A Two-Parameter Ratio-Product-Ratio Type Exponential Estimator for Finite Population Mean in Sample Surveys

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Abstract

This paper suggests a two-parameter ratio-product-ratio type exponential estimator for a finite population mean in simple random sampling without replacement (SRSWOR) following the methodology in the studies of Singh and Espejo (2003) and Chami et al (2012). The bias and mean squared error of the suggested estimator are obtained to the first degree of approximation. The conditions are obtained in which suggested estimator is more efficient than the sample mean, classical ratio and product estimators, ratio-type and product type exponential estimators. An empirical study is given in support of the present study.

Keywords: Finite populations mean, Study Variable, Auxiliary Variable, Bias, Mean Squared Error.

1. Introduction

In sample surveys it is well established fact that the improvement in the precision of an estimator of the population mean $\bar{Y}$ of the study variable y is possible by using information on an auxiliary variable x (highly correlated with the study variable y) at the estimation stage. It is known that if the correlation between study variate y and the auxiliary variate x is positive (high) the usual ratio estimator is employed. On the other hand if this correction is negative (high), the product method of estimation can be employed. Consider the finite population $U=(U_1,U_2,...,U_N)$ of size N. Let (y,x) be the study, auxiliary variates respectively likely values (y_i,x_i), $i=1,2,...,N$ respectively. Let $\left(\bar{Y}=\frac{1}{N}\sum_{i=1}^{N}y_i, \bar{X}=\frac{1}{N}\sum_{i=1}^{N}x_i\right)$ be the population means of the variates (y,x) respectively. It is desired to estimate the population means of the variates (y,x) respectively. It is desired to estimate the population mean $\bar{Y}$ of the study variable y using known population mean $\bar{X}$ of the auxiliary variable x. For estimating the population mean $\bar{Y}$, a simple random sample of size n is drawn without replacement from the population U.
The usual unbiased estimator for population mean $\bar{Y}$ is defined by

$$\bar{y}=\frac{1}{n}\sum_{i=1}^{n}y_i \tag{1}$$

The classical ratio and product estimators for population mean $\bar{Y}$ are respectively given by

$$\bar{y}_R = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right) \tag{1.2}$$

and

$$\bar{y}_P = \bar{y}\left(\frac{\bar{x}}{\bar{X}}\right) \tag{1.3}$$

It can be shown that the ratio estimator $\bar{y}_R$ is more efficient than the unbiased estimator $\bar{y}$ if

$$C > \frac{1}{2} \tag{1.4}$$

and the product estimator $\bar{y}_P$ is more efficient than the usual unbiased estimator $\bar{y}$ if

$$C < -\frac{1}{2}, \tag{1.5}$$

where $C = \frac{C_y}{C_x}, C_y = \frac{S_y}{\bar{Y}},\ C_x = \frac{S_x}{\bar{X}},\ \rho = \frac{S_{yx}}{S_y S_x},$

$S_y^2 = (N-1)^{-1}\sum_{i=1}^{N}\left(y_i - \bar{Y}\right)^2,\ S_x^2 = (N-1)^{-1}\sum_{i=1}^{N}\left(x_i - \bar{X}\right)^2$ and

$S_{yx} = (N-1)^{-1}\sum_{i=1}^{N}\left(y_i - \bar{Y}\right)\left(x_i - \bar{X}\right).$

Bahl and Tuteja (1991) suggested the ratio-type and product-type exponential estimators for the population $\bar{Y}$ respectively as

$$\bar{y}_{Re} = \bar{y}\,exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{1.6}$$

and

$$\bar{y}_{Pe} = \bar{y}\,exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right). \tag{1.7}$$

The ratio-type exponential estimator $\bar{y}_{Re}$ is better than unbiased estimator $\bar{y}$ if

$$C > \frac{1}{4} \tag{1.8}$$

and the product-type exponential estimator $\bar{y}_{Pe}$ is better than the unbiased estimator $\bar{y}$ if

