Static Mean-Variance Portfolio Optimization under General Sources of Uncertainty

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Abstract

The only source of uncertainty in the standard Markowitz's static Mean-Variance portfolio selection model is the future price of assets. This paper studies the static Mean-Variance portfolio selection model under general sources of uncertainty which generalizes the Markowitz's model. It is shown that how the generalized problem can be reformulated as a quadratic program. Sufficient conditions are provided under which the standard and the generalized models produce the same set of optimal portfolios. Some sources of uncertainty and relevant examples are investigated. An illustrative example is provided to demonstrate the model.

Keywords: Static Mean-Variance portfolio selection, Optimal portfolio, Uncertain exit-time, Uncertain quantity.

1. Introduction

Harry Markowitz (1952,1959) laid the foundation of the modern portfolio theory in the 1950's by introducing his static Mean-Variance (M-V) portfolio selection model. This model suggests portfolio diversification to reduce the risk of the investment which is measured by the variance of the portfolio return, and tries to compute *efficient* portfolios. A portfolio is efficient if, in compare to it, there is no portfolio with the same risk and a higher expected return, or there is no portfolio with the same expected return and a lower risk. Analytical expression of the efficient portfolios with and without no-shorting constraint was derived by Markowitz (1952,1959) and Merton (1972), respectively.

The only source of uncertainty in a static M-V portfolio selection model is the asset returns. However, in the real-world, there are other sources of uncertainty, such as exit-time, recovery rate and cash flows which are considered by other researchers. Considering the time of death as the uncertain exit-time, Yari (1965) investigated the optimal consumption problem in a market with one riskless asset only. The Yari's model (1965) was generalized by Hakansson (1969,1971) to the case with multi-period setting with risky assets and an uncertain lifetime. Merton (1971) considered an investment-consumption problem when the exit-time is the first jump time of a Poisson process. Liu and Loewenstein (2002) studied a portfolio selection model with uncertain exit-time and transaction cost. Martellini and Urosevic (2006) generalized the standard static M-V model of Markowitz to the case with uncertain exit-time. Keykhaei (2016) extended this model to the case where each asset has individual uncertain exit-time. The multi-period

M-V portfolio optimization problem with uncertain exit-time is considered by Guo and Hu (2005). Zhang and Li (2012) studied a multi-period M-V portfolio selection model with uncertain exit-time where the asset returns are serially correlated. Wu and Li (2011) investigated a multi-period M-V model in regime switching markets when exit-time is uncertain. Wu et al. (2014) considered a multi-period M-V model with state-dependent exit probability and regime switching. Using mean-field formulation, Yi et al. (2014) studied a multi-period M-V model with an uncertain exit-time. Yi et al. (2008) considered an asset-liability management model when the investment horizon is uncertain. Blanchet-Scalliet et al. (2008) studied an optimal investment problem when the uncertain time-horizon depends on the asset returns. Lv et al. (2016) studied a M-V portfolio optimization problem where market parameters and exit-time are random and market is incomplete.

Some researchers have considered the notions of bankruptcy and recovery rate in their models. For instance, Cheung and Yang (2007) proposed a multi-period investment-consumption portfolio selection model in a Markovian regime switching market with a bankruptcy state. Also, Wu and Zeng (2013) studied a M-V portfolio selection model in a regime-switching market with a bankruptcy state. In these models, when bankruptcy occurs, the investor only receive a random fraction, that is the recovery rate, of his/her wealth.

Moreover, some researchers have considered other uncertainties in their models. For example, Merton (1971) proposed an investment-consumption model with stochastic wage income. Wu and Li (2012) and Wu (2013) considered discrete-time and continuous-time M-V portfolio selection problems, respectively, with stochastic cash flow. Yao et al. (2013) considered a M-V asset-liability management model with an uncontrolled cash flow and an uncertain exit-time. Yao et al. (2014) investigated a M-V asset allocation model for defined contribution pension funds with mortality risk and stochastic income. Zhou et al. (2016) studied the pre-commitment and time-consistent investment strategies in the M-V framework with stochastic cash flows. Yao et al. (2016) proposed a M-V asset-liability management model with stochastic cash flows for wealth and liability. Also, see Tsai and Wu (2015).

In this study we were motivated by the models proposed by Martellini and Urosevic's (2006) and Keykhaei (2016). These model considers two types of uncertainty: asset returns and exit-time. This paper studies the static M-V portfolio selection model under general sources of uncertainty, which generalizes the Markowitz's model (1952) and covers the models of Martellini and Urosevic (2006) and Keykhaei (2016). Inspired by Keykhaei and Jahandideh (2012), we investigate a third type of uncertainty which affects the quantity of each asset, and combine it with the other types. In fact, we assume that the quantity of each asset is uncertain during the investment period. It is shown how the generalized problem can be reformulated as a quadratic program. Also, some sufficient conditions are provided under which the standard and the generalized models produce the same set of optimal portfolios.

The rest of this paper proceeds as follows. In Section 2, we review some definitions, basic notations, and the standard static M-V portfolio selection problem formulation. In Section 3, we describe portfolio optimization problem under general sources of uncertainty and reformulate it into a standard quadratic program. The main results are given in this section. Two types of examples for the generalized model are given, distinctly, in Sections 4 and 5. In the last section, an illustrative example is provided.

2. Standard Model

In this section we explain some basic notions (see, for example, Chapter 1 of Korn and Korn (2001)). We consider a market consisting of $n \geq 2$ risky assets. Consider an investor who joins the market at initial time $t = 0$ with an initial wealth $W(0)$ and investment time-horizon T. At the beginning of the investment period ($t = 0$), the investor invests his/her wealth as $W(0) = \sum_{i=1}^{n} \varphi_i S_i(0)$, where $S_i(0)$ and φ_i are the value and the quantity of asset i, respectively. At the end of the investment period ($t = T$), the wealth is $W(T) = \sum_{i=1}^{n} \varphi_i S_i(T)$, where $S_i(T)$ is the value of asset i. Here, φ_i's remain fixed during the investment period.

