Upper Bound of Ruin Probability for an Insurance Discrete-Time Risk Model with Proportional Reinsurance and Investment

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Abstract

Two upper bounds for ruin probability under the discrete time risk model for insurance controlled by two factors: proportional reinsurance and surplus investment are presented. The latter is of interest because of the assumption that insurers invest some or their entire financial surplus on both the stock and bond markets, for which bond interest rates follow a time – homogeneous Markov chain. In addition, the control of reinsurance and stock investment in each time period are assumed to be constant values. The first upper bound for finite time ruin probability and ultimate ruin probability was derived under the condition that the Lundberg coefficient exists. The second upper bound is for finite time ruin probability and was developed from a new worse than used function. Numerical examples are used to illustrate these results, and the upper bound of ruin probability using real-life motor insurance claims data from a broker is also presented.

Keywords: Discrete-time Risk Model, Upper bound of ruin probability, Lundberg coefficient, new worse than used.

1. Introduction

There is increasing attention on ruin probability for insurance discrete time risk models with reinsurance and investment of financial surplus over the last decade due to the fact that insurance companies can purchase reinsurance, invest in the stock market, and receive dividends, among other transactions. However, obtaining an explicit solution of a company's ruin probability is actually a difficult task. One alternative method commonly used in ruin theory is deriving bounds for the ruin probabilities (Diasparra and Romera, 2009, and Lin et al., 2015) thus the focus in this paper is on upper bounds for ruin probability.

Lundberg's inequality provides a well-known upper bound for the probability of ultimate ruin in the classical risk model when the moment generating function of the claim size random variable exists. However, in many practical distributions, the moment generating function does not exist, so the Lundberg inequality is not available in these cases (Cai and Wu, 1997, and Cai and Garrido, 1999). Thus, there are many researches in which the upper bound of ruin probability has been derived (see, for example, Dickson (1994), Willmot (1994), Kalashnikov (1999), and other researchers) and can be applied to more general claim size distributions.

In this paper, two upper bounds of ruin probability for a discrete time risk model controlled by reinsurance and investment are presented. The former was derived under the condition that the moment generating function of a claim size exists (the Lundberg coefficient exists). This upper bound can be viewed as an extension of the results from the studying of Diasparra and Romera (2009), and Jasiulewicz and Kordecki (2015) by adding investment to a risk model. The latter was developed from the idea of Willmot (1994) by providing the upper bound in terms of the new worse than used (NWU) function. This upper bound can be applied for more general claim size distributions.

2. Model description

The typical discrete time risk model for insurance can be written as

$$U_n = U_{n-1} + X_n - Y_n \quad ; n = 1, 2, 3, \ldots, \tag{1}$$

where U_n denotes the insurer's surplus at the end of time period n with initial constant $U_0 = u$, X_n being the total premiums amount during time period n (i.e. from time $n-1$ to n), and Y_n is the total claims amount during n. We assume that this sequence consists of independent and identically distributed (i.i.d.) random variables with a common distribution function $P(y) = Pr(Y_n \leq y); y \geq 0$. In this study, the above risk model is expanded upon by adding proportional reinsurance and investment.

Under proportional reinsurance contracts, the reinsurer agrees to cover a fraction of each claim equal to the fraction of premiums that it receives from the insurer. Throughout this study, $b_n \in (0,1]$ is defined as the retention level of a reinsurance contract for time period n. This means that the insurer pays $b_n Y_n$ of total claim amount Y_n while the reinsurer is liable for $(1-b_n)Y_n$, and if the retention level $b_n = 1$, this means that there is no reinsurance. Let $h(b_n, Y_n)$ denote the fraction of the total claim Y_n paid by the insurer, $0 < h(b_n, Y_n) \leq Y_n$, with $G(y_b) = Pr\left[h(b_n, Y_n) \leq y_b\right]$, $y_b \geq 0$, then $h(b_n, Y_n)$ can be evaluated by $h(b_n, Y_n) = b_n Y_n$ (this is the case throughout this paper). In addition, the insurance premium during time period *n*, X_n, is assumed to be a fixed constant c for all *n*. Subsequently, by the expected value principle with safety loading factor $\theta > 0$, the premium constant is calculated as $c = (1+\theta)E(Y_n)$ and paid at the end of every time period unit $(n-1, n]$. Let δ be the safety loading factor added by the reinsurer and c_{re} be the premium constant for the reinsurer. Thus, by the expected value principle, the constant premium for reinsurer in a unit period is given by

$$\begin{aligned} c_{re} &= (1+\delta)E\left[Y_n - h(b_n, Y_n)\right] \\ &= (1+\delta)(1-b_n)E(Y_n). \end{aligned} \tag{2}$$

Next, the constant premium which is retained by the insurer in a unit period denoted by $c(b_n)$, when $0 \le c(b_n) \le c$, can be calculated as

$$\begin{aligned} c(b_n) &= c - c_{re} \\ &= \left[(1+\theta) - (1+\delta)(1-b_n)\right] E(Y_n). \end{aligned} \tag{3}$$

For the effect of an investment on a risk model, we assume that the insurer can invest in two assets. One is a bond with a known interest rate at the initial time (I_0); the interest rate at time n $(I_n, n=1,2,3...)$ has a finite countable number (d_n) of possible values ($I_n = i_k$, where $k \in 1,2,3,...,d_n$), and we assume that $d_n = d$ for all n throughout this dissertation. In addition, I_n is assumed to follow a time-homogeneous Markov chain, i.e. both the transition probabilities and the time are independent, and are denoted by

$$\begin{aligned} Pr\{I_n = i_b \mid I_{n-1} = i_a, ..., I_0 = i_s\} &= Pr\{I_n = i_b \mid I_{n-1} = i_a\} \\ &= p_{ab}, \end{aligned} \tag{4}$$

where $\sum_{b=0}^{d} p_{ab} = 1$ for all $a, b \in \{0,1,2,...,d\}$.

The other investment asset is a stock with simple net return R_n and the price of one share of stock S_n at time n is defined as

$$S_n = S_{n\text{-}1}(1+R_n) = S_{n\text{-}1}W_n \qquad ; n = 1,2,3,... \tag{5}$$

A standard assumption on the stock market is $1 + R_n = W_n > 0$, which is called the gross return. Throughout this paper, W_n is assumed to be a sequence of i.i.d. nonnegative random variables with the distribution function $F(w) = Pr(W_n \le w)$, $w \ge 0$.

Based on the risk model in Equation (1), if at the beginning of n^{th} period the insurer has the chance to decide the amount of stock investment α_n $(\alpha_n \ge 0)$ using the information from $\{I_j \text{ and } W_j : j = 0,1,2,...,n-1\}$, and the retention level b_n of a reinsurance contract, our risk model is finally formulated as

$$U_n = U_{n-1}(1+I_n) + \alpha_n W_n + c(b_n) - h(b_n, Y_n), \tag{6}$$

with the assumption that the sequences $\{I_n\}$, $\{W_n\}$, and $\{Y_n\}$ are mutually independent.

