Characterization and Estimation of the Length Biased Nakagami Distribution

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Abstract

In this paper, we introduce the length biased form of the Nakagami distribution known as length biased Nakagami distribution (LBND). Some properties of the model were studied such as moments, reliability function, and the hazard rate function. Maximum likelihood and Bayes estimators of the scale parameter are derived. Also, the Posterior risk under different loss functions is obtained. A real life data set is used and the results are compared through R-software.

Keywords: Length biased nakagami distribution; Moments; Maximum likelihood estimates; Bayesian estimates; Posterior risk.

1. Introduction

Nakagami distribution propounded by M.Nakagami (1960) is a flexible lifetime distribution. This distribution has been employed to model attenuation of wireless signals travelling manifold paths (see Hoffman (1960)), fading of radio signals, data regarding communication engineering, etc. It may also be used to model failure times of diversity of products such as vacuum tubes, ball bearing and electrical insulation. It finds wide applicability in biomedical fields such as to model the time to the occurrence of tumors and appearance of lung cancer. This distribution has applications in medical imaging studies to model the ultrasounds especially in Echo (heart efficiency test). Shanker et al. (2005) and Tsui et al. (2006) use this distribution to model ultrasound data in medical imaging studies. Abdi and Kaveh (2000) have shown that Nakagami distribution is useful for modeling multipath faded envelope in wireless channels and also estimated the shape parameter of the distribution. Yang and Lin (2000) studied and extracted the statistical model of spatial-chromatic distribution of images. Through extensive evaluation of large image data bases, they revealed that a two parameter Nakagami distribution suits well the purpose. Kim and Latchman (2009) used the Nakagami distribution in their analysis of multimedia. Azam Zaka and Ahmad Saeed Akhter (2014) obtained the Bayes estimators of the Nakagami distribution.

The probability density function of Nakagami distribution is given as

$$f(x;\lambda,\beta)=\frac{2\lambda^{\lambda}}{\Gamma(\lambda)\beta^{\lambda}}x^{2\lambda-1}\exp\left\{-\left(\frac{\lambda x^2}{\beta}\right)\right\}, x>0 \tag{1}$$

Where $\lambda \geq 0$ is the shape parameter and $\beta > 0$ is the scale parameter.

2. Derivation of Length Biased Nakagami Distribution

Length biased distribution is a special case of weighted distribution which were first introduced by Fisher (1934) to model ascertainment bias. These were later formalized in a unifying theory by Rao (1965). Weighted distributions arise when the observations generated from a stochastic process are not given equal chances of being recorded; rather they are recorded in accordance to some weighted function. When the weight function depends on the length of the units of interest, the resulting distribution is called length biased. Weighted distributions help us to deal with model specification and data interpretation problems. Much work was carried out to characterize relationships between original distributions and their length biased versions. Patil and Rao (1978) have given a table for some basic distributions and their length biased forms such as Lognormal, Gamma, Pareto, Beta distributions. Loppi and Baily (1987) used weighted distributions to analyses HPS diameter increment data. Sofi Mudasir and S.P. Ahmad (2015) studied the structural properties of length biased Nakagami distribution.

If the random variable X has distribution function $f(x;\sigma)$, with unknown parameter σ, then the corresponding weighted distribution is of the form:

$$f(x;\sigma) = \frac{x^c f(x;\sigma)}{\mu'_c}$$

Where $\mu'_c = \int x^c f(x;\sigma)dx$

When c=1,2 we obtain length biased and area biased distributions respectively.

By applying the weight x^c, where c=1, we obtain the length biased Nakagami distribution with the pdf given as

$$f_l(x;\lambda,\beta) = \frac{2\lambda^{\lambda+\frac{1}{2}} x^{2\lambda} \exp\left\{-\left(\frac{\lambda x^2}{\beta}\right)\right\}}{\beta^{\lambda+\frac{1}{2}} \Gamma(\lambda+\frac{1}{2})} \tag{2}$$

The distribution function corresponding to (2) is given as

$$F_X(x) = \frac{2\lambda\beta}{\Gamma\left(\lambda+\frac{1}{2}\right)} \gamma\left(\lambda+\frac{1}{2}, \frac{\lambda x^2}{\beta}\right) \tag{3}$$

The reliability and hazard functions corresponding to equation (2) are given by

$$R(t,\lambda,\beta) = 1 - \frac{2\lambda\beta}{\rho_1} \gamma\left(\lambda+\frac{1}{2}, \frac{\lambda t^2}{\beta}\right)$$

And $h(t,\lambda,\beta)=\dfrac{2\lambda^{\lambda+\frac{1}{2}}t^{2\lambda}\exp\left\{-\left(\dfrac{\lambda t^2}{\beta}\right)\right\}}{\beta^{\lambda+\frac{1}{2}}\left(\rho_1-2\lambda\beta\ \gamma\left(\lambda+\dfrac{1}{2},\dfrac{\lambda t^2}{\beta}\right)\right)}$

Special case:

If λ =1/2 in equation (2), we get

$$f(x;\beta)=\frac{x}{\beta}\exp\left\{-\left(\frac{x^2}{2\beta}\right)\right\}, x>0, \beta>0$$

This is the probability density function of Rayleigh distribution.

