concomitants of *gos*. Concomitants of *gos* for Gumbel Bivariate Exponential distribution has been studied by Ahsanullah and Beg (2006). Further Beg and Ahsanullah (2008) has studied concomitants of GOS for Gumbel bivariate family of distributions. Nayabuddin (2013) has studied concomitants of GOS for bivariate Lomax distribution. Hanif Shahbaz and Shahbaz (2016) have studied the concomitants of *gos* for a bivariate exponential distribution.

In this paper we have obtained the distribution of the concomitants of upper record statistics for Bivariate Pseudo–Weibull distribution. Firstly, we have defined the Bivariate Pseudo–Weibull distribution in the following section.

2. Bivariate Pseudo–Weibull Distribution

The bivariate pseudo Weibull distribution has been defined by Hanif Shahbaz and Ahmad (2009) as compound distribution of Weibull random variables. The density function of bivariate Weibull distribution is given in (1.2). From the density function we can readily see that the marginal density function of X is

$$f\left(x\right)=\beta\alpha_1 x^{\alpha_1-1}exp\left(-\beta x^{\alpha_1}\right);\ \beta>0,\ \alpha_1>0\ ,x>0. \tag{2.1}$$

The conditional distribution of Y given $X = x$ is

$$f\left(y\,|\,x\right)=\alpha_2 x^{\alpha_1} y^{\alpha_2-1}exp\left(-x^{\alpha_1}y^{\alpha_2}\right);\quad \alpha_1>0,\ \alpha_2>0\ ,x>0\ ,\ y>0. \tag{2.2}$$

The marginal and conditional distributions are useful in studying the distribution of concomitants of *gos* for bivariate Weibull distribution.

In the following section the distribution of concomitant of record statistics has been derived for (1.2).

3. Distribution of *r*–th Concomitant and its Properties

The Bivariate Pseudo–Weibull distribution has been given in (1.1) and (1.2). In this section the distribution of *r*–th concomitants of *gos* for Bivariate Pseudo–Weibull distribution, given in (1.2), has been obtained.

In order to obtain the distribution of concomitant of *gos* we first need the distribution of *r*th *gos* for the marginal distribution of X given in (2.1). The distribution of *gos* for X can be obtained by using (1.4). For this we first see that

$$F\left(x\right)=1-\exp\left(-\beta x^{\alpha_1}\right);\ \beta>0,\ \alpha_1>0,x>0.$$

Also

$$g_m\left[F\left(x\right)\right]=\frac{1}{m+1}\left[1-\left\{1-F\left(x\right)\right\}^{m+1}\right]=\frac{1}{m+1}\left[1-\exp\left\{-\left(m+1\right)\beta x^{\alpha_1}\right\}\right].$$

So

$$g_m^{r-1}\left[F\left(x\right)\right]=\frac{1}{\left(m+1\right)^{r-1}}\left[1-\exp\left\{-\left(m+1\right)\beta x^{\alpha_1}\right\}\right]^{r-1}$$

$$=\frac{1}{\left(m+1\right)^{r-1}}\sum_{i=0}^{r-1}\left(-1\right)^i\binom{r-1}{i}\exp\left\{-\left(m+1\right)\beta i x^{\alpha_1}\right\}. \tag{3.1}$$

Now, using (2.1) and (3.1) in (1.4), the distribution of *r*th *gos* for *X* is

$$f_{r:n,m,k}(x)=\frac{C_{r-1}}{(r-1)!}f(x)\{1-F(x)\}^{\gamma_r-1}g_m^{r-1}\left[F(x)\right]$$

$$f_{r:n,m,k}(x)=\frac{C_{r-1}}{(r-1)!}\beta\alpha_1x^{\alpha_1-1}\exp\left(-\beta x^{\alpha_1}\right)\exp\left\{-\beta x^{\alpha_1-1}(\gamma_r-1)\right\}$$
$$\times\frac{1}{(m+1)^{r-1}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\exp\left\{-(m+1)\beta ix^{\alpha_1}\right\}$$

or $$f_{r:n,m,k}(x)=\frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\beta\alpha_1x^{\alpha_1-1}\exp\left(-\beta w_1x^{\alpha_1}\right), \tag{3.2}$$

where $w_1=\{(m+1)i+\gamma_r\}$.