From the risk model in Equation (6), if we replace the values of n as $n = 1,2,..,m$, then the output from this action is another form of the former model as follows:

$$U_1 = U_0(1+I_1) + \alpha_1 W_1 + c(b_1) - h(b_1, Y_1)$$

$$\begin{aligned} U_2 &= U_1(1+I_2) + \alpha_2 W_2 + c(b_2) - h(b_2, Y_2) \\ &= U_0 \prod_{j=1}^{2}(1+I_j) + \alpha_1 W_1(1+I_2) + c(b_1)(1+I_2) - h(b_1, Y_1)(1+I_2) \\ &\quad + \alpha_2 W_2 + c(b_2) - h(b_2, Y_2), \end{aligned}$$

$$\begin{aligned} U_3 &= U_2(1+I_3) + \alpha_3 W_3 + c(b_3) - h(b_3, Y_3) \\ &= U_0 \prod_{j=1}^{3}(1+I_j) + \alpha_1 W_1 \prod_{k=2}^{3}(1+I_k) + c(b_1)\prod_{k=2}^{3}(1+I_k) - h(b_1, Y_1)\prod_{k=2}^{3}(1+I_k) \\ &\quad + \alpha_2 W_2(1+I_3) + c(b_2)(1+I_3) - h(b_2, Y_2)(1+I_3) + \alpha_3 W_3 + c(b_3) - h(b_3, Y_3), \end{aligned}$$

$$U_m = U_0 \prod_{j=1}^{m}(1+I_j) + \sum_{j=1}^{m}\left[\left(\alpha_j W_j + c(b_j) - h(b_j, Y_j)\right)\prod_{k=j+1}^{m}(1+I_k)\right]$$

Therefore, the other form of U_n is

$$U_n = U_0 \prod_{j=1}^{n}(1+I_j) + \sum_{j=1}^{n}\left[\left(\alpha_j W_j + c(b_j) - h(b_j, Y_j)\right)\prod_{k=j+1}^{n}(1+I_k)\right] \qquad ; n = 1, 2, 3, \ldots \tag{7}$$

Remark 1: In the case where the value of *k* is greater than *n*, I_k does not exist in $\prod_{k=j+1}^{n}(1+I_k)$, thus we assume $I_k = 0$, i.e. $\prod_{k=j+1}^{n}(1+I_k) = \prod_{k=j+1}^{n}(1) = 1$.

From the definition of ruin probability in the article of Cai and Dickson (2004) and Jasiulewicz and Kordecki (2015), and given the initial values $U_0 = u$ and $I_0 = i_s$, the ruin probability for the insurance risk model can be written as follows.

The ruin probability for finite time is given by

$$\begin{aligned} \psi_n(u, i_s) &= \Pr\left\{\bigcup_{k=1}^{n}(U_k < 0) \mid U_0 = u, I_0 = i_s\right\} \\ &= \Pr\left\{U_k < 0 \text{ for some } 1 \le k \le n \mid U_0 = u, I_0 = i_s\right\}, \end{aligned} \tag{8}$$

and the ultimate ruin probability is also given by

$$\begin{aligned} \psi(u, i_s) &= \Pr\left\{\bigcup_{k=1}^{\infty}(U_k < 0) \mid U_0 = u, I_0 = i_s\right\} \\ &= \Pr\left\{U_k < 0 \text{ for some } k \ge 1 \mid U_0 = u, I_0 = i_s\right\}. \end{aligned} \tag{9}$$

Consider that the ruin probabilities are the cumulative probability from Equations (8) and (9), then

$$\psi_1(u, i_s) \le \psi_2(u, i_s) \le \psi_3(u, i_s) \le \ldots \tag{10}$$

and

$$\lim_{n\to\infty}\psi_n(u,i_s)=\psi(u,i_s) \tag{11}$$

Remark 2:

1) When the insurance risk model in Equation (6) is neither reinsurance ($b_n=1$) nor investment ($I_n=0$ and $\alpha_n=0$), then the model is reduced to the classical discrete time risk model

$$U_n=U_{n-1}+c-Y_n. \tag{12}$$

Subsequently, the famous Lundberg inequality for the ultimate ruin probability, $\psi(u)$, for Equation (12) states that if $E(Y_n)<c$ and the constant (Lundberg coefficient) $R_0>0$ exists such that

$$E\left[e^{R_0(Y_n-c)}\right]=1, \tag{13}$$

then

$$\psi(u)\le e^{-R_0u} \quad ;\ U_0=u>0, \tag{14}$$

2) If we omit the investment factor ($I_n=0$ and $\alpha_n=0$) from the insurance risk model in Equation (6), the model is reduced to

$$U_n=U_{n-1}+c(b_n)-h(b_n,Y_n). \tag{15}$$

Again, the ultimate ruin probability, $\psi(u)$, for Equation (15) with constant values of reinsurance in each time period, i.e. $b_n=b$ for all $n=1,2,3,\ldots$ and if $E[bY_n]<c(b)$, in which the Lundberg constant $R_0>0$ exists, then

$$E\left[e^{R_0(bY_n-c(b))}\right]=1 \tag{16}$$

becomes

$$\psi(u)\le e^{-R_0u} \quad ;\ U_0=u>0 \tag{17}$$

(see Diasparra and Romera, 2009, p. 102).

3. Recursive and Integral Equations form for Ruin Probability

The recursive form of ruin probability for finite and ultimate time under discrete time risk model for insurance as in Equation (6), in which ruin probabilities are defined as in Equations (8) and (9), are derived as follows.

Lemma 1: The recursive form of ruin probability for finite time and the integral equation of the ultimate ruin probability under the discrete time insurance risk model as in Equation (6) are given as

$$\psi_1(u,i_s)=\sum_{t=0}^{d}p_{st}\int_0^{\infty}\bar{G}(\pi)\mathrm{dF}(\mathrm{w}), \tag{18}$$

where $\bar{G}(\pi)=1-G(\pi)=\Pr\left(h(b_1,Y_1)\geq\pi\right)$ and $\pi=u(1+i_t)+\alpha_1 w+c(b_1)$,

$$\psi_{n+1}(u,i_s)=\sum_{t=0}^{d}p_{st}\int_0^{\infty}\int_0^{\pi}\psi_n\left(u(1+i_t)+\alpha_1 w-z(y_b),i_t\right)dG(y_b)\,dF(w)$$
$$+\sum_{t=0}^{d}p_{st}\int_0^{\infty}\bar{G}(\pi)\mathrm{dF}(\mathrm{w}), \tag{19}$$

and

$$\psi(u,i_s)=\sum_{t=0}^{d}p_{st}\int_0^{\infty}\int_0^{\pi}\psi\left(u(1+i_t)+\alpha_1 w-z(y_b),i_t\right)dG(y_b)\,dF(w)$$
$$+\sum_{t=0}^{d}p_{st}\int_0^{\infty}\bar{G}(\pi)\,dF(w). \tag{20}$$

Proof:

Let $Z_n=z\left[h(b_n,Y_n)\right]=h(b_n,Y_n)-c(b_n)$, $n=1,2,3,\ldots$, and suppose that $I_1=i_t$, $t\in\{0,1,2,\ldots,d\}$, $W_1=w, w\geq 0$, and $h(b_1,Y_1)=y_b, y_b\geq 0$. Thus, $Z_1=z(y_b)=h(b_1,Y_1)-c(b_1)$.