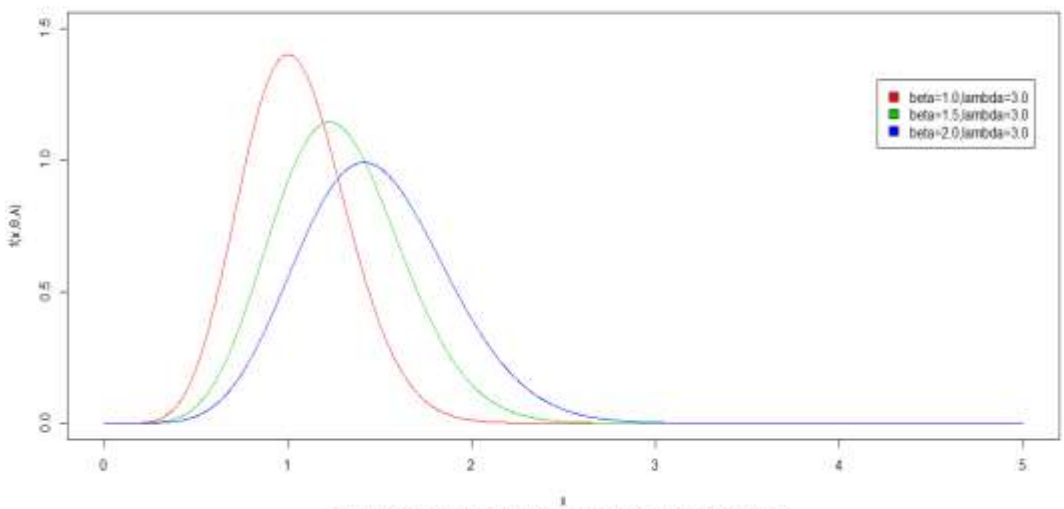

Figure 1:The pdf of Weibull distribution under various values of Scale Parameter β

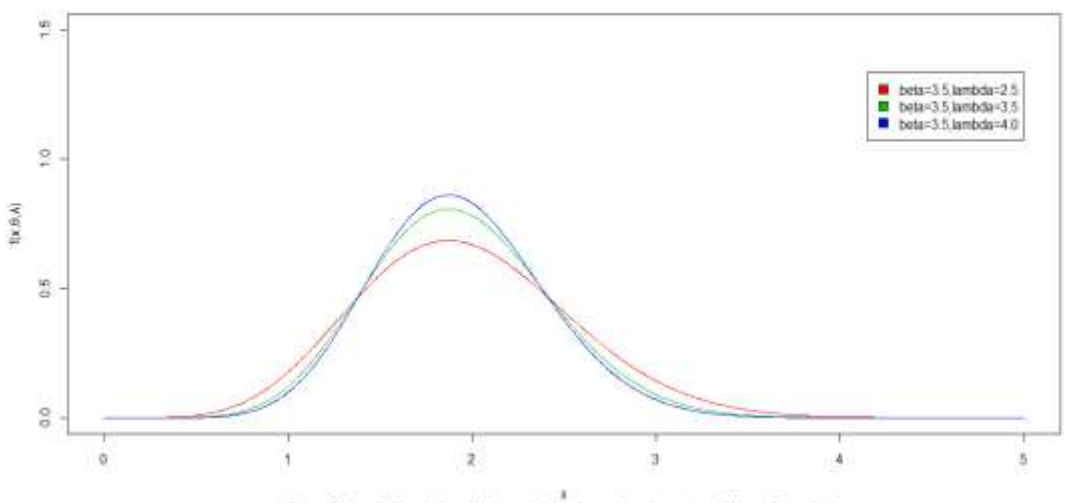

Figure 2:The pdf of length biased Nakagami distribution under various values of Shape Parameter λ

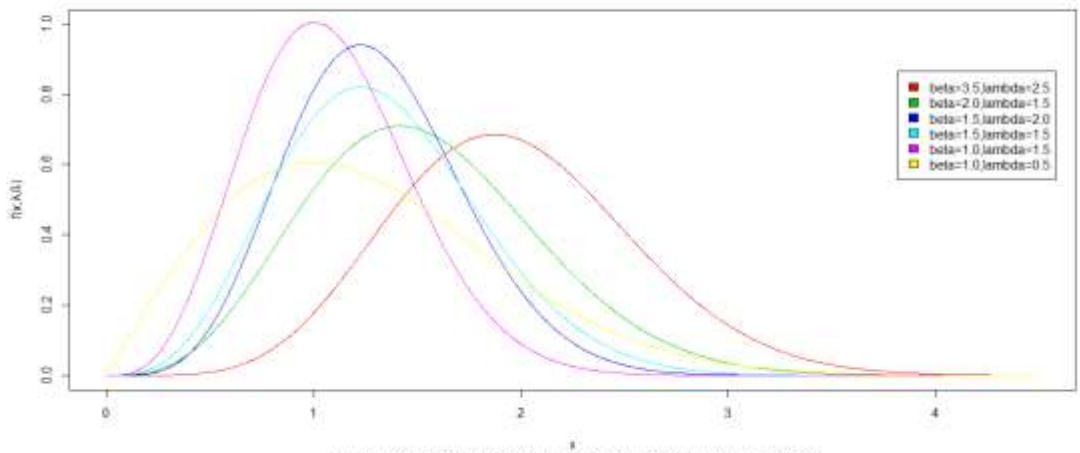

Figure 3:The pdf of length biased Nakagami distribution under various values of β and λ

Theorem 1: If X follows length biased Nakagami distribution with parameters λ and β, then the kth moment about origin is given as follows:

$$\mu'_k = \left(\frac{\beta}{\lambda}\right)^{\frac{k}{2}} \frac{\rho_{k+1}}{\rho_1}, \text{ where } \rho_s = \Gamma\left(\lambda + \frac{s}{2}\right)$$

Proof: The kth moment of length biased Nakagami distribution about origin is obtained as

$$\mu'_k = \int_0^\infty x^k f_l(x;\lambda,\beta)dx$$

$$\mu'_k = \int_0^\infty x^k \frac{2\lambda^{\lambda+\frac{1}{2}} x^{2\lambda} \exp\left\{-\left(\frac{\lambda x^2}{\beta}\right)\right\}}{\Gamma(\lambda+\frac{1}{2})\beta^{\lambda+\frac{1}{2}}} dx$$