The return of each asset is defined by $r_i = \frac{S_i(T)}{S_i(0)}$ for $i = 1,2,\dots,n$. Similarly, the return of the portfolio is defined by $Z = \frac{W(T)}{W(0)}$. Then, the portfolio return can be expressed in terms of asset returns as follows

$$Z = \frac{\sum_{i=1}^{n} \varphi_i S_i(T)}{W(0)} = \sum_{i=1}^{n} \frac{\varphi_i S_i(0)}{W(0)} \frac{S_i(T)}{S_i(0)} = \sum_{i=1}^{n} x_i r_i,$$

in which

$$x_i = \frac{\varphi_i S_i(0)}{W(0)}, \quad (i = 1,2,\dots,n)$$

is the weight allocated to asset i in the portfolio. Each portfolio is denoted by the weight vector $\mathbf{X} := (x_1,\dots,x_n)' \in R^n$. We assume that the random vector of asset returns $\mathbf{R} = (r_1,\dots,r_n)'$ has the known mean vector $\bar{\mathbf{R}} = (\bar{r}_1,\dots,\bar{r}_n)'$ and covariance matrix $\mathbf{V} = (\sigma_{ij})_{n\times n}$. Also, $\mathbf{V}$ is positive definite and $\bar{\mathbf{R}} \neq k\mathbf{1}$ for all $k \in \mathbb{R}$, where $\mathbf{1} = (1,1,\dots,1)' \in R^n$. Obviously, the mean and the variance of the portfolio return can be rewritten by $\mathbb{E}(Z) = \mathbf{X}'\bar{\mathbf{R}}$ and $\mathbb{V}ar\,(Z) = \mathbf{X}'\mathbf{V}\mathbf{X}$, respectively. The static M-V portfolio selection problem for a desired return μ^* is formulated as follows:

$$P(\mu^*): \begin{cases} \min\limits_{\mathbf{X}} & \frac{1}{2}\mathbf{X}'\mathbf{V}\mathbf{X} \\ \text{s.t.} & \mathbf{X}'\bar{\mathbf{R}} = \mu^*, \\ & \mathbf{X}'\mathbf{1} = 1. \end{cases}$$

Define

$$A := \mathbf{1}'\mathbf{V}^{-1}\mathbf{1}, B := \mathbf{1}'\mathbf{V}^{-1}\bar{\mathbf{R}}, C := \bar{\mathbf{R}}'\mathbf{V}^{-1}\bar{\mathbf{R}}.$$

Theorem 2.1 *Problem $P(\mu^*)$ has the following unique optimal solution*

$$\mathbf{X} = \frac{C - \mu^* B}{AC - B^2}\mathbf{V}^{-1}\mathbf{1} + \frac{\mu^* A - B}{AC - B^2}\mathbf{V}^{-1}\bar{\mathbf{R}}.$$

Proof. See Chapter 4 of Ingersoll (1987).

3. Portfolio Selection with Uncertain Parameters

In this section, we generalize the standard model to the case where the wealth of the investor is affected by some sources of uncertainty denoted by $\Theta = (\Theta_1, \dots, \Theta_m)' \in \mathbb{R}^m$. Under the random parameter vector Θ, the return of the investment is $Z^\Theta = \frac{W(\Theta)}{W(0)}$, where $W(\Theta)$ is the wealth of the investor at the end of the investment period. Actually, in the standard case we have $\Theta = (S_1, \dots, S_n)$, where S_i denotes the price of asset i, for $i = 1, \dots, n$. The generalized static M-V portfolio selection problem corresponding to the uncertain parameter vector Θ for a desired return μ is defined as follows:

$$P(\mu, \Theta): \begin{cases} \min_{\mathbf{X}} & \mathbb{V}ar\,(Z^\Theta) \\ \text{s.t.} & \mathbb{E}(Z^\Theta) = \mu, \\ & \mathbf{X}'\mathbf{1} = 1. \end{cases}$$

In the following, compared to the standard case, we exclude the S_i's from Θ, as the trivial sources of uncertainty, and consider Θ to denote the other sources. In this case the return of asset i is defined in terms of S_i and Θ as follows:

$$r_i^\Theta := r_i^\Theta(S_i, \Theta) \quad (i = 1,2, \dots, n).$$

The aim is to describe the return of the portfolio by a *linear* combination

$$Z^\Theta = \mathbf{X}'\mathbf{R}^\Theta, \tag{1}$$

where $\mathbf{R}^\Theta = (r_1^\Theta, \dots, r_n^\Theta)'$ is the random vector of returns with mean vector $\bar{\mathbf{R}}^\Theta = (\bar{r}_1^\Theta, \dots, \bar{r}_n^\Theta)'$ and covariance matrix $\mathbf{V}^\Theta$. Equation (1) yields that $\mathbb{E}(Z^\Theta) = \mathbf{X}'\bar{\mathbf{R}}^\Theta$ and $\mathbb{V}ar\,(Z^\Theta) = \mathbf{X}'\mathbf{V}^\Theta\mathbf{X}$. Therefore, problem $P(\mu, \Theta)$ can be reformulated as a standard quadratic program which is equivalent to the following problem:

$$\tilde{P}(\mu, \Theta): \begin{cases} \min_{\mathbf{X}} & \frac{1}{2}\mathbf{X}'\mathbf{V}^\Theta\mathbf{X} \\ \text{s.t.} & \mathbf{X}'\bar{\mathbf{R}}^\Theta = \mu, \\ & \mathbf{X}'\mathbf{1} = 1. \end{cases}$$

In the following we try to provide some conditions for which the generalized problem $\tilde{P}(\mu, \Theta)$ and the standard problem $P(\mu^*)$ produce the same set of optimal portfolios obtained from various desired expected returns. This means that the other sources of uncertainty does not affect on the set of optimal portfolios.