Consider that

$$\begin{aligned}U_1&=U_0(1+I_1)+\alpha_1 W_1+c(b_1)-h(b_1,Y_1)\\&=u(1+i_t)+\alpha_1 w-z(y_b)\\&=h-z(y_b),\end{aligned} \tag{21}$$

where $h=u(1+i_t)+\alpha_1 w$. Thus, if $z(y_b)>h$, then

$$\Pr\left\{U_1<0\,|\,W_1=w,h(b_1,Y_1)=y_b,I_1=i_t,I_0=i_s,U_0=u\right\}=1,$$

implying that for $z(y_b)>h$,

$$\Pr\left\{\bigcup_{k=1}^{n+1}(U_k<0)\,|\,W_1=w,h(b_1,Y_1)=y_b,I_1=i_t,I_0=i_s,U_0=u\right\}=1. \tag{22}$$

However, if $0\leq z(y_b)\leq h$, then

$$\Pr\left\{U_1<0\,|\,W_1=w,h(b_1,Y_1)=y_b,I_1=i_t,I_0=i_s,U_0=u\right\}=0,$$

implying that for $0\leq z(y_b)\leq h$,

$$
\begin{aligned}
&\Pr\left\{\bigcup_{k=1}^{n+1}(U_k<0)\,|\,W_1=w,h(b_1,Y_1)=y_b,I_1=i_t,I_0=i_s,U_0=u\right\}\\
&=\Pr\left\{\bigcup_{k=2}^{n+1}(U_k<0)\,|\,W_1=w,h(b_1,Y_1)=y_b,I_1=i_t,I_0=i_s,U_0=u\right\}\\
&=\Pr\left\{\begin{array}{l}\bigcup_{k=2}^{n+1}\left\{\left[\left(h-z(y_b)\right)\prod_{j=1}^{k}(1+I_j)+\sum_{j=1}^{k}\left(\left(\alpha_j W_j-Z_j\right)\prod_{m=j+1}^{k}(1+I_m)\right)\right]<0\right\}\\ |\,U_1=h-z(y_b),I_1=\ i_t\end{array}\right\}\\
&=\Pr\left\{\begin{array}{l}\bigcup_{r=1}^{n}\left\{\left[\left(h-z(y_b)\right)\prod_{j=1}^{r}(1+I_j)+\sum_{j=1}^{r}\left(\left(\alpha_j W_j-Z_j\right)\prod_{m=j+1}^{r}(1+I_m)\right)\right]<0\right\}\\ |\,U_0=h-z(y_b),I_0=\ i_t\end{array}\right\}\\
&=\psi_n\left(h-z(y_b),i_t\right)\\
&=\psi_n\left(u(1+i_t)+\alpha_1 w-z(y_b),i_t\right). \qquad (23)
\end{aligned}
$$

Consider $\psi_{n+1}(u,i_s)$ from Equation (8) is as follows:

$$
\psi_{n+1}(u,i_s)=\Pr\left\{\bigcup_{k=1}^{n+1}(U_k<0)\,|\,U_0=u,I_0=i_s\right\},
$$

we can therefore rewrite $\psi_{n+1}(u,i_s)$ as

$$
\begin{aligned}
\psi_{n+1}(u,i_s)&=\sum_{t=0}^{d}p_{st}\int_0^\infty\int_0^\infty\Pr\left\{\begin{array}{l}\bigcup_{k=1}^{n+1}(U_k<0)\,|\,U_0=u,I_0=i_s,I_1=i_t,\\ h(b_1,Y_1)=y_b,W_1=w\end{array}\right\}dG(y_b)\,dF(w)\\
&=\sum_{t=0}^{d}p_{st}\int_0^\infty\int_0^\infty\Pr\left\{\bigcup_{k=1}^{n+1}(U_k<0)\,|\,\beta\right\}dG(y_b)\,dF(w), \qquad (24)
\end{aligned}
$$

where $\beta=\{U_0=u,I_0=i_s,I_1=i_t,h(b_1,Y_1)=y_b,W_1=w\}$

From Equation (21), consider that ruin will occur in the first period if $z(y_b)>h$ or $h(b_1,Y_1)>u(1+i_t)+\alpha_1 w+c(b_1)$ and will occur in another period if $h(b_1,Y_1)\le u(1+i_t)+\alpha_1 w+c(b_1)$. Since $h(b_1,Y_1)=y_b$ is defined at the beginning of the proof, we now define $u(1+i_t)+\alpha_1 w+c(b_1)=\pi$ for the short term. In order to use Equations (22) and (23) to derive the recursive form, we need to rewrite $\psi_{n+1}(u,i_s)$ in Equation (24) as

$$\psi_{n+1}(u,i_s)=\sum_{t=0}^{d}p_{st}\int_0^{\infty}\left\{\begin{array}{l}\int_0^{\pi}Pr\left[\bigcup_{k=1}^{n+1}(U_k<0)\,|\,\beta\right]dG(y_b)\\+\int_{\pi}^{\infty}Pr\left[\bigcup_{k=1}^{n+1}(U_k<0)\,|\,\beta\right]dG(y_b)\end{array}\right\}dF(w)$$

$$=\sum_{t=0}^{d}p_{st}\int_0^{\infty}\int_0^{\pi}\psi_n\left(u(1+i_t)+\alpha_1 w-z(y_b),i_t\right)dG(y_b)\,dF(w)$$

$$+\sum_{t=0}^{d}p_{st}\int_0^{\infty}\int_{\pi}^{\infty}dG(y_b)\,dF(w).$$

Since $\bar{G}(\pi)=Pr\left(h(b_1,Y_1)\geq\pi\right)=\int_{\pi}^{\infty}dG(y_b)$, then we can rewrite $\psi_{n+1}(u,i_s)$ as

$$\psi_{n+1}(u,i_s)=\sum_{t=0}^{d}p_{st}\int_0^{\infty}\int_0^{\pi}\psi_n\left(u(1+i_t)+\alpha_1 w-z(y_b),i_t\right)dG(y_b)\,dF(w)$$

$$+\sum_{t=0}^{d}p_{st}\int_0^{\infty}\bar{G}(\pi)\,dF(w). \qquad (25)$$

By using Equation (11) and the Lebesgue dominated convergence theorem, the result of taking $n\to\infty$ in Equation (25) becomes

