Nonparametric Confidence Interval for Quantiles

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Abstract

Quantiles, which are also known as values-at-risk in finance, frequently arise in practice as measures of risk. Confidence intervals for quantiles are typically constructed via large sample theory or the sectioning.

One of the ways for achieving the confidence interval for quantiles is direct use of a central limit theorem. In this approach, we require a good estimator of the quantile density function. In this paper, we consider the nonparametric estimator of the quantile density function from Soni et al. (2012) and we obtain confidence interval for quantiles. In the following, by using simulation, the coverage probability and mean square error of this confidence interval is calculated. Also, we compare our proposed approach with alternative approaches such as sectioning and jackknife.

Keywords: Bandwidth, Jackknife Estimator, Kernel Function, Nonparametric Confidence Interval, Quantile Density Function, Quantile Function, Sectioning.

1. Introduction

The quantile function $Q \equiv F^{-1}$ associated with a distribution function F defined as

$$Q(u) = F^{-1}(u) = \inf\{x; F(x) \geq u\}, \quad \text{for} \quad 0 < u < 1,$$

is sometimes the object of more direct interest than the F itself. The use of quantiles spans a wide range of fields, especially in finance, which are known as values-at-risk, nuclear engineering and project planning (Nakayama, 2012).

Assuming F has a positive derivative $F'(x) = f(x)$ on its domain, define

$$q(u) = Q'(u) = \frac{1}{f(Q(u))},$$

to be the *quantile density function* by Parzen (1979), and earlier dubbed the *sparsity index* by Tukey (1965). The quantile density function is of much practical relevance mainly because it appears as part of the asymptotic variance of empirical quantiles.

The basic concept for deriving the confidence Interval (CI) for quantiles is well known. The first time Woodruff (1952) presented a method for estimating a CI for quantiles. The approach consists of three steps: a point estimate of the cumulative distribution function, a CI for the point estimate, and converting it into CI for the quantile.

Nowadays, there are several approaches for deriving CI for quantiles, but no consensus about the best practice. In the following, we will express three attempts on the problem. The first attempt is direct use of a central limit theorem (CLT). We try to improve it by using nonparametric estimator of the quantile density function. The second attempt is to use sectioning or batching to calculate both a point estimate of the quantile and an estimate of the variance of the point estimate. The last attempt is to apply a jackknife approach to reducing the bias of previously method (based on sectioning).

In this paper, we consider the first approach and by using nonparametric estimator of the quantile density function introduced by Soni et al. (2012) construct a CI for quantile. Then we compare results of three methods numerically.

The layout of the paper is as follows: In Section 2, we express three methods to achieve CI for quantile. In Section 3, we represent smooth estimator of the quantile density function from Soni et al. (2012) and using it construct a CI for quantile. Section 4 contains the results of the simulation. In this section, we compare coverage probability (CP), mean square error (MSE) and the expected length of CI in our introduced approach with those in two other approaches. The conclusion is given in Section 5 and all tables appear in the appendix.

2. Confidence Interval Attempts

In the following we will review three attempts to achieve CI for quantiles.

2.1. Direct use of a CLT

A common approach to developing a CI is to first show that the quantile estimator satisfies a CLT, and then replace the variance constant in the CLT with a consistent estimator of it to construct a CI. We will use this approach to introduce nonparametric CI for quantile. For this purpose, we use the CLT for quantile estimator from Serfling (1980).

Suppose $X_1,\ldots,X_n$ be n identically independent distribution (i.i.d.) random variables from a distribution F and $X_{(1)} \le \ldots \le X_{(n)}$ be the order statistics corresponding to $X_1,\ldots,X_n$. Let $F_n(x)$ be the usual empirical distribution function, i.e.,

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \le x),$$

the sample estimator corresponding to $Q(u)$ is defined by

$$\hat{Q}_n(u) = F_n^{-1}(u) = \sum_{i=1}^{n} X_{(i)} I_{(\frac{i-1}{n},\frac{i}{n}]}(u), \quad \text{for} \quad 0 < u < 1.$$

Serfling (1980) shows that $\hat{Q}_n(u)$ is asymptotically normal with mean $Q(u)$ and variance $\frac{\sigma^2}{n}$, where

$$\sigma = \frac{\sqrt{u(1-u)}}{f(Q(u))} = \sqrt{u(1-u)}\,q(u).$$

This leads to a large sample $100(1-\alpha)\%$ CI for quantile $Q(u)$ of the form

$$\hat{Q}_n(u) \pm z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \qquad (1)$$

Unfortunately, this CI is not implementable in practice since $q(u)$ is typically unknown, and then so is σ. If we have a consistent estimator for $q(u)$ (i.e., $q_n(u) \to q(u)$ as $n \to \infty$), then we can replace σ in Equation (1) by $S_n = \sqrt{u(1-u)}q_n(u)$ to obtain

$$\hat{Q}_n(u) \pm z_{1-\frac{\alpha}{2}} \frac{S_n}{\sqrt{n}},$$

as another approximate $100(1-\alpha)\%$ CI for quantile $Q(u)$.

Generality, in this method there are two sources of error to build the CI for $Q(u)$. The first source, estimator of the quantile function and another source, estimator of the quantile density function. To view the various estimators for the *u*-th quantile refer to Steitnberg (1983). The estimation of the quantile density function for the first time have been suggested by Siddiqui (1960) and studied by Bloch and Gastwirth (1968), Bofinger (1975), Reiss (1978), Sheather and Maritz (1983), Babu (1986), Falk (1986), Welsh (1988), Jones (1992), Soni et al. (2012) and Chesneau et al. (2016).