$$C < -\frac{1}{4} \tag{1.9}$$

In this paper taking motivation from Singh and Ruiz Espejo (2003) and Chami et al (2012), we have suggested a class of ratio-product-ratio-type exponential estimators for estimating the population mean $\bar{Y}$ and its properties are studied under large sample approximation. We have compared the proposed class of ratio-product-ratio-type exponential estimators with the three traditional estimators $(\bar{y}, \bar{y}_R, \bar{y}_P)$, ratio-type exponential estimator $\bar{y}_{Re}$ and product-type exponential estimator $\bar{y}_{Pe}$ and conditions are obtained in which the proposed class of ratio-product-ratio-type exponential estimators is preferred. We carry out an empirical study showing that the proposed class of ratio-product-ratio-type exponential estimators out performs the estimators $\bar{y}$, $\bar{y}_R$, $\bar{y}_{Re}$, $\bar{y}_P$ and $\bar{y}_{Pe}$. In this context reader is referred to Shirley et al (2014), Singh and Pal (2015, 2017) Singh et al (2016) and Sharma and Singh (2015).

2. The Suggested Two Parameter Ratio-Product-Ratio Type Exponential Estimator

For estimating the population mean $\bar{Y}$ of the main variable y, we suggest the following two-parameter ratio-product-ratio type exponential estimator:

$$T_{e(\alpha,\beta)} = \bar{y}\left[\alpha\, exp\left\{\frac{(1-2\beta)(\bar{x}-\bar{X})}{(\bar{x}+\bar{X})}\right\} + (1-\alpha)exp\left\{\frac{(1-2\beta)(\bar{X}-\bar{x})}{(\bar{x}+\bar{X})}\right\}\right] \tag{2.1}$$

where α,β are real constants. The aim of this paper is to derive values for these constants α,β such that the bias and / or the mean squared error (MSE) of $T_{e(\alpha,\beta)}$ is minimal.

We mention that $T_{e(\alpha,\beta)} = T_{e(1-\alpha,1-\beta)}$, that is the estimator $T_{e(\alpha,\beta)}$ is invariant under point reflection through the point $(\alpha,\beta)=\left(\frac{1}{2},\frac{1}{2}\right)$. In the point of symmetry $(\alpha,\beta)=\left(\frac{1}{2},\frac{1}{2}\right)$, the estimator reduces to the sample mean $\bar{y}$, that is, we have $T_{e\left(\frac{1}{2},\frac{1}{2}\right)} = \bar{y}$. In fact, on the whole line $\beta=\frac{1}{2}$ our proposed estimator reduces to the sample mean estimator $\bar{y}$, that is $T_{e\left(\alpha,\frac{1}{2}\right)} = \bar{y}$.

For $(\alpha,\beta)=(1,1)$ the proposed estimator $T_{e(\alpha,\beta)}$ reduces to the Bahl and Tuteja (1991) ratio-type exponential estimator $\bar{y}_{Re}$ defined at (1.6) while for $(\alpha,\beta)=(1,0),(0,1)$ it reduces to the Bahl and Tuteja (1991) product-type exponential estimator $\bar{y}_{Pe}$ defined at (1.7). Owing to the simplicity of the proposed estimator $T_{e(\alpha,\beta)}$ and that all three known estimators $\bar{y}$, $\bar{y}_{Re}$, $\bar{y}_{Pe}$ can be obtained from it by choosing appropriate parameters why we study the estimator in (2.1) and compare it to the three known estimators ($\bar{y}$, $\bar{y}_{Re}$, $\bar{y}_{Pe}$), usual ratio estimator $\bar{y}_R$ and product estimator $\bar{y}_P$.

2.1 Bias and Mean Squared Error (*MSE*) of the Proposed Estimator

Applying the standard techniques we evaluate the first degree of approximation (upto terms of order n^{-1}) to the bias and mean squared error *(MSE)* of the suggested estimator $T_{e(\alpha,\beta)}$.

Let $\bar{y}=\bar{Y}(1+e_0)$, and $\bar{x}=\bar{X}(1+e_1)$

so that $E(e_0)=E(e_1)=0$, $E(e_0^2)=\frac{(1-f)}{n}C_y^2$, $E(e_1^2)=\frac{(1-f)}{n}C_x^2$, $E(e_0e_1)=\frac{(1-f)}{n}\rho C_y C_x$, where, $f=\frac{n}{N}$ is the sampling fraction. Further, it is assumed that the sample is so large as to make $|e_0|$ and $|e_1|$ small, justifying the first degree approximation considerd wherein we ignore the terms involving e_0 and/or e_1 in a degree greater than two, see Sahai and Ray(1980, p.272).