On solving the above integral, we get

$$\mu'_k = \left(\frac{\beta}{\lambda}\right)^{\frac{k}{2}} \frac{\rho_{k+1}}{\rho_1} \tag{4}$$

Using the equation (4), the first and second moment of length biased Nakagami distribution is given by

$$\mu'_1 = \left(\frac{\beta}{\lambda}\right)^{\frac{1}{2}} \frac{\rho_2}{\rho_1} = mean \tag{5}$$

$$\mu'_2 = \left(\frac{\beta}{\lambda}\right) \frac{\rho_3}{\rho_1} \tag{6}$$

Using equations (5) and (6), the variance of length biased Nakagami distribution is given by $\mu_2 = \beta\left\{\frac{\rho_3}{\lambda\rho_1} - \frac{\lambda\Gamma^2(\lambda)}{\rho_1^{\,2}}\right\}$

3. Estimation of the Scale Parameter

In this section an attempt will be made to estimate the scale parameter of length biased Nakagami distribution.

3.1 Maximum Likelihood Estimator

Theorem 2: If $x_1, x_2, \ldots, x_n$ is a random sample from LBND given by (2), then the maximum likelihood estimator is given by

$$\hat{\beta} = \frac{\lambda\eta}{n\left(\lambda + \frac{1}{2}\right)}, \quad \text{Where } \eta = \sum_{i=1}^{n} x_i^{\,2}$$

Proof: The likelihood function of equation (2) is

$$l(\lambda,\beta) = \left(\frac{2\lambda^{\lambda+\frac{1}{2}}}{\Gamma(\lambda+\frac{1}{2})\beta^{\lambda+\frac{1}{2}}}\right)^n \prod_{i=1}^{n} x_i^{\,2\lambda} \exp\left\{-\left(\frac{\lambda\eta}{\beta}\right)\right\} \tag{7}$$

By taking logarithm of equation (7) on both sides, we obtain the log likelihood function as

$$\log\{l(\lambda,\beta)\} = n\log\left(\frac{2\lambda^{\lambda+\frac{1}{2}}}{\Gamma(\lambda+\frac{1}{2})}\right) - n\left(\lambda+\frac{1}{2}\right)\log\beta + 2\lambda\sum_{i=1}^{n}\log x_i - \frac{\lambda\eta}{\beta} \tag{8}$$

Differentiating equation (8) w.r.t. β and equate to zero, we get

$$\hat{\beta} = \frac{\lambda\eta}{n\left(\lambda+\frac{1}{2}\right)} \tag{9}$$

3.2 Bayes Estimator

Recently Bayesian estimation approach has received great consideration by most researchers. In this approach, parameters are treated as random variables and data is treated fixed. An important pre-requisite in Bayesian estimation is the appropriate choice of prior for the parameters. However, Bayesian analysts have pointed out that there is no clear cut way from which one can conclude that one prior is better than the other. Very often, priors are chosen according to ones subjective knowledge and beliefs. However, if one has adequate information about the parameter(s), one should use informative prior(s);

otherwise it is preferable to use non informative prior(s). The other integral part of Bayesian inference is the choice of loss function. As there is no specific analytical procedure that allows us to identify the appropriate loss function to be used; a number of symmetric and asymmetric loss functions have been shown to be functional; see Varian (1975), Zellner (1986), Kifayat et al. (2012), Ahmad and Kaisar (2013), etc.

We now derive the Bayes estimator of the parameter β in length biased Nakagami distribution when the parameter λ is known. We consider two different priors and three different loss functions.

(i) Jeffrey's prior: Jeffrey (1946) proposed a formal rule for obtaining a non-informative prior as

$$g(\theta) \propto \sqrt{I(\theta)}$$

Where θ is k-vector valued parameter and $I(\theta)$ is the Fishers information matrix of order k*k. Thus in our problem the prior distribution of β to be

$$\pi_1(\beta) \propto \frac{1}{\beta}$$

(ii) Extension of Jeffrey's prior: The extension of Jeffery's prior relating to the scale parameter β is given as

$$\pi_2(\beta) \propto \left(\sqrt{\det(I(\beta))}\right)^C, C \in R^+$$

$$\pi_2(\beta) \propto \frac{1}{\beta^{2C}}$$

(iii) Precautionary loss function (PLF): Norstrom (1996) introduced an alternative asymmetric precautionary loss function, and also presented a general class of precautionary loss functions as a special case. These loss functions approach infinitely near the origin to prevent underestimation, thus giving conservative estimators, especially when low failure rates are being estimated. These estimators are very useful when underestimation may lead to serious consequences. A very useful and simple asymmetric precautionary loss function is

$$L(\hat{\beta}, \beta) = \frac{(\hat{\beta} - \beta)^2}{\hat{\beta}}$$

(ii) Al-Bayyati's new loss function (ALF): The Al-Bayyati's new loss function also called new loss function is of the form

$$L\left(\hat{\beta}, \beta\right) = \beta^{C_1}\left(\hat{\beta} - \beta\right)^2; \quad C_1 \varepsilon R.$$

Which is an asymmetric loss function, β and $\hat{\beta}$ represent the true and estimated values of the parameter.