2.2. Sectioning

Sectioning or batching is a very general methodology for constructing CI for a performance measure α. All the performance measures considered thus far in the course can be handled via the use of sectioning (for a detailed discussion of sectioning see Lewis and Orav, 1988). To illustrate the idea, suppose that we wish to construct a CI for quantile $Q(u)$. We suppose that the total number of independent replications n of our simulation experiment takes the form $n = mk$, where m corresponds to the number of sections. Think of m as being relatively small (say 10 to 20), with the number of observations k per section being very large.

Let $Q_{n(i)}(u)$ be the estimator for quantile $Q(u)$ based on all the observations $\{X_j : (i-1)k+1 \le j \le ik\}$ associated with the *i*-th section. Observe that the estimators $Q_{n(1)}(u),\ldots,Q_{n(m)}(u)$ are i.i.d. (since each estimator is based on a statistically identical, independent section of observations). By the previous CLT,

$$Q_{n(i)}(u) \overset{D}{\sim} N_i\left(Q(u), \frac{\sigma^2}{k}\right) \quad for\ 1 \le i \le m,$$

where the normal random variables $N_1,\ldots,N_m$ are independent. In other words, the section estimators $Q_{n(1)}(u),\ldots,Q_{n(m)}(u)$ behave, for large n (or equivalently, large k), like i.i.d. normal random variables with mean $Q(u)$ (the quantity we wish to estimate) and unknown variance $\frac{\sigma^2}{k}$. Therefore if n is large (so that the CLTs are good approximations), it follows that an approximate $100(1-\alpha)\%$ CI for quantile $Q(u)$ is

$$\bar{Q}_n \pm t_{m-1,1-\frac{\alpha}{2}} \sqrt{\frac{V_n}{m}},$$

where

$$\bar{Q}_n = \frac{1}{m}\sum_{i=1}^{m} Q_{n(i)}.$$

$$V_n = \frac{1}{m-1}\sum_{i=1}^{m}(Q_{n(i)} - \bar{Q}_n)^2.$$

and $t_{m-1,1-\frac{\alpha}{2}}$ is percentile $(1-\frac{\alpha}{2})$ from Student-t distribution with $m-1$ degrees of freedom.

Remark 1. *It is important, in using the above Student-t CI, to use the divisor* $m-1$ *(rather than* m*) in computing* V_n*, because the number of sections* m *will typically be relatively small.*

2.3. The Jackknife Estimator and Sectioning

$\hat{Q}_n$ and $\bar{Q}_n$ in CLT and sectioning methods are two estimators of quantile $Q(u)$ respectively, but $\bar{Q}_n$ has a bias that is roughly m times as large as that of $\hat{Q}_n$. The large bias of $\bar{Q}_n$ makes it an undesirable estimator for quantile $Q(u)$ and renders this approach to constructing CI for quantile $Q(u)$ unsuitable without some further modification. We can apply a jackknife approach to reducing the bias of previously described CI methodology based on sectioning (for more details see Lewis and Orav (1988) and Nelson (1990)).

Suppose that Q_n be the sample estimator for $Q(u)$ based on all n observations and $\tilde{Q}_n(i)$ be the sample quantile associated with all the n replications, except those in the *i*-th section (i.e., replications indexed from $(i-1)k+1$ through ik, for $i=1,\ldots,m$). We compute the m pseudo-values

$$Q_n(i) = mQ_n - (m-1)\tilde{Q}_n(i) \quad for\ 1 \le i \le m.$$

Jackknife sectioning estimator for quantile $Q(u)$ defined by

$$Q_n^J = \frac{1}{m}\sum_{i=1}^{m} Q_n(i),$$

and Jackknife variance estimator computed by

$$V_n^J = \frac{1}{m-1}\sum_{i=1}^{m}(Q_n(i) - Q_n^J)^2.$$

Therefore

$$Q_n^J \pm t_{m-1,1-\frac{\alpha}{2}}\sqrt{\frac{V_n^J}{m}},$$

is an approximate $100(1-\alpha)\%$ CI for quantile $Q(u)$, where $t_{m-1,1-\frac{\alpha}{2}}$ is percentile $(1-\frac{\alpha}{2})$ from Student-t distribution with $m-1$ degrees of freedom.

3. Nonparametric Confidence Interval

In this section, we represent kernel estimator of the quantile density function from Soni et al. (2012). Then we express introduction to the kernel function and bandwidth and finally by using estimator of the quantile density function, we introduce a CI for the quantile $Q(u)$.

3.1. Estimator of the Quantile Density Function

Soni et al. (2012) introduce a smooth estimator of the quantile density function. This estimator is made based on kernel function. Based on sample $X_1,\ldots,X_n$, they propose a smooth estimator of the quantile density function as

$$q_n(u)=\frac{1}{h(n)}\int_0^1\frac{K\left(\frac{t-u}{h(n)}\right)}{f_n(\hat{Q}_n(t))}dt, \tag{2}$$

where $K(.)$ is a kernel and $h(n)$ is the bandwidth sequence. The estimator (2) can also be written as

$$q_n(u)=\frac{1}{h(n)}\sum_{i=1}^{n}\frac{1}{f_n(X_{(i)})}\int_{S_{i-1}}^{S_i}K\left(\frac{t-u}{h(n)}\right)dt,$$

where S_i is the proportion of observations less than or equal to $X_{(i)}$.

