Inference for Exponentiated General Class of Distributions Based on Record Values

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Abstract

The main objective of this paper is to suggest and study a new exponentiated general class (EGC) of distributions. Maximum likelihood, Bayesian and empirical Bayesian estimators of the parameter of the EGC of distributions based on lower record values are obtained. Furthermore, Bayesian prediction of future records is considered. Based on lower record values, the exponentiated Weibull distribution, its special cases of distributions and exponentiated Gompertz distribution are applied to the EGC of distributions.

Keywords: Lower record values, Maximum likelihood estimation, Bayesian estimation, empirical Bayesian estimation, Bayesian prediction.

1. Introduction

Bayesian prediction of record values has a special interest, as the best approach found until now, in many fields such as clinical trials, insurance, industry, architecture, physics, marketing, geography, and other fields. Recently, few studies are concerned with Bayesian prediction and both Bayesian and classical inferences for record values. Among others are Hsieh (2001), Ali-Mousa (2001,2003), Ali-Mousa et al. (2002), AL-Hussaini and Ahmed (2003a,b), Jaheen (2003), Madi and Raqab (2004), Ahmadi et al. (2005), Ahmadi and Doostparast (2006), Abdel-Aty et al. (2007), Al-Aboud and Soliman (2008), Ahmadi and MirMostafaee (2009).

A general form of the Burr X distribution is known as the exponentiated Weibull distribution. Sartawi and Abu-Salih (1991) and Jaheen (1995, 1996) considered the problem of Bayesian estimation in the one-parameter Burr X case. Ahmad et al. (1997) considered inference for the stress-strength reliability $R = P(Y < X)$ where X and Y are independent Burr X random variables. While, Surles and Padget (1998) considered inference for $R = P(Y < X)$ for the scaled Burr X distribution. Surles and Padgett (1998) considered inference for a scaled version of the Burr X distribution. Ali-Mousa (2001) obtained the maximum likelihood, minimum variance unbiased and Bayes estimators of the one parameter of the Burr X distribution. In addition, he presented the Bayesian and non-Bayesian confidence intervals for the parameter in closed forms. Furthermore, he obtained a Bayesian prediction interval for the k^{th} future record in a closed form. For more details about exponentiated Weibull see (Nadarajah and Kotz (2006)).

Raqab and Ahsanullah (2001) derived exact expressions for the single and product moments of order statistics from the generalized exponential distribution. They used these moments to obtain estimators of the location and scale parameters of the model. Raqab (2002) derived exact expressions for means, variances and covariances of record values from the generalized exponential distribution. Jaheen (2004) derived Bayesian and empirical Bayesian estimators for the unknown parameter of the generalized exponential distribution based on record values. He obtained the estimates based on the squared error and the LINEX loss function. Furthermore, he obtained the prediction bounds for future lower record values by using both Bayesian and empirical Bayesian techniques. He, also, gave a numerical example to illustrate the results.

Ahmadi et al. (2009) studied the prediction of future k-record based on observed record which come from a general class of continuous distributions. They obtained the Bayes predictors of the s^{th} future k-record under balanced type loss function. Nadar et al. (2012) reviewed and derived some results on record values for some well known distributions and based on m records from Kumaraswamy's distribution. They also obtained the estimators of the parameters and for the future s^{th} record value, when the m past record values have been observed. They used the maximum likelihood and Bayesian approaches. Then, they illustrated the findings with actual and computer generated data.

Describing the layout of the paper; Section 2 is devoted to the construction of the new EGC of distributions. ML, Bayesian and empirical Bayesian estimators of a new introduced parameter are obtained. In addition, Bayesian prediction of future records, based on lower record values, is found. Section 3 deals with the application of all the mathematical work done in Section 2. The exponentiated Weibull distribution (Burr X) is considered in details. Furthermore, same special cases of the exponentiated Weibull distribution and other distributions are also been applied.

2. Exponentiated General Class

In this section the idea of the new EGC of distributions is displayed. ML, Bayesian and empirical Bayesian estimators based on lower record values are obtained. In addition, Bayesian predictions of future records are discussed.

AL-Hussaini and Ahmad (2003a) have introduced a general class denoted by (GC) of distributions. It includes the Weibull, compound Weibull, Pareto, beta, Gompertz and compound Gompertz among other distributions. The GC of distributions presents the populations with distribution function of the form

$$> 0,\ x, \quad F(x) = 1 - \exp[-\lambda(x)] \tag{1}$$

where, $\lambda(x) = \lambda(x,\gamma)$ is a non-negative function of x, such that $\lambda(x,\gamma) \to 0$ as $x \to 0^{+}$ and $\lambda(x,\gamma) \to \infty$ as $x \to \infty$ and γ is a known parameter or (vector of parameters).