(iii) Weighted loss function (WLF): As compared to other loss functions, WLF may be mathematically more convenient to obtain Bayes estimates. An important thing is that the

bayes estimates under WLF may not exist if the weighted function increases too fast to infinity. The WLF is defined as

$$L(\hat{\beta},\beta)=\frac{(\hat{\beta}-\beta)^2}{\beta}$$

3.2.1 Posterior Distribution under Jeffrey's prior $\pi_1(\beta)$

Under $\pi_1(\beta)$, the posterior distribution is given by

$$P(\beta/\underline{x})=k\left(\frac{2\lambda^{\lambda+\frac{1}{2}}}{\Gamma\left(\lambda+\frac{1}{2}\right)}\right)^n\beta^{-n\lambda-\frac{n}{2}-1}\prod_{i=1}^{n}x_i^{2\lambda}\exp\left\{-\left(\frac{\lambda\eta}{\beta}\right)\right\} \tag{10}$$

where k is independent of β and is given by

$$k=\left(\frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{2\lambda^{\lambda+\frac{1}{2}}}\right)^n\frac{(\lambda\eta)^{n\lambda+\frac{n}{2}}}{\prod_{i=1}^{n}x_i^{2\lambda}\Gamma\left(n\lambda+\frac{n}{2}\right)}$$

Thus, from equation (10), posterior distribution is given by

$$P(\beta/\underline{x})=\frac{(\lambda\eta)^{n\lambda+\frac{n}{2}}}{\Gamma\left(n\lambda+\frac{n}{2}\right)}\beta^{-n\lambda-\frac{n}{2}-1}\exp\left\{-\left(\frac{\lambda\eta}{\beta}\right)\right\}$$

Bayesian Estimation using Jeffrey's prior under different loss function

Theorem 3: Assuming the Precautionary loss function $L(\hat{\beta},\beta)=\frac{(\hat{\beta}-\beta)^2}{\hat{\beta}}$, the Bayes estimator of the scale parameter β, when the shape parameter λ is known, is of the form

$$\beta_{PLF}=\frac{\lambda\eta}{\left[\left(n\lambda+\frac{n}{2}-1\right)\left(n\lambda+\frac{n}{2}-2\right)\right]^{\frac{1}{2}}}$$

Proof: By using Precautionary loss function $L(\hat{\beta},\beta)=\frac{(\hat{\beta}-\beta)^2}{\hat{\beta}}$, the risk function is given by

$$R(\hat{\beta})=\hat{\beta}+\frac{(\lambda\eta)^2}{\hat{\beta}\Gamma\left(n\lambda+\frac{n}{2}\right)}\Gamma\left(n\lambda+\frac{n}{2}-2\right)-2\frac{\lambda\eta}{\Gamma\left(n\lambda+\frac{n}{2}\right)}\Gamma\left(n\lambda+\frac{n}{2}-1\right)$$

Now, Solving $\frac{R(\hat{\beta})}{\partial \hat{\beta}} = 0$, we obtain the Baye's estimator as

$$\beta_{PLF} = \frac{\lambda\eta}{\left[\left(n\lambda + \frac{n}{2} - 1\right)\left(n\lambda + \frac{n}{2} - 2\right)\right]^{\frac{1}{2}}} \tag{11}$$

Theorem 4: Assuming Al-Bayyati's new loss function $L\left(\hat{\beta}, \beta\right) = \beta^{C_1}\left(\hat{\beta} - \beta\right)^2$, the Bayes estimator of the scale parameter β, when the shape parameter λ is known, is of the form

$$\beta_{ALF} = \frac{\lambda\eta}{\left(n\lambda + \frac{n}{2} - C_1 - 1\right)}$$

Proof: By using Al-Bayyati's new loss function $L\left(\hat{\beta}, \beta\right) = \beta^{C_1}\left(\hat{\beta} - \beta\right)^2$, the risk function is given by

$$R(\hat{\beta}) = \frac{(\lambda\eta)^{C_1}}{\Gamma\left(n\lambda + \frac{n}{2}\right)}\hat{\beta}^2\Gamma\left(n\lambda + \frac{n}{2} - C_1\right) + \frac{(\lambda\eta)^{C_1+2}}{\Gamma\left(n\lambda + \frac{n}{2}\right)}\Gamma\left(n\lambda + \frac{n}{2} - C_1 - 2\right) - 2\hat{\beta}\frac{(\lambda\eta)^{C_1+1}}{\Gamma\left(n\lambda + \frac{n}{2}\right)}\Gamma\left(n\lambda + \frac{n}{2} - C_1 - 1\right)$$

Now, Solving $\frac{R(\hat{\beta})}{\partial \hat{\beta}} = 0$, we obtain the Baye's estimator as

$$\beta_{ALF} = \frac{\lambda\eta}{\left(n\lambda + \frac{n}{2} - C_1 - 1\right)} \tag{12}$$

Theorem 5: Assuming weighted loss function $L\left(\hat{\beta}, \beta\right) = \frac{(\hat{\beta} - \beta)^2}{\beta}$, the Bayes estimator of the scale parameter β, when the shape parameter λ is known, is of the form

$$\beta_{WLF} = \frac{\lambda\eta}{\left(n\lambda + \frac{n}{2}\right)}$$

Proof: By using weighted loss function $L(\hat{\beta}, \beta) = \frac{(\hat{\beta} - \beta)^2}{\beta}$, the risk function is given by

$$R(\hat{\beta}) = \frac{\lambda\eta}{n\lambda + \frac{n}{2} - 1} + \frac{n\lambda + \frac{n}{2}}{\lambda\eta}\hat{\beta}^2 - 2\hat{\beta}$$

Now, Solving $\frac{R(\hat{\beta})}{\partial \hat{\beta}} = 0$, we obtain the Baye's estimator as

$$\beta_{WLF} = \frac{\lambda\eta}{\left(n\lambda + \frac{n}{2}\right)} \tag{13}$$