Generality, for small S_i-S_{i-1}, we use the mean value theorem to get

$$q_n(u)=\frac{1}{nh(n)}\sum_{i=1}^{n}\frac{K\left(\frac{S_i-u}{h(n)}\right)}{f_n(X_{(i)})}. \tag{3}$$

Kernel estimator is characterized by the kernel function, $K(.)$, which determines the shape of the weighting function, and the bandwidth, $h(n)$, which determines the "width" of the weighting function and hence the amount of smoothing. The two components determine the properties of the estimator. Considerable research has been carried out (and continues to be done) on the question of how one should select $K(.)$ and $h(n)$ in order to optimize the properties of the estimator. This issue will be discussed in the following.
The kernel $K(.)$ is a real valued function satisfying the following properties:

i. $K(u)\geq 0$ for all u;

ii. $\int_{-\infty}^{\infty}K(u)du=1$;

iii. $K(.)$ has finite support, that is $K(u) = 0$ for $|u|>c$ where $c>0$ is some constant;

iv. $K(.)$ is symmetric about zero;

v. $K(.)$ satisfies Lipschitz condition, viz there exists a positive constant M such that $|K(u)-K(v)|\leq M\,|u-v|$.

Several types of kernel functions are commonly used. Prakasa Rao (1983) shows that Epanechnikov kernel is the optimal kernel and efficiency of kernel functions relative to Epanechnikov kernel measure. This kernel is defined as

$$K(u)=\frac{3}{4}(1-u^2)I(|u|\leq 1).$$

In Equation (3), the parameter h is called the bandwidth or smoothing constant. It determines the amount of smoothing applied in estimating kernel. Selection of the bandwidth of kernel estimator is a subject of considerable research. Four popular methods commonly used are: subjective selection, selection with reference to some given distribution, cross-validation and "plug-in" estimator.

These bandwidth selectors represent only a sample of the many suggestions that have been offered in the recent literature. Some alternatives are described in Wand and Jones (1995) in which the theory is given in more details.

Unlike the above methods, Soni et al. (2012) did not try to optimize the bandwidth h, but they choose it arbitrarily to be a constant 0.15, 0.19 and 0.25 which led to similar results. In this paper, we also use the constant bandwidth.

3.2. Confidence Interval for Quantile

In the following, we present two theorems from Soni et al. (2012), in the first theorem they show consistency of the estimator of the quantile density function and in the other one they prove asymptotic normality of the proposed estimator.

Theorem 1. *Suppose* $q_n(u)$ *given by Equation (2) is the proposed estimator of the quantile density function* $q(u)$*, we have*

$$\sup_{0<u<1}|q_n(u)-q(u)|\rightarrow 0 \quad as \quad n\rightarrow\infty.$$

Theorem 2. *Suppose* $q_n(u)$ *given by Equation (2) is the proposed estimator of the quantile density function* $q(u)$*. Then* $\sqrt{n}(q_n(u)-q(u))$ *is asymptotically normal with mean zero and variance*

$$\sigma^2(u)=\frac{n}{(h(n))^2}E\left(\int_0^1 dK^*(u,t)F_n(\hat{Q}_n(t))\right)^2,$$

where $K^*(u,t)=K\left(\frac{t-u}{h(n)}\right)q(t)$.

Corollary 1. *According to the two theorems, proposed estimator of the quantile density function (2) is a consistent estimator and asymptotically normal, therefore we can use it as a good estimator to estimating* σ *in Equation (1). Hence we use the CLT and Soni's estimator of the quantile density function for constructing a CI for quantile* $Q(u)$.

4. Simulation

In this section, we report the results of the simulation. All studies that follow were carried out with the R statistical software package (R Core Team, 2014). We consider some well-known distribution functions, and therefore we can easily achieve $Q(u)$. These distributions are: Uniform(0, 1), Exponential(1), Normal(0, 1), Lognormal(0, 1) and Weibull(1, 1). By using the simulation, samples from these distributions were generated and then by using the estimator of the quantile density function, we obtain a CI from this sample. By repeating this process, we compute CP, MSE and the expected length of CI. Also, we obtain CI by sectioning and jackknife methods and we compute their CP, MSE and the expected length similarly.

In the simulation, we use the samples size $n = 50, 100, 200$ and 1000 at the level $\alpha = 0.05$. For finding the estimator of the quantile density function, we consider Epanechnikov kernel and reporting the results for $h(n) = 0.15$ (As mentioned, in this paper we are not trying to optimize the bandwidth $h(n)$ and we just want to show that even with constant bandwidth, the efficiency of this method is desirable). To determine CP, MSE and the expected length of CI, we generate 10000 samples of sizes 50, 100, 200 and 1000 from the specified distributions. We have taken the number of sections 10 in sectioning and jackknife methods. It is obvious that the best approach is the one providing the CP nearest to the nominal value (0.95, in this case) and have the shortest length. Results are reported in Tables 1-20.

We choose sample size $n = 50$ in Tables 1-5. As seen in tables for $u = 0.5$, the sectioning approach has more CP than jackknife and nonparametric methods, but in some cases, MSE of this method is more than the two other methods and usually the length of CI of sectioning method is more than nonparametric approach. However for $u = 0.1, 0.25, 0.75, 0.9$ results of sectioning method is not satisfactory. In almost all distributions (except lognormal distribution for $u = 0.9$) CP of the nonparametric method is better than sectioning and jackknife methods, also MSE of this method is low and CI have the shortest length. Even in some cases for $u = 0.05$ or 0.95, the performance of the nonparametric method is better than the two other methods. As described in Section 2.2, in sectioning method, m should be relatively small and the number of observations k per section should be very large. When sample size is 50, with choosing m=10, the number of observations per section is equal to 5, where is very small, hence CI is violated in extreme quantiles. It should be noted that, as expressed in Section 2.3, we applied the jackknife approach to reducing the bias of sectioning method (where the simulation results show this), but the CP of sectioning method may be better than other approaches, as quantiles near the median.

