The Extended Fréchet Distribution: Properties and Applications

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Abstract

In this paper, we study a new model called the Burr X exponentiated Fréchet Distribution. The new model exhibits unimodal, unimodal then buthtab and buthtab hazard rates. Various properties of the new model are explored including moments, generating function, probability weighted moments, Stress-strength model and order statisics. The maximum likelihood method is used to estimate the model parameters. Simulation results to assess the performance of the maximum likelihood estimates are discussed. We compare the flexibility of the proposed model with other extensions of the Fréchet distribution by means of two real data sets.

Keywords: Burr X Family, Fréchet distribution, Maximum likelihood, Simulation.

1. Introduction

The probability distributions have a great importance in modelling data in several areas like economics, biomedical sciences, finance and engineering, among others. It is known that data sets following the classical distributions are more often the exception rather than the reality. This motivated statisticians to develop the generalization of some classical distributions by adding one or more shape parameter(s) to the existing probability distribution to improve the flexibility and goodness of fits of the generated distribution.

Recently, Okorie et al. (2016) defined the exponentiated Gumbel type-2 distribution. It is also can be known as the exponentiated Fréchet (EF) distribution.

Consider the cumulative distribution function (CDF) and probability density function (PDF) of the EF distribution given, respectively, by

$$G_{EF}(x;\lambda,\beta,\alpha) = 1 - (1 - \mathrm{e}^{-\alpha x^{-\beta}})^{\lambda}$$

and

$$g_{EF}(x;\lambda,\beta,\alpha) = \alpha\beta\lambda x^{-\beta-1}\mathrm{e}^{-\alpha x^{-\beta}}(1 - \mathrm{e}^{-\alpha x^{-\beta}})^{\lambda-1},$$

where $\alpha > 0$ is a scale parameter and $\beta > 0$ and $\lambda > 0$ are shape parameters.

Our aim in this article, is to define and study a new four-parameter model called the *Burr-X exponentiated Fréchet* (BXEF) distribution. Using the Burr-X generator (BX-G) introduced by Yousof et al. (2016), we construct the BXEF model. The CDF of the BX-G family is defined by

$$F(x;\theta,\xi) = \left\{1 - \exp\left[-\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^2\right]\right\}^{\theta} \qquad (1)$$

The corresponding PDF of the BX-G is given by $f(x;\theta,\xi)=\frac{2\theta g(x;\xi)G(x;\xi)}{\overline{G}(x;\xi)^3}\exp\left[-\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^2\right]\left\{1-\exp\left[-\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^2\right]\right\}^{\theta-1}$, (2)

where θ is a positive shape parameter and $\overline{G}(x;\xi)=1-G(x;\xi)$. In general a random variable X with PDF ((2)) is denoted by $X\sim$BX-G(θ,ξ).

The BXEF distribution contains several lifetime distributions, such as Fréchet, inverse exponential, exponentiated Fréchet, Burr X Fréchet and Burr X inverse exponential distributions as special cases. We are motivated to introduce the BXEF distribution because (1) it contains a number aforementioned of known lifetime sub models; (2) The BXEF distribution exhibits unimodal as well as bathtub hazard rates; (3) It is shown in Section 3 that the BXEF distribution can be viewed as a mixture of one-parameter Fréchet distribution introduced by Fréchet (1924); (4) it can be viewed as a suitable model for fitting the right skewed data; and (5) The BXEF distribution outperforms several of the well-known lifetime distributions with respect to two real data applications.

The rest of the article is organized as follows. In Sections 2 and 3, we introduce the BXEF distribution, and provide a useful linear representation for its PDF. We discuss some properties of this distribution in Section 4. Section 5 describes the maximum likelihood estimation of the model parameters. In Section 6, a simulation study is carried out to assess the performance of the maximum likelihood estimates. In Section 7, the usefulness of the BXEF distribution is illustrated by means of two real data sets. Finally, Section 8 is devoted to some concluding remarks.

2. The BXEF distribution

Based on Equation ((1)), the CDF of the BXEF distribution is defined (for $x>0$) by

$$F(x;\theta,\lambda,\beta,\alpha)=\left(1-\exp\left\{-\left[\frac{1-(1-\mathrm{e}^{-\alpha x^{-\beta}})^{\lambda}}{(1-\mathrm{e}^{-\alpha x^{-\beta}})^{\lambda}}\right]^2\right\}\right)^{\theta} \quad (3)$$

Using Equation ((2)), we have PDF of the BXEF

$$f(x;\theta,\lambda,\beta,\alpha)=\frac{2\alpha\beta\lambda\theta 1-(1-\mathrm{e}^{-\alpha x^{-\beta}})^{\lambda}]}{x^{\beta+1}\mathrm{e}^{\alpha x^{-\beta}}(1-\mathrm{e}^{-\alpha x^{-\beta}})^{2\lambda+1}}\exp\left[-\left(\frac{1-(1-\mathrm{e}^{-\alpha x^{-\beta}})^{\lambda}}{(1-\mathrm{e}^{-\alpha x^{-\beta}})^{\lambda}}\right)^2\right]$$

$$\times\left\{1-\exp\left[-\left(\frac{1-(1-\mathrm{e}^{-\alpha x^{-\beta}})^{\lambda}}{(1-\mathrm{e}^{-\alpha x^{-\beta}})^{\lambda}}\right)^2\right]\right\}^{\theta-1} \quad (4)$$

Henceforth, let $X\sim$BXEF$(\theta,\lambda,\beta,\alpha)$ be a random variable having the PDF ((4)).

The quantile function (qf) of the BXEF distribution, $Q(.)$, follows as

$$Q(u)=\left[\frac{-1}{\alpha}\log\left(1-\left\{\left[-\log\left(1-u^{\frac{1}{\theta}}\right)\right]^{\frac{1}{2}}+1\right\}^{-\frac{1}{\lambda}}\right)\right]^{-\frac{1}{\beta}},\ 0<u<1. \quad (5)$$

Simulating the BXEF random variable is straightforward. If U is a uniform variate on the unit interval (0,1), then the random variable $X=Q(U)$ has PDF ((4)).

Figures 1, 2 and 3 show the curves for PDF and hazard rate function (HRF), respectively, of the BXEF distribution. These figures reveal that the hazard function of the BXEF distribution can be decreasing, increasing, bathtub or unimodal and then bathtub shapes. One of the advantages of the BXEF distribution over the EF distribution is that the latter cannot model phenomenon showing bathtub or an unimodal and then bathtub shape failure rates.