The conditional distribution of *Y* given *X* is given in (2.2). Now using (2.2) and (3.2) in (1.6), the distribution of *r*th concomitant of *gos* for bivariate Weibull distribution is

$$f_{[r:n,m,k]}(y)=\int_0^\infty\alpha_2x^{\alpha_1}y^{\alpha_2-1}\exp\left(-x^{\alpha_1}y^{\alpha_2}\right)\frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}$$
$$\times\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\beta\alpha_1x^{\alpha_1-1}\exp\left(-\beta w_1x^{\alpha_1}\right)dx$$

or $$f_{[r:n,m,k]}(y)=\frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\frac{\beta\alpha_2y^{\alpha_2-1}}{\left(y^{\alpha_2}+\beta w_1\right)^2};y>0. \tag{3.3}$$

The distribution of concomitants for special cases when sample is available from a bivariate Weibull distribution can be obtained from (3.3) by using specific values of the parameters involved.

The *r*–th moment of the distribution given in (3.3) is obtained as:

$$\mu_{[r:n,m,k]}^p=\frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\int_0^\infty y^p\frac{\beta\alpha_2y^{\alpha_2-1}}{\left(y^{\alpha_2}+\beta w_1\right)^2}dy,$$

which after simplifications become

$$\mu_{[r:n,m,k]}^p=\frac{\beta pC_{r-1}\Gamma(p/\alpha_1)\Gamma(1-p/\alpha_2)}{\alpha_2(r-1)!(m+1)^{r-1}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}(\beta w_1)^{p/\alpha_2-1} \tag{3.4}$$

which exist for $p<\alpha_2$. We can see that the moment expression given in (3.5) reduces to expression for moments of concomitants of order statistics given by Shahbaz et al. (2009) for m=0 and k=1. The table of means for m = 2, k = 2 and for various values of n, β and α_2 is given below.

Table 1: Mean of *gos* for Selected Values of Parameters

β	α_2	***n***	***r***									
			1	2	3	4	5	6	7	8	9	10
1.5	2.5	5	4.499	16.873	23.200	13.533	2.538					
		6	4.947	23.499	44.060	40.388	17.670	2.650				
		7	5.345	30.737	72.999	91.249	62.734	21.957	2.745			
		8	5.706	38.518	110.739	175.338	164.379	90.408	26.369	2.825		
		9	6.038	46.792	157.923	302.685	359.439	269.579	123.557	30.889	2.896	
		10	6.345	55.518	215.131	484.045	695.815	661.024	413.140	162.305	35.504	2.959
2.0	3.0	5	3.579	13.422	18.455	10.765	2.018					
		6	3.893	18.494	34.675	31.786	13.906	2.086				
		7	4.168	23.968	56.924	71.155	48.919	17.122	2.140			
		8	4.414	29.794	85.658	135.625	127.148	69.931	20.397	2.185		
		9	4.637	35.934	121.276	232.446	276.029	207.022	94.885	23.721	2.224	
		10	4.841	42.358	164.137	369.308	530.881	504.337	315.211	123.833	27.088	2.257
2.5	3.5	5	3.018	11.318	15.563	9.078	1.702					
		6	3.258	15.477	29.019	26.601	11.638	1.746				
		7	3.466	19.927	47.326	59.158	40.671	14.235	1.779			
		8	3.649	24.628	70.805	112.108	105.101	57.806	16.860	1.806		
		9	3.813	29.550	99.732	191.153	226.994	170.246	78.029	19.507	1.829	
		10	3.962	34.672	134.353	302.294	434.548	412.820	258.013	101.362	22.173	1.848

From above table we can see that the expected values shows an interesting trend. For even values of *n*, the mean increases for $1 \le r \le (n/2)$ and decreases for $(n/2) < r \le n$. For odd values of *n*, the mean increases for $1 \le r \le (n+1)/2$ and decreases for $(n+1)/2 < r \le n$. It can be further seen that for fixed value of *n*, the mean decreases with increase in β and α_2.