As the sample size increases to 100 (Tables 6-10) or 200 (Tables 11-15), results of nonparametric method improve in center and tails (especially in the lower tail), such that for $u = 0.05, 0.95$ in most cases the nonparametric method outperforms the two other methods.

In Tables 16-20, we report results for the sample size 1000. By comparison CP, MSE and the length of CI in various methods we conclude, except in tails, the nonparametric method has good results and in almost all cases CP, MSE and the expected length of CI in this method is better than sectioning and jackknife methods.

5. Conclusion

According to the simulation results, for small sample size, in estimation of CI for the median, sectioning method has better performance, and in tails the results of jackknife method is favorable. But for other quantiles, the nonparametric method has higher CP (in this work, close to 0.95), lower MSE and the shortest length in comparison with the two other methods. As the sample size increases, results of nonparametric method improve in center and tails (especially in the lower tail).

We emphasize that the results of nonparametric method have been achieved for constant bandwidth and we expect that the performance of this method increase by using optimal bandwidth.

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Appendix

Table 1: CP, MSE and the length of CI for Various Choices of u in sampling from Uniform(0,1) Distribution (n=50, replicate=10000, $\alpha = 0.05$).

Method	Index \ u	0.01	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.99
Nonparametric	CP	0.6001	0.8722	0.9615	0.9346	0.9299	0.9252	0.9599	0.8963	0.6484
	MSE	0.0005	0.0011	0.0024	0.0037	0.0048	0.0037	0.0025	0.0013	0.0005
	Length	0.0393	0.1117	0.1818	0.2357	0.2707	0.2342	0.1855	0.1214	0.0455
Sectioning	CP	0.0085	0.0346	0.1735	0.7797	0.9475	0.7839	0.1663	0.0364	0.0081
	MSE	0.0285	0.0245	0.0198	0.0103	0.0036	0.0102	0.0199	0.0243	0.0286
	Length	0.1733	0.1714	0.1752	0.2206	0.2344	0.2207	0.1752	0.1714	0.1733
Jackknife	CP	0.8727	0.8119	0.6977	0.8201	0.8367	0.8231	0.7011	0.8127	0.8734
	MSE	0.0007	0.0017	0.0026	0.0053	0.0067	0.0054	0.0026	0.0017	0.0007
	Length	0.1026	0.1634	0.1979	0.2848	0.3212	0.2875	0.1981	0.1633	0.1032

Table 2: CP, MSE and the length of CI for Various Choices of u in sampling from Exp(1) Distribution (n=50, replicate=10000, $\alpha = 0.05$).

Method	Index \ u	0.01	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.99
Nonparametric	CP	0.6289	0.8861	0.9611	0.9278	0.9175	0.8914	0.7374	0.4107	0.0736
	MSE	0.0005	0.0014	0.0031	0.0069	0.0194	0.0475	0.0643	0.0483	0.3823
	Length	0.0433	0.1242	0.2070	0.3187	0.5441	0.8534	0.9596	0.6436	0.2445
Sectioning	CP	0.0408	0.0611	0.1854	0.7739	0.9464	0.8672	0.6695	0.3160	0.0100
	MSE	0.0441	0.0433	0.0430	0.0369	0.0298	0.0574	0.2584	0.9426	5.7174
	Length	0.2490	0.2545	0.2776	0.3982	0.5743	0.8399	1.1305	1.3007	1.4608
Jackknife	CP	0.8779	0.8168	0.7269	0.8176	0.8417	0.8152	0.6784	0.7583	0.7210
	MSE	0.0008	0.0022	0.0035	0.0100	0.0279	0.0851	0.2031	0.4013	0.9157
	Length	0.1099	0.1827	0.2329	0.3928	0.6550	1.1432	1.7632	2.4787	3.7280

Table 3: CP, MSE and the length of CI for Various Choices of u in sampling from Normal(0, 1) Distribution (n=50, replicate=10000, $\alpha = 0.05$).

Method	Index \ u	0.01	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.99
Nonparametric	CP	0.1363	0.5713	0.8610	0.9285	0.9228	0.9043	0.8516	0.6117	0.1529
	MSE	0.0737	0.0235	0.0395	0.0360	0.0296	0.0327	0.0398	0.0251	0.0754
	Length	0.1742	0.4814	0.7510	0.7399	0.6740	0.7032	0.7522	0.5173	0.1990
Sectioning	CP	0.0025	0.1927	0.5051	0.8445	0.9488	0.8520	0.5086	0.1941	0.0036
	MSE	1.4592	0.4186	0.1820	0.0633	0.0284	0.0625	0.1793	0.4179	1.4544
	Length	0.8142	0.7561	0.7002	0.6902	0.6610	0.6930	0.7004	0.7563	0.8137
Jackknife	CP	0.7512	0.7837	0.6923	0.8254	0.8424	0.8176	0.6951	0.7774	0.7554
	MSE	0.1517	0.1015	0.0676	0.0519	0.0433	0.0522	0.0676	0.1014	0.1487
	Length	1.5143	1.2477	1.0185	0.8927	0.8160	0.8958	1.0193	1.2472	1.4991

Table 4: CP, MSE and the length of CI for Various Choices of u in sampling from Lognormal(0, 1) Distribution (n=50, replicate=10000, $\alpha = 0.05$).