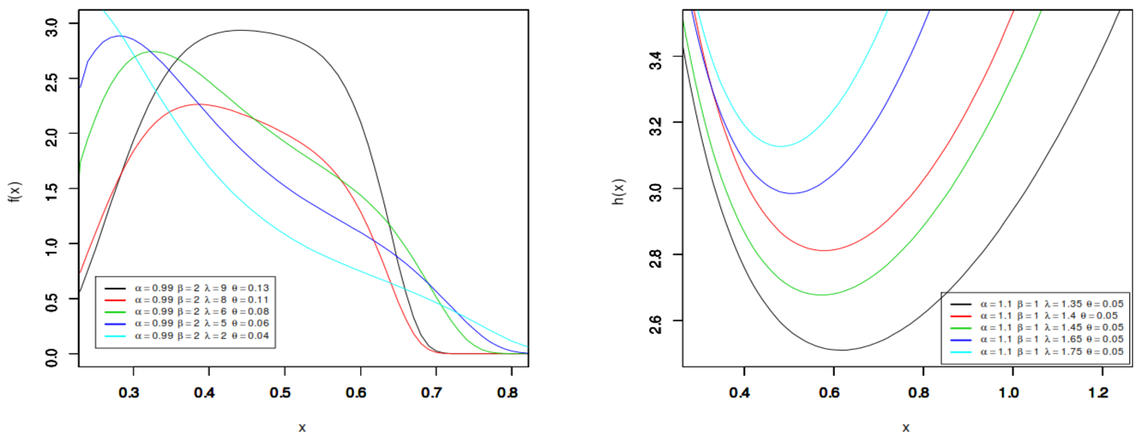

Figure 1: The PDF and HRF plots of the BXEF distribution

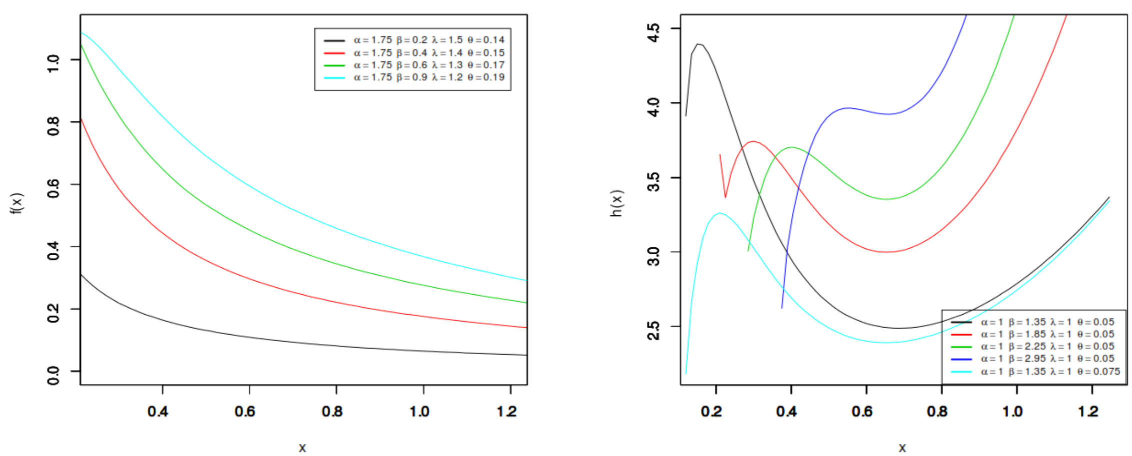

Figure 2: The PDF and HRF plots of the BXEF distribution

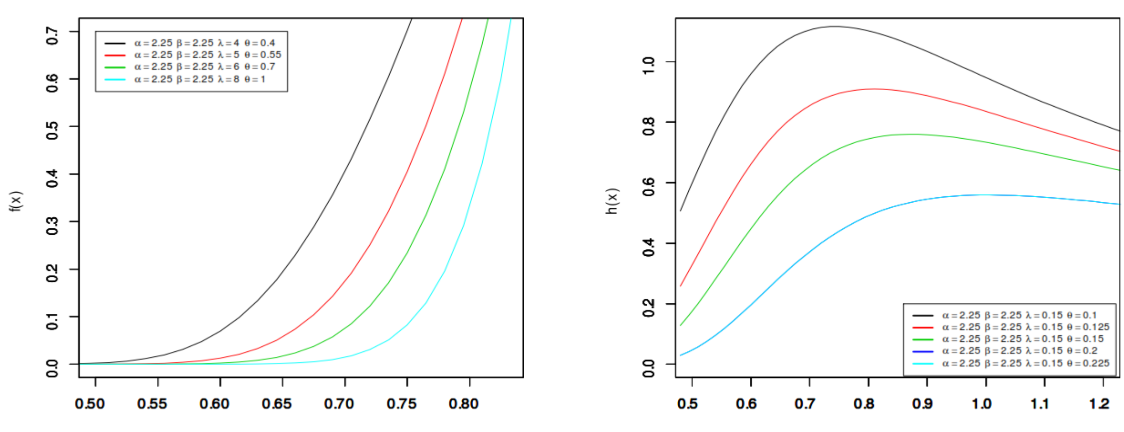

Figure 3: The PDF and HRF plots of the BXEF distribution

3. Linear representation

Using the generalized binomial series, we can write ((3)) as

$$F(x) = \sum_{j=0}^{\infty} (-1)^j \binom{\theta}{j} \exp\left\{-j\left[\frac{1-(1-\mathrm{e}^{-\alpha x^{-\beta}})^{\lambda}}{(1-\mathrm{e}^{-\alpha x^{-\beta}})^{\lambda}}\right]^2\right\}.$$

Applying the exponential series, we obtain

$$F(x) =_{j=0}^{\infty} \frac{(-1)^{j+k} j^k}{k!} \binom{\theta}{j} \frac{\left[1-(1-\mathrm{e}^{-\alpha x^{-\beta}})^{\lambda}\right]^{2k}}{(1-\mathrm{e}^{-\alpha x^{-\beta}})^{2k\lambda}}.$$

Applying the generalized binomial series, we have

$$F(x) == \sum_{j=0}^{\infty} \sum_{i=0}^{2k} \frac{(-1)^{j+k+i} j^k}{k!} \binom{\theta}{j} \binom{2k}{i} (1-\mathrm{e}^{-\alpha x^{-\beta}})^{-\lambda(2k-i)}.$$

For $|z| < 1$, the power series holds

$$(1-z)^{-q} = \sum_{j=0}^{\infty} (-1)^j \binom{-q}{j} z^j. \quad (6)$$

Applying ((6)) to $(1-\mathrm{e}^{-\alpha x^{-\beta}})^{-\lambda(2k-i)}$, we can write

$$F(x) = \sum_{j,l=0}^{\infty} \sum_{i=0}^{2k} \frac{(-1)^{j+\mathrm{k}+i+l} j^k}{k!} \binom{\theta}{j} \binom{2k}{i} \binom{-\lambda(2k-i)}{l} \mathrm{e}^{-l\alpha x^{-\beta}}.$$

Then, the CDF of the BXEF distribution reduces to

$$F(x) = \sum_{l=0}^{\infty} d_l G_{l\alpha,\beta}(x), \quad (7)$$

where

$$d_l = \sum_{j=0}^{\infty} \sum_{i=0}^{2k} \frac{(-1)^{j+k+i+l} j^k}{k!} \binom{\theta}{j} \binom{2k}{i} \binom{-\lambda(2k-i)}{l}$$

and $G_{l\alpha,\beta}(x)$ is the CDF of the Fréchet distribution with scale parameter $l\alpha$ and shape parameter β.