The idea of the construction of the new EGC of distributions is to raise the cdf of the GC, presented in relation (1), to the power θ, where θ is assumed to be a positive parameter. That is, the cdf of the EGC of distributions is expressed in the form

$$F(x) = \{1 - \exp[-\lambda(x)]\}^{\theta}, \qquad x, \theta > 0. \tag{2}$$

Thus, the density function is given by

$$f(x) = \theta\lambda'(x)\exp[-\lambda(x)]\{1 - \exp[-\lambda(x)]\}^{\theta-1}, \qquad x, \theta > 0, \tag{3}$$

where, $\lambda(x) = \lambda(x,\gamma)$ is a non-negative continuous function of x and γ. We assume that γ is a known parameter or (vector of parameters). That is, the unique unknown parameter of the EGC of distribution is θ. Where, $\lambda(x,\gamma) \to 0$ as $x \to 0^{+}$ and $\lambda(x,\gamma) \to \infty$ as $x \to \infty$.

2.1 Estimation of the Parameter

This subsection concerns with the ML and Bayesian and empirical Bayesian estimation of the parameter θ of the EGC of distributions, based on lower record values. Furthermore, Bayesian prediction of future records is explained.

Suppose that m lower record values, $X_{L(1)} = x_1, X_{L(2)} = x_2, \ldots, X_{L(m)} = x_m$, are observed from the distribution with cdf and pdf given, respectively by (2) and (3). ML, Bayesian and empirical Bayesian estimators are obtained as follows.

2.1.1 Maximum Likelihood Estimation

The likelihood function is given by (see Arnold et al. (1998))

$$L(\theta|\underline{x}) = f(x_m) \prod_{i=0}^{m-1} r(x_i),$$

where, $\underline{x} = (x_1, x_2, \ldots, x_m)$ and $r(x) = \frac{f(x)}{F(x)}$,

It follows that,

$$L(\theta|\underline{x}) = \theta^m \exp(-\theta\eta(x_m))\, u(\underline{x}), \tag{4}$$

where,

$$u(\underline{x}) = \prod_{i=1}^{m} \lambda'(x_i) e^{-\lambda(x_i)} \left(1 - e^{-\lambda(x_i)}\right)^{-1},$$

and

$$\eta(x_m) = -ln\left(1 - e^{-\lambda(x_m)}\right). \tag{5}$$

Assuming that the parameter θ is unknown, the ML estimator (MLE) of θ is given by

$$\hat{\theta}_{ML} = m/\eta(x_m), \tag{6}$$

where, $\eta(x_m)$ is expressed in (5).

To study the properties of this estimator, the distribution of $\eta(x_m)$ is needed. The pdf of X_m is given by

$$f_{X_m}(x) = \frac{1}{\Gamma(m)} f(x)[-lnF(x)]^{m-1}.$$

That is,

$$f_{X_m}(x) = \frac{\theta^m}{\Gamma(m)} \lambda'(x) e^{-\lambda(x)} \left(1 - e^{-\lambda(x)}\right)^{\theta-1} \left(\eta(x)\right)^{m-1}, x > 0, \tag{7}$$

where, $\eta(x)$ is given by (5).

The pdf of $Z = {}^{m}/_{\eta(x_m)}$ can be obtained by using simple transformation on (7) and is given as,

$$f_Z(z) = \frac{(m\theta)^m}{\Gamma(m)} e^{-\theta m/z} \left(\frac{1}{z}\right)^{m+1}, >0 \ , \tag{8}$$

which is the inverted gamma distribution with parameters $(m, m\theta)$.