$$\psi(u,i_s)=\lim_{n\to\infty}\psi_{n+1}(u,i_s)$$

$$=\sum_{t=0}^{d}p_{st}\int_0^{\infty}\int_0^{\pi}\psi\left(u(1+i_t)+\alpha_1 w-z(y_b),i_t\right)dG(y_b)\,dF(w)$$

$$+\sum_{t=0}^{d}p_{st}\int_0^{\infty}\bar{G}(\pi)\,dF(w).$$

Furthermore, following on from Equation (18), we obtain

$$\psi_1(u,i_s)=\Pr\{U_1<0\,|\,U_0=u,I_0=i_s\}$$

$$=\Pr\{h-z(y_b)<0\,|\,U_0=u,I_0=i_s\}$$

$$=\Pr\{h(b_1,Y_1)>u(1+i_t)+\alpha_1 w+c(b_1)\,|\,U_0=u,I_0=i_s\}$$

$$=\sum_{t=0}^{d}p_{st}\int_0^{\infty}\Pr\{h(b_1,Y_1)>u(1+i_t)+\alpha_1 w+c(b_1)$$

$$|\,U_0=u,I_0=i_s,I_1=i_t,W_1=w\}\,dF(w)$$

$$=\sum_{t=0}^{d}p_{st}\int_0^{\infty}\bar{G}(\pi)\,dF(w).$$

4. Upper bounds for ruin probability

In this section, two upper bounds for ruin probability are derived. The first is the case when the Lundberg coefficient exists, and the second is based on the NWU function.

4.1 The upper bound for ruin probability when the Lundberg coefficient exists

Theorem 1: If the Lundberg coefficient, $R_0 > 0$ satisfies (16), suppose that the reinsurance and investment in stock in each time period are controlled to be constant values, i.e. $b_n = b$ and $\alpha_n = \alpha$, for $n = 1,2,3,...$, then the upper bound of ruin probability for finite time and the ultimate ruin probability from Lemma1 is

$$\psi_{n+1}(u,i_s) \le \psi(u,i_s) \le \beta_0 E\left(e^{-R_0\left[u(1+I_1)+\alpha W_1\right]} \mid I_0 = i_s\right), \tag{26}$$

where

$$\beta_0^{-1} = \inf_{m\ge 0} \frac{\int_m^\infty e^{R_0 y_b} dG(y_b)}{e^{R_0 m}\bar{G}(m)}, \qquad \text{for all } m \ge 0. \tag{27}$$

Proof:

Let $\bar{G}(m) = 1 - G(m) = \Pr\left[h(b,Y_1) > m\right]$, for all $m \ge 0$, then the $\bar{G}(m)$ can be rewritten as

$$\begin{aligned}
\bar{G}(m) &= \left(\frac{\int_m^\infty e^{R_0 y_b} dG(y_b)}{e^{R_0 m}\bar{G}(m)}\right)^{-1} e^{-R_0 m}\int_m^\infty e^{R_0 y_b} dG(y_b) \\
&\le \beta_0 e^{-R_0 m}\int_m^\infty e^{R_0 y_b} dG(y_b) \text{ ; where } \beta_0^{-1} = \inf_{m\ge 0} \frac{\int_m^\infty e^{R_0 y_b} dG(y_b)}{e^{R_0 m}\bar{G}(m)} \\
&\le \beta_0 e^{-R_0 m}\int_{-\infty}^\infty e^{R_0 y_b} dG(y_b) \\
&= \beta_0 e^{-R_0 m} E\left(e^{R_0 h(b,Y_1)}\right).
\end{aligned} \tag{28}$$

From (18) in Lemma 1, we have

$$\psi_1(u,i_s) = \sum_{t=0}^{d} p_{st} \int_0^\infty \bar{G}(\pi)\, dF(w)$$

where $\pi = u(1+i_t) + \alpha_1 w + c(b)$. Subsequently,

$$\begin{aligned}
\psi_1(u,i_s) &\le \sum_{t=0}^{d} p_{st} \int_0^\infty \beta_0 e^{-R_0\{u(1+i_t)+\alpha w + c(b)\}} E\left(e^{R_0 h(b,Y_1)}\right) dF(w) \qquad \text{(from Equation(28))} \\
&= \beta_0 E\left(e^{R_0 h(b,Y_1)}\right) E\left(e^{-R_0\{u(1+I_1)+\alpha W_1 + c(b)\}} \mid I_0 = i_s\right)
\end{aligned}$$

$$= \beta_0 E\left(e^{-R_0\{c(b)-h(b,Y_1)\}}\right)E\left(e^{-R_0\{u(1+I_1)+\alpha W_1\}} \mid I_0 = i_s\right)$$

$$= \beta_0 E\left(e^{-R_0\{u(1+I_1)+\alpha W_1\}} \mid I_0 = i_s\right). \qquad \text{(from Equation(18))} \qquad (29)$$

By using the inductive method, we obtain

$$\psi_n(u,i_s) \le \beta_0 E\left(e^{-R_0\{u(1+I_1)+\alpha W_1\}} \mid I_0 = i_s\right). \qquad (30)$$

Replace u and i_s by $u\left(1+i_t\right)+\alpha w - z\left(y_b\right)$ and i_t in Equation (30), and consider Equation (19) in that $u\left(1+i_t\right)+\alpha w - z\left(y_b\right) > 0$ when $u\left(1+i_t\right)+\alpha w + c\left(b\right) > h\left(b,Y_1\right)$, then

$$\psi_n\left(u(1+i_t)+\alpha w - z(y_b), i_t\right) \le \beta_0 E\left(e^{-R_0\{\left[u(1+i_t)+\alpha w - z(y_b)\right](1+I_1)+\alpha W_1\}} \mid I_0 = i_t\right)$$

$$\le \beta_0 e^{-R_0\left[u(1+i_t)+\alpha w - z(y_b)\right]}. \qquad (31)$$

From Equation (19) in Lemma 1, we obtain

$$\psi_{n+1}(u,i_s) = \sum_{t=0}^{d} p_{st}\int_0^{\infty}\int_0^{\pi}\psi_n\left(u\left(1+i_t\right)+\alpha w - z\left(y_b\right), i_t\right)dG\left(y_b\right)dF\left(w\right)$$

$$+\sum_{t=0}^{d} p_{st}\int_0^{\infty}\bar{G}\left(\pi\right)dF\left(w\right). \qquad (32)$$