By differentiating equation ((7)), we obtain

$$f(x) = \sum_{l=0}^{\infty} d_l g_{l\alpha,\beta}(x), \quad (8)$$

where $g_{l\alpha,\beta}(x)$ is the PDF of the Fréchet distribution with scale parameter $l\alpha$ and shape parameter β. Equation (8) reveals that the PDF of the BXEF model can be expressed as a linear mixture of Fréchet densities. So, several mathematical properties of the new model can be obtained by knowing those of the Fréchet distribution.

Let a random variable Z have the Fréchet distribution with two parameters $\alpha > 0$ and $\beta > 0$. Then, the PDF of Z is given (for $z > 0$) by $g(x;\alpha,\beta) = \alpha\beta x^{-\beta-1}\mathrm{e}^{-\alpha x^{-\beta}}$. The rth ordinary and incomplete moments of Z are given by

$$\mu_r' = E(Z^r) = \alpha^{\frac{r}{\beta}} \Gamma\left(1-\frac{r}{\beta}\right) \text{and} \varphi_r(t) = \alpha^{\frac{r}{\beta}} \gamma\left(1-\frac{r}{\beta}, \alpha t^{-\beta}\right), \forall r < \beta\,,$$

respectively, where $\Gamma(.)$ is the complete gamma function $\gamma(.,.)$ is the lower incomplete gamma function.

4. Properties

4.1 Moments and generating function

The rth ordinary moment of X is given by $\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx$. Then we obtain

$$\mu'_r = \sum_{l=0}^{\infty} d_l (l\alpha)^{\frac{r}{\beta}} \Gamma\left(1-\frac{r}{\beta}\right), \forall r < \beta. \quad (9)$$

The sth incomplete moment of X is given (for $s < \beta$) by

$$\varphi_s(t) = \sum_{l=0}^{\infty} d_l (l\alpha)^{\frac{s}{\beta}} \gamma\left(1-\frac{s}{\beta}, l\alpha t^{-\beta}\right).$$

The moment generating function (mgf) $M_X(t) = E(e^{t\,X})$ of X.

Afify et al. (2016) provided a simple formula for the Fréchet distribution using the Wright generalized hypergeometric function. According to Afify et al. (2016), the mgf of the Fréchet distribution is defined by

$$M(t;\alpha,\beta) = {}_1\Psi_0\left[\begin{matrix}(1,-\beta^{-1})\\ -\end{matrix}; \alpha^{1/\beta}\, t\right]$$

Combining the last equation and equation ((8)), we obtain the mgf of X as

$$M_X(t) = \sum_{l=0}^{\infty} d_l\, {}_1\Psi_0\left[\begin{matrix}(1,-\beta^{-1})\\ -\end{matrix}; (l\alpha)^{1/\beta}\, t\right].$$

4.2 Probability weighted moments

The (s,r)th probability weighted moments (PWM)of X following the BXEF distribution, say $\rho_{s,r}$, is formally defined by

$$\rho_{s,r} = E\{X^s\, F(X)^r\} = \int_{-\infty}^{\infty} x^s\, F(x)^r\, f(x)\, dx. \quad (10)$$

Using equations (3), (4) and (8) we can write

$$f(x)\,F(x)^r = \sum_{i,j,k=0}^{\infty} \frac{2\theta(-1)^{i+j}(i+1)^j\Gamma(2j+k+3)}{j!\,k!\,\Gamma(2j+3)} \binom{\theta(r+1)-1}{i}$$
$$\times \alpha\beta\lambda x^{-\beta-1} \mathrm{e}^{-\alpha x^{-\beta}} (1-\mathrm{e}^{-\alpha x^{-\beta}})^{\lambda-1} \left[1-(1-\mathrm{e}^{-\alpha x^{-\beta}})^{\lambda}\right]^{2j+k+1}.$$

Applying the generalized binomial series to $\left[1-(1-\mathrm{e}^{-\alpha x^{-\beta}})^{\lambda}\right]^{2j+k+1}$, we have

$$f(x)\,F(x)^r = \sum_{i,j,k,m=0}^{\infty} \frac{2\theta(-1)^{i+j+m}(i+1)^j\Gamma(2j+k+3)}{j!\,k!\,\Gamma(2j+3)} \binom{\theta(r+1)-1}{i}$$
$$\times \binom{2j+k+1}{m} \alpha\beta\lambda x^{-\beta-1} \mathrm{e}^{-\alpha x^{-\beta}} (1-\mathrm{e}^{-\alpha x^{-\beta}})^{\lambda(m+1)-1}.$$

Applying the generalized binomial series to $(1-\mathrm{e}^{-\alpha x^{-\beta}})^{\lambda(m+1)-1}$, we can write

$$f(x)\,F(x)^r = \sum_{w=0}^{\infty} d_w \underbrace{\alpha(w+1)\beta\lambda x^{-\beta-1}\mathrm{e}^{-\alpha(w+1)x^{-\beta}}}_{g_{(w+1)\alpha,\beta}(x)},$$

Where

$$d_w = \sum_{i,j,k,m=0}^{\infty} \frac{2\theta(-1)^{i+j+m+w}(i+1)^j\Gamma(2j+k+3)}{j!\,k!\,\Gamma(2j+3)(w+1)} \times \binom{\theta(r+1)-1}{i}\binom{2j+k+1}{m}\binom{\lambda(m+1)-1}{w}.$$

Then, the (s,r)th PWM of X can be expressed as

$$\rho_{s,r} = \sum_{w=0}^{\infty} d_w[(w+1)\alpha]^{\frac{r}{\beta}}\Gamma\left(1-\frac{r}{\beta}\right), \forall r < \beta.$$

4.3 Stress-strength model

Let X_1 and X_2 be two independent random variables with BXEF$(\theta_1,\lambda,\beta,\alpha)$ and BXEF$(\theta_2,\lambda,\beta,\alpha)$ distributions, respectively. Then, the reliability is defined by

$$\mathbf{R} = \int_0^{\infty} f_1(x;\theta_1,\lambda,\beta,\alpha)F_2(x;\theta_2,\lambda,\beta,\alpha)dx.$$

Thus, **R** can be expressed as

$$\mathbf{R} = \sum_{j,k,w,m=0}^{\infty} s_{j,k,w,m},$$

where

$$s_{j,k,w,m} = 4\theta_1\theta_2 \sum_{j,k,w,m=0}^{\infty} \frac{(-1)^{j+w}\,\Gamma(2j+k+3)\Gamma(2w+m+3)}{j!\,k!\;w!\,m!\,\Gamma(\theta_2-h)\Gamma(2j+3)\Gamma(2w+3)} \sum_{i,h=0}^{\infty} \frac{(-1)^{i+h}(i+1)^j\,(h+1)^w\binom{\theta_1-1}{i}\binom{\theta_2-1}{h}}{(2w+m+2)(2j+k+2w+m+4)}.$$