Let $g, h: \mathbb{R}^m \to \mathbb{R}$ are real-valued functions such that

$$\mathbb{E}(r_i^\Theta | \Theta = \theta) = \bar{r}_i g(\theta), \tag{2}$$

$$\mathbb{C}ov\,(r_i^\Theta, r_j^\Theta | \Theta = \theta) = \sigma_{ij} h(\theta). \tag{3}$$

Then, under condition (2), the expected return of asset i is

$$\begin{aligned} \mathbb{E}(r_i^\Theta) &= \int_{\mathbb{R}^m} \mathbb{E}(r_i^\Theta | \Theta = \theta) dF(\theta) \\ &= \int_{\mathbb{R}^m} \bar{r}_i g(\theta) dF(\theta) \\ &= \bar{r}_i \mathbb{E}(g(\Theta)), \end{aligned}$$

where F denotes the distribute function of Θ. Moreover,

$$
\begin{aligned}
\mathbb{C}\,ov\,(\mathbb{E}(r_i^{\Theta}|\Theta),\mathbb{E}(r_j^{\Theta}|\Theta)) &= \mathbb{E}(\mathbb{E}(r_i^{\Theta}|\Theta)\mathbb{E}(r_j^{\Theta}|\Theta)) - \mathbb{E}(\mathbb{E}(r_i^{\Theta}|\Theta))\mathbb{E}(\mathbb{E}(r_j^{\Theta}|\Theta)) \\
&= \int_{\mathbb{R}^m} \mathbb{E}(r_i^{\Theta}|\Theta=\theta)\mathbb{E}(r_j^{\Theta}|\Theta=\theta)dF(\theta) - \bar{r}_i\bar{r}_j\mathbb{E}^2(g(\Theta)) \\
&= \int_{\mathbb{R}^m} \bar{r}_i\bar{r}_j g^2(\theta)dF(\theta) - \bar{r}_i\bar{r}_j\mathbb{E}^2(g(\Theta)) \\
&= \bar{r}_i\bar{r}_j\mathbb{E}(g^2(\Theta)) - \bar{r}_i\bar{r}_j\mathbb{E}^2(g(\Theta)) \\
&= \bar{r}_i\bar{r}_j\mathbb{V}\,ar\,(g(\Theta)).
\end{aligned}
$$

Using (3) we can obtain

$$
\begin{aligned}
\mathbb{E}(\mathbb{C}\,ov\,(r_i^{\Theta},r_j^{\Theta}|\Theta)) &= \int_{\mathbb{R}^m} \mathbb{C}\,ov\,(r_i^{\Theta},r_j^{\Theta}|\Theta=\theta)dF(\theta) \\
&= \int_{\mathbb{R}^m} \sigma_{ij}h(\theta)dF(\theta) \\
&= \sigma_{ij}\mathbb{E}(h(\Theta)).
\end{aligned}
$$

Applying the equation

$$\mathbb{C}\,ov\,(r_i^{\Theta},r_j^{\Theta}) = \mathbb{E}(\mathbb{C}\,ov\,(r_i^{\Theta},r_j^{\Theta}|\Theta)) + \mathbb{C}\,ov\,(\mathbb{E}(r_i^{\Theta}|\Theta),\mathbb{E}(r_j^{\Theta}|\Theta)),$$

we can calculate the covariation between r_i^{Θ} and r_j^{Θ} as follows:

$$\mathbb{C}\,ov\,(r_i^{\Theta},r_j^{\Theta}) = \sigma_{ij}\mathbb{E}(h(\Theta)) + \bar{r}_i\bar{r}_j\mathbb{V}\,ar\,(g(\Theta)).$$

Then

$$\mathbf{V}^{\Theta} = \mathbb{E}(h(\Theta))\mathbf{V} + \mathbb{V}\,ar\,(g(\Theta))\mathbf{E}, \tag{4}$$

where $\mathbf{E} = (e_{ij})_{n\times n}$ and $e_{ij} = \bar{r}_i\bar{r}_j$, for $i,j = 1,\dots,n$. If $\mathbf{X}'\bar{\mathbf{R}} = \mu^*$ then we have

$$\mu := \mathbb{E}(Z^{\Theta}) = \mathbf{X}'\bar{\mathbf{R}}^{\Theta} = \mathbb{E}(g(\Theta))\mathbf{X}'\bar{\mathbf{R}} = \mathbb{E}(g(\Theta))\mu^*,$$

$$\sigma^2 := \mathbb{V}\,ar\,(Z^{\Theta}) = \mathbf{X}'\mathbf{V}^{\Theta}\mathbf{X} = \mathbb{E}(h(\Theta))\mathbf{X}'\mathbf{V}\mathbf{X} + \mathbb{V}\,ar\,(g(\Theta))\mathbf{X}'\mathbf{E}\mathbf{X}.$$

Lemma 3.1 *The matrix $\boldsymbol{V}^{\Theta}$ is positive definite if $\mathbb{E}(h(\Theta)) > 0$.*

Proof. Let $\mathbf{X} \in \mathbb{R}^n$ and $\mathbf{X} \neq 0$. Since $\mathbb{V}\,ar\,(g(\Theta))$ is non-negative and $\mathbf{V}$ is positive definite, then

$$\mathbf{X}'\mathbf{V}^{\Theta}\mathbf{X} = \mathbb{E}(h(\Theta))\mathbf{X}'\mathbf{V}\mathbf{X} + \mathbb{V}\,ar\,(g(\Theta))(\mathbf{X}'\bar{\mathbf{R}})^2 > 0.$$

The relation between problems $P(\mu^*)$ and $\tilde{P}(\mu,\Theta)$ are given in the next theorem.