The expected value of $\hat{\theta}_{ML}$ is

$$E(\hat{\theta}_{ML}) = E(Z) = \frac{m}{m-1}\theta. \tag{9}$$

Therefore, the MLE of θ is a biased estimator and the unbiased estimator is $(m-1)/\eta(x_m)$. The variance of $\hat{\theta}_{ML}$ can be shown to be

$$Var(\hat{\theta}_{ML}) = Var(Z) = \frac{m^2}{(m-1)^2(m-2)}\theta^2 \tag{10}$$

2.1.2 Bayesian Estimation

Under the assumption that the parameter θ is unknown, we can use the conjugate gamma prior with pdf

$$g(\theta) = \frac{\beta^\delta}{\Gamma(\delta)} \theta^{\delta-1} e^{-\beta\theta}, \quad \theta > 0, \ \ (\beta, \delta > 0). \tag{11}$$

The posterior density function of θ given the data, denoted by $q(\theta|\underline{x})$ is

$$q(\theta|\underline{x}) = \frac{[\beta+\eta(x_m)]^{m+\delta}}{\Gamma(m+\delta)} \theta^{m+\delta-1} e^{-\theta[\beta+\eta(x_m)]}. \tag{12}$$

Under the squared error loss function, the Bayesian estimator of θ (denoted by $\hat{\theta}_{BS}$) is the mean of the posterior density and given by

$$\hat{\theta}_{BS} = \int_0^\infty \theta \, q(\theta|\underline{x})\, d\theta.$$

Therefore,

$$\hat{\theta}_{BS} = \frac{m+\delta}{[\beta+\eta(x_m)]}. \tag{13}$$

Under the LINEX loss function, the Bayes estimator of θ (denoted by $\hat{\theta}_{BL}$) is given by

$$\hat{\theta}_{BL} = -\frac{1}{a} ln\left(E\left(e^{-a\theta}\right)\right).$$

Hence,

$$\hat{\theta}_{BL} = \frac{m+\delta}{a} ln\left(1 + \frac{a}{\beta+\eta(x_m)}\right), a \neq 0. \tag{14}$$

2.1.3 Empirical Bayesian Estimation

When the prior parameters β and δ are unknown, we may use the empirical Bayesian approach to get their estimators. Here, we assume that the prior density (11) is belonging to a parametric family with unknown parameters. Such parameters are to be estimated using past samples. Applying the estimators of θ given, previously, by (13) and (14). We obtain the empirical Bayesian estimators of the parameter θ based on the squared error and the LINEX loss functions, respectively. For more details on the empirical Bayesian approach, see Maritz and Lwin (1989).

When the current (informative) sample is observed, suppose that n past similar samples $X_{j,L(1)}, X_{j,L(2)}, \dots, X_{j,L(m)}$, $j = 1,2, \dots, n$ are available with past realization $\theta_1, \theta_2, \dots, \theta_n$ of the random variable θ. Each sample is assumed to be lower record sample of size m obtained from the distribution with pdf given by (3). The likelihood function of the j^{th} sample is given by (4) with x_m being replaced by$x_{j,m}$. For a sample *j*, *j* = 1, 2,., *n* the MLE of the parameter θ_j can be rewritten by using (6).

$$\hat{\theta}_j = Z_j = m/\eta(x_{j,m}). \tag{15}$$

The conditional pdf of Z_j, *j* = 1,2,…, *n*, for a given θ_j is obtained from (8) in the form

$$f(z_j|\theta_j) = \frac{(m\theta_j)^m}{\Gamma(m)} \left[\frac{1}{z_j}\right]^{m+1} exp - (m\theta_j/z_j), \quad z_j > 0. \tag{16}$$

Which is the inverted gamma distribution with parameters (m, $m\theta_j$). The marginal pdf of z_j , $j = 1,2, \dots, n$, is the compound distribution

$$f_{Z_j}(z_j) = \int_0^\infty f(z_j|\theta_j)\, g(\theta_j)\, d\theta_j ,$$

where $f(z_j|\theta_j)$ and $g(\theta_j)$ are given by (16) and (11), respectively, after indexing θ in (11) by *j*. That is,

$$f_{Z_j}(z_j) = \frac{\beta^\delta m^m}{B(m,\delta)} \times \frac{z_j^{\delta-1}}{(m+\beta z_j)^{m+\delta}} \;, \; z_j > 0. \tag{17}$$

The method of moments estimators of the parameters β and δ are given respectively by

$$\hat{\beta} = \frac{m s_1}{(s_2 - s_1^2)(m-1)} \text{ and } \hat{\delta} = \frac{s_1^2}{s_2 - s_1^2}. \tag{18}$$

Then, the empirical Bayesian estimators of the parameter θ under the squared error and the LINEX loss functions are given, respectively, by

$$\hat{\theta}_{EBS} = \frac{m+\hat{\delta}}{\hat{\beta}+\eta(x_m)}, \tag{19}$$

and

$$\hat{\theta}_{EBL} = \frac{m+\hat{\delta}}{a} \ln\left(1 + \frac{a}{\hat{\beta}+\eta(x_m)}\right), \quad a \neq 0, \tag{20}$$

where, $\hat{\delta}$ and $\hat{\beta}$ are the estimators of β and δ given by (18).