4.4 Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Let $X_1,\dots,X_n$ be a random sample from the BXEF of distributions and let $X_{(1)},\dots,X_{(n)}$ be the corresponding order statistics. The PDF of ith order statistic, say $X_{i:n}$, can be written as

$$f_{i:\mathrm{n}}(x) = \frac{f(x)}{\mathrm{B}(i,n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F^{j+i-1}(x), \qquad (11)$$

where $B(\cdot,\cdot)$ is the beta function. Using (3), (4) and (9) we get

$$f(x)\,F(x)^r = \sum_{w=0}^{\infty} b_w\, g_{(w+1)\alpha,\beta}(x),$$

where

$$b_w = \sum_{l,h,k,m=0}^{\infty} \frac{2\theta(-1)^{l+h+m+w}(l+1)^h\Gamma(2h+k+3)}{h!\,k!\,\Gamma(2h+3)(w+1)} \times \binom{\theta(r+1)-1}{l}\binom{2h+k+1}{m}\binom{\lambda(m+1)-1}{w}.$$

The PDF of $X_{i:n}$ can be expressed as

$$f_{i:n}(x)=\sum_{w=0}^{\infty}\sum_{j=0}^{n-i}\frac{(-1)^j}{\mathrm{B}(i,n-i+1)}\binom{n-i}{j}b_w g_{(w+1)\alpha,\beta}(x).$$

Based on the last equation, we note that the properties of $X_{i:n}$ follow from those properties of Fréchet model. For example, the moments of $X_{i:n}$ can be expressed (for $q<\beta$) as

$$E\left(X_{i:n}^q\right)=\sum_{w=0}^{\infty}\sum_{j=0}^{n-i}\frac{(-1)^j}{\mathrm{B}(i,n-i+1)}\binom{n-i}{j}b_w[(w+1)\alpha]^{\frac{q}{\beta}}\Gamma\left(1-\frac{q}{\beta}\right).$$

5. Parameter estimation

Several approaches for parameter estimation were proposed in the literature but the maximum likelihood method is the most commonly employed. So, we consider the estimation of the unknown parameters of this family from complete samples only by maximum likelihood. Let $x_1,\ldots,x_n$ be a random sample from the BXEF model with parameters θ and ξ. Let $\Theta=(\theta,\xi^{\mathrm{T}})^T$ be the $p\times 1$ parameter vector. For determining the MLE of Θ, we have the log-likelihood function

$$\begin{aligned}\ell=\ell(\Theta)&=n\log2+n\log\theta+n\log\alpha+n\log\beta+n\log\lambda\\&+\sum_{i=1}^{n}\log(1-z_i)-(\beta+1)\sum_{i=1}^{n}\log(x_i)-\alpha\sum_{i=1}^{n}x_i^{-\beta}\\&-(2\lambda+1)\sum_{i=1}^{n}\log(1-\mathrm{e}^{-\alpha x_i^{-\beta}})-\sum_{i=1}^{n}s_i^2\\&+(\theta-1)\textstyle\sum_{i=1}^{n}\log[1-\exp(-s_i^2)],\end{aligned}$$

where $s_i=\frac{1-z_i}{z_i}$ and $z_i=(1-\mathrm{e}^{-\alpha x_i^{-\beta}})^{\lambda}$. The components of the score vector, $\mathbf{U}(\Theta)=\frac{\partial\ell}{\partial\Theta}=(\frac{\partial\ell}{\partial\theta},\frac{\partial\ell}{\partial\lambda},\frac{\partial\ell}{\partial\beta},\frac{\partial\ell}{\partial\alpha})^{\mathrm{T}}$, are given by

$$U_\theta=\frac{n}{\theta}+\sum_{i=1}^{n}\log[1-\exp(-s_i^2)],$$

$$\begin{aligned}U_\lambda=\frac{n}{\lambda}-\sum_{i=1}^{n}\frac{\log(1-\mathrm{e}^{-\alpha x_i^{-\beta}})}{(1-z_i)(1-\mathrm{e}^{-\alpha x_i^{-\beta}})^{-\lambda}}-2\sum_{i=1}^{n}\log(1-\mathrm{e}^{-\alpha x_i^{-\beta}})\\-2\sum_{i=1}^{n}a_is_i+2(\theta-1)\sum_{i=1}^{n}\frac{a_is_i\exp(-s_i^2)}{1-\exp(-s_i^2)}\end{aligned}$$

$$U_\beta = \frac{n}{\beta} + \lambda\alpha \sum_{i=1}^{n} \frac{x_i^{-\beta} e^{-\alpha x_i^{-\beta}} \log(x_i) z_i}{(1-z_i)(1-e^{-\alpha x_i^{-\beta}})^{1-\lambda}} - \sum_{i=1}^{n} \log(x_i) + \alpha \sum_{i=1}^{n} x_i^{-\beta} \mathrm{loglog}(x_i)$$

$$+\alpha(2\lambda+1)\sum_{i=1}^{n} \frac{x_i^{-\beta}\log(x_i) e^{-\alpha x_i^{-\beta}}}{1-e^{-\alpha x_i^{-\beta}}} - 2\sum_{i=1}^{n} b_i s_i + 2(\theta-1)\sum_{i=1}^{n} \frac{b_i s_i \exp(-s_i^2)}{1-\exp(-s_i^2)},$$

and

$$U_\alpha = \frac{n}{\alpha} + \lambda \sum_{i=1}^{n} \frac{x_i^{-\beta} e^{-\alpha x_i^{-\beta}} z_i}{(1-z_i)(1-e^{-\alpha x_i^{-\beta}})^{1-\lambda}} - \sum_{i=1}^{n} x_i^{-\beta}$$

$$+(2\lambda+1)\sum_{i=1}^{n} \frac{x_i^{-\beta} e^{-\alpha x_i^{-\beta}}}{1-e^{-\alpha x_i^{-\beta}}} - 2\sum_{i=1}^{n} c_i s_i + 2(\theta-1)\sum_{i=1}^{n} \frac{c_i s_i \exp(-s_i^2)}{1-\exp(-s_i^2)}$$

where

$$a_i = \frac{-\log(1-e^{-\alpha x_i^{-\beta}})}{z_i^2(1-e^{-\alpha x_i^{-\beta}})^{-\lambda}}, b_i = \frac{\lambda\alpha x_i^{-\beta}\log(x_i) e^{-\alpha x_i^{-\beta}}}{z_i^2(1-e^{-\alpha x_i^{-\beta}})^{1-\lambda}} \text{and} c_i = \frac{\lambda x_i^{-\beta} e^{-\alpha x_i^{-\beta}}}{z_i^2(1-e^{-\alpha x_i^{-\beta}})^{1-\lambda}}.$$

Setting the nonlinear system of equations $U_\theta = U_\lambda = U_\beta = 0$ and $U_\alpha = 0$ and solving them simultaneously yields the MLE $\widehat{\Theta} = (\hat{\theta}, \hat{\lambda}, \hat{\beta}, \alpha)^{\mathrm{T}}$. To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize ℓ.