Theorem 3.2 *If $\mathbb{E}(h(\Theta)) > 0$, then problems $P(\mu^*)$ and $\tilde{P}(\mu,\Theta)$ have the same unique optimal solution when $\mu^* = \frac{\mu}{\mathbb{E}(g(\Theta))}$.*

Proof. Considering (4), the Lagrangian associated with problem $\tilde{P}(\mu,\Theta)$ is

$$L(\mathbf{X},\lambda_1,\lambda_2) := \frac{\mathbb{E}(h(\Theta))}{2}\mathbf{X}'\mathbf{V}\mathbf{X} + \frac{\mathbb{V}\,ar\,(g(\Theta))}{2}\mathbf{X}'\mathbf{E}\mathbf{X} + \lambda_1(\mathbf{X}'\bar{\mathbf{R}}^{\Theta} - \mu) + \lambda_2(\mathbf{X}'\mathbf{1} - 1).$$

At the optimal solution we have

$$\mathbb{E}(h(\Theta))\mathbf{V}\mathbf{X} + \mathbb{V}\,ar\,(g(\Theta))\mathbf{E}\mathbf{X} + \lambda_1\bar{\mathbf{R}}^{\Theta} + \lambda_2\mathbf{1} = \mathbf{0}, \tag{5}$$

$$\mathbf{X}'\bar{\mathbf{R}}^{\Theta} - \mu = 0, \tag{6}$$

$$\mathbf{X}'\mathbf{1} - 1 = 0. \tag{7}$$

If $\mu^* = \frac{\mu}{\mathbb{E}(g(\Theta))}$ then $\mathbf{X}'\bar{\mathbf{R}}^{\Theta} = \mu$ yields that $\mathbf{X}'\bar{\mathbf{R}} = \mu^*$. Then we can obtain $\mathbb{E}(g(\Theta))\mathbf{E}\mathbf{X} = \mu\bar{\mathbf{R}}$ and rewrite equation (5) as

$$\mathbb{E}(h(\Theta))\mathbf{VX} + \mathbb{V}ar\,(g(\Theta))\frac{\mu}{\mathbb{E}(g(\Theta))}\overline{\mathbf{R}} + \lambda_1\mathbb{E}(g(\Theta))\overline{\mathbf{R}} + \lambda_2\mathbf{1} = \mathbf{0}.$$

Therefore the optimal portfolio is

$$\mathbf{X} = \alpha\mathbf{V}^{-1}\mathbf{1} + \beta\mathbf{V}^{-1}\overline{\mathbf{R}}, \tag{8}$$

where

$$\alpha = -\frac{\lambda_2}{\mathbb{E}\big(h(\Theta)\big)}, \qquad \beta = -\Big(\frac{\mathbb{V}ar\,(g(\Theta))\mu}{\mathbb{E}(h(\Theta))\mathbb{E}(g(\Theta))} + \frac{\lambda_1\mathbb{E}(g(\Theta))}{\mathbb{E}(h(\Theta))}\Big).$$

Replace $\mathbf{X}$ obtained from (8) into (6) and (7) to obtain

$$1 = \mathbf{1}'\mathbf{X} = \alpha\mathbf{1}'\mathbf{V}^{-1}\mathbf{1} + \beta\mathbf{1}'\mathbf{V}^{-1}\overline{\mathbf{R}} = \alpha A + \beta B, \tag{9}$$

$$\mu^* = \frac{\mu}{\mathbb{E}(g(\Theta))} = \overline{\mathbf{R}}'\mathbf{X} = \alpha\overline{\mathbf{R}}'\mathbf{V}^{-1}\mathbf{1} + \beta\overline{\mathbf{R}}'\mathbf{V}^{-1}\overline{\mathbf{R}} = \alpha B + \beta C. \tag{10}$$

Considering equations (9)-(10) we can obtain

$$\alpha = \frac{C-\mu^* B}{AC-B^2}, \quad \beta = \frac{\mu^* A - B}{AC-B^2}. \tag{11}$$

Now the result follows from equations (8) and (11) and theorem 2.1. Since $\mathbf{V}^{\Theta}$ is positive definite, the solution is unique.

Theorem 3.2 guarantees that the sets of optimal portfolios corresponding to the standard model and the generalized model are the same, under conditions (2) and (3). In the next two sections, two examples of the generalized problem are presented. Also, in the following, we combine these two examples together.

4. Uncertain Exit-Time

Martellini and Urosevic (2006) considered the static M-V portfolio selection model when the exit-time is uncertain. In fact, they considered the asset prices and the exit-time as the sources of uncertainty in their model. They showed that when asset returns follow a random walk and the exit-time is independent of the asset returns, the standard model and the generalized model produce the same set of optimal portfolios. Also, this result does not necessarily hold when the exit-time and asset returns are dependant. In this model, it is assumed that all assets have the same exit-time which is the market exit-time. In fact, by denoting τ as the exit-time, they set $\Theta = \tau$ and $Z^{\tau} = \mathbf{X}'\mathbf{R}^{\tau}$, where

$$r_i^{\tau} := r_i^{\tau}(S_i, \tau) = \frac{S_i(\tau)}{S_i(0)}, \quad (i = 1, \dots, n)$$

and $S_i(\tau)$ is the price of asset i at the exit-time. Keykhaei (2016) considered a model where each asset has individual uncertain exit-time. In this model $\Theta = \{\tau_1, \dots, \tau_n\}$ and

$$r_i^{\Theta} := r_i^{\Theta}(S_i, \Theta) = \frac{S_i(\tau_i)}{S_i(0)}, \quad (i = 1, \dots, n)$$

where τ_i and $S_i(\tau_i)$ are the exit-time and the price of asset i, respectively, at its exit-time. In fact, the model of Martellini and Urosevic (2006) is an extension of the Keykhaei's (2016) model when

$$\tau_1 = \tau_2 = \cdots = \tau_n = \tau.$$

Let $\tau_i = \tau$ for $i = 1, \ldots, n$. Without loss of generality we assume that $T = 1$. When the exit-time is independent of asset returns and the asset returns follow a random walk we have

$$\mathbb{E}(r_i^\tau | \tau = t) = \mathbb{E}(r_i(t)) = \bar{r}_i t,$$
$$\mathbb{C}\,ov\,(r_i^\tau, r_j^\tau | \tau = t) = \mathbb{C}\,ov\,(r_i(t), r_j(t)) = \sigma_{ij} t.$$

Using theorem 3.2 and considering the functions $g, h: \mathbb{R}^m \to \mathbb{R}$ where $g(x_1, \ldots, x_m) = h(x_1, \ldots, x_m) = x_1$ in (2)) and (3)), we have the following theorem.