The mean and the variance of the concomitants of *gos* for other values of parameters can also be tabulated.

The distribution function of *r*th concomitant of *gos* for bivariate Weibull distribution is given as

$$F_{[r:n,m,k]}(y) = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \int_0^y \frac{\beta\alpha_2 y^{\alpha_2-1}}{\left(y^{\alpha_2}+\beta w_1\right)^2} dy$$

$$= \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{y^{\alpha_2}}{w_1\left(y^{\alpha_2}+\beta w_1\right)}. \tag{3.5}$$

The hazard rate function for concomitant of *gos* for bivariate Weibull distribution can be easily written by using (3.3) and (3.5) as

$$h_{[r:n,m,k]}(y) = \frac{\dfrac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \displaystyle\sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{\beta\alpha_2 y^{\alpha_2-1}}{\left(y^{\alpha_2}+\beta w_1\right)^2}}{1-\dfrac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \displaystyle\sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{y^{\alpha_2}}{w_1\left(y^{\alpha_2}+\beta w_1\right)}};\ y > 0. \tag{3.6}$$

The hazard rate function can be computed for given values of the parameters involved. In the following section we have obtained the joint distribution of two concomitants of *gos* for bivariate Weibull distribution.

4. Joint Distribution of the Concomitants and Moments

In this section we have derived the joint distribution of the concomitants of *gos* for bivariate Weibull distribution given in (1.2). The joint distribution is obtained by using expression (1.7). In order to obtain the joint distribution we first obtain the joint distribution of two *gos* by using (1.5) and is given as

$$f_{r,s:n,m,k}(x_1,x_2)=\frac{C_{s-1}}{(r-1)!(s-r-1)!}\beta\alpha_1 x_1^{\alpha_1-1}exp\left(-\beta x_1^{\alpha_1}\right)\beta\alpha_1 x_2^{\alpha_1-1}exp\left(-\beta x_2^{\alpha_1}\right)\exp\left(-\beta m x_1^{\alpha_1}\right)$$
$$\times\frac{1}{(m+1)^{r-1}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\exp\left\{-\beta(m+1)ix_1^{\alpha_1}\right\}\exp\left\{-\beta x_1^{\alpha_1}(\gamma_s-1)\right\}$$
$$\times\frac{1}{(m+1)^{s-r-1}}\left[\exp\left\{-\beta(m+1)x_1^{\alpha_1}\right\}-\exp\left\{-\beta(m+1)x_2^{\alpha_1}\right\}\right]^{s-r-1},$$

which after simplification becomes

$$f_{r,s:n,m,k}(x_1,x_2)=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}$$
$$\times\beta^2\alpha_1^2x_1^{\alpha_1-1}x_2^{\alpha_1-1}\exp\left(-\beta w_2x_1^{\alpha_1}\right)\exp\left(-\beta w_3x_2^{\alpha_1}\right);\quad 0<x_1<x_2<\infty, \tag{4.1}$$

where $w_2=\{(m+1)(s-r-j+i)\}$; $w_3=\{(m+1)j+\gamma_s\}$.