And by replacing Equation (28) and, (31) in Equation (32), we can achieve

$$\psi_{n+1}(u,i_s) \le \sum_{t=0}^{d} p_{st}\int_0^{\infty}\int_0^{\pi}\beta_0 e^{-R_0\left[u(1+i_t)+\alpha w - z(y_b)\right]}dG\left(y_b\right)dF\left(w\right)$$

$$+\sum_{t=0}^{d} p_{st}\int_0^{\infty}\beta_0 e^{-R_0\left[u(1+i_t)+\alpha w + c(b)\right]}\int_{\pi}^{\infty}e^{R_0 y_b}dG\left(y_b\right)dF\left(w\right)$$

$$= \sum_{t=0}^{d} p_{st}\int_0^{\infty}\beta_0 e^{-R_0\left[u(1+i_t)+\alpha w + c(b)\right]}\int_0^{\infty}e^{R_0 y_b}dG\left(y_b\right)dF\left(w\right)$$

$$= \sum_{t=0}^{d} p_{st}\int_0^{\infty}\beta_0 e^{-R_0\left[u(1+i_t)+\alpha w + c(b)\right]}E\left(e^{R_0 h(b,Y_1)}\right)dF\left(w\right)$$

$$= \beta_0 E\left(e^{-R_0\left[c(b)-h(b,Y_1)\right]}\right)E\left(e^{-R_0\left[u(1+I_1)+\alpha W_1\right]} \mid I_0 = i_s\right)$$

$$= \beta_0 E\left(e^{-R_0\left[u(1+I_1)+\alpha W_1\right]} \mid I_0 = i_s\right) \qquad \text{(from Equation (16))}$$

$$\psi(u,i_s) = \lim_{n\to\infty}\psi_{n+1}(u,i_s)$$

$$\le \beta_0 E\left(e^{-R_0\left[u(1+I_1)+\alpha W_1\right]} \mid I_0 = i_s\right).$$

4.2 The upper bound for finite time ruin probability based on the NWU function.

The upper bound of ruin probability in Theorem 1 is derived using the condition that "the Lundberg coefficient, R_0 exists, which satisfies Equation (16)." However, this condition is not true for some distributions of Y_n claims, especially in heavy-tailed distributions such as Pareto and Weibull, among others, because R_0 cannot be found due to the fact that the moment generating function is not present in these distributions. Hence, the next theorem for the upper bound of ruin probability is derived based on another term of R_0, the NWU function. However, our derivation of this upper bound only restricts the results where Y_n is a compound distribution in order to use the outcome of Willmot (1994) to support this procedure. Therefore, the additional assumptions for the next theorem are as follows.

Let $B(x)$ be the distribution of a non-negative random variable and $\overline{B}(x)=1-B(x)$, then $B(x)$ is the NWU if $\overline{B}(x)\overline{B}(y)\le\overline{B}(x+y)$, for $x\ge 0, y\ge 0$ (Willmot, 1994)

Let $Y_n=\sum_{i=1}^{N_n}V_{ni}$; $n=1,2,3,...$, and $i=1,2,3,...,N_n$, (33)

where

V_{ni} is the ith claim amount occur during time period n (i.e. from $n-1$ to n) which is assumed to be an i.i.d. sequence with common distribution function $O(v)=\Pr(V_{ni}\le v)$, $v\ge 0$; and

N_n is the number of claims occurring during time period n, which is assumed to be an i.i.d. sequence with

$$j_{nm}=\Pr(N_n=m) \qquad ; \; m=0,1,2,... \tag{34}$$

and

$$a_{nm}=\sum_{k=m+1}^{\infty}p_{nk} \qquad ; \; m=0,1,2,... \tag{35}$$

Suppose there exist positive numbers $0<\phi_n<1$ (see Willmot and Lin (1994) for more details) such that

$$a_{n(m+1)}\le\phi_n a_{nm} \qquad ; \; m=0,1,2,..., \tag{36}$$

and since the sequence of $N_n, n=1,2,3,...$ is assumed to be i.i.d., then the values of j_{nm}, a_{nm}, and ϕ_n as defined in Equations (34) –(36) are constant for all values of n. To make this easier, we define

$$j_m=j_{nm}=\Pr(N_n=m) \qquad ; \; m=0,1,2,... \tag{37}$$

and

$$a_m = a_{nm} = \sum_{k=m+1}^{\infty} j_{nk} = \sum_{k=m+1}^{\infty} j_k \quad ; \; m = 0,1,2,... \tag{38}$$

Suppose there exist positive numbers $0 < \phi_n = \phi < 1$ such that

$$a_{m+1} \le \phi a_m \qquad ; \; m = 0,1,2,... \tag{39}$$

From the afore mentioned additional assumptions, Willmot (1994) shows us that if the non-negative, non-increasing function $B(x)$, $x \ge 0$ exists (which is NWU) such that

$$\int_0^{\infty} \left\{ \overline{B}(v) \right\}^{-1} dO(v) \le \phi^{-1} \tag{40}$$

and in addition,

$$\overline{O}(y) = \int_y^{\infty} dO(v) \le \overline{B}(y) \int_y^{\infty} \left\{ \overline{B}(v) \right\}^{-1} dO(v) \qquad ; \; y \ge 0 , \tag{41}$$

then the upper bound for $\overline{P}(y) = 1 - P(y)$, where $P(y) = Pr(Y_n \le y); \; y \ge 0$ is the common distribution of the total claims Y_n, is written as

$$\overline{P}(y) \le \phi^{-1}(1 - j_0)\overline{B}(y), \tag{42}$$

where j_0 is defined as in Equation (37).

Since the total claims $Y_n = \sum_{i=1}^{N_n} V_{ni}$; $n = 1,2,3,...$; $i = 1,2,3,...,N_n$ is assumed and the fraction of the total claims paid by insurer when the company signed the reinsurance contract is $h(b_n, Y_n) = b_n Y_n$; $n = 1,2,3,...$, then $h(b_n, Y_n)$ can be rewritten as

$$h(b_n, Y_n) = \sum_{i=1}^{N_n} b_n V_{ni} = \sum_{i=1}^{N_n} h(b_n, V_{ni}), \tag{43}$$

where $h(b_n, V_{ni})$ is an i.i.d. sequence for which the common distribution function is assumed to be $Q(v_b) = \Pr\{h(b_n, V_{ni}) \le v_b\}$, $v_b \ge 0$.

Similar to Equations (37) - (39), if the non-negative, non-increasing function $D(x), x \ge 0$ (which is NWU) exists such that

$$\int_0^{\infty} \left\{ \overline{D}(v_b) \right\}^{-1} dQ(v_b) \le \phi^{-1} \tag{44}$$

and in addition,

$$\overline{Q}(y)=\int_{y}^{\infty}dQ(v_b)\le\overline{D}(y)\int_{y}^{\infty}\left\{\overline{D}(v_b)\right\}^{-1}dQ(v_b)\qquad ;y\ge 0, \tag{45}$$

then the upper bound for $\overline{G}(y)=1-G(y)$, where $G(y)=Pr\left[h(b_n,Y_n)\le y\right];\ y\ge 0$ is the common distribution of claim $h(b_n,Y_n)$, is given by

$$\overline{G}(y)\le\phi^{-1}(1-j_0)\overline{D}(y). \tag{46}$$

The next theorem is derived from the above information.