Theorem 4.1 *(Proposition 4 of Martellini and Urosevic (2006) and Theorem 4.1 of Keykhaei (2016)) Assume that the asset returns follow a random walk. If τ is independent of asset returns, then problems $P(\mu^*)$ and $\tilde{P}(\mu, \tau)$ have the same unique optimal solution when $\mu^* := \frac{\mu}{\mathbb{E}(\tau)}$.*

Martellini and Urosevic (2006) demonstrated that when the uncertain exit-time depends on the asset returns, efficient portfolios in problem $P(\mu^*)$ might be inefficient in problem $\tilde{P}(\mu, \tau)$ and vice versa.

5. Uncertain Quantities

Bankruptcy is not a rare occurrence and many companies are going bankrupt every year (for example, see Wu and Zeng (2013)). Cheung and Yang (2007) and Wu and Zeng (2013) considered regime-switching markets with a bankruptcy state and applied the notion of *recovery rate* in their portfolio selection models. when bankruptcy occurs, the investor only get a random fraction $\delta \in [0,1]$ (the recovery rate) of his/her wealth. So, if investors neglect the possibility of bankruptcy, and consequently neglect the recovery rate, they can not get an accurate estimation of their expected wealth.

In the standard portfolio selection problems, it is assumed that the portfolios are self-financing, i.e., capital additions and withdrawals are forbidden during the investment period. Wu and Li (2012) investigated a multi-period non-self-financing M-V portfolio selection model with regime-switching and a stochastic cash flow. We can justify stochastic cash flow by capital additions or withdrawals. For instance, the insurers might face premium income and claim payout in each period; or, the investors decide to make an additional investment on some assets to earn a better income or decide to withdraw some of their wealth to stop their losses (see Wu and Li (2012)).

In the following, we investigate a general static model which covers possibility of bankruptcy, asset additions and withdrawals.

5.1 Weight Coefficients

In this section, unlike Section 2, we assume that asset quantities are non-constant and change randomly during the investment period. With this assumption, for some i we might have $\varphi_i(0) \neq \varphi_i(T)$, where $\varphi_i(0)$ and $\varphi_i(T)$ are the quantities of asset i at the beginning and at the end of the investment period, respectively. Thus $\varphi_i(T)$ is uncertain. Then, the values of portfolio, at the beginning and at the end of the investment period, are $W(0) = \sum_{i=1}^{n} \varphi_i(0)S_i(0)$ and $W(T) = \sum_{i=1}^{n} \varphi_i(T)S_i(T)$, respectively.

Changes in asset quantities can be interpreted as follows,

- Consider an investment opportunity in commodities. In this case, some activities such as holding and transporting affect asset quantities which is caused by natural or accidental events such as decay, fire or earthquake.
- Consider an investment opportunity which is associated with a Birth and Death process.
- Consider an investment opportunity with asset additions and withdrawals possibilities.

Under these conditions, we try to reformulate problem $P(\mu, \Theta)$ into a quadratic form like problem $\tilde{P}(\mu, \Theta)$. Following Keykhaei (2016), we give the following two definitions. Firstly, we define the notion of weight coefficient.

Definition 5.1 *The weight coefficient of asset i is*

$$\Lambda_i := \frac{\varphi_i(T)}{\varphi_i(0)}, \quad (i = 1, \dots, n).$$

In fact, the vectors of quantities, at the beginning and at the end of the investment period, are $(\varphi_1(0), \dots, \varphi_n(0))'$ and $(\Lambda_1\varphi_1(0), \dots, \Lambda_n\varphi_n(0))'$, respectively. Using the notion of weight coefficient, we can formulate some portfolio selection problems in terms of uncertain quantities, see examples 5.3-5.6.

Note that, in discrete multi-period portfolio selection models, it is common that the quantity of each asset is not the same at the beginning and at the end of the investment period. This is the result of rebalancing the portfolio during the period. Actually, the quantities to be bought at time zero are deterministic, whereas the quantities to be held at future trading dates, and therefore at the end of the investment period, are random from the time-zero perspective. In this case, the investor changes the quantity of each asset, whereas in our model, an exogenous factor can also change the quantities, and this is the main difference between these two models. Actually, in a multi-period model, the investor solves a new optimization problem at the beginning of each period to obtain new optimal quantities. In fact, the investor changes the quantities according to the optimal strategy. However, in our model, the investor solves an optimization problem only once at the beginning, and there are no other optimization problems to solve in the future time periods. In other words, changes in asset quantities are out of investor's hands. We can develop this model to multi-period settings. Consider a multi-period portfolio optimization problem with the exit-time T. Let $\varphi_i(j)$ be the quantity of asset i held at the beginning of period j (i.e., the time interval $[j, j+1)$) after rebalancing, and $S_i(j)$ be the value of the asset i at time j, for $i = 1, \dots, n$ and $j = 0,1, \dots, T-1$. Then, at the end of period j and before rebalancing, the quantity of asset i is $\Lambda_i(j)\varphi_i(j)$, where $\Lambda_i(j)$ is the weight coefficient of asset i during the period j. In this settings, the wealth at time $j \geq 1$

before rebalancing is $W(j) = \sum_{i=1}^{n} \Lambda_i(j-1)\varphi_i(j-1)S_i(j)$ and the wealth at time j after it is $W(j) = \sum_{i=1}^{n} \varphi_i(j)S_i(j)$.

In the following we go back to static model. Applying the notion of weight coefficients for $\Theta = (\Lambda_1, \dots, \Lambda_n)$, we have

$$\begin{aligned} Z^{\Theta} &= \frac{\sum_{i=1}^{n} \varphi_i(T)S_i(T)}{W(0)} \\ &= \sum_{i=1}^{n} \frac{\varphi_i(0)S_i(0)}{W(0)} \frac{\varphi_i(T)}{\varphi_i(0)} \frac{S_i(T)}{S_i(0)} \\ &= \sum_{i=1}^{n} x_i \Lambda_i r_i, \end{aligned}$$

where $x_i = \varphi_i(0)S_i(0)/W(0)$ is the initial weight allocated to asset i.