3.2.2 Posterior Distribution under Extension of Jeffrey's prior $\pi_2(\beta)$

Under $\pi_2(\beta)$**,** the posterior distribution is given by

$$P(\beta / \underline{x}) = k\left(\frac{2\lambda^{\lambda+\frac{1}{2}}}{\Gamma\left(\lambda+\frac{1}{2}\right)}\right)^n \beta^{-n\lambda-\frac{n}{2}-2C} \prod_{i=1}^{n} x_i^{2\lambda} \exp\left\{-\left(\frac{\lambda\eta}{\beta}\right)\right\} \tag{14}$$

where $$k = \left(\frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{2\lambda^{\lambda+\frac{1}{2}}}\right)^n \frac{(\lambda\eta)^{n\lambda+\frac{n}{2}+2C-1}}{\prod_{i=1}^{n} x_i^{2\lambda}\Gamma\left(n\lambda+\frac{n}{2}+2C-1\right)}$$

By substituting the value of k in equation (14), the posterior distribution is given by

$$P(\beta / \underline{x}) = \frac{(\lambda\eta)^{n\lambda+\frac{n}{2}+2C-1}}{\Gamma\left(n\lambda+\frac{n}{2}+2C-1\right)} \beta^{-n\lambda-\frac{n}{2}-2C} \exp\left\{-\left(\frac{\lambda\eta}{\beta}\right)\right\}$$

Bayesian estimation using extension of Jeffrey's prior under different loss function

Theorem 6: Assuming the Precautionary loss function $L(\hat{\beta},\beta) = \frac{(\hat{\beta}-\beta)^2}{\hat{\beta}}$, the Bayes estimator of the scale parameter β, when the shape parameter λ is known, is of the form.

$$\beta_{PLF} = \frac{\lambda\eta}{\left[\left(n\lambda+\frac{n}{2}+2C-2\right)\left(n\lambda+\frac{n}{2}+2C-3\right)\right]^{\frac{1}{2}}}$$

Proof: By using Precautionary loss function $L(\hat{\beta},\beta) = \frac{(\hat{\beta}-\beta)^2}{\hat{\beta}}$, the risk function is given by

$$R(\hat{\beta}) = \hat{\beta} + \frac{(\lambda\eta)^2}{\hat{\beta}\Gamma\left(n\lambda + \frac{n}{2} + 2C - 1\right)}\Gamma\left(n\lambda + \frac{n}{2} + 2C - 3\right) - 2\frac{\lambda\eta}{\Gamma\left(n\lambda + \frac{n}{2} + 2C - 1\right)}\Gamma\left(n\lambda + \frac{n}{2} + 2C - 2\right)$$

Now, Solving $\frac{R(\hat{\beta})}{\partial\hat{\beta}} = 0$, we obtain the Baye's estimator as

$$\beta_{PLF} = \frac{\lambda\eta}{\left[\left(n\lambda + \frac{n}{2} + 2C - 2\right)\left(n\lambda + \frac{n}{2} + 2C - 3\right)\right]^{\frac{1}{2}}} \tag{15}$$

Remark 1. If C=1/2 in equation (15), we get the Bayes estimator which is same as in equation (11).

Theorem 7: Assuming Al-Bayyati's new loss function $L\left(\hat{\beta}, \beta\right) = \beta^{C_1}\left(\hat{\beta} - \beta\right)^2$, the Bayes estimator of the scale parameter β, when the shape parameter λ is known, is of the form

$$\beta_{ALF} = \frac{\lambda\eta}{\left(n\lambda + \frac{n}{2} + 2C - C_1 - 2\right)}$$

Proof: By using Al-Bayyati's new loss function $L\left(\hat{\beta}, \beta\right) = \beta^{C_1}\left(\hat{\beta} - \beta\right)^2$, the risk function is given by

$$R(\hat{\beta}) = \hat{\beta}^2 \frac{(\lambda\eta)^{C_1}}{\Gamma\left(n\lambda + \frac{n}{2} + 2C - 1\right)}\Gamma\left(n\lambda + \frac{n}{2} - C_1 + 2C - 1\right) + \frac{(\lambda\eta)^{2C_1 - n\lambda - \frac{n}{2} - 2C + 2}}{\Gamma\left(n\lambda + \frac{n}{2} + 2C - 1\right)}\Gamma\left(n\lambda + \frac{n}{2} - C_1 + 2C - 3\right) -$$
$$2\hat{\beta}\frac{(\lambda\eta)^{C_1+1}}{\Gamma\left(n\lambda + \frac{n}{2} + 2C - 1\right)}\Gamma\left(n\lambda + \frac{n}{2} - C_1 + 2C - 2\right)$$

Now, Solving $\frac{R(\hat{\beta})}{\partial\hat{\beta}} = 0$, we obtain the Baye's estimator as

$$\beta_{ALF} = \frac{\lambda\eta}{\left(n\lambda + \frac{n}{2} + 2C - C_1 - 2\right)} \tag{16}$$

Remark 2. If C=1/2 in equation (16), we get the Bayes estimator which is same as in equation (12).

Theorem 8: Assuming weighted loss function $L\left(\hat{\beta},\beta\right)=\frac{(\hat{\beta}-\beta)^2}{\beta}$, the Bayes estimator of the scale parameter β, when the shape parameter λ is known, is of the form

$$\beta_{WLF}=\frac{\lambda\eta}{\left(n\lambda+\frac{n}{2}+2C-1\right)}$$

Proof: By using weighted loss function $L(\hat{\beta},\beta)=\frac{(\hat{\beta}-\beta)^2}{\beta}$, the risk function is given by

$$R(\hat{\beta})=\frac{\lambda\eta}{n\lambda+\frac{n}{2}+2C-2}+\frac{n\lambda+\frac{n}{2}+2C-1}{\lambda\eta}\hat{\beta}^2-2\hat{\beta}$$

Now, Solving $\frac{R(\hat{\beta})}{\partial\hat{\beta}}=0$, we obtain the Bayes estimator as

$$\beta_{WLF}=\frac{\lambda\eta}{n\lambda+\frac{n}{2}+2C-1} \tag{17}$$

Remark 3. If C=1/2 in equation (17), we get the Bayes estimator which is same as in equation (13).