Theorem 2: Let the total claims Y_n, $n=1,2,3,...$ and the quantity $0<\phi_n=\phi<1$ satisfy Equations (33) and (39); suppose that the reinsurance and investment in stock in each time period are controlled to be constant values, i.e. $b_n=b$ and $\alpha_n=\alpha$, for $n=1,2,3,...$; and suppose that the non-negative, non-increasing function $\overline{D}(x)=1-D(x)$ for $x\ge 0$ exists, in which $D(x)$ is NWU, $\overline{D}(0)=1$, and $\overline{D}(x)$ satisfies Equations (44) and (45), then the upper bound for the finite time ruin probability in Equation (19) can be written as

$$\psi_{n+1}(u,i_s)\le\phi^{-1}(1-j_0)\left[E\left\{\overline{D}(y_b)\right\}^{-1}\right]^{n}E\left[\overline{D}(\pi)\,|\,I_0=i_s\right], \tag{47}$$

where (as before) $\pi=u(1+i_t)+\alpha w+c(b)$.

Proof

From Equation (18) in Lemma 1, we obtain

$$\psi_1(u,i_s)=\sum_{t=0}^{d}p_{st}\int_{0}^{\infty}\overline{G}(\pi)dF(w),$$

where $\pi=u(1+i_t)+\alpha w+c(b)$.

We can rewrite $\psi_1(u,i_s)$ using Equation (46) as

$$\psi_1(u,i_s)\le\sum_{t=0}^{d}p_{st}\int_{0}^{\infty}\left[\phi^{-1}(1-j_0)\overline{D}(\pi)\right]dF(w)$$

$$=\phi^{-1}(1-j_0)\sum_{t=0}^{d}p_{st}\int_{0}^{\infty}\left[\overline{D}(\pi)\right]dF(w)$$

$$=\phi^{-1}(1-j_0)\sum_{t=0}^{d}p_{st}\int_{0}^{\infty}\left[\overline{D}\left\{u(1+i_t)+\alpha_1 w+c(b)\right\}\right]dF(w)$$

$$=\phi^{-1}(1-j_0)E\left[\overline{D}\left\{u(1+i_t)+\alpha w+c(b)\right\}|\,I_0=i_s\right]$$

$$= \phi^{-1}(1-j_0)E\left[\overline{D}(\pi) \mid I_0 = i_s\right]$$

$$= \phi^{-1}(1-j_0)\left[E\left\{\overline{D}(y_b)\right\}^{-1}\right]^{(0)} E\left[\overline{D}(\pi) \mid I_0 = i_s\right].$$

By using the inductive method, we arrive at

$$\psi_n(u,i_s) \le \phi^{-1}(1-j_0)\left[E\left\{\overline{D}(y_b)\right\}^{-1}\right]^{(n-1)} E\left[\overline{D}(\pi) \mid I_0 = i_s\right]$$

$$= \phi^{-1}(1-j_0)\left[E\left\{\overline{D}(y_b)\right\}^{-1}\right]^{(n-1)} E\left[\overline{D}\left\{u(1+i_t)+\alpha w + c(b)\right\} \mid I_0 = i_s\right] \quad (48)$$

If we replace u and i_s by $\pi - y_b$ and i_t, we can rewrite $\psi_n(u,i_s)$ in Equation (48) as

$$\psi_n(\pi - y_b, i_t) \le \phi^{-1}(1-j_0)\left[E\left\{\overline{D}(y_b)\right\}^{-1}\right]^{(n-1)} E\left[\overline{D}\left\{(\pi - y_b)(1+i_t)+\alpha w + c(b)\right\} \mid I_0 = i_t\right]$$

$$\le \phi^{-1}(1-j_0)\left[E\left\{\overline{D}(y_b)\right\}^{-1}\right]^{(n-1)} E\left[\overline{D}(\pi - y_b) \mid I_0 = i_t\right]$$

$$= \phi^{-1}(1-j_0)\left[E\left\{\overline{D}(y_b)\right\}^{-1}\right]^{(n-1)} \overline{D}(\pi - y_b). \quad (49)$$

From Equation (19) in Lemma 1, we obtain

$$\psi_{n+1}(u,i_s) = \sum_{t=0}^{d} p_{st} \int_0^{\infty} \left\{ \int_0^{\pi} \psi_n\left(u(1+i_t)+\alpha w - z(y_b), i_t\right) dG(y_b) + \overline{G}(\pi) \right\} dF(w) \quad (50)$$

By considering $u(1+i_t)+\alpha w - z(y_b)$ in Equation (49) as

$$u(1+i_t)+\alpha w - z(y_b) = u(1+i_t)+\alpha w + c(b) - h(b,Y_1)$$

$$= \pi - y_b, \quad (51)$$

And by replacing Equation (51) in Equation (50), we can achieve

$$\psi_{n+1}(u,i_s)=\sum_{t=0}^{d}p_{st}\int_0^{\infty}\left\{\int_0^{\pi}\psi_n(\pi-y_b,i_t)\,dG(y_b)+\bar{G}(\pi)\right\}dF(w)$$

$$\leq\sum_{t=0}^{d}p_{st}\int_0^{\infty}\left\{\begin{array}{l}\int_0^{\pi}\left\{\phi^{-1}(1-j_0)\left[E\left\{\overline{D}(y_b)\right\}^{-1}\right]^{(n-1)}\overline{D}(\pi-y_b)\right\}dG(y_b)\\+\phi^{-1}(1-j_0)\overline{D}(\pi)\end{array}\right\}dF(w)$$

$$\leq\sum_{t=0}^{d}p_{st}\int_0^{\infty}\left\{\begin{array}{l}\int_0^{\pi}\left\{\phi^{-1}(1-j_0)\left[E\left\{\overline{D}(y_b)\right\}^{-1}\right]^{(n-1)}\overline{D}(\pi-y_b)\right\}dG(y_b)\\+\phi^{-1}(1-j_0)\overline{D}(\pi-y_b)\end{array}\right\}dF(w)$$

$$\leq\sum_{t=0}^{d}p_{st}\int_0^{\infty}\left\{\begin{array}{l}\int_0^{\pi}\left\{\phi^{-1}(1-j_0)\left[E\left\{\overline{D}(y_b)\right\}^{-1}\right]^{(n-1)}\overline{D}(\pi-y_b)\right\}dG(y_b)\\+\phi^{-1}(1-j_0)\int_{\pi}^{\infty}\overline{D}(\pi-y_b)\,dG(y_b)\end{array}\right\}dF(w)$$

$$\leq\sum_{t=0}^{d}p_{st}\int_0^{\infty}\left\{\begin{array}{l}\phi^{-1}(1-j_0)\left[E\left\{\overline{D}(y_b)\right\}^{-1}\right]^{(n-1)}\overline{D}(\pi)\int_0^{\pi}\left\{\overline{D}(y_b)\right\}^{-1}dG(y_b)\\+\ \phi^{-1}(1-j_0)\overline{D}(\pi)\int_{\pi}^{\infty}\left\{\overline{D}(y_b)\right\}^{-1}dG(y_b)\end{array}\right\}dF(w)$$