2.2 Bayesian Prediction of Future Records

Suppose that we have m lower records $X_{L(1)} = x_1, X_{L(2)} = x_2, \dots, X_{L(m)} = x_m$ from the distribution with cdf and pdf given, respectively, by (2) and (3). Based on such a record sample, Bayesian prediction is needed for the s^{th} lower record, 1 < m < s. The conditional pdf of *Y* given the parameter θ is given by

$$f(y|\theta) = \frac{[H(y) - H(x_m)]^{s-m-1}}{\Gamma(s-m)} \times \frac{f(y)}{F(x_m)} \;, 0 < y < x_m < \infty, \tag{21}$$

where *H*(.) = - ln *F*(.).

For the pdf of the EGC of distribution, $f(y|\theta)$ in (21) takes the form

$$f(y|\theta)=\frac{\theta^{s-m}\lambda'(y)}{\Gamma(s-m)}\left[ln\left(\frac{1-e^{-\lambda(x_m)}}{1-e^{-\lambda(y)}}\right)\right]^{s-m-1}\left[\frac{1-e^{-\lambda(y)}}{1-e^{-\lambda(x_m)}}\right]^{\theta}\frac{e^{-\lambda(y)}}{1-e^{-\lambda(y)}}, \quad (22)$$

$0<y<x_m<\infty.$

The Bayes predictive density function of $Y=X_{L(s)}$ given the past *m* lower records is

$$f^*(y|\underline{x})=\int_\theta f(y|\theta)\,q(\theta|\underline{x})\,d\theta, \quad (23)$$

Where $f(y|\theta)$ and $q(\theta|\underline{x})$ are given by (22) and (12), respectively. Therefore, the result of the integral in (23) is

$$f^*(y|\underline{x})=\frac{\lambda'(y)}{B(s-m,m+\delta)}\left[ln\left(\frac{1-e^{-\lambda(x_m)}}{1-e^{-\lambda(y)}}\right)\right]^{s-m-1}\frac{e^{-\lambda(y)}}{1-e^{-\lambda(y)}}\frac{[\beta+\eta(x_m)]^{m+\delta}}{[\beta+\eta(y)]^{s+\delta}},\ 0<y<x_m<\infty \quad (24)$$

Hence, the Bayesian prediction bounds $Y=X_{L(s)}$, are obtained by evaluating $P(Y\geq t|\underline{x})$, for some positive *t*. It follows, from (24) that

$$P(Y\geq t|\underline{x})=\int_t^{x_m}f^*(y|\underline{x})\,dy.$$

That is,

$$P(Y\geq t|\underline{x})=1-\frac{1}{B(m+\delta,s-m)}IB_\varepsilon(m+\delta,s-m). \quad (25)$$

And

$$\varepsilon=\frac{\beta+\eta(x_m)}{\beta+\eta(t)}<1.$$

For the special case, when $s=m+1$, which is practically of special interest. The Bayesian prediction for a future lower record value X_{m+1} can be obtained by using (25) as

$$P(X_{m+1}\geq t_1|\underline{x})=1-\left[\frac{\beta+\eta(x_m)}{\beta+\eta(t_1)}\right]^{m+\delta}. \quad (26)$$

A 100 τ% Bayesian prediction interval for $Y=X_{m+1}$ is

$$P[LL(\underline{X})<Y<UL(\underline{X})]=\tau,$$

where $LL(\underline{X})$ and $UL(\underline{X})$ are the lower and upper limits satisfying the following equations

$$P(Y>LL(\underline{X})|\underline{x})=(1+\tau)/2, \quad (27)$$

and

$$P(Y>UL(\underline{X})|\underline{x})=(1-\tau)/2\,. \quad (28)$$

It follows from (26), (27) and (5) that

$$\lambda\left(LL(\underline{X})\right)=-ln\left\{1-exp\left[\beta-(\beta+\eta(x_m))\left(\frac{1-\tau}{2}\right)^{-1/m+\delta}\right]\right\}. \quad (29)$$

Similarly, from (26), (28) and (5), it can be shown that

$$\lambda\left(UL(\underline{X})\right)=-ln\left\{1-exp\left[\beta-(\beta+\eta(x_m))\left(\frac{1+\tau}{2}\right)^{-1/m+\delta}\right]\right\}. \quad (30)$$

3. Applications

In this section, we apply the exponentiated Weibull distribution to the classical Bayesian and empirical Bayesian estimation of the parameter θ of the EGC of distributions based on lower record values. In addition, Bayesian prediction of the future records is applied. However, the application of some special cases of the exponentiated Weibull distribution and the exponentiated Gompertz distribution are presented in Table 1 and Table 2.