From Equation (12) and for fixed α, β and λ, we can obtain $\hat{\theta}(\alpha, \beta, \lambda)$ as

$$\hat{\theta}(\alpha,\beta,\lambda) = -n \Bigg/ \sum_{i=1}^{n} \log\left(1 - \exp\left\{-\left[\frac{1-\left(1-e^{-\alpha x_i^{-\beta}}\right)^{\lambda}}{\left(1-e^{-\alpha x_i^{-\beta}}\right)^{\lambda}}\right]^2\right\}\right).$$

6. Simulation study

In this section, various simulations are considered for different sample sizes (n=350,500,1000 and 7500) to evaluate the performance of the MLEs for the Burr X Exponentiated Fréchet distribution's parameters. The parameter values are set at α=1.5, (β=0.5,0.75,1.5), (λ=0.5, 0.75,1.5) and (θ=0.5,1.5,2.5). Each sample size is replicated 1500 times. Average estimates (AEs) and standard deviations of the estimates are given in Table 1. Results indicate that the SDs of the MLEs of the parameters approaches toward zero with increase in sample size.

Table1: Simulation results: mean estimates, sd of estimates at various sample sizes

α=1.5 Parameter Values	n=350								n=500							
	$\hat{\alpha}$	$sd(\hat{\alpha})$	$\hat{\beta}$	$sd(\hat{\beta})$	$\hat{\lambda}$	$sd(\hat{\lambda})$	$\hat{\theta}$	$sd(\hat{\theta})$	$\hat{\alpha}$	$sd(\hat{\alpha})$	$\hat{\beta}$	$sd(\hat{\beta})$	$\hat{\lambda}$	$sd(\hat{\lambda})$	$\hat{\theta}$	$sd(\hat{\theta})$
β=0.5 ,λ=0.5 ,θ=0.5	1.503	0.063	0.521	0.013	0.513	0.025	0.498	0.002	1.503	0.063	0.521	0.013	0.513	0.025	0.498	0.002
β=0.5 ,λ=0.5 ,θ=1.5	1.559	0.139	0.596	0.180	0.609	0.256	1.503	0.006	1.559	0.139	0.596	0.180	0.609	0.256	1.503	0.006
β=0.5 ,λ=0.5 ,θ=2.5	1.602	0.161	0.651	0.230	0.668	0.919	2.505	0.019	1.602	0.161	0.651	0.230	0.668	0.919	2.505	0.019
β=0.5 ,λ=0.75 ,θ=0.5	1.524	0.099	0.514	0.012	0.802	0.092	0.499	0.002	1.524	0.099	0.514	0.012	0.802	0.092	0.499	0.002
β=0.5 ,λ=0.75 ,θ=1.5	1.551	0.257	0.591	0.213	0.971	0.911	1.502	0.006	1.551	0.257	0.591	0.213	0.971	0.911	1.502	0.006
β=0.5 ,λ=0.75 ,θ=2.5	1.573	0.344	0.656	0.288	1.126	3.020	2.505	0.019	1.573	0.344	0.656	0.288	1.126	3.020	2.505	0.019
β=0.5 ,λ=1.5 ,θ=0.5	1.518	0.157	0.516	0.011	1.666	0.731	0.500	0.001	1.518	0.157	0.516	0.011	1.666	0.731	0.500	0.001
β=0.5 ,λ=1.5 ,θ=1.5	1.622	0.609	0.555	0.075	2.940	80.250	1.503	0.006	1.622	0.609	0.555	0.075	2.940	80.250	1.503	0.006
β=0.5 ,λ=1.5 ,θ=2.5	1.678	0.968	0.628	0.404	4.206	133.708	2.505	0.018	1.678	0.968	0.628	0.404	4.206	133.708	2.505	0.018
β=0.75 ,λ=0.5 ,θ=0.5	1.509	0.068	0.787	0.033	0.514	0.028	0.493	0.005	1.509	0.068	0.787	0.033	0.514	0.028	0.493	0.005
β=0.75 ,λ=0.5 ,θ=1.5	1.544	0.134	0.962	0.668	0.587	0.214	1.503	0.006	1.544	0.134	0.962	0.668	0.587	0.214	1.503	0.006
β=0.75 ,λ=0.5 ,θ=2.5	1.614	0.163	1.028	0.687	0.666	0.669	2.500	0.019	1.614	0.163	1.028	0.687	0.666	0.669	2.500	0.019
β=0.75 ,λ=0.75 ,θ=0.5	1.514	0.094	0.783	0.031	0.789	0.089	0.492	0.005	1.514	0.094	0.783	0.031	0.789	0.089	0.492	0.005
β=0.75 ,λ=0.75 ,θ=1.5	1.558	0.276	0.904	0.611	1.013	2.512	1.503	0.006	1.558	0.276	0.904	0.611	1.013	2.512	1.503	0.006
β=0.75 ,λ=0.75 ,θ=2.5	1.590	0.411	1.009	0.816	1.246	5.184	2.502	0.017	1.590	0.411	1.009	0.816	1.246	5.184	2.502	0.017
β=0.75 ,λ=1.5 ,θ=0.5	1.527	0.160	0.798	0.048	1.702	0.879	0.483	0.010	1.527	0.160	0.798	0.048	1.702	0.879	0.483	0.010
β=0.75 ,λ=1.5 ,θ=1.5	1.637	0.644	0.838	0.255	2.891	36.400	1.505	0.007	1.637	0.644	0.838	0.255	2.891	36.400	1.505	0.007
β=0.75 ,λ=1.5 ,θ=2.5	1.703	1.059	1.003	1.422	4.153	82.160	2.500	0.018	1.703	1.059	1.003	1.422	4.153	82.160	2.500	0.018
β=1.5 ,λ=0.5 ,θ=0.5	1.504	0.069	1.551	0.086	0.518	0.028	0.501	0.001	1.504	0.069	1.551	0.086	0.518	0.028	0.501	0.001
β=1.5 ,λ=0.5 ,θ=1.5	1.567	0.155	1.889	2.670	0.622	0.330	1.503	0.006	1.567	0.155	1.889	2.670	0.622	0.330	1.503	0.006
β=1.5 ,λ=0.5 ,θ=2.5	1.663	0.230	2.099	3.888	0.777	2.008	2.504	0.018	1.663	0.230	2.099	3.888	0.777	2.008	2.504	0.018
β=1.5 ,λ=0.75 ,θ=0.5	1.514	0.096	1.544	0.087	0.797	0.086	0.501	0.001	1.514	0.096	1.544	0.087	0.797	0.086	0.501	0.001
β=1.5 ,λ=0.75 ,θ=1.5	1.559	0.282	1.804	2.349	1.014	1.734	1.506	0.006	1.559	0.282	1.804	2.349	1.014	1.734	1.506	0.006
β=1.5 ,λ=0.75 ,θ=2.5	1.603	0.513	2.276	6.589	1.407	8.990	2.508	0.019	1.603	0.513	2.276	6.589	1.407	8.990	2.508	0.019
β=1.5 ,λ=1.5 ,θ=0.5	1.525	0.155	1.549	0.098	1.669	0.651	0.497	0.002	1.525	0.155	1.549	0.098	1.669	0.651	0.497	0.002
β=1.5 ,λ=1.5 ,θ=1.5	1.706	0.804	1.658	1.477	3.810	174.887	1.503	0.006	1.706	0.804	1.658	1.477	3.810	174.887	1.503	0.006
β=1.5 ,λ=1.5 ,θ=2.5	1.698	1.121	2.069	6.557	4.432	106.702	2.504	0.018	1.698	1.121	2.069	6.557	4.432	106.702	2.504	0.018