Bias of the Estimator $T_{e(\alpha,\beta)}$

Expressing (2.1) in terms of *e's* we have

$$T_{e(\alpha,\beta)}=\bar{Y}(1+e_0)\left[\alpha\, exp\left\{\frac{(1-2\beta)e_1}{(2+e_1)}\right\}+(1-\alpha)exp\left\{\frac{-(1-2\beta)e_1}{(2+e_1)}\right\}\right]$$

$$=\bar{Y}(1+e_0)\left[\alpha\, exp\left\{\frac{(1-2\beta)e_1}{2}\left(1+\frac{e_1}{2}\right)^{-1}\right\}+(1-\alpha)exp\left\{\frac{-(1-2\beta)e_1}{2}\left(1+\frac{e_1}{2}\right)^{-1}\right\}\right] \quad (2.2)$$

Expanding the right hand side of (2.2) and neglecting terms of *e's* having power greater than two we have

$$T_{e(\alpha,\beta)}\cong\bar{Y}\left\{1+e_0-\frac{(1-2\alpha)(1-2\beta)}{2}e_1+\frac{(3-4\alpha-2\beta)}{8}(1-2\beta)e_1^2-\frac{(1-2\alpha)(1-2\beta)}{2}e_0e_1\right\} \quad (2.3)$$

or

$$\left(T_{e(\alpha,\beta)}-\bar{Y}\right)=\bar{Y}\left\{e_0-\frac{(1-2\alpha)(1-2\beta)}{2}(e_1+e_0e_1)+\frac{(3-4\alpha-2\beta)(1-2\beta)}{8}e_1^2\right\} \quad (2.4)$$

Taking expectation of both sides of (2.4) we get the bias of $T_{e(\alpha,\beta)}$ to the first degree of approximation as

$$B\left(T_{e(\alpha,\beta)}\right)=\frac{(1-f)}{8n}\bar{Y}C_x^2(1-2\beta)[3-4\alpha-2\beta-4C+8\alpha C] \quad (2.5)$$

The bias of $T_{e(\alpha,\beta)}$ would be zero if

$$\beta=\frac{1}{2} \text{ or } \qquad \beta=1.5-2\alpha-2C+4\alpha C. \quad (2.6)$$

The suggested ratio-product –ratio-type exponential estimator $T_{e(\alpha,\beta)}$, substituted with the values of β from (2.6), becomes an (approximately) unbiased estimator for the population mean $\hat{\bar{Y}}$.In the three dimensional parameter space $(\alpha,\beta,C)\in R^3$, these unbiased estimators lie on a plane (in the case $\beta=\frac{1}{2}$). We mention further that as the sample size n approaches the population size *N*, the bias $T_{e(\alpha,\beta)}$ becomes negligible, since the factor $\frac{(1-f)}{n}$ tends to zero.

Mean Squared Error of $T_{e(\alpha,\beta)}$

Squaring both sides of (2.4) and neglecting terms of $e's$ having power greater than two we have

$$\left(T_{e(\alpha,\beta)}-\bar{Y}\right)^2=\bar{Y}^2\left\{e_0^2-(1-2\alpha)(1-2\beta)e_0e_1+\frac{(1-2\alpha)^2(1-2\beta)^2}{4}e_1^2\right\} \tag{2.7}$$

Taking expectation of both sides of (2.7) we get the *MSE* of the estimator $T_{e(\alpha,\beta)}$ to the first degree of approximation as

$$MSE\left(T_{e(\alpha,\beta)}\right)=E\left[T_{e(\alpha,\beta)}-\bar{Y}\right]^2$$
$$=\frac{(1-f)}{n}\bar{Y}^2\left[C_y^2+\frac{(1-2\alpha)(1-2\beta)}{4}C_x^2\{(1-2\alpha)(1-2\beta)-4C\}\right] \tag{2.8}$$

Taking the gradient $\nabla=\left(\frac{\delta}{\delta\alpha},\frac{\delta}{\delta\beta}\right)$ of (2.6), we get

$$\nabla MSE(T_e)=\frac{2(1-f)}{n}\bar{Y}^2C_x^2\left[\frac{(1-2\alpha)(1-2\beta)}{2}-C\right](1-2\alpha,1-2\beta) \tag{2.9}$$