3.1 Maximum Likelihood Estimation

The cdf and the pdf of a random variable having the exponentiated Weibull distribution are given, respectively, by

$$F(x) = [1 - \exp(-(\mu x)^{\alpha})]^{\theta}, \qquad x > 0\,,\ \mu > 0\,,\ \alpha > 0\ and\ \theta > 0 \qquad (31)$$

and

$$f(x) = \theta\alpha\mu^{\alpha}x^{\alpha-1}e^{-(\mu x)^{\alpha}}[1 - \exp(-(\mu x)^{\alpha})]^{\theta-1},\ x > 0, \alpha > 0, \mu > 0\ and\ \theta > 0 \qquad (32)$$

Where, $\theta\ and\ \alpha$ are shape parameters and μ is a scale parameter.

Suppose that the lower record values, $X_{L(1)} = x_1, X_{L(2)} = x_2, \dots, X_{L(m)} = x_m$, are from the exponentiated Weibull distribution with cdf and pdf given respectively by (31), (32). Recall equation (2), one can see that $\lambda(x) = (\mu x)^{\alpha}$, $\gamma = (\mu, \alpha)$, and $\lambda'(x) = \alpha\mu^{\alpha}x^{\alpha-1}$. The likelihood function is

$$L(\theta|\underline{x}) = (\theta\alpha\mu^{\alpha})^m e^{\theta ln\left(1-e^{-(\mu x_m)^{\alpha}}\right)} \prod_{i=1}^{m} \frac{e^{-(\mu x_i)^{\alpha}}}{1-e^{-(\mu x_i)^{\alpha}}} x_i^{\alpha-1}. \qquad (33)$$

Assuming that the parameter θ is unknown, the MLE of θ is given by (6), which can be rewritten as

$$\hat{\theta}_{ML} = \frac{m}{-ln\left(1-e^{-(\mu x_m)^{\alpha}}\right)}\,. \qquad (34)$$

To study the properties of this estimate the distribution of $\eta(x_m)$ is needed. Applying equation (7) the pdf of X_m is then given by

$$f_{x_m}(x) = \frac{\theta^m\alpha\mu^{\alpha}}{\Gamma(m)} x^{\alpha-1}e^{-(x\mu)^{\alpha}}\left(1 - e^{-(\mu x)^{\alpha}}\right)^{\theta-1}\left(-ln\left(1 - e^{-(\mu x)^{\alpha}}\right)\right)^{m-1},\ x > 0. \qquad (35)$$

Hence, the pdf of $Z = {}^{m}/_{\eta(x_m)}$ is given by (8).Which is the inverted gamma distribution with parameters $(m, m\theta)$. That is, the expected value and the variance of $\hat{\theta}_{ML}$ are given respectively, by (9) and (10).

3.2 Bayesian Estimation

Under the assumption that the parameter θ is unknown, we can use the conjugate gamma prior with pdf which is given by (11). The posterior density function of θ given the data, denoted by $q(\theta|\underline{x})$, can be obtained by using (11) and (33) as

$$q(\theta|\underline{x}) = \frac{\left[\beta - ln\left(1-e^{-(\mu x_m)^{\alpha}}\right)\right]^{m+\delta}}{\Gamma(m+\delta)} \theta^{m+\delta-1}e^{-\theta\left[\beta - ln\left(1-e^{-(\mu x_m)^{\alpha}}\right)\right]}. \qquad (36)$$

Under a squared error loss function, the Bayesian estimator of θ, $\hat{\theta}_{BS}$ is given by

$$\hat{\theta}_{BS} = \frac{m+\delta}{\left[\beta - ln\left(1-e^{-(\mu x_m)^{\alpha}}\right)\right]}. \qquad (37)$$

Under the LINEX loss function, the Bayesian estimator of θ, $\hat{\theta}_{BL}$ is given by

$$\hat{\theta}_{BL} = \frac{m+\delta}{a} ln\left(1 + \frac{a}{\beta - ln\left(1-e^{-(\mu x_m)^{\alpha}}\right)}\right), a \neq 0. \qquad (38)$$

3.3 Empirical Bayesian Estimation

When the prior parameters β and δ are unknown, we may use the empirical Bayesian approach to get their estimates. Since the prior density (11) belongs to a parametric family with unknown parameters, such parameters are to be estimated using past samples. Applying these estimates in (37) and (38), we obtain the empirical Bayes estimate of the parameter θ based on the squared error and the LINEX loss functions, respectively.