Table1: Simulation results: mean estimates, sd of estimates at various sample sizes (Continued)

α=1.5 Parameter Values	n=1000								n=7500							
	$\hat{\alpha}$	$sd(\hat{\alpha})$	$\hat{\beta}$	$sd(\hat{\beta})$	$\hat{\lambda}$	$sd(\hat{\lambda})$	$\hat{\theta}$	$sd(\hat{\theta})$	$\hat{\alpha}$	$sd(\hat{\alpha})$	$\hat{\beta}$	$sd(\hat{\beta})$	$\hat{\lambda}$	$sd(\hat{\lambda})$	$\hat{\theta}$	$sd(\hat{\theta})$
β=0.5 ,λ=0.5 ,θ=0.5	1.501	0.044	0.516	0.008	0.507	0.016	0.499	0.002	1.499	0.003	0.502	0.000	0.499	0.001	0.499	0.000
β=0.5 ,λ=0.5 ,θ=1.5	1.531	0.084	0.570	0.108	0.552	0.096	1.500	0.004	1.502	0.004	0.501	0.001	0.503	0.002	1.500	0.000
β=0.5 ,λ=0.5 ,θ=2.5	1.564	0.102	0.604	0.130	0.594	0.304	2.498	0.013	1.503	0.003	0.501	0.002	0.504	0.003	2.500	0.001
β=0.5 ,λ=0.75 ,θ=0.5	1.507	0.058	0.511	0.006	0.774	0.048	0.499	0.001	1.502	0.005	0.501	0.000	0.749	0.004	0.496	0.002
β=0.5 ,λ=0.75 ,θ=1.5	1.539	0.170	0.548	0.066	0.894	0.489	1.499	0.005	1.499	0.007	0.503	0.001	0.753	0.007	1.499	0.000
β=0.5 ,λ=0.75 ,θ=2.5	1.556	0.219	0.590	0.143	0.967	0.826	2.496	0.013	1.502	0.007	0.503	0.002	0.756	0.009	2.499	0.001
β=0.5 ,λ=1.5 ,θ=0.5	1.500	0.112	0.516	0.008	1.591	0.431	0.500	0.001	1.503	0.007	0.501	0.000	1.507	0.018	0.498	0.001
β=0.5 ,λ=1.5 ,θ=1.5	1.588	0.367	0.534	0.049	2.112	5.076	1.502	0.004	1.500	0.017	0.503	0.002	1.514	0.052	1.500	0.000
β=0.5 ,λ=1.5 ,θ=2.5	1.624	0.602	0.587	0.249	2.763	30.137	2.505	0.013	1.499	0.022	0.505	0.003	1.518	0.074	2.499	0.001
β=0.75 ,λ=0.5 ,θ=0.5	1.506	0.042	0.777	0.022	0.505	0.017	0.494	0.005	1.502	0.004	0.753	0.002	0.499	0.002	0.497	0.002
β=0.75 ,λ=0.5 ,θ=1.5	1.536	0.091	0.870	0.305	0.564	0.148	1.503	0.004	1.505	0.004	0.750	0.003	0.505	0.003	1.499	0.000
β=0.75 ,λ=0.5 ,θ=2.5	1.574	0.110	0.942	0.416	0.602	0.249	2.501	0.013	1.502	0.003	0.755	0.004	0.502	0.003	2.499	0.001
β=0.75 ,λ=0.75 ,θ=0.5	1.514	0.065	0.773	0.020	0.774	0.055	0.493	0.004	1.502	0.005	0.753	0.002	0.747	0.006	0.496	0.002
β=0.75 ,λ=0.75 ,θ=1.5	1.538	0.159	0.824	0.166	0.879	0.336	1.501	0.005	1.502	0.008	0.753	0.003	0.756	0.008	1.499	0.000
β=0.75 ,λ=0.75 ,θ=2.5	1.559	0.263	0.925	0.471	1.015	1.459	2.501	0.012	1.505	0.009	0.753	0.005	0.759	0.011	2.498	0.001
β=0.75 ,λ=1.5 ,θ=0.5	1.513	0.104	0.789	0.036	1.611	0.383	0.486	0.008	1.502	0.010	0.755	0.002	1.497	0.019	0.496	0.002
β=0.75 ,λ=1.5 ,θ=1.5	1.562	0.363	0.814	0.129	2.104	10.266	1.503	0.005	1.503	0.020	0.754	0.004	1.521	0.063	1.499	0.000
β=0.75 ,λ=1.5 ,θ=2.5	1.616	0.660	0.915	0.726	2.887	31.955	2.508	0.013	1.501	0.028	0.758	0.007	1.528	0.097	2.500	0.001
β=1.5 ,λ=0.5 ,θ=0.5	1.504	0.042	1.533	0.053	0.510	0.015	0.499	0.002	1.510	0.007	1.509	0.010	0.492	0.006	0.489	0.005
β=1.5 ,λ=0.5 ,θ=1.5	1.533	0.088	1.756	1.406	0.555	0.090	1.501	0.005	1.501	0.005	1.510	0.016	0.502	0.003	1.500	0.000
β=1.5 ,λ=0.5 ,θ=2.5	1.585	0.117	2.018	2.904	0.607	0.252	2.502	0.013	1.505	0.005	1.510	0.027	0.505	0.005	2.499	0.001
β=1.5 ,λ=0.75 ,θ=0.5	1.503	0.062	1.537	0.057	0.773	0.049	0.499	0.001	1.504	0.008	1.519	0.014	0.740	0.010	0.489	0.006
β=1.5 ,λ=0.75 ,θ=1.5	1.534	0.171	1.681	1.001	0.884	0.398	1.500	0.004	1.507	0.009	1.501	0.016	0.761	0.010	1.500	0.000
β=1.5 ,λ=0.75 ,θ=2.5	1.572	0.296	1.920	2.937	1.075	2.333	2.500	0.012	1.506	0.013	1.511	0.031	0.763	0.017	2.500	0.001
β=1.5 ,λ=1.5 ,θ=0.5	1.531	0.107	1.525	0.064	1.636	0.431	0.497	0.003	1.503	0.014	1.528	0.024	1.515	0.025	0.482	0.009
β=1.5 ,λ=1.5 ,θ=1.5	1.588	0.472	1.628	0.421	2.441	23.867	1.498	0.005	1.507	0.021	1.505	0.018	1.529	0.068	1.499	0.000
β=1.5 ,λ=1.5 ,θ=2.5	1.659	0.716	1.779	2.695	3.133	43.358	2.500	0.012	1.504	0.033	1.516	0.034	1.537	0.116	2.501	0.001

7. Data analysis

In this section, the BXEF distribution is fitted to two real data sets and compared with other some competitive models. In order to compare the fits of the distributions, we consider some measures of goodness-of-fit including the maximized log-likelihood under the model ($-\hat{\ell}$), Anderson-Darling (A^*), Cramér-Von Mises (W^*) and Kolmogorov Smirnov (KS) statistics (with its p-value). The smaller these statistics are, the better the fit is.