4. Posterior Risks under Different Loss Functions

4.1 Posterior risk of the Bayes estimator under Jeffrey's prior

Under precautionary loss function, the posterior risk is

$$P\left(\beta_{PLF}\right)=2\left[\sqrt{E(\beta)^2}-E(\beta)\right] \tag{18}$$

Now, $$E\left(\beta^k\right)=\frac{\Gamma\left(n\lambda+\frac{n}{2}-k\right)}{\Gamma\left(n\lambda+\frac{n}{2}\right)}(\lambda\eta)^k \tag{19}$$

From equation (19), we have

$$E(\beta)=\frac{\lambda\eta}{n\lambda+\frac{n}{2}-1} \tag{20}$$

$$E\left(\beta^{2}\right)=\frac{(\lambda\eta)^{2}}{\left(n\lambda+\frac{n}{2}-1\right)\left(n\lambda+\frac{n}{2}-2\right)} \tag{21}$$

Substitute the value of equations (20) and (21) in equation (18), we get

$$P(\beta_{PLF})=2\lambda\eta\left[\frac{1}{\left\{\left(n\lambda+\frac{n}{2}-1\right)\left(n\lambda+\frac{n}{2}-2\right)\right\}^{\frac{1}{2}}}-\frac{1}{n\lambda+\frac{n}{2}-1}\right] \tag{22}$$

Under Al-Bayyati's new loss function, the posterior risk is

$$P(\beta_{ALF})=E\left(\beta^{C_1+2}\right)-\frac{\left\{E\left(\beta^{C_1+1}\right)\right\}^{2}}{E\left(\beta^{C_1}\right)} \tag{23}$$

Again, from equation (19), we have

$$E\left(\beta^{C_1}\right)=\frac{\Gamma\left(n\lambda+\frac{n}{2}-C_1\right)}{\Gamma\left(n\lambda+\frac{n}{2}\right)}(\lambda\eta)^{C_1} \tag{24}$$

$$E\left(\beta^{C_1+1}\right)=\frac{\Gamma\left(n\lambda+\frac{n}{2}-C_1-1\right)}{\Gamma\left(n\lambda+\frac{n}{2}\right)}(\lambda\eta)^{C_1+1} \tag{25}$$

$$E\left(\beta^{C_1+2}\right)=\frac{\Gamma\left(n\lambda+\frac{n}{2}-C_1-2\right)}{\Gamma\left(n\lambda+\frac{n}{2}\right)}(\lambda\eta)^{C_1+2} \tag{26}$$

Substitute the value of equations (24), (25) and (26) in equation (23), we get

$$P(\beta_{ALF})=\frac{(\lambda\eta)^{C_1+2}}{\Gamma\left(n\lambda+\frac{n}{2}\right)}\left[\Gamma\left(n\lambda+\frac{n}{2}-C_1-2\right)-\frac{\Gamma\left(n\lambda+\frac{n}{2}-C_1-1\right)}{n\lambda+\frac{n}{2}-C_1-1}\right] \tag{27}$$

Under weighted loss function, the posterior risk is

$$P(\beta_{WLF}) = E(\beta) - \frac{1}{E(\beta^{-1})} \tag{28}$$

From equation (19), we have

$$E(\beta^{-1}) = \frac{n\lambda + \frac{n}{2}}{\lambda\eta} \tag{29}$$

By Substituting the value of equations (20) and (29) in equation (28), we get

$$P(\beta_{WLF}) = \frac{\lambda\eta}{\left(n\lambda + \frac{n}{2}\right)\left(n\lambda + \frac{n}{2} - 1\right)} \tag{30}$$

4.2 Posterior risk of the Bayes estimator under extension of Jeffrey's prior

Under precautionary loss function, the posterior risk is

$$P(\beta_{PLF}) = 2\left[\sqrt{E(\beta)^2} - E(\beta)\right] \tag{31}$$

Now, $$E\left(\beta^k\right) = \frac{\Gamma\left(n\lambda + \frac{n}{2} + 2C - k - 1\right)}{\Gamma\left(n\lambda + \frac{n}{2} + 2C - 1\right)}(\lambda\eta)^k \tag{32}$$

From equation (32), we have

$$E(\beta) = \frac{\lambda\eta}{n\lambda + \frac{n}{2} + 2C - 2} \tag{33}$$

$$E(\beta^2) = \frac{(\lambda\eta)^2}{\left(n\lambda + \frac{n}{2} + 2C - 2\right)\left(n\lambda + \frac{n}{2} + 2C - 3\right)} \tag{34}$$

Substitute the value of equations (33) and (34) in equation (31), we get

$$P(\beta_{PLF}) = 2\lambda\eta\left[\frac{1}{\left\{\left(n\lambda + \frac{n}{2} + 2C - 2\right)\left(n\lambda + \frac{n}{2} + 2C - 3\right)\right\}^{\frac{1}{2}}} - \frac{1}{n\lambda + \frac{n}{2} + 2C - 2}\right] \tag{35}$$

Remark 4. If C=1/2 in equation (35), we get the posterior risk which is same as in equation (22).