$$\leq\sum_{t=0}^{d}p_{st}\int_0^{\infty}\left[\phi^{-1}(1-j_0)\left[E\left\{\overline{D}(y_b)\right\}^{-1}\right]^{(n-1)}\overline{D}(\pi)\int_0^{\infty}\left\{\overline{D}(y_b)\right\}^{-1}dG(y_b)\right]dF(w)$$

$$=\sum_{t=0}^{d}p_{st}\int_0^{\infty}\left\{\phi^{-1}(1-j_0)\left[E\left\{\overline{D}(y_b)\right\}^{-1}\right]^{n}\overline{D}(\pi)\right\}dF(w)$$

$$=\phi^{-1}(1-j_0)\left[E\left\{\overline{D}(y_b)\right\}^{-1}\right]^{n}\sum_{t=0}^{d}p_{st}\int_0^{\infty}\left\{\overline{D}(\pi)\right\}dF(w)$$

$$=\phi^{-1}(1-j_0)\left[E\left\{\overline{D}(y_b)\right\}^{-1}\right]^{n}E\left[\overline{D}(\pi)\,|\,I_0=i_s\right].$$

5. Numerical example

The upper bound of ruin probability in Theorems 1 and 2 are illustrated in Examples 1 and 2, respectively, using R programming.

Table 1: The upper bound of ruin probability at each initial surplus level U_0, reinsurance contract retention level b, stock investment value α, and initial bond interest rate I_0

U_0	b	α	The Proposed Upper Bound $I_0 = 0.02$	The Proposed Upper Bound $I_0 = 0.05$	Lundberg's Upper bound
50	0.2	2.5	0.075979	0.074323	0.376198
		5	0.061892	0.060542	
		10	0.042261	0.041339	
	0.6	2.5	0.168041	0.165655	
		5	0.146599	0.144518	
		10	0.113097	0.111491	
	1.0	2.5	0.281212	0.278484	
		5	0.255835	0.253353	
		10	0.213189	0.211121	
100	0.2	5	0.006351	0.006079	0.141525
		10	0.004337	0.004151	
		15	0.003078	0.003024	
	0.6	5	0.033602	0.032659	
		10	0.025923	0.025196	
		15	0.020308	0.019738	
	1.0	5	0.093840	0.092034	
		10	0.078197	0.076692	
		15	0.065688	0.064424	
500	0.2	50	0.000000	0.000000	0.000057
		100	0.000000	0.000000	
		150	0.000000	0.000000	
	0.6	50	0.000000	0.000000	
		100	0.000000	0.000000	
		150	0.000000	0.000000	
	1.0	50	0.000007	0.000007	
		100	0.000002	0.000002	
		150	0.000001	0.000001	

Example 1: Suppose that total claims $Y_n \sim \exp\left(\frac{1}{9}\right)$ in time periods $n = 1, 2, 3, \ldots$, and that the insurance company has the chance to invest its surplus in both the bond and stock markets. The bond interest rates during time periods $n = 1, 2, 3, \ldots$ are $I_n \in \{0.02, 0.03, 0.05\}$, respectively, which are forced to follow a time-homogeneous Markov chain with transition probability matrix

$$\begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}.$$

Gross stock return (W_n) is assumed to be $W_n = e^{\left(\mu - \frac{\sigma^2}{2}\right) + \sigma B_n}$ with $B_n \sim B_n \sim N(0,1)$, $\mu = 0.7$, and $\sigma = 0.5$ for time periods $n = 1, 2, 3, \ldots$. The safety loading factors given by the insurer and the reinsurer are 10% and 12%, respectively.

The results from Table 1 show that the upper bound's value decreased when either initial surplus U_0 or the investment value in the stock α increased whereas it increased when the reinsurance contract retention level b increased. In addition, the results show that the upper bound's value from Theorem 1 was sharper than the Lundberg upper bound.

Example 2: Here, it is assumed that the total claims amount Y_n; $n = 1,2,3,...$ is a summation of i.i.d. claim amounts V_{ni}, $V_{ni} \sim pareto(1.5, 0.5)$; $i = 1, 2, 3, ..., N_n$. The number of claims N_n during time period n is an i.i.d. Poisson distribution with mean $\lambda = 3$. It is also assumed that the bond interest rate during time periods $n = 1,2,3,...$ are $I_n \in \{0.02, 0.03, 0.05\}$ with initial value $I_0 = 0.02$. Furthermore, they follow a time-homogeneous Markov chain with transition probability matrix

$$\begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}.$$

Gross stock return (W_n) is assumed to be W_n = $W_n = e^{\left(\mu - \frac{\sigma^2}{2}\right) + \sigma B_n}$ with $B_n \sim N(0,1)$, $\mu = 0.7$, and $\sigma = 0.5$ for time periods $n = 1,2,3,\ldots$. The safety loading factors given by the insurer and the reinsurer are 10% and 12%, respectively. There exists $\overline{D}(x) = (1 + kx)^{-1}$ which is NWU. Thus, the upper bound of ruin probability in Theorem 2 is appropriate to this case.

The results from Table 2 show that the upper bound values for finite time ruin probability increased as the number of time periods (n) increased corresponding with Equation (12) that values for finite time ruin probability do not decrease as n increases. The effects of variations of the other factors, i.e. the initial surplus (U_0), the stock investment value (α), and the reinsurance contract retention level (b) on the upper bound values of finite time ruin probability were the same as Experiment 1.

Table 2: The upper bounds of finite time ruin probability (time period $n=$ 1,2,3,4, and 5) at each initial surplus level U_0, reinsurance contract retention level b, and stock investment value α

U_0	b	α	Upper Bound $n=1$	$n=2$	$n=3$	$n=4$	$n=5$
50	0.2	2.5	0.048653	0.075875	0.118328	0.184534	0.287784
		5	0.044982	0.070150	0.109399	0.170610	0.266068
		10	0.039266	0.061236	0.095498	0.148930	0.232258
	0.6	2.5	0.136791	0.213327	0.332687	0.518830	0.809121
		5	0.127043	0.198126	0.308980	0.481858	0.751464
		10	0.111676	0.174160	0.271605	0.423572	0.660566
	1	2.5	0.214514	0.334537	0.521714	0.813619	1.000000
		5	0.200033	0.311953	0.486495	0.758695	1.000000
		10	0.176944	0.275946	0.430341	0.671121	1.000000
100	0.2	5	0.024735	0.038574	0.060157	0.093815	0.146306
		10	0.022841	0.035620	0.055551	0.086632	0.135103
		15	0.021256	0.033148	0.051695	0.080619	0.125727
	0.6	5	0.071759	0.111909	0.174523	0.272171	0.424454
		10	0.066424	0.103589	0.161548	0.251936	0.392897
		15	0.061937	0.096591	0.150635	0.234918	0.366357
	1	5	0.115783	0.180564	0.281592	0.439147	0.684855
		10	0.107418	0.167519	0.261248	0.407419	0.635375
		15	0.100350	0.156497	0.244059	0.380613	0.593571
500	0.2	50	0.004626	0.007214	0.011250	0.017545	0.027361
		100	0.004024	0.006276	0.009788	0.015264	0.023805
		150	0.003575	0.005575	0.008694	0.013558	0.021144
	0.6	50	0.013789	0.021504	0.033536	0.052299	0.081562
		100	0.012006	0.018723	0.029199	0.045536	0.071014
		150	0.010670	0.016640	0.025951	0.040470	0.063114
	1	50	0.022836	0.035613	0.055540	0.086615	0.135077
		100	0.019898	0.031032	0.048394	0.075472	0.117699
		150	0.017695	0.027595	0.043035	0.067113	0.104664