Equating (2.9) to zero to obtain the critical points, we get the following solutions:

$$\alpha=\frac{1}{2},\ \ \beta=\frac{1}{2} \tag{2.10}$$

or

$$2C=(1-2\alpha)(1-2\beta) \tag{2.11}$$

It can be easily shown that the critical point in (2.10) is a saddle point unless $C=0$, in which case we get a local minimum. However, the critical points determined by (2.11) are always local minima; for a given C, (2.11) is the equation of a hyperbola symmetric through $(\alpha,\beta)=\left(\frac{1}{2},\frac{1}{2}\right)$. Inserting (2.10) into the estimator $T_{e(\alpha,\beta)}$ gives the unbiased estimator $\bar{y}$ (sample mean) of the population mean $\bar{Y}$. Thus we get the *MSE* of the sample mean $\bar{y}$ as

$$MSE\left(T_{e\left(\frac{1}{2},\frac{1}{2}\right)}\right)=MSE(\bar{y})=\frac{(1-f)}{n}\bar{Y}^2C_y^2=\frac{(1-f)}{n}S_y^2\ . \tag{2.12}$$

For $\alpha=\frac{1}{2}$ in (2.1), the class of estimators $T_{e(\alpha,\beta)}$ reduces to the estimator for the population mean $\bar{Y}$ as

$$T_{e\left(\frac{1}{2},\beta\right)}=\frac{\bar{y}}{2}\left[exp\left\{\frac{(1-2\beta)(\bar{x}-\bar{X})}{(\bar{x}+\bar{X})}\right\}+exp\left\{\frac{-(1-2\beta)(\bar{x}-\bar{X})}{(\bar{x}+\bar{X})}\right\}\right] \tag{2.13}$$

whose bias and *MSE* to the first degree of approximation are respectively given by

$$B\left(T_{e\left(\frac{1}{2},\beta\right)}\right)=\bar{Y}\frac{(1-f)}{8n}(1-2\beta)^2\ C_x^2 \tag{2.14}$$

and

$$MSE\left(T_{e\left(\frac{1}{2},\beta\right)}\right)=\frac{(1-f)}{n}S_y^2=MSE(\bar{y}) \tag{2.15}$$

It is observed from (2.14) and (2.15) that the proposed estimator $T_{e(\alpha,\beta)}$ at $\alpha=\frac{1}{2}$ is positively biased though it will be negligible for sufficiently large sample size n. However it has *MSE* equal to the sample mean $\bar{y}$. So the estimator $T_{e(1/2,\beta)}$ at (2.13) is not advisable to use in practice.

By substituting (2.10) into the estimator, an asymptotically optimum estimator (*AOE*) $T^{(0)}_{e(\alpha,\beta)}$ is obtained. For the first degree approximation of the *MSE*, we find (independent of α and β)

$$MSE\left(T^{(0)}_{e(\alpha,\beta)}\right)=\frac{(1-f)}{n}S_y^2\left(1-\rho^2\right). \tag{2.16}$$

In fact Srivastava (1971, 1980) has shown that $\frac{(1-f)}{n}S_y^2\left(1-\rho^2\right)$ is the minimal possible *MSE* up to first degree of approximation for a large class of estimators to which the estimator (2.1) also belongs, for example, for estimators of the form $\bar{y}_g=\bar{y}.g\left(\frac{\bar{x}}{\bar{X}}\right)$ where g is a C^2 function with $g(1)=1$. Further Srivastava and Jhajj (1981) incorporating sample and population variances of the auxiliary variable *x* might yield an estimator that has a lower *MSE* than $\frac{(1-f)}{n}S_y^2\left(1-\rho^2\right)$ especially when the relationship between the study variate *y* and the auxiliary variate *x* is markedly non-linear. Thus irrespective of value of *C*, we are always able to select an *AOE* $\bar{y}^{(0)}_{e(\alpha,\beta)}$ from the two-parameter family in (2.1).