The first data set was studied by Lee and Wang (2003), which represents the remission times (in months) of a random sample of 128 bladder cancer patients. The second data set represents the exceedances of flood peaks (in m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. The data consist of 72 exceedances for the years 1958-1984, rounded to one decimal place.

For the two data sets, we shall compare the fits of the BXEF model with other models: the EF (Okorie et al., 2016), Kumaraswamy Fréchet (KF) (Mead and Abd-Eltawab, 2014), Weibull Fréchet (WF) (Afify et al., 2016), Marshall-Olkin Fréchet (MOF) (Krishna et al., 2013) and Gumbel (Gu) (Gumbel, 1935) distributions, whose PDF's (for $x > 0$) are given by

$$\text{KF: } f(x) = ab\beta\alpha^{\beta}\, x^{-\beta-1}\mathrm{e}^{-a\left(\frac{\alpha}{x}\right)^{\beta}}\left[1-\mathrm{e}^{-a\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{b-1}, \alpha,\beta,a,b > 0.$$

$$\text{WF: } f(x) = ab\beta\alpha^{\beta}\, x^{-(\beta+1)}\,\mathrm{e}^{-a\left(\frac{\alpha}{x}\right)^{\beta}}\left[1-\mathrm{e}^{-a\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{b-1}, \alpha,\beta,a,b > 0.$$

$$\text{MOF: } f(x) = \theta\beta\alpha^{\beta}x^{-\beta-1}\mathrm{e}^{-\left(\frac{\alpha}{x}\right)^{\beta}\left[\theta+(1-\theta)\mathrm{e}^{-\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{-2}}, \alpha,\beta,\theta > 0.$$

$$\text{Gu: } f(x) = \alpha\beta x^{-\beta-1}\mathrm{e}^{-\alpha x^{-\beta}}, \alpha,\beta > 0.$$

Tables 2 and 3 list the numerical values of $-\hat{\ell}$, W^*, A^*, KS and P-value for the models fitted to both data sets. The MLEs and their corresponding standard errors (in parentheses) of the model parameters are also given in the same tables.

In Tables 2 and 3, we compare the fits of the BXEF model with the EF, KF, WF, MOF and Gu models. The figures in these tables indicate that the BXEF distribution has the lowest values for all goodness-of-fit statistics, for both data sets, among all fitted models.

The histogram and the fitted BXEF distribution of both data sets are displayed in Figures 4 and 6. Also, the plots of the estimated CDFs and QQ plots of the two data sets are displayed in Figures 5 and 7, respectively.

Table 2: Goodness-of-fit statistics, MLEs and standard errors (SE) (in parentheses) for cancer data

Model		Estimates (SE)	$-\hat{\ell}$	W^*	A^*	KS	P-value
BXEF	α	3.2950(2.9901)	410.841	0.04674	0.30997	0.046	0.946
	β	0.0722(0.0847)					
	λ	14.1871(45.6238)					
	θ	3.0313(1.8456)					
EF	α	9.3672(2.6452)	411.112	0.05312	0.34928	0.051	0.897
	β	0.1577(0.0633)					
	λ	63.471(2094.01)					
KF	α	356.3106(1533.8)	411.112	0.05322	0.34967	0.051	0.897
	β	0.1570(0.0628)					
	a	3.7344(3.5460)					
	b	787.21(2159.708)					
WF	α	1462.8238(1866.973)	411.350	0.05835	0.38234	0.054	0.847
	β	0.1858(0.0583)					
	a	150.4306(221.1002)					
	b	2.0023(1.0212)					
MOF	α	0.0498(0.0188)	411.459	0.04426	0.31989	0.0399	0.9869
	β	1.7229(0.1255)					
	θ	3951.9479(740.0426)					
Gu	α	2.4311(0.2193)	444.001	0.74432	4.54642	0.1408	0.0125
	β	0.7521(0.0424)					

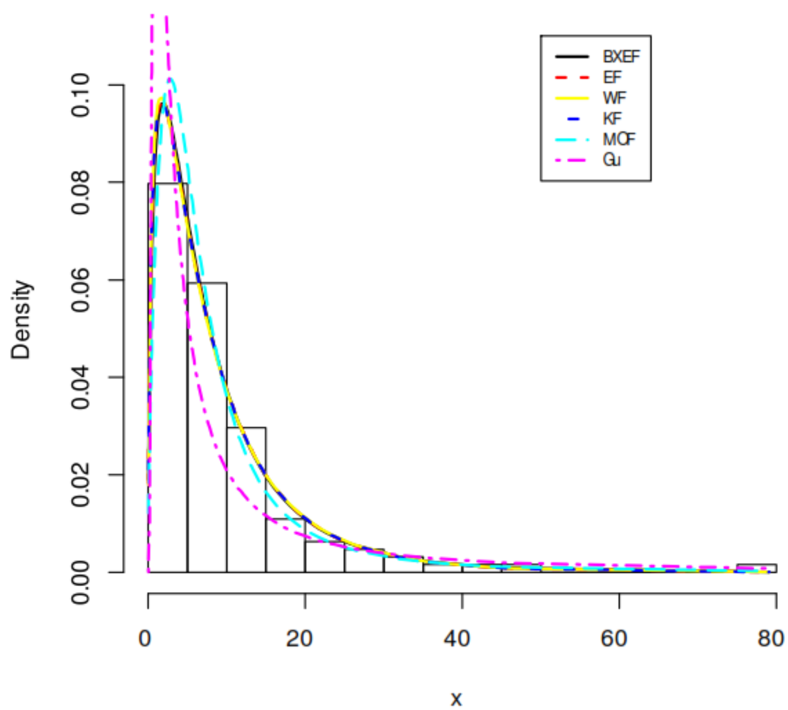

Figure 4: Fitted densities for cancer data

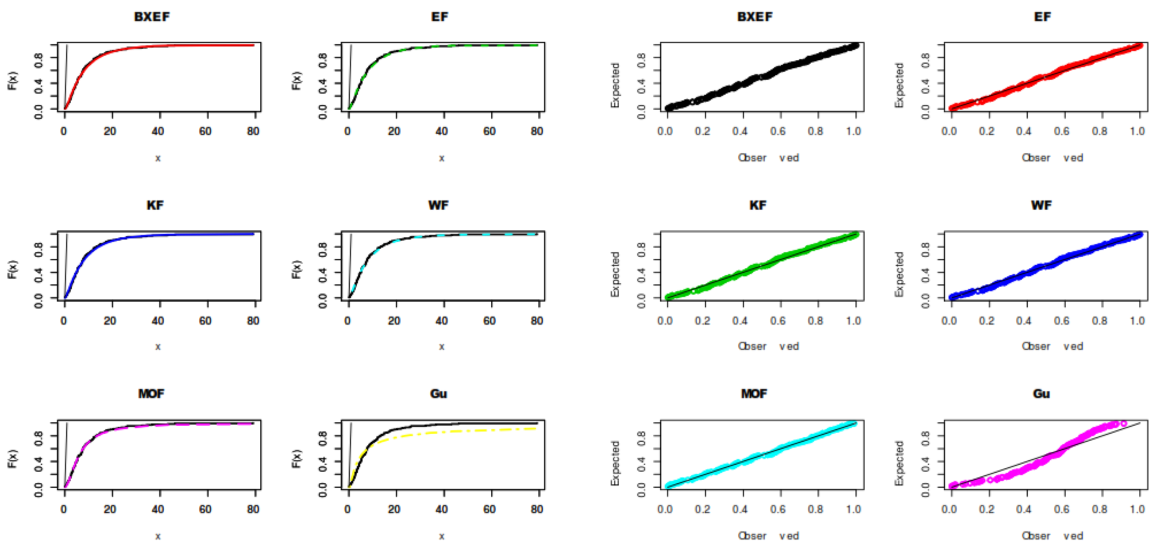

Figure 5: Estimated CDFs (left panel) and QQ plots (right panel) for cancer data

Table 3: Goodness-of-fit statistics, MLEs and standard errors (SE) (in parentheses) for Wheaton River data