When the current (informative) sample is observed, suppose that n past similar samples $X_{j,L(1)}, X_{j,L(2)}, \ldots, X_{j,L(m)}$, $j = 1,2, \ldots, n$ are available with past realization $\theta_1, \theta_2, \ldots, \theta_n$of the random variable θ. Each sample is assumed to be lower record sample of size m obtained from the distribution with pdf given by (32). The likelihood function of the j^{th} sample is given by (33) with x_m being replaced by $x_{j,m}$. For a sample j, j = 1, 2,…, n , the MLE of the parameter θ_j can be written by using (15) as

$$\hat{\theta}_j = Z_j = \frac{m}{-ln\left(1-e^{-\left(\mu x_{j,m}\right)^{\alpha}}\right)}. \qquad (39)$$

The conditional pdf of Z_j , j = 1,2,…, n , for a given θ_j is given by (16).Which is the inverted gamma distribution with parameters (m , $m\theta_j$). The marginal pdf of z_j , $j = 1,2, \ldots, n$, is given by (17).

Then, the empirical Bayesian estimates of the parameter θ under the squared error and the LINEX loss function are given by (19) and (20), respectively, as

$$\hat{\theta}_{EBS} = \frac{m+\hat{\delta}}{\hat{\beta} - ln\left(1-e^{-(\mu x_m)^{\alpha}}\right)}, \qquad (40)$$

and

$$\hat{\theta}_{EBL} = \frac{m+\hat{\delta}}{a}\, ln\left(1 + \frac{a}{\hat{\beta} - ln\left(1-e^{-(\mu x_m)^{\alpha}}\right)}\right), \quad a \neq 0, \qquad (41)$$

where, $\hat{\beta}$ and $\hat{\delta}$ are the moment estimates of β and δ given by (18).

3.4 Bayesian Prediction of Future Records

Suppose that we have m lower records $X_{L(1)} = x_1, X_{L(2)} = x_2, \ldots, X_{L(m)} = x_m$ from the distribution with cdf and pdf are given respectively by (31) and (32). based on such a record sample, Bayesian prediction is needed for the s^{th} lower record, $1 < m < s$. The conditional pdf of Y given the parameter θ is given by (22), it can be written as

$$f(y|\theta) = \frac{\theta^{s-m}\alpha\mu^{\alpha}y^{\alpha-1}}{\Gamma(s-m)} \frac{e^{-(\mu y)^{\alpha}}}{1-e^{-(\mu y)^{\alpha}}}\left[ln\left(\frac{1-e^{-(\mu x_m)^{\alpha}}}{1-e^{-(\mu y)^{\alpha}}}\right)\right]^{s-m-1}\left[\frac{1-e^{-(\mu y)^{\alpha}}}{1-e^{-(\mu x_m)^{\alpha}}}\right]^{\theta}, \ 0 < y < x_m < \infty \qquad (42)$$

The Bayes predictive density function of $Y = X_{L(s)}$ given the past m lower records is given by (24), it can be written in the form

$$f^*(y|\underline{x}) = \frac{\alpha\mu^\alpha y^{\alpha-1}}{\mathrm{B}(s-m,m+\delta)}\left[ln\left(\frac{1-e^{-(\mu x_m)^\alpha}}{1-e^{-(\mu y)^\alpha}}\right)\right]^{s-m-1} \frac{e^{-(\mu y)^\alpha}}{1-e^{-(\mu y)^\alpha}} \frac{\left[\beta-ln\left(1-e^{-(\mu x_m)^\alpha}\right)\right]^{m+\delta}}{\left[\beta-ln\left(1-e^{-(\mu y)^\alpha}\right)\right]^{s+\delta}},$$

$$0 < y < x_m < \infty. \quad (43)$$

Bayesian prediction bounds $Y = X_{L(s)}$, are obtained by evaluating $P(Y \geq t|\underline{x})$, for some positive t. Which is given by (25).