Under Al-Bayyati's new loss function, the posterior risk is

$$P(\beta_{ALF}) = E\left(\beta^{C_1+2}\right) - \frac{\left\{E\left(\beta^{C_1+1}\right)\right\}^2}{E\left(\beta^{C_1}\right)} \tag{36}$$

Again, from equation (32), we have

$$E\left(\beta^{C_1}\right) = \frac{\Gamma\left(n\lambda + \frac{n}{2} + 2C - C_1 - 1\right)}{\Gamma\left(n\lambda + \frac{n}{2} + 2C - 1\right)}(\lambda\eta)^{C_1} \tag{37}$$

$$E\left(\beta^{C_1+1}\right) = \frac{\Gamma\left(n\lambda + \frac{n}{2} + 2C - C_1 - 2\right)}{\Gamma\left(n\lambda + \frac{n}{2} + 2C - 1\right)}(\lambda\eta)^{C_1+1} \tag{38}$$

$$E\left(\beta^{C_1+2}\right) = \frac{\Gamma\left(n\lambda + \frac{n}{2} + 2C - C_1 - 3\right)}{\Gamma\left(n\lambda + \frac{n}{2} + 2C - 1\right)}(\lambda\eta)^{C_1+2} \tag{39}$$

Substitute the value of equations (37), (38) and (39) in equation (36), we get

$$P(\beta_{ALF}) = \frac{(\lambda\eta)^{C_1+2}}{\Gamma\left(n\lambda + \frac{n}{2} + 2C - 1\right)}\left[\Gamma\left(n\lambda + \frac{n}{2} + 2C - C_1 - 3\right) - \frac{\Gamma\left(n\lambda + \frac{n}{2} + 2C - C_1 - 2\right)}{n\lambda + \frac{n}{2} + 2C - C_1 - 2}\right] \tag{40}$$

Remark 5. If C=1/2 in equation (40), we get the posterior risk which is same as in equation (27).

Under weighted loss function, the posterior risk is

$$P(\beta_{WLF}) = E(\beta) - \frac{1}{E\left(\beta^{-1}\right)} \tag{41}$$

From equation (32), we have

$$E\left(\beta^{-1}\right) = \frac{n\lambda + \frac{n}{2} + 2C - 1}{\lambda\eta} \tag{42}$$

By Substituting the value of equations (33) and (42) in equation (41), we get

$$P(\beta_{WLF}) = \frac{\lambda\eta}{\left(n\lambda + \frac{n}{2} + 2C - 2\right)\left(n\lambda + \frac{n}{2} + 2C - 1\right)} \tag{43}$$

Remark 6. If C=1/2 in equation (43), we get the posterior risk which is same as in equation (30).

Table 1: Bayes estimator under Jeffrey's and Extension of Jeffrey`s Priors, using different Loss Functions

Loss function	Prior	
	Jeffrey's prior	Extension of Jeffrey's prior
Precautionary loss function	$\dfrac{\lambda\eta}{\left[\left(n\lambda+\frac{n}{2}-1\right)\left(n\lambda+\frac{n}{2}-2\right)\right]^{\frac{1}{2}}}$	$\dfrac{\lambda\eta}{\left[\left(n\lambda+\frac{n}{2}+2c-2\right)\left(n\lambda+\frac{n}{2}+2c-3\right)\right]^{\frac{1}{2}}}$
Al-Bayyati's new loss function	$\dfrac{\lambda\eta}{\left(n\lambda+\frac{n}{2}-C_1-1\right)}$	$\dfrac{\lambda\eta}{\left(n\lambda+\frac{n}{2}+2c-C_1-2\right)}$
Weighted loss function	$\dfrac{\lambda\eta}{\left(n\lambda+\frac{n}{2}\right)}$	$\dfrac{\lambda\eta}{\left(n\lambda+\frac{n}{2}+2c-1\right)}$

Table 2: Posterior risks under Jeffrey's and Extension of Jeffrey`s Priors, using different Loss Functions

Loss function	Prior	
	Jeffrey's prior	Extension of Jeffrey's prior
Precautionary loss function	$2\lambda\eta\left[\dfrac{1}{\left\{\left(n\lambda+\frac{n}{2}-1\right)\left(n\lambda+\frac{n}{2}-2\right)\right\}^{\frac{1}{2}}}-\dfrac{1}{n\lambda+\frac{n}{2}-1}\right]$	$2\lambda\eta\left[\dfrac{1}{\left\{\left(n\lambda+\frac{n}{2}+2c-2\right)\left(n\lambda+\frac{n}{2}+2c-3\right)\right\}^{\frac{1}{2}}}-\dfrac{1}{n\lambda+\frac{n}{2}+2c-2}\right]$
Al-Bayyati's new loss function	$\dfrac{(\lambda\eta)^{C_1+2}}{\Gamma\left(n\lambda+\frac{n}{2}\right)}\left[\Gamma\left(n\lambda+\frac{n}{2}-C_1-2\right)-\dfrac{\Gamma\left(n\lambda+\frac{n}{2}-C_1-1\right)}{n\lambda+\frac{n}{2}-C_1-1}\right]$	$\dfrac{(\lambda\eta)^{C_1+2}}{\Gamma\left(n\lambda+\frac{n}{2}+2c-1\right)}\left[\Gamma\left(n\lambda+\frac{n}{2}+2c-C_1-3\right)-\dfrac{\Gamma\left(n\lambda+\frac{n}{2}+2c-C_1-2\right)}{n\lambda+\frac{n}{2}+2c-C_1-2}\right]$
Weighted loss function	$\dfrac{\lambda\eta}{\left(n\lambda+\frac{n}{2}\right)\left(n\lambda+\frac{n}{2}-1\right)}$	$\dfrac{\lambda\eta}{\left(n\lambda+\frac{n}{2}+2c-2\right)\left(n\lambda+\frac{n}{2}+2c-1\right)}$

5. Real Life Data

The following real data set is considered for illustration of the proposed methodology. The data below are from an accelerated life test of 59 conductors, failure times are in hours, and there are no censored observations Lawless (2003).