Model	Estimates (SE)		$-\hat{\ell}$	W^*	A^*	KS	P-value
BXEF	α	7.0886(0.2304)	248.904	0.07064	0.40946	0.0795	0.7531
	β	0.3387(0.0142)					
	λ	5.9206(0.2432)					
	θ	0.2009(0.0257)					
WF	α	0.2513(0.2161)	250.791	0.12839	0.71754	0.0971	0.5061
	β	1.3329(1.3429)					
	a	0.03535(0.0363)					
	b	0.6568(0.6325)					
EF	α	13.0118(0.3630)	252.124	0.16511	0.91345	0.0967	0.5116
	β	0.0798(0.0073)					
	λ	46472.32(10343.08)					
KF	α	0.0158(0.0189)	252.365	0.17241	0.95153	0.1011	0.4539
	β	0.0969(0.0209)					
	a	16.4824(2.5774)					
	b	6334.144(11706.52)					
MOF	α	0.1175(0.1789)	257.715	0.29379	1.63579	0.1146	0.3006
	β	1.1966(0.1217)					
	θ	126.1731(255.1551)					
Gu	α	1.9928(0.2366)	267.019	0.44261	2.59659	0.1532	0.0682
	β	0.6521(0.0538)					

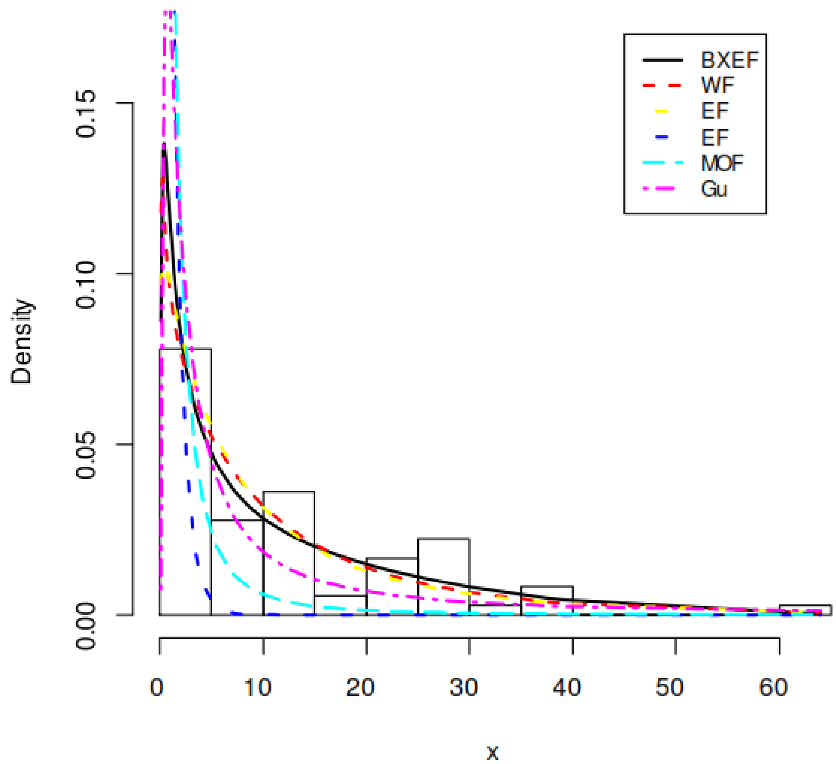

Figure 6: Fitted densities for Wheaton River data

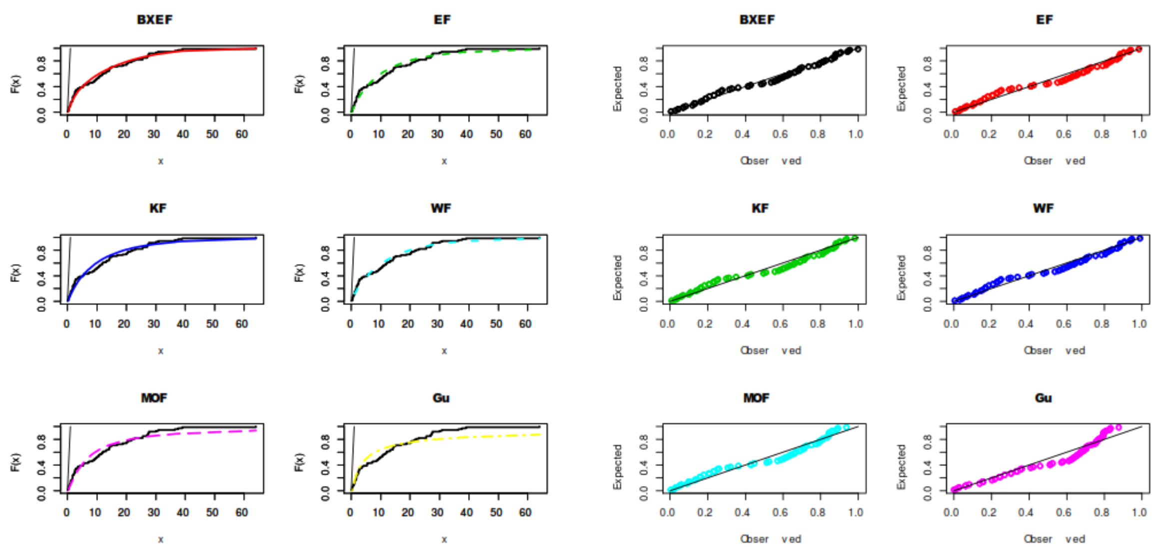

Figure 7: Estimated CDFs (left panel) and QQ plots (right panel) for Wheaton River data

8. Conclusion

In this article, we propose a new four-parameter model, called the transmuted Burr X exponentiated Fréchet (BXEF) distribution, which extends the exponentiated Fréchet (EF) (Okorie et al., 2016) distribution introduced by Okorie et al. (2016). The density function of BXEF can be expressed as a mixture of Fréchet densities. We derive explicit expressions for the ordinary moments, quantile and generating functions, probability weighted moments, stress-strength model and order statistics. We discuss the maximum likelihood estimation of the model parameters. Two applications illustrate that the BXEF distribution provides consistently better fit than other nested and non-nested models.

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