Method	Index \ u	0.01	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.99
Nonparametric	CP	0.4176	0.8382	0.9502	0.9335	0.9128	0.8479	0.4995	0.2100	0.0232
	MSE	0.0021	0.0028	0.0059	0.0102	0.0298	0.0851	0.0969	0.0980	2.2224
	Length	0.0650	0.1783	0.2867	0.3890	0.6740	1.1437	1.1757	0.7661	0.2825
Sectioning	CP	0.0151	0.0861	0.2559	0.7916	0.9405	0.8920	0.7781	0.4626	0.0464
	MSE	0.0969	0.0733	0.0658	0.0580	0.0697	0.1469	0.7388	3.2679	40.377
	Length	0.3252	0.3317	0.3601	0.5182	0.8320	1.4925	2.9955	3.7199	4.1655
Jackknife	CP	0.8249	0.8158	0.7215	0.8260	0.8363	0.8084	0.6654	0.7450	0.6716
	MSE	0.0031	0.0052	0.0063	0.0152	0.0464	0.2122	0.9051	2.8640	15.912
	Length	0.2166	0.2834	0.3116	0.4840	0.8441	1.8055	3.7060	6.6068	15.636

Table 5: CP, MSE and the length of CI for Various Choices of u in sampling from Weibull(1, 1) Distribution (n=50, replicate=10000, $\alpha = 0.05$).

Method	Index \ u	0.01	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.99
Nonparametric	CP	0.6028	0.8849	0.9596	0.9306	0.9138	0.8894	0.7395	0.4113	0.0759
	MSE	0.0005	0.0014	0.0032	0.0069	0.0194	0.0475	0.0639	0.0473	0.3659
	Length	0.0432	0.1236	0.2070	0.3187	0.5426	0.8545	0.9569	0.6465	0.2445
Sectioning	CP	0.0369	0.0644	0.1785	0.7753	0.9457	0.8755	0.6686	0.3235	0.0104
	MSE	0.0443	0.0436	0.0430	0.0367	0.0297	0.0572	0.2587	0.9404	5.7220
	Length	0.2486	0.2557	0.2768	0.3960	0.5743	0.8448	1.1212	1.3101	1.4562
Jackknife	CP	0.8812	0.8107	0.7108	0.8203	0.8382	0.8122	0.6791	0.7611	0.7204
	MSE	0.0008	0.0021	0.0035	0.0105	0.0283	0.0835	0.2019	0.4161	0.9365
	Length	0.1092	0.1789	0.2318	0.4024	0.6589	1.1319	1.7579	2.5219	3.7683

Table 6: CP, MSE and the length of CI for Various Choices of u in sampling from Uniform(0,1) Distribution (n=100, replicate=10000, $\alpha = 0.05$).

Method	Index \ u	0.01	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.99
Nonparametric	CP	0.6978	0.8973	0.9651	0.9420	0.9411	0.9400	0.9676	0.9113	0.7188
	MSE	0.0001	0.0005	0.0011	0.0019	0.0024	0.0019	0.0012	0.0005	0.0002
	Length	0.0286	0.0799	0.1287	0.1681	0.1935	0.1680	0.1302	0.0834	0.0308
Sectioning	CP	0.0214	0.1533	0.5356	0.8626	0.9465	0.8652	0.5373	0.1495	0.0213
	MSE	0.0087	0.0074	0.0065	0.0037	0.0019	0.0037	0.0064	0.0074	0.0087
	Length	0.1025	0.1082	0.1317	0.1562	0.1707	0.1565	0.1310	0.1080	0.1022
Jackknife	CP	0.7100	0.6962	0.7856	0.8297	0.8423	0.8192	0.7757	0.7006	0.7133
	MSE	0.0003	0.0007	0.0013	0.0024	0.0030	0.0023	0.0013	0.0007	0.0003
	Length	0.0689	0.1038	0.1414	0.1903	0.2135	0.1891	0.1410	0.1058	0.0698

Table 7: CP, MSE and the length of CI for Various Choices of u in sampling from Exp(1) Distribution (n=100, replicate=10000, $\alpha = 0.05$).

Method	Index \ u	0.01	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.99
Nonparametric	CP	0.7132	0.9063	0.9663	0.9385	0.9360	0.9345	0.8811	0.5594	0.1054
	MSE	0.0002	0.0006	0.0015	0.0034	0.0100	0.0302	0.0621	0.0375	0.1781
	Length	0.0312	0.0883	0.1454	0.2273	0.3920	0.6808	0.9665	0.6830	0.2665
Sectioning	CP	0.0432	0.1697	0.5495	0.8433	0.9480	0.9206	0.7537	0.5906	0.0565
	MSE	0.0110	0.0112	0.0108	0.0112	0.0122	0.0267	0.1324	0.3526	3.2361
	Length	0.1250	0.1393	0.1746	0.2474	0.3848	0.6208	0.9278	1.1441	1.4566
Jackknife	CP	0.7166	0.6991	0.7888	0.8211	0.8396	0.8288	0.7686	0.6622	0.5796
	MSE	0.0003	0.0008	0.0017	0.0044	0.0122	0.0367	0.1166	0.2125	0.9036
	Length	0.0723	0.1127	0.1633	0.2605	0.4324	0.7511	1.3386	1.7934	3.4441

Table 8: CP, MSE and the length of CI for Various Choices of u in sampling from Normal(0, 1) Distribution (n=100, replicate=10000, $\alpha = 0.05$).