For the special case, when s=m+1, which is practically of special interest. The Bayesian prediction for a future lower record value X_{m+1} can be obtained by using (26) as

$$P(X_{m+1} \geq t_1|\underline{x}) = 1 - \left[\frac{\beta-ln\left(1-e^{-(\mu x_m)^\alpha}\right)}{\beta-ln\left(1-e^{-(\mu t_1)^\alpha}\right)}\right]^{m+\delta}. \quad (44)$$

A 100 τ% Bayesian prediction interval for $Y = X_{m+1}$ is such that

$$P[LL(\underline{X}) < Y < UL(\underline{X})] = \tau,$$

where $LL(\underline{X})$ and $UL(\underline{X})$ are the lower and upper limits satisfying (27) and (28), it follows by using (29) that

$$LL(\underline{X}) = \frac{1}{\mu}\left\{ln\left\{1 - exp\left[\beta - \left(\beta - ln\left(1 - e^{-(\mu x_m)^\alpha}\right)\right)\left(\frac{1-\tau}{2}\right)^{-1/m+\delta}\right]\right\}^{-1}\right\}^{1/\alpha} \quad (45)$$

Similarly, from (30), one can find that

$$UL(\underline{X}) = \frac{1}{\mu}\left\{ln\left\{1 - exp\left[\beta - \left(\beta - ln\left(1 - e^{-(\mu x_m)^\alpha}\right)\right)\left(\frac{1+\tau}{2}\right)^{-1/m+\delta}\right]\right\}^{-1}\right\}^{1/\alpha} \quad (46)$$

3.5 Special cases of the Exponentiated Weibull Distribution and the Exponentiated Gompertz Distribution

This subsection concerns with some special cases of the exponentiated Weibull distribution and the exponentiated Gompertz distribution. The ML, Bayesian and empirical Bayesian estimation, based on lower record values, of the parameter of each of the considered special cases and the exponentiated Gompertz distribution are obtained. In addition, Bayesian prediction of future records are found.

- Some special cases of the exponentiated Weibull distribution can be seen as follows:

  1. When the parameters α and μ equal the value 1, the exponentiated Weibull distribution reduces to the generalized exponential distribution with density function

$$f(x) = \theta\, e^{-x}\left(1-e^{-x}\right)^{\theta-1}; \quad x > 0, \quad \theta > 0.$$

2. When the parameter μ equals $\frac{1}{\sqrt{2}\sigma}$ and the parameter α is 2, the exponentiated Weibull distribution reduces to the exponentiated Reyleigh with density function given by

$$f(x)=\frac{\theta}{\sigma^2}x e^{\frac{-1}{2}\left(\frac{x}{\sigma}\right)^2}\left[1-e^{\frac{-1}{2}\left(\frac{x}{\sigma}\right)^2}\right]^{\theta-1};\quad x>0,\quad \theta>0.$$

3. When the parameter μ is equal $\frac{1}{\sigma}$ and the parameter α is 2, the exponentiated Weibull distribution reduces to the Burr X distribution with density function given by

$$f(x)=\frac{2\theta}{\sigma^2}x e^{-\left(\frac{x}{\sigma}\right)^2}\left[1-e^{-\left(\frac{x}{\sigma}\right)^2}\right]^{\theta-1};\quad x>0,\quad \theta>0.$$

Table 1 represents the cdf, $\lambda(x)$, $\hat{\theta}_{ML}$, $\hat{\theta}_{BS}$, $\hat{\theta}_{BL}$, and the Bayesian prediction, lower and upper limits of future records of the above special cases of the exponentiated Weibull distribution.

- One can write the cdf of the exponentiated Gompertz distribution as follows
  $F(x)=(1-e^{-c(e^{ax}-1)})^{\theta};\ x>0,\quad a,c,\theta>0.$

Since, the exponentiated Gompertz distribution is a member of the EGC of distributions, we write

$$\lambda(x)=c\left(e^{ax}-1\right).$$

The ML estimator of the parameter θ is

$$\hat{\theta}_{ML}=\frac{m}{-\ln\left(1-e^{-c\left(e^{ax}-1\right)}\right)}.$$

However, the Bayesian estimators of the parameter θ under the squared error loss function and the LINEX loss function are given , respectively, by

$$\hat{\theta}_{BS}=\frac{m+\delta}{\beta-\ln\left(1-e^{-c\left(e^{ax}-1\right)}\right)},$$

and

$$\hat{\theta}_{BL}=\frac{m+\delta}{a}\ln\left(1+\frac{a}{\beta-\ln\left(1-e^{-c\left(e^{ax}-1\right)}\right)}\right).$$