3. Comparison of Mean Squared Errors and Choice of Parameters

3.1 Comparing the *MSE* of the Sample Mean $\bar{y}$ to the Suggested Estimator $T_{e(\alpha,\beta)}$

From (2.8) and (2.12) we have

$$MSE\left(T_{e(\alpha,\beta)}\right)-MSE(\bar{y})=\frac{(1-f)}{4n}\bar{Y}^2C_x^2\left[(1-2\alpha)(1-2\beta)\{(1-2\alpha)(1-2\beta)-4C\}\right]$$

which is less than zero if

$$(1-2\alpha)(1-2\beta)\{(1-2\alpha)(1-2\beta)-4C\}<0 \tag{3.1}$$

Therefore, either

(i) $\alpha>\frac{1}{2},\beta>\frac{1}{2}\,and\,C>\frac{(1-2\alpha)(1-2\beta)}{4}$

(ii) $\alpha<\frac{1}{2},\beta>\frac{1}{2}\,and\,C<\frac{(1-2\alpha)(1-2\beta)}{4}$

(iii) $\alpha>\frac{1}{2},\beta<\frac{1}{2}\,and\,C<\frac{(1-2\alpha)(1-2\beta)}{4}$

(iv) $\alpha < \frac{1}{2}, \beta < \frac{1}{2}\, and\, C > \frac{(1-2\alpha)(1-2\beta)}{4}$

Combining these with the condition

$$-\frac{1}{4} \le C \le \frac{1}{4},$$

we get the following explicit ranges **:**

(i) If $0 < C \le \frac{1}{4}\, and\, \beta > \frac{1}{2}$, then

$$\frac{1}{4} < \alpha < \frac{(2\beta + 4C - 1)}{2(2\beta - 1)}, \text{ (from (i))}$$

(ii) If $0 \le C \le \frac{1}{4}\, and\, \beta < \frac{1}{2}$, then

$$\frac{(2\beta + 4C - 1)}{2(2\beta - 1)} < \alpha < \frac{1}{2}, \text{ (from (iv))}$$

(iii) If $-\frac{1}{2} \le C \le 0\, and\, \beta > \frac{1}{2}$, then

$$\frac{(2\beta + 4C - 1)}{2(2\beta - 1)} < \alpha < \frac{1}{2}, \text{ (from (ii))}$$

(iv) If $-\frac{1}{2} \le C \le 0\, and\, \beta < \frac{1}{2}$, then

$$\frac{1}{2} < \alpha < \frac{(2\beta + 4C - 1)}{2(2\beta - 1)}, \text{ (from (iii))}$$

We note that the case *C=0* implies $\rho = 0$, and thus the sample mean $\bar{y}$ is the estimator with minimal MSE.

3.2 Comparing the *MSE* of the Ratio-Type Exponential Estimator $\bar{y}_{Re}$ to the Suggested Estimator $T_{e(\alpha,\beta)}$

For $C > \frac{1}{4}$, the ratio-type exponential estimator $\bar{y}_{Re}$ due to Bahl and Tuteja (1991) is used instead of the sample mean $\bar{y}$ or product-type exponential estimator $\bar{y}_{Pe}$. Thus, we are concerned with a range of plausible values for α and β, where the suggested estimator $T_{e(\alpha,\beta)}$ performs better than the ratio-type exponential estimator $\bar{y}_{Re}$.

Putting $(\alpha, \beta) = (1,1)$ in (2.8) we get the *MSE* of the ratio-type exponential estimator $\bar{y}_{Re}$ as

$$MSE(\bar{y}_{Re}) = \frac{(1-f)}{n}\bar{Y}^2\left[C_y^2 + \frac{C_x^2}{4}(1-4C)\right] \tag{3.2}$$

From (2.8) and (3.2) we have

$$MSE(\bar{y}_{Re}) - MSE(T_{e(\alpha,\beta)}) = \frac{(1-f)}{4n}\bar{Y}^2 C_x^2 [1-(1-2\alpha)(1-2\beta)]\,[1+(1-2\alpha)(1-2\beta)-4C] \tag{3.3}$$

which is non negative if

$$(2\alpha\beta - \alpha - \beta)[2C - 1 - (2\alpha\beta - \alpha - \beta)] > 0 \quad (3.4)$$

Therefore,

(i) $(2C-1)>(2\alpha\beta-\alpha-\beta)>0$ or

(ii) $(2C-1)<(2\alpha\beta-\alpha-\beta)<0$

Hence, from solution (i), where $C>\frac{1}{2}$, we have the following:

(i) If $\beta<\frac{1}{2}$, then $\frac{(\beta+2C-1)}{(2\beta-1)}<\alpha<\frac{\beta}{(2\beta-1)}$.