Method	Index \ u	0.01	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.99
Nonparametric	CP	0.1931	0.7234	0.9467	0.9459	0.9389	0.9359	0.9456	0.7459	0.1990
	MSE	0.0314	0.0171	0.0313	0.0197	0.0156	0.0188	0.0304	0.0175	0.0316
	Length	0.1832	0.4759	0.6852	0.5491	0.4901	0.5362	0.6771	0.4869	0.1938
Sectioning	CP	0.0267	0.4620	0.6929	0.8990	0.9516	0.8941	0.6900	0.4715	0.0269
	MSE	0.7267	0.1443	0.0716	0.0233	0.0139	0.0240	0.0723	0.1432	0.7364
	Length	0.6964	0.5965	0.5639	0.4938	0.4622	0.4928	0.5658	0.5975	0.6950
Jackknife	CP	0.6095	0.6743	0.7860	0.8206	0.8462	0.8296	0.7783	0.6739	0.6075
	MSE	0.1318	0.0507	0.0384	0.0233	0.0189	0.0233	0.0387	0.0507	0.1323
	Length	1.3467	0.8804	0.7679	0.5983	0.5388	0.5977	0.7710	0.8814	1.3560

Table 9: CP, MSE and the length of CI for Various Choices of u in sampling from Lognormal(0, 1) Distribution (n=100, replicate=10000, $\alpha = 0.05$).

Method	Index \ u	0.01	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.99
Nonparametric	CP	0.4481	0.8672	0.9640	0.9430	0.9393	0.9251	0.7201	0.3229	0.0340
	MSE	0.0007	0.0013	0.0029	0.0051	0.0161	0.0672	0.1182	0.0770	1.3345
	Length	0.0487	0.1319	0.2055	0.2791	0.4960	1.0163	1.3342	0.9051	0.3417
Sectioning	CP	0.0130	0.2511	0.5834	0.8466	0.9427	0.9278	0.8257	0.7261	0.1317
	MSE	0.0303	0.0187	0.0167	0.0165	0.0251	0.0714	0.3698	1.2822	25.193
	Length	0.1764	0.1861	0.2233	0.3065	0.5189	1.0470	2.0936	3.4594	4.8413
Jackknife	CP	0.6684	0.6947	0.7828	0.8271	0.8484	0.8163	0.7629	0.6519	0.5288
	MSE	0.0019	0.0022	0.0032	0.0063	0.0195	0.0904	0.5000	1.3808	15.718
	Length	0.1689	0.1857	0.2224	0.3123	0.5468	1.1788	2.7717	4.5789	13.605

Table 10: CP, MSE and the length of CI for Various Choices of u in sampling from Weibull(1, 1) Distribution (n=100, replicate=10000, $\alpha = 0.05$).

Method	Index \ u	0.01	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.99
Nonparametric	CP	0.7238	0.9156	0.9667	0.9366	0.9317	0.9345	0.8849	0.5541	0.1071
	MSE	0.0002	0.0006	0.0015	0.0034	0.0100	0.0304	0.0620	0.0382	0.1737
	Length	0.0311	0.0880	0.1451	0.2267	0.3919	0.6831	0.9677	0.6820	0.2667
Sectioning	CP	0.0427	0.1712	0.5345	0.8410	0.9431	0.9210	0.7503	0.5819	0.0524
	MSE	0.0111	0.0110	0.0110	0.0109	0.0124	0.0266	0.1306	0.3535	3.2660
	Length	0.1256	0.1388	0.1749	0.2453	0.3844	0.6202	0.9276	1.1349	1.4512
Jackknife	CP	0.7219	0.7029	0.7855	0.8276	0.8419	0.8314	0.7781	0.6688	0.5788
	MSE	0.0003	0.0008	0.0017	0.0044	0.0122	0.0375	0.1195	0.2124	0.8850
	Length	0.0725	0.1119	0.1629	0.2586	0.4321	0.7587	1.3552	1.8002	3.4177

Table 11: CP, MSE and the length of CI for Various Choices of u in sampling from Uniform(0,1) Distribution (n=200, replicate=10000, $\alpha = 0.05$).

Method	Index \ u	0.01	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.99
Nonparametric	CP	0.7691	0.9174	0.9629	0.9469	0.9427	0.9433	0.9670	0.9225	0.7738
	MSE	0.0001	0.0002	0.0006	0.0009	0.0012	0.0009	0.0006	0.0002	0.0001
	Length	0.0204	0.0567	0.0908	0.1198	0.1378	0.1197	0.0914	0.0580	0.0212
Sectioning	CP	0.0431	0.5081	0.7378	0.9086	0.9505	0.9101	0.7288	0.5051	0.0404
	MSE	0.0024	0.0022	0.0020	0.0014	0.0011	0.0014	0.0020	0.0022	0.0024
	Length	0.0567	0.0761	0.0895	0.1150	0.1287	0.1153	0.0897	0.0752	0.0565
Jackknife	CP	0.6870	0.7797	0.8103	0.8434	0.8603	0.8489	0.8080	0.7736	0.6927
	MSE	0.0001	0.0004	0.0006	0.0011	0.0015	0.0011	0.0006	0.0004	0.0001
	Length	0.0405	0.0748	0.0969	0.1306	0.1494	0.1316	0.0960	0.0746	0.0407

Table 12: CP, MSE and the length of CI for Various Choices of u in sampling from Exp(1) Distribution (n=200, replicate=10000, $\alpha = 0.05$).

Method	Index \ u	0.01	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.99
Nonparametric	CP	0.7841	0.9279	0.9677	0.9437	0.9432	0.9538	0.9567	0.7153	0.1538
	MSE	0.0001	0.0003	0.0007	0.0017	0.0051	0.0165	0.0518	0.0306	0.0483
	Length	0.0221	0.0624	0.1025	0.1610	0.2799	0.5040	0.8901	0.6657	0.2704
Sectioning	CP	0.0563	0.5089	0.7335	0.9018	0.9504	0.9311	0.8383	0.6968	0.1866
	MSE	0.0028	0.0028	0.0030	0.0036	0.0056	0.0140	0.0598	0.1765	1.5697
	Length	0.0635	0.0871	0.1097	0.1674	0.2756	0.4556	0.7331	0.9607	1.3786
Jackknife	CP	0.6935	0.7830	0.8090	0.8481	0.8623	0.8419	0.8040	0.7605	0.6058
	MSE	0.0001	0.0004	0.0008	0.0020	0.0059	0.0179	0.0565	0.1249	0.4935
	Length	0.0415	0.0800	0.1077	0.1774	0.3005	0.5245	0.9314	1.3849	2.6603

Table 13: CP, MSE and the length of CI for Various Choices of u in sampling from Normal(0, 1) Distribution (n=200, replicate=10000, $\alpha = 0.05$).