Definition 5.2 *For each $i = 1, \dots, n$, we define $r_i^{\Theta} := \Lambda_i r_i$ as the total-return of asset i.*

Replacing r_i^{Θ} in Z^{Θ} we have

$$Z^{\Theta} = \mathbf{X}'\mathbf{R}^{\Theta},$$

as we desire.

Example 5.3 *(Bankruptcy) Following Cheung and Yang (2007) and Wu and Zeng (2013), assume that there are only one risk-free asset and one risky asset in the market. Let r_1 and r_2 denote the return of risk-free asset and the return of risky asset, respectively. As it is mentioned in Cheung and Yang (2007) and Wu and Zeng (2013), after occurrence of bankruptcy, the dynamic of wealth, corresponding to the time interval $[n, n+1]$, can be represented by*

$$W(n+1) = \delta(W(n)r_1 + \varphi_2(n)S_2(n)(r_2 - r_1)),$$

where δ is the recovery rate. Then we have

$$Z^{\Theta} = x_1\Lambda_1 r_1 + x_2\Lambda_2 r_2,$$

where $\Lambda_1 = \Lambda_2 = \delta$. In fact, each weight coefficient can be interpreted as the recovery rate.

Example 5.4 *(An asset with a birth-death process) Consider an investment opportunity in an asset in which the size of its population follows a discrete time birth-death process. If $\varphi(t)$ denotes the population size of this asset in the portfolio at time t, then $\varphi(t+1)$ is a random variable for which $\varphi(t+1) = \varphi(t) + i$, where $i = -1,0,1$. So $\varphi(T)$ is a random variable with the sample space $S = \{(\varphi(0) - T)^+, (\varphi(0) - T)^+ + 1, \dots, \varphi(0) + T\}$, where $(.)^+ := max(0, .)$.*

Example 5.5 *(Portfolio optimization with specific stop-loss level for assets) Consider an investor who determines a stop-loss level for the price of an asset, for example the first one, in the portfolio. If the price at the (deterministic or stochastic) time interval $[\tau_1, \tau_2]$ ($\tau_1 \leq \tau_2 \leq T$) falls below this level, he sells the asset at time $\tau := \tau_2$ to avoid more losses and invests the income between other assets. Let A denotes the mentioned event. Then, for each $i \geq 2$, the investor invests the amount $\alpha_i \varphi_1 S_1(\tau)$ in asset i where each portion $\alpha_i \geq$*

0 *is predetermined and* $\sum_{i=2}^{n} \alpha_i = 1$. *So, his wealth at time* $t = \tau$ *(after the reinvestment) and at the exit time* $t = T$ *are equal to*

$$X(\tau) = \varphi_1 \mathbf{1}_A S_1(\tau) + \sum_{i=2}^{n} [\varphi_i + \alpha_i \varphi_1 \mathbf{1}_{A^c} S_1(\tau)/S_i(\tau)] S_i(\tau)$$

and

$$\begin{aligned} W(T) &= \varphi_1 \mathbf{1}_A S_1(T) + \sum_{i=2}^{n} [\varphi_i + \alpha_i \varphi_1 \mathbf{1}_{A^c} S_1(\tau)/S_i(\tau)] S_i(T) \\ &= \varphi_1 [\mathbf{1}_A + \mathbf{1}_{A^c} \sum_{i=2}^{n} \alpha_i \frac{S_1(\tau) S_i(T)}{S_1(T) S_i(\tau)}] S_1(T) + \sum_{i=2}^{n} \varphi_i S_i(T), \end{aligned}$$

respectively, where $\mathbf{1}_A$ denotes the indicator function of A. Then

$$Z^{\Theta} = \sum_{i=1}^{n} x_i r_i^{\Theta} = \sum_{i=1}^{n} x_i \Lambda_i r_i,$$

where

$$\Lambda_1 = [\mathbf{1}_A + \mathbf{1}_{A^c} \sum_{i=2}^{n} \alpha_i \frac{S_1(\tau) S_i(T)}{S_1(T) S_i(\tau)}], \quad \varphi_1(T) = \Lambda_1 \varphi_1,$$

and $\Lambda_i = 1$ for $i = 2, \dots, n$.

Example 5.6 *(Portfolios of options) Let* $C_i(0)$ *denotes the initial price of a European call option on a stock with the expiration price* $S_i(T)$ *and the strike price* K_i, *for* $i = 1, \dots, n$. *Now the value of each option at the time of expiration is*

$$C_i(T) = \max(0, S_i(T) - K_i).$$

A portfolio containing these options has the initial value $W(0) = \sum_{i=1}^{n} \varphi_i C_i(0)$ and the final value $W(T) = \sum_{i=1}^{n} \varphi_i \Lambda_i (S_i(T) - K_i)$, where

$$\Lambda_i = \begin{cases} 1, & if\ S_i(T) \geq K_i, \\ 0, & if\ S_i(T) < K_i. \end{cases}$$

Now the portfolio return is $Z^{\Theta} = \sum_{i=1}^{n} x_i \Lambda_i r_i$, where $r_i = (S_i(T) - K_i)/C_i(0)$.

Let all assets have the same weight coefficient, that is $\Lambda_i = \Lambda$ for $i = 1, \dots n$, and independent of the asset prices. Then, we can see that

$$\begin{aligned} \mathbb{E}(r_i^{\Theta} | \Theta = (\lambda, \dots, \lambda)) &= \mathbb{E}(\lambda r_i | \Theta = (\lambda, \dots, \lambda)) \\ &= \lambda \bar{r}_i \end{aligned} \quad (12)$$

and

$$\begin{aligned} \mathbb{E}(r_i^{\Theta} r_j^{\Theta} | \Theta = (\lambda, \dots, \lambda)) &= \mathbb{E}(\lambda^2 r_i r_j | \Theta = (\lambda, \dots, \lambda)) \\ &= \lambda^2 \mathbb{E}(r_i r_j). \end{aligned}$$

Therfore

$$\mathbb{C}\,ov\,(r_i^{\Theta}, r_j^{\Theta} | \Theta = (\lambda, \dots, \lambda)) = \lambda^2 \sigma_{ij}. \quad (13)$$

Theorem 5.7 *If all assets have the same weight coefficient* Λ *and independent of asset returns, then problems* $P(\mu^*)$ *and* $\tilde{P}(\mu, \tau)$ *have the same unique optimal solution when* $\mu^* := \frac{\mu}{\mathbb{E}(\Lambda)}$.