(ii) If $\beta>\frac{1}{2}$, then $\frac{\beta}{(2\beta-1)}<\alpha<\frac{(\beta+2C-1)}{(2\beta-1)}$.

Further, from solution (ii), where $\frac{1}{4}<C<\frac{1}{2}$, we obtain the following**:**

(i) If $\beta<\frac{1}{2}$, then $\frac{\beta}{(2\beta-1)}<\alpha<\frac{(\beta+2C-1)}{(2\beta-1)}$.

(ii) If $\beta>\frac{1}{2}$, then $\frac{(\beta+2C-1)}{(2\beta-1)}<\alpha<\frac{\beta}{(2\beta-1)}$.

3.3 Comparing the *MSE* of the Product-Type Exponential Estimator $\bar{y}_{Pe}$ to the Suggested Estimator $T_{e(\alpha,\beta)}$

It is known that $C<-\frac{1}{4}$, the product-type exponential estimator $\bar{y}_{Pe}$ is preferred to the sample mean estimator $\bar{y}$ and the ratio-type exponential estimator $\bar{y}_{Re}$. Putting $(\alpha,\beta)=(0,1)$ or $(1,0)$ in (2.9) we get the *MSE* of the product-type exponential estimator $\bar{y}_{Pe}$ to the first degree of approximation as

$$MSE(\bar{y}_{Pe})=\frac{(1-f)}{n}\bar{Y}^2\left[C_y^2+\frac{C_x^2}{4}(1+4C)\right] \tag{3.5}$$

We, therefore, seek a range of α and β values, where the suggested estimator $T_{e(\alpha,\beta)}$ has lesser *MSE* than the product-type exponential estimator $\bar{y}_{Pe}$ due to Bahl and Tuteja (1991).

From (2.8) and (3.5) we have

$$MSE(\bar{y}_{Pe})-MSE(T_{e(\alpha,\beta)})=\frac{(1-f)}{n}\bar{Y}^2C_x^2[1+2\alpha\beta-\alpha-\beta][2C-(2\alpha\beta-\alpha-\beta)] \tag{3.6}$$

which is positive if

$$[1+2\alpha\beta-\alpha-\beta][2C-(2\alpha\beta-\alpha-\beta)]>0 \tag{3.7}$$

We obtain the following two cases:

(i) $2C>(2\alpha\beta-\alpha-\beta)>-1$ (if both factors in (3.7) are positive) or

(ii) $2C < (2\alpha\beta - \alpha - \beta) < -1$ (if both factors in (3.7) are negative)

Observing that $C < -\frac{1}{4}$, so we get from (i),

$$-\frac{1}{2} \geq 2C > (2\alpha\beta - \alpha - \beta) > -1 \tag{3.8}$$

We mention that this implies $-\frac{1}{2} < C < -\frac{1}{4}$, and the range for α and β where these inequalities hold are explicitly given by the following two cases:

(i) If $\beta < \frac{1}{2}$, then $\frac{(\beta-1)}{(2\beta-1)} < \alpha < \frac{(\beta+2C)}{(2\beta-1)}$.

(ii) If $\beta > \frac{1}{2}$, then $\frac{(\beta+2C)}{(2\beta-1)} < \alpha < \frac{(\beta-1)}{(2\beta-1)}$.

For any given C, we again note that the two regions determined here are symmetric through $(\alpha,\beta) = \left(\frac{1}{2},\frac{1}{2}\right)$. We also note that the parameters (α,β) which yields an *AOE* [see (2.11)] which for a fixed *C* lie on a hyperbola, are contained in these regions.

In case (ii), where $2C < -1 \left(i.e.\, C < -\frac{1}{2}\right)$ (and therefore automatically $C < -\frac{1}{4}$), the range of α and β are given by

(i) If $\beta < \frac{1}{2}$, then $\frac{(\beta-1)}{(2\beta-1)} < \alpha < \frac{(\beta+2C)}{(2\beta-1)}$.

(ii) If $\beta > \frac{1}{2}$, then $\frac{(\beta+2C)}{(2\beta-1)} < \alpha < \frac{(\beta-1)}{(2\beta-1)}$.