Method	Index \ u	0.01	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.99
Nonparametric	CP	0.2469	0.8218	0.9769	0.9557	0.9489	0.9470	0.9744	0.8370	0.2569
	MSE	0.0106	0.0122	0.0212	0.0101	0.0080	0.0099	0.0208	0.0123	0.0109
	Length	0.1698	0.4207	0.5682	0.3943	0.3513	0.3889	0.5621	0.4238	0.1736
Sectioning	CP	0.1297	0.6518	0.8187	0.9249	0.9502	0.9218	0.8150	0.6478	0.1309
	MSE	0.3218	0.0612	0.0268	0.0108	0.0073	0.0106	0.0270	0.0616	0.3229
	Length	0.5915	0.4935	0.4310	0.3626	0.3358	0.3615	0.4310	0.4911	0.5924
Jackknife	CP	0.6171	0.7737	0.8124	0.8425	0.8651	0.8498	0.8025	0.7601	0.6186
	MSE	0.0722	0.0294	0.0187	0.0110	0.0091	0.0110	0.0184	0.0296	0.0722
	Length	1.0251	0.6725	0.5353	0.4119	0.3739	0.4113	0.5323	0.6746	1.0321

Table 14: CP, MSE and the length of CI for Various Choices of u in sampling from Lognormal(0, 1) Distribution (n=200, replicate=10000, $\alpha = 0.05$).

Method	Index \ u	0.01	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.99
Nonparametric	CP	0.4892	0.8930	0.9690	0.9447	0.9438	0.9511	0.8857	0.4900	0.0613
	MSE	0.0002	0.0006	0.0014	0.0026	0.0082	0.0409	0.1262	0.0687	0.4003
	Length	0.0350	0.0944	0.1462	0.1976	0.3541	0.7921	1.3900	1.0015	0.3919
Sectioning	CP	0.0432	0.5612	0.7638	0.9041	0.9435	0.9391	0.8636	0.7529	0.3159
	MSE	0.0097	0.0055	0.0046	0.0051	0.0102	0.0353	0.1971	0.7507	12.566
	Length	0.1055	0.1266	0.1435	0.2036	0.3571	0.7360	1.5649	2.5280	5.3603
Jackknife	CP	0.6639	0.7805	0.8098	0.8478	0.8647	0.8479	0.8054	0.7549	0.5813
	MSE	0.0009	0.0012	0.0015	0.0029	0.0093	0.0436	0.2461	0.7921	8.0904
	Length	0.1155	0.1358	0.1508	0.2110	0.3776	0.8188	1.9445	3.4881	10.635

Table 15: CP, MSE and the length of CI for Various Choices of u in sampling from Weibull(1, 1) Distribution (n=200, replicate=10000, $\alpha = 0.05$).

Method	Index \ u	0.01	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.99
Nonparametric	CP	0.7916	0.9300	0.9696	0.9465	0.9460	0.9565	0.9552	0.7110	0.1532
	MSE	0.0001	0.0003	0.0007	0.0017	0.0051	0.0166	0.0518	0.0309	0.0518
	Length	0.0222	0.0623	0.1023	0.1608	0.2797	0.5056	0.8907	0.6658	0.2695
Sectioning	CP	0.0533	0.5091	0.7253	0.8959	0.9487	0.9364	0.8414	0.6931	0.1852
	MSE	0.0028	0.0029	0.0030	0.0037	0.0056	0.0141	0.0581	0.1781	1.5710
	Length	0.0628	0.0875	0.1092	0.1681	0.2755	0.4553	0.7329	0.9599	1.3819
Jackknife	CP	0.6856	0.7760	0.8151	0.8497	0.8613	0.8394	0.8065	0.7585	0.6079
	MSE	0.0001	0.0004	0.0008	0.0020	0.0059	0.0180	0.0556	0.1241	0.4870
	Length	0.0416	0.0794	0.1080	0.1770	0.2998	0.5261	0.9246	1.3802	2.6343

Table 16: CP, MSE and the length of CI for Various Choices of u in sampling from Uniform(0,1) Distribution (n=1000, replicate=10000, $\alpha = 0.05$).

Method	Index \ u	0.01	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.99
Nonparametric	CP	0.8327	0.9319	0.9681	0.9534	0.9420	0.9405	0.9744	0.9421	0.8543
	MSE	0.00001	0.00004	0.0001	0.0002	0.0002	0.0002	0.0001	0.00004	0.00001
	Length	0.0092	0.0255	0.0405	0.0536	0.0619	0.0538	0.0407	0.0256	0.0093
Sectioning	CP	0.4786	0.8532	0.9117	0.9391	0.9489	0.9439	0.9094	0.8532	0.4839
	MSE	0.00011	0.00013	0.0002	0.0002	0.0002	0.0002	0.0002	0.00013	0.00011
	Length	0.0170	0.0288	0.0379	0.0531	0.0610	0.0533	0.0379	0.0288	0.0169
Jackknife	CP	0.7693	0.8444	0.8685	0.8951	0.8993	0.8942	0.8614	0.8443	0.7751
	MSE	0.00002	0.00006	0.0001	0.0002	0.0003	0.0002	0.0001	0.00006	0.00002
	Length	0.0155	0.0299	0.0404	0.0565	0.0644	0.0564	0.0400	0.0301	0.0156

Table 17: CP, MSE and the length of CI for Various Choices of u in sampling from Exp(1) Distribution (n=1000, replicate=10000, $\alpha = 0.05$).