Using the distribution (4.1) and (2.2) in (1.5), the joint distribution of two concomitants of *gos* for bivariate Weibull distribution is

$$f_{[r,s:n,m,k]}(y_1,y_2)=\int_0^\infty\int_{x_1}^\infty\alpha_2x_1^{\alpha_1}y_1^{\alpha_2-1}exp\left(-x_1^{\alpha_1}y_1^{\alpha_2}\right)\alpha_2x_2^{\alpha_1}y_2^{\alpha_2-1}exp\left(-x_2^{\alpha_1}y_2^{\alpha_2}\right)$$
$$\times\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}$$
$$\times\beta^2\alpha_1^2x_1^{\alpha_1-1}x_2^{\alpha_1-1}\exp\left(-\beta w_2x_1^{\alpha_1}\right)\exp\left(-\beta w_3x_2^{\alpha_1}\right)dx_2dx_1$$

$$f_{[r,s:n,m,k]}(y_1,y_2)=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}$$
$$\times\beta^2\alpha_1^2\alpha_2^2\int_0^\infty x_1^{2\alpha_1-1}y_1^{\alpha_2-1}exp\left\{-x_1^{\alpha_1}\left(y_1^{\alpha_2}+\beta w_2\right)\right\}I(x_2)dx_1 \tag{4.2}$$

where $I(x_2)=\int_{x_1}^\infty x_2^{2\alpha_1-1}y_2^{\alpha_2-1}exp\left\{-x_2^{\alpha_1}\left(y_2^{\alpha_2}+\beta w_3\right)\right\}dx_2$

Using the transformation $x_2^{\alpha_1}\left(y_2^{\alpha_2}+\beta w_3\right)=t$ and simplifying we have

$$I(x_2)=\frac{y_2^{\alpha_2-1}}{\left(y_2^{\alpha_2}+\beta w_3\right)^2}\left\{1+x_1^{\alpha_1}\left(y_2^{\alpha_2}+\beta w_3\right)x_1\right\}\exp\left\{-x_1^{\alpha_1}\left(y_2^{\alpha_2}+\beta w_3\right)\right\}.$$

Using this in (4.2) we have

$$f_{[r,s:n,m,k]}(y_1,y_2)=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}$$
$$\times\beta^2\alpha_1\alpha_2^2\frac{y_1^{\alpha_2-1}y_2^{\alpha_2-1}}{\left(y_2^{\alpha_2}+\beta w_3\right)^2}\int_0^\infty x_1^{2\alpha_1-1}\left\{1+x_1^{\alpha_1}\left(y_2^{\alpha_2}+\beta w_3\right)x_1\right\}$$
$$\times exp\left[-x_1^{\alpha_1}\left\{y_1^{\alpha_2}+y_2^{\alpha_2}+\beta\left(w_2+w_3\right)\right\}\right]dx_1$$

Simplifying, the joint density function of two concomitants of *gos* for bivariate Weibull distribution is

$$f_{[r,s:n,m,k]}(y_1,y_2)=\frac{\beta^2\alpha_1\alpha_2C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}$$
$$\times\binom{s-r-1}{j}\frac{y_1^{\alpha_2-1}y_2^{\alpha_2-1}\left\{y_1^{\alpha_2}+3y_2^{\alpha_2-1}+\beta\left(w_2+w_3\right)\right\}}{\left(y_2^{\alpha_1}+\beta w_3\right)^2\left\{y_1^{\alpha_2}+y_2^{\alpha_2}+\beta\left(w_2+w_3\right)\right\}^3}. \tag{4.3}$$

The product moments can be numerically computed from (4.3).

5. Conclusions and Recommendations

In this paper we have studied the distribution of concomitants of generalized order statistics when a sample is available from a bivariate Weibull distribution. The study has been conducted when $\phi(x)=x^{\alpha_1}$. We have obtained the distribution of single concomitant and joint distribution of two concomitants. We have seen that the distribution of single concomitant of *gos* for bivariate Weibull distribution is weighted sum of Burr XII distributions. We have also seen that the mean of concomitants of *gos* increase with increase in the value of *r* until a specific point and then starts decreasing. This study can be extended by using some other choices of $\phi(x)$.

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