Proof. Consider the functions $g, h: \mathbb{R}^n \to \mathbb{R}$ where $g(x_1, \dots, x_n) = x_1$ and $h(x_1, \dots, x_n) = x_1^2$ in (2) and (3). Now the result follows from theorem 3.2 and equations (14) and (15).

In Section 6, we will show that, by means of an example, the assertion of Theorem 5.7 is not true in general without the assumption that the weight coefficients and asset returns are independent. In fact efficient portfolios in the standard case (problem $P(\mu^*)$) might be inefficient in the generalized case (problem $\tilde{P}(\mu, \tau)$) and vice versa.

5.2 Uncertain Quantities and Uncertain Exit-Time

In this section we combine two sources of uncertainty, exit-time and weight coefficient, together. In fact we consider a static portfolio selection model with uncertain weights and an uncertain exit-time. Here, we follow the same manner as presented in Section 5.1.

Definition 5.8 *The weight coefficient of asset i is*

$$\Lambda_i^\tau := \frac{\varphi_i(\tau)}{\varphi_i(0)}, \quad (i = 1, \dots, n).$$

Definition 5.8 indicates that each weight coefficient depends on the exit-time. Now the portfolio return is

$$\begin{aligned} Z^\Theta &= \frac{\sum_{i=1}^n \varphi_i(\tau) S_i(\tau)}{W(0)} \\ &= \sum_{i=1}^n \frac{\varphi_i(0) S_i(0)}{W(0)} \frac{\varphi_i(\tau)}{\varphi_i(0)} \frac{S_i(\tau)}{S_i(0)} \\ &= \sum_{i=1}^n x_i \Lambda_i^\tau r_i^\tau, \end{aligned}$$

where $\Theta := ((\Lambda_1^\tau, \dots, \Lambda_n^\tau), \tau) = (\Lambda_1^\tau, \dots, \Lambda_n^\tau, \tau)$.

Definition 5.9 *The total-return of asset i is $r_i^\Theta := \Lambda_i^\tau r_i^\tau$, for $i = 1, \dots, n$.*

Replacing r_i^Θ in Z^Θ we have

$$Z^\Theta = \mathbf{X}'\mathbf{R}^\Theta,$$

as we wish.

Consider the random walk case. If all assets have the same weight coefficient Λ and the same exit-time τ, when both are independent of asset returns, then

$$\mathbb{E}(r_i^\Theta | \Theta = ((\lambda, \dots, \lambda), t)) = \lambda \mathbb{E}(r_i(t)) = \lambda t \bar{r}_i \tag{14}$$

and

$$\begin{aligned} \mathbb{E}(r_i^\Theta r_j^\Theta | \Theta = ((\lambda, \dots, \lambda), t)) &= \lambda^2 \mathbb{E}(r_i(t) r_j(t)) \\ &= \lambda^2 (\mathbb{C}ov\,(r_i(t), r_j(t)) + \mathbb{E}(r_i(t))\mathbb{E}(r_j(t))) \\ &= \lambda^2 (t\sigma_{ij} + t^2 \bar{r}_i \bar{r}_j). \end{aligned}$$

Therefore

$$\mathbb{C}ov\,(r_i^{\Theta}, r_j^{\Theta} | \Theta = ((\lambda, \ldots, \lambda), t)) = \lambda^2 t \sigma_{ij}. \tag{15}$$

Theorem 5.10 *Suppose that asset returns follow a random walk and all assets have the same weight coefficient Λ and exit-time τ. If Λ and τ are independent of asset returns, then problems $P(\mu^*)$ and $\tilde{P}(\mu,\tau)$ have the same unique optimal solution when $\mu^* := \frac{\mu}{\mathbb{E}(\Lambda\tau)}$. This holds for $\mu^* := \frac{\mu}{\mathbb{E}(\Lambda)\mathbb{E}(\tau)}$ if Λ and τ are independent.*

Proof. Consider the functions $g, h: \mathbb{R}^{n+1} \to \mathbb{R}$ where $g(x_1, \ldots, x_{n+1}) = x_1 x_{n+1}$ and $h(x_1, \ldots, x_{n+1}) = x_1^2 x_{n+1}$ in (2) and (3). Now the result follows from theorem 3.2 and equations (14) and (15). The last part of the theorem is obvious.

6. An Illustrative Example

In this section we refer back to example 5.5 and investigate a portfolio selection problem with a specific stop-loss criterion for an asset. We consider a four week (T=20) portfolio optimization problem with three stocks: NEM, KO and IBM from S&P 500. Let S_1, S_2 and S_3 denote the price of the stocks NEM, KO and IBM, respectively. We chose the historical stock prices from 1/2/2013 to 1/2/2015 to estimate the required parameters. Considering the historical data, we can see that NEM does not have a good performance during this period (see Figure 1). Consider an investor who worried about the price of this stock and specifies a stop-loss criterion for it. He would withdraw the investment in NEM at the end of the second week ($\tau = 10$) if S_1 falls below the level k throughout the first two weeks of the investment period, and invest the income in KO for the remainder of the period (two weeks). Let A denotes the mentioned event. Then

$$W(T) = \varphi_1[\mathbf{1}_A + \mathbf{1}_{A^c}\frac{S_1(\tau)S_2(T)}{S_1(T)S_2(\tau)}]S_1(T) + \sum_{i=2}^{3} \varphi_i S_i(T).$$

Therefore

$$Z^{\Theta} = \sum_{i=1}^{3} x_i \Lambda_i r_i,$$

where

$$\Lambda_1 = [\mathbf{1}_A + \mathbf{1}_{A^c}\frac{S_1(\tau)S_2(T)}{S_1(T)S_2(\tau)}]$$

and $\Lambda_i = 1$ for $i = 2,3$. It is assumed that borrowing is not allowed, i.e., $\mathbf{X} \geq 0$. Figure 2 shows the efficient frontiers for some levels of k. Note that the efficient frontier of the standard model is corresponding to $k = 0$ which is displayed by the blue curve ($k \leq 19$).