2.997, 4.137, 4.288, 4.531, 4.700, 4.706, 5.009, 5.381, 5.434, 5.459, 5.589, 5.640, 5.807, 5.923, 6.033, 6.071, 6.087, 6.129, 6.352, 6.369, 6.476, 6.492, 6.515, 6.522, 6.538, 6.545, 6.573, 6.725, 6.869, 6.923, 6.948, 6.956, 6.958, 7.024, 7.224, 7.365, 7.398, 7.459, 7.489, 7.495, 7.496, 7.543, 7.683, 7.937, 7.945, 7.974, 8.120, 8.336, 8.532, 8.591, 8.687, 8.799, 9.218, 9.254, 9.289, 9.663, 10.092, 10.491, 11.038

Programs have been developed in R language to obtain the Bayes estimates and posterior risks and are presented in the tables below:

Table 3: Estimates and (Posterior risk) of β under Jeffrey's prior

λ	MLE	PLF	ALF				WLF
			$c_1 = 1$	$c_1 = 2$	$c_1 = -1$	$c_1 = -2$	
0.312	19.704	20.343	20.563	21.021	19.704	19.302	19.704
		(0.436)	(14.001)	(83.648)	(1.545)	(**0.277**)	(0.420)
0.467	24.766	25.436	25.666	26.140	24.766	24.340	24.766
		(0.456)	(22.699)	(167.370)	(1.625)	(**0.233**)	(0.442)
0.564	27.183	27.851	28.079	28.548	27.183	26.757	27.183
		(0.453)	(26.920)	(58.717)	(1.619)	(**0.212**)	(0.440)
0.785	31.328	31.961	32.177	32.619	31.328	30.920	31.328
		(0.429)	(33.355)	(82.631)	(1.540)	(**0.176**)	(0.419)

MLE=Maximum Likelihood Estimate, PLF= Precautionary Loss Function, ALF= Al-Bayyati's New loss Function, WLF=Weighted Loss Function

Table 4: Bayes Estimates and (Posterior risk) of β under extension of Jeffrey's prior

λ	c	PLF	ALF				WLF
			$c_1=1$	$c_1=2$	$c_1=-1$	$c_1=-2$	
	0.5	20.343	20.563	21.021	19.704	19.302	19.704
		(0.436)	(14.001)	(83.648)	(1.545)	(**0.277**)	(0.420)
	1.0	19.913	20.125	20.563	19.302	18.915	19.302
0.312		(0.418)	(12.845)	(74.983)	(1.482)	(**0.271**)	(0.403)
	2.0	19.107	19.302	19.704	18.543	18.186	18.543
		(0.385)	(10.869)	(60.697)	(1.367)	(**0.260**)	(0.371)
	2.5	18.728	18.915	19.301	18.186	17.842	18.186
		(0.370)	(10.023)	(54.784)	(1.314)	(**0.256**)	(0.357)
	0.5	25.436	25.666	26.104	24.766	24.340	24.766
		(0.456)	(22.699)	(167.379)	(1.625)	(**0.233**)	(0.442)
	1.0	24.986	25.208	25.666	24.340	23.927	24.340
0.467		(0.440)	(21.122)	(152.823)	(1.569)	(**0.229**)	(0.427)
	2.0	24.132	24.340	24.766	23.529	23.143	23.529
		(0.410)	(18.358)	(127.990)	(1.466)	(**0.221**)	(0.398)
	2.5	23.723	23.927	24.340	23.143	22.770	23.143
		(0.397)	(17.146)	(117.415)	(1.418)	(**0.218**)	(0.385)
	0.5	27.851	28.079	28.548	27.183	26.757	27.183
		(0.453)	(26.920)	(58.717)	(1.619)	(**0.212**)	(0.440)
	1.0	27.402	27.623	28.078	26.757	26.344	26.757
0.564		(0.438)	(25.218)	(198.918)	(1.568)	(**0.209**)	(0.426)
	2.0	26.550	26.757	27.183	25.944	25.555	25.944
		(0.411)	(22.200)	(169.356)	(1.473)	(**0.203**)	(0.400)
	2.5	26.143	26.344	26.757	25.555	25.178	25.555
		(0.399)	(20.860)	(156.563)	(1.429)	(**0.199**)	(0.388)
	0.5	31.961	32.177	32.619	31.328	30.920	31.328
		(0.429)	(33.355)	(82.631)	(1.540)	(**0.176**)	(0.419)
	1.0	31.537	31.747	32.177	30.920	30.523	30.920
0.785		(0.417)	(31.607)	(77.211)	(1.500)	(**0.174**)	(0.409)
	2.0	30.720	30.920	31.328	30.136	29.758	30.136
		(0.396)	(28.441)	(67.597)	(1.425)	(**0.169**)	(0.387)
	2.5	30.327	30.523	30.920	29.758	29.390	29.758
		(0.386)	(27.007)	(63.330)	(1.389)	(**0.167**)	(0.377)

6. Conclusion

In this paper, we define length biased Nakagami distribution and study its various characteristics. The estimates and the posterior risk of the scale parameter of the model have been obtained. The application of the new model has been demonstrated with the help of real life data set. The results are shown in the tables above.

Table 3 and table 4 shows the Bayes estimates and posterior risk of the scale parameter β for different values of the shape parameter λ under the Jeffrey's and extension of Jeffrey's priors. The initial value of the shape parameter λ is obtained through R-software. The value of the loss parameter c_1 is taken as ±1 and ±2. We also take different values of the hyper-parameter c as 0.5, 1.0, 2.0, and 2.5 respectively. From both the tables it is clear that as we increase the value of λ, the value of estimates of β increases. The estimates obtained under extension of Jeffrey's prior coincides with the estimates obtained under Jeffrey's prior when the value of c is 0.5. It is also observed that as we increase the value of hyper-parameter c, the value of posterior risk decreases. Also the value of posterior risk obtained under extension of Jeffrey's prior coincides with the values of posterior risk obtained under Jeffrey's prior when the value of c is 0.5. We also observe that it is the Al-Bayyati's new loss function which has the minimum value of posterior risk as compared to other loss functions. So we can say that the Al-Bayyati's new loss function is the better loss function as compared to other loss functions which are compared in this manuscript.

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