Remark 3: The term $\left[E\left\{\overline{D}\left(y_b\right)\right\}^{-1}\right]^n$ at the right-hand side of Equation (49) in theorem 2 affects the variation of the upper bound values for finite time ruin probability in this theorem as the time period n increases, the upper bound values increase. The values of the upper bound from Theorem 2 depend on not only the change of factors in the risk model mentioned in Example 2, but also the NWU function selected. Based on running the results from data in Example 2 and resembling data, we found that the upper bound from Theorem 2 is appropriate when the initial surplus u is sufficient large and the time period n should not be set too many. In other cases, overrated values of upper bound might lead to a misunderstanding of the risk level that makes insurers more nervous about the risk than is really necessary.

6. Real-life example

Data from 334 real-life motor insurance claims for a broker with three branches in the year 2012 were used to analyze the upper bound for ruin probability. The real-life claims dataset were fitted to a lognormal distribution with maximum likelihood estimation of log data parameter $\hat{\mu} = 2.467120$ (thousand baht) and $\hat{\sigma} = 1.039443$ (thousand baht). The moment generating function of the lognormal distribution was infinite at any positive number, thus the upper bound of ruin probability in Theorem 2 was appropriate in this situation. We used the dataset in December to find the upper bound of ruin probability for the next 5 months. This dataset fit to lognormal distribution with maximum likelihood estimation of log data parameter $\hat{\mu} = 2.385506$ (thousand baht) and $\hat{\sigma} = 1.089433$ (thousand baht). The number of claims occurring in each month was assumed to be i.i.d. with a Poisson distribution for which the mean was estimated as the average value of 12 months of real-life claims data (the result was 27.75). The other factors for finding the upper bound of ruin probability for this broker were an initial surplus of 5 million baht and an initial bond interest rate at 0.03 (based on Example 1), and the NWU function used was $\overline{D}(x) = (1+kx)^{-1}$, $x \geq 0$. Assuming the dealer kept 0.6 of the reinsurance retention level and invested 500,000 baht on the stock market under previous assumptions, then the upper bounds of ruin probability in the next 5 months were 0.670366, 0.670384, 0.670401, 0.670419, and 0.670436 respectively. High values of the upper bound for ruin probability indicate that there is a high risk (value of ruin probability) of the company going bust under these conditions. The amount and frequency of claims with respect to the initial capital seem to be the main cause of this situation.

7. Conclusions

In this study, we propose two upper bounds of ruin probability under a discrete time risk model for reinsurance by generalizing the classic model for two controlling factors: proportional reinsurance and investment. The insurer can invest in the bond and stock markets, and we assume that the interest rates of the bond have a finite number of possible values and follow a time-homogenous Markov chain. Moreover, we assume that the controlling reinsurance and stock investment values in each time period are constant values.

The ruin probability for finite time is presented in a recursive form while the ultimate ruin probability is given as integral equations. The first upper bound for finite time and ultimate ruin probability is derived under the condition that the Lundberg coefficient exists. This upper bound can be view as an extension of the ideas of Diasparra and Romera (2009) and Jasiulewicz and Kordecki (2015). The second upper bound for finite time ruin probability is developed from the idea of Willmot (1994) in terms of NWU. Numerical examples show the results for the two proposed upper bounds. In the first example, the total claims amount in each time period were assumed to follow an exponential distribution so that the Lundberg coefficient can be found in this case, thus Theorem 1 was applied. In the second example, the claims amount in each time period was assumed to be an i.i.d. Pareto distribution, under which circumstances the Lundberg coefficient does not exist, thus, Theorem 2 was applied in this case.

References

1. Cai, J. (2002). Discrete time risk models under rates of interest. Prob. Eng. Inf. Sci., 16, 309-324.
2. Cai, J. and Dickson, D.C.M. (2004). Ruin probabilities with a Markov chain interest model. Insur. Math. Econ., 35, 513-525.
3. Cai, J. and Garrido, J. (1999). A unified approach to the study of tail probabilities of compound distributions. J. Appl. Prob., 36, 1058-1073.
4. Cai, J. and Wu, Y. (1997). Some improvements on the Lundberg bound for the ruin probability. Stat. Probabil. Lett., 33, 395-403.
5. Diasparra, M.A. and Romera, R. (2009). Bounds for the ruin probability of a discrete-time risk process. J. Appl. Prob., 46, 99-112.
6. Dickson, D.C.M. (1994). An upper bound for the probability of ultimate ruin. Scand. Actuar. J., 2, 131-138.
7. Jasiulewicz, H. and Kordecki, W. (2015). Ruin probability of a discrete-time risk process with proportional reinsurance and investment for exponential and Pareto distribution. Oper. Res. Decisions, 25(3), 17-38.
8. Kalashnikov, V. (1999). Bounds for ruin probabilities in the presence of large claims and their comparison. N. Am. Actuar. J, 3(2), 116-128.
9. Lin, X, Dongjin, Z., and Yanru, Z. (2015). Minimizing upper bound of ruin probability under discrete risk model with Markov chain interest rate. Comm. Stat. Theory Methods, 44, 810-822.
10. Liu, C.S. and Yang, H. (2013). Optimal investment for an insurer to minimize its probability of ruin. N. Am. Actuar. J., 8(2), 11-31.
11. Mennis, E.A. and Ryals, S.D. (1992). Insurance company investment management handbook. New York Institute of Finance, New York.
12. Outreville, J.F. (1998). Theory and practice of insurance. Springer, New York.
13. Willmot, G.E. (1994). Refinements and distributional generalizations of Lundberg's inequality. Insur. Math. Econ, 15, 49-63.
14. Willmot, G.E. and Lin, X. (1994). Lundberg bounds on the tails of compound distributions. J. Appl. Prob., 31, 743-756.
15. Willmot, G.E. and Lin, X. (1996). Bounds on the tails of convolutions of compound distributions. Insur. Math. Econ., 18, 29-33.