This example illustrates a portfolio selection problem in which the uncertain quantities depend on the asset returns. The estimated mean vectors and covariance matrices of total-returns are presented in Table 1 for $k = 0,25,32$. We can see that in the standard case, corresponding to $k = 0$, the portfolio $\mathbf{X}_0 = (0.07641, 0.69596, 0.22763)'$ is the optimal portfolio corresponding to the pair $(0.03241, 1)$ in the Mean-Standard Deviation plane. For $k = 25$, $\mathbf{X}_0$ corresponds to the pair $(0.03199, 1.00005)$. In this case, the optimal portfolio for $\mu = 1.00005$ is $\mathbf{X}_{25} = (0.08441, 0.7098, 0.20578)'$ which is corresponding to the pair $(0.03197, 1.00005)$. So $\mathbf{X}_0$ is not optimal in the generalized model when $k = 25$. On the other hand, the portfolio $\mathbf{X}_{32} = (0.0754, 0.6634, 0.26121)'$ associated with the pair $(0.0326, 1)$ is optimal when $k = 32$. But $\mathbf{X}_{32}$ is not optimal in the standard case. Actually, $\mathbf{X}_{32}$ corresponds to the pair $(0.03234, 0.99959)$, whereas for $\mu = 0.99959$ the optimal portfolio $\widetilde{\mathbf{X}}_0 = (0.08595, 0.68212, 0.23185)'$ has a lower risk, that is 0.03231. This implies that efficient portfolios in the standard case might be inefficient in the generalized case and vice versa. Martellini and Urosevic (2006) demonstrated this result, by means of an example, when the uncertain exit-time depends on the asset returns.

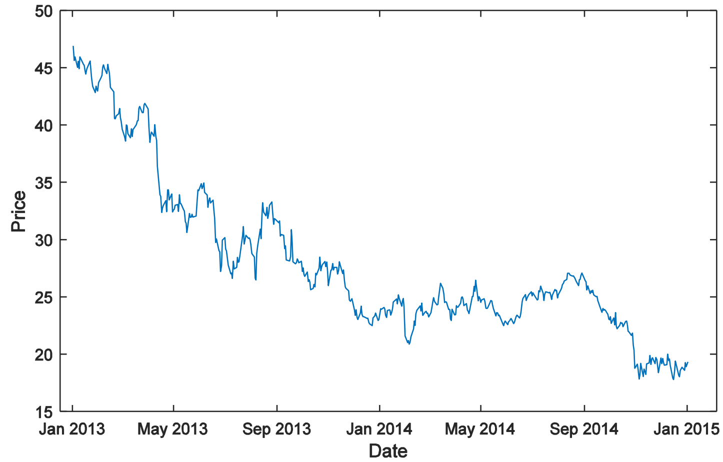

Figure 1. NEM stock chart.

Table 1: Estimated mean vectors and covariance matrices of total-returns for $k = 0, 25, 32$

	$\overline{\mathbf{R}}^{\Theta}$	$\mathbf{V}^{\Theta}$
$k = 0$	$\begin{pmatrix} 0.96879 \\ 1.00587 \\ 0.99254 \end{pmatrix}$	$\begin{pmatrix} 0.00737 & -0.00018 & 0.00138 \\ -0.00018 & 0.00144 & 0.00051 \\ 0.00138 & 0.00051 & 0.00228 \end{pmatrix}$
$k = 25$	$\begin{pmatrix} 0.96948 \\ 1.00587 \\ 0.99254 \end{pmatrix}$	$\begin{pmatrix} 0.0065 & -0.00025 & 0.00094 \\ -0.00025 & 0.00144 & 0.00051 \\ 0.00094 & 0.00051 & 0.00228 \end{pmatrix}$
$k = 32$	$\begin{pmatrix} 0.97422 \\ 1.00587 \\ 0.99254 \end{pmatrix}$	$\begin{pmatrix} 0.00615 & 0.00023 & 0.00095 \\ 0.00023 & 0.00144 & 0.00051 \\ 0.00095 & 0.00051 & 0.00228 \end{pmatrix}$

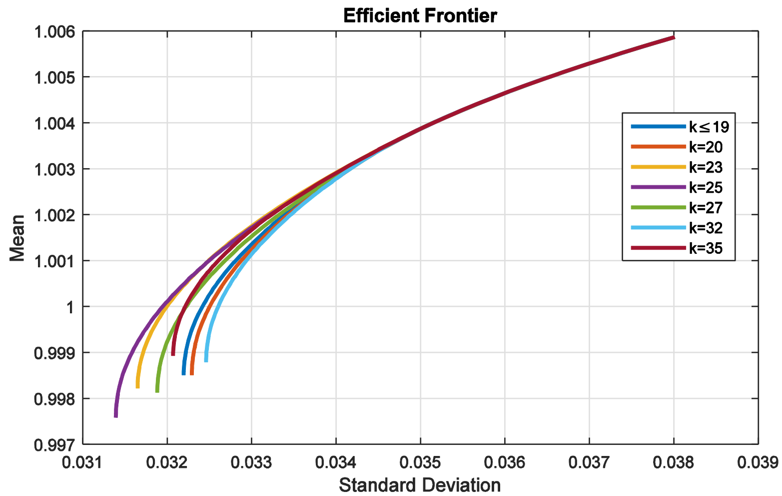

Figure 2. Efficient frontiers.

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