Method	Index \ u	0.01	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.99
Nonparametric	CP	0.8772	0.9475	0.9516	0.9440	0.9520	0. 9512	0.9593	0.9224	0.2692
	MSE	0.00001	0.00005	0.0001	0.0003	0.0010	0.0035	0.0228	0.0159	0.0056
	Length	0.0100	0.0279	0.0456	0.0721	0.1255	0.2316	0.5920	0.4921	0.2145
Sectioning	CP	0.4829	0.8577	0.9109	0.9415	0.9523	0.9427	0.9268	0.8940	0.6379
	MSE	0.0001	0.0002	0.0002	0.0004	0.0010	0.0030	0.0096	0.0238	0.2325
	Length	0.0174	0.0309	0.0430	0.0722	0.1242	0.2118	0.3613	0.5152	0.9896
Jackknife	CP	0.7693	0.8420	0.8664	0.8920	0.9003	0.8911	0.8670	0.8389	0.7522
	MSE	0.00002	0.00007	0.0001	0.0004	0.0011	0.0033	0.0103	0.0228	0.1300
	Length	0.0159	0.0317	0.0450	0.0757	0.1291	0.2250	0.3983	0.5919	1.4131

Table 18: CP, MSE and the length of CI for Various Choices of u in sampling from Normal(0, 1) Distribution (n=1000, replicate=10000, $\alpha = 0.05$).

Method	Index \ u	0.01	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.99
Nonparametric	CP	0.3791	0.9437	0.9597	0.9519	0.9520	0.9617	0.9428	0.9474	0.3783
	MSE	0.0011	0.0044	0.0066	0.0020	0.0016	0.0020	0.0066	0.0044	0.0010
	Length	0.1120	0.2592	0.3194	0.1774	0.1576	0.1769	0.3183	0.2603	0.1117
Sectioning	CP	0.6065	0.8784	0.9237	0.9472	0.9474	0.9457	0.9219	0.8882	0.6155
	MSE	0.0403	0.0065	0.0035	0.0019	0.0016	0.0019	0.0035	0.0066	0.0402
	Length	0.3827	0.2512	0.2080	0.1671	0.1543	0.1677	0.2081	0.2521	0.3842
Jackknife	CP	0.7580	0.8420	0.8645	0.8838	0.9054	0.8910	0.8682	0.8405	0.7593
	MSE	0.0187	0.0052	0.0033	0.0021	0.0017	0.0021	0.0034	0.0054	0.0187
	Length	0.5358	0.2839	0.2249	0.1782	0.1633	0.1784	0.2282	0.2868	0.5364

Table 19: CP, MSE and the length of CI for Various Choices of u in sampling from Lognormal(0, 1) Distribution (n=1000, replicate=10000, $\alpha = 0.05$).

Method	Index \ u	0.01	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.99
Nonparametric	CP	0.5124	0.8903	0.9547	0.9582	0.9601	0.9518	0.9417	0.8451	0.1840
	MSE	0.00002	0.0001	0.0003	0.0005	0.0017	0.0090	0.0894	0.0613	0.0422
	Length	0.0159	0.0426	0.0655	0.0884	0.1595	0.3716	1.1719	0.9685	0.4159
Sectioning	CP	0.5597	0.8691	0.9206	0.9413	0.9541	0.9428	0.9299	0.9003	0.6887
	MSE	0.0007	0.0004	0.0004	0.0006	0.0017	0.0071	0.0384	0.1317	2.3786
	Length	0.0448	0.0516	0.0596	0.0872	0.1564	0.3309	0.7473	1.3006	3.8243
Jackknife	CP	0.7645	0.8459	0.8698	0.8951	0.9002	0.8885	0.8661	0.8397	0.7488
	MSE	0.0002	0.0002	0.0003	0.0005	0.0017	0.0079	0.0436	0.1451	1.9471
	Length	0.0537	0.0560	0.0636	0.0903	0.1625	0.3490	0.8183	1.4929	5.4697

Table 20: CP, MSE and the length of CI for Various Choices of u in sampling from Weibull(1, 1) Distribution (n=1000, replicate=10000, $\alpha = 0.05$).

Method	Index \ u	0.01	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.99
Nonparametric	CP	0.8605	0.9561	0.9520	0.9461	0.9463	0.9564	0.9512	0.9232	0.2619
	MSE	0.00001	0.00005	0.0001	0.0003	0.0010	0.0035	0.0228	0.0158	0.0055
	Length	0.0100	0.0279	0.0456	0.0722	0.1258	0.2327	0.5922	0.4915	0.2149
Sectioning	CP	0.4825	0.8549	0.9065	0.9419	0.9489	0.9512	0.9291	0.8894	0.6206
	MSE	0.00012	0.00016	0.0002	0.0004	0.0010	0.0030	0.0097	0.0240	0.2354
	Length	0.0175	0.0308	0.0430	0.0720	0.1238	0.2136	0.3618	0.5103	0.9828
Jackknife	CP	0.7747	0.8422	0.8631	0.8898	0.8967	0.8918	0.8662	0.8380	0.7522
	MSE	0.00002	0.00007	0.0001	0.0004	0.0011	0.0033	0.0103	0.0227	0.1310
	Length	0.0156	0.0318	0.0447	0.0749	0.1299	0.2248	0.3971	0.5905	1.4186