Finally, Bayesian prediction lower and upper limits of future records of the exponentiated Gompertz distribution are given by

$$LL(\underline{X})=-\ln\left\{1-\exp\left[\beta-\left(\beta-\ln\left[1-e^{-c\left(e^{ax}-1\right)}\right]\right)\times\left(\frac{1-r}{2}\right)^{\frac{-1}{m+\delta}}\right]\right\},$$

and

$$UL(\underline{X}) = -\ln\left\{1-\exp\left[\beta - \left(\beta - \ln\left[1-e^{-c\left(e^{ax}-1\right)}\right]\right)\times\left(\frac{1+r}{2}\right)^{\frac{-1}{m+\delta}}\right]\right\}.$$

The results can be used to estimate parameters of other exponentiated distaributions that can be written as (2).

Table 1: Estimators and Bayesian prediction of the parameter θ of the special cases of the exponentiated Weibull distribution

	Generalized exponential	Exponentiated Reyleigh	Burr X
Cdf	$F(x)=\left(1-e^{-x}\right)^{\theta}$	$F(x)=(1-e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2})^{\theta}$	$F(x)=(1-e^{-\left(\frac{x}{\sigma}\right)^2})^{\theta}$
$\lambda(x)$	X	$\frac{1}{2}\left(\frac{x}{\sigma}\right)^2$	$\left(\frac{x}{\sigma}\right)^2$
$\hat{\theta}_{ML}$	$\frac{m}{-\ln\left(1-e^{-x_m}\right)}$	$\frac{m}{-\ln\left(1-e^{-\frac{1}{2}\left(\frac{x_m}{\sigma}\right)^2}\right)}$	$\frac{m}{-\ln\left(1-e^{-\left(\frac{x_m}{\sigma}\right)^2}\right)}$
$\hat{\theta}_{BS}$	$\frac{m+\delta}{\beta-\ln\left(1-e^{-x_m}\right)}$	$\frac{m+\delta}{\beta-\ln\left(1-e^{-\frac{1}{2}\left(\frac{x_m}{\sigma}\right)^2}\right)}$	$\frac{m+\delta}{\beta-\ln\left(1-e^{-\left(\frac{x_m}{\sigma}\right)^2}\right)}$
$\hat{\theta}_{BL}$	$\frac{m+\delta}{a}\ln\left(1+\frac{a}{\beta-\ln\left(1-e^{-x_m}\right)}\right)$	$\frac{m+\delta}{a}\times \ln\left(1+\frac{a}{\beta-\ln\left(1-e^{-\frac{1}{2}\left(\frac{x_m}{\sigma}\right)^2}\right)}\right)$	$\frac{m+\delta}{a}\times \ln\left(1+\frac{a}{\beta-\ln\left(1-e^{-\left(\frac{x_m}{\sigma}\right)^2}\right)}\right)$
Lower limit	$\ln\{1-\exp[\beta-(\beta-\ln(1-e^{-x_m}))\times\left(\frac{1-\tau}{2}\right)^{-1/m+\delta}]\}^{-1}$	$\sigma\{\ln\{1-\exp[\beta-\left(\frac{1-\tau}{2}\right)^{-1/m+\delta}\times(\beta-\ln(1-e^{-\frac{1}{2}\left(\frac{x_m}{\sigma}\right)^2}))]\}^{-1}\}^{1/2}$	$\sigma\{\ln\{1-\exp[\beta-\left(\frac{1-\tau}{2}\right)^{-1/m+\delta}\times(\beta-\ln(1-e^{-\left(\frac{x_m}{\sigma}\right)^2}))]\}^{-1}\}^{1/2}$
Upper limit	$\ln\{1-\exp[\beta-(\beta-\ln(1-e^{-x_m}))\times\left(\frac{1+\tau}{2}\right)^{-1/m+\delta}]\}^{-1}$	$\sigma\{\ln\{1-\exp[\beta-\left(\frac{1+\tau}{2}\right)^{-1/m+\delta}\times(\beta-\ln(1-e^{-\frac{1}{2}\left(\frac{x_m}{\sigma}\right)^2}))]\}^{-1}\}^{1/2}$	$\sigma\{\ln\{1-\exp[\beta-\left(\frac{1+\tau}{2}\right)^{-1/m+\delta}\times(\beta-\ln(1-e^{-\left(\frac{x_m}{\sigma}\right)^2}))]\}^{-1}\}^{1/2}$

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