3.4 Comparing the *MSE* of the Classical Ratio Estimator $\bar{y}_R$ to the Suggested Estimator $T_{e(\alpha,\beta)}$

To the first degree of approximation, the *MSE* of the usual ratio estimator $\bar{y}_R$ is given by

$$MSE(\bar{y}_R) = \frac{(1-f)}{n}\bar{Y}^2\left[C_y^2 + C_x^2(1-2C)\right] \tag{3.9}$$

From (2.8) and (3.9) we have

$$MSE(\bar{y}_R) - MSE(T_{e(\alpha,\beta)}) = \frac{(1-f)}{n}\bar{Y}^2 C_x^2\left[1 - \frac{(1-2\alpha)(1-2\beta)}{2}\right]\left[1 + \frac{(1-2\alpha)(1-2\beta)}{2} - 2C\right] \tag{3.10}$$

which is non-negative if

$$\left[1 - \frac{(1-2\alpha)(1-2\beta)}{2}\right]\left[1 - 2C + \frac{(1-2\alpha)(1-2\beta)}{2}\right] > 0 \tag{3.11}$$

Therefore,

(i) $(2C-1) > \frac{(1-2\alpha)(1-2\beta)}{2} > 1$ or

(ii) $(2C-1) < \frac{(1-2\alpha)(1-2\beta)}{2} < 1$

Hence from solution (i), where $C > 1$,we have the following:

(i) If $\beta < \frac{1}{2}$, then $\frac{(2\beta+4C-3)}{2(2\beta-1)} < \alpha < \frac{(2\beta+1)}{2(2\beta-1)}$.

(ii) If $\beta > \frac{1}{2}$, then $\frac{(2\beta+1)}{2(2\beta-1)} < \alpha < \frac{(2\beta+4C-3)}{2(2\beta-1)}$.

Also from solution (ii), where $\frac{1}{2} < C < 1$,we obtain the following:

(i) If $\beta < \frac{1}{2}$, then $\frac{(2\beta+1)}{2(2\beta-1)} < \alpha < \frac{(2\beta+4C-3)}{2(2\beta-1)}$.

(ii) If $\beta > \frac{1}{2}$, then $\frac{(2\beta+4C-3)}{2(2\beta-1)} < \alpha < \frac{(2\beta+1)}{2(2\beta-1)}$.

3.5 Comparing the *MSE* of the classical Product Estimator $\bar{y}_P$ to the Suggested Estimator $T_{e(\alpha,\beta)}$

It is well known that, for $C < -\frac{1}{2}$, the product estimator $\bar{y}_p$ is more efficient than the sample mean $\bar{y}$ and ratio estimator $\bar{y}_R$. Thus, we are concerned with a range of plausible values for α and β where the proposed estimator $T_{e(\alpha,\beta)}$ acts better than the classical product estimator $\bar{y}_P$.

The *MSE* of the usual product estimator $\bar{y}_p$ to the first degree of approximation as

$$MSE(\bar{y}_P) = \frac{(1-f)}{n}\bar{Y}^2\left[C_y^2 + C_x^2(1+2C)\right] \tag{3.12}$$

From (2.8) and (3.12) we have

$$MSE(\bar{y}_P) - MSE(T_{e(\alpha,\beta)}) = \frac{(1-f)}{n}\bar{Y}^2 C_x^2\left[1+\frac{(1-2\alpha)(1-2\beta)}{2}\right]\left[1+2C-\frac{(1-2\alpha)(1-2\beta)}{2}\right] \tag{3.13}$$

which is positive if

$$\left[1+\frac{(1-2\alpha)(1-2\beta)}{2}\right]\left[1+2C-\frac{(1-2\alpha)(1-2\beta)}{2}\right] > 0 \tag{3.14}$$

Thus the proposed estimator $T_{e(\alpha,\beta)}$ is more efficient than the classical product estimator $\bar{y}_p$ if

$$\left.\begin{array}{l} either\ -1<\dfrac{(1-2\alpha)(1-2\beta)}{2}<(2C+1) \\ \\ or\ (2C+1)<\dfrac{(1-2\alpha)(1-2\beta)}{2}<-1 \end{array}\right\} \tag{3.15}$$

or equivalently,

$$min.\{-1,(2C+1)\}<\frac{(1-2\alpha)(1-2\beta)}{2}<max.\{-1,(2C+1)\} \tag{3.16}$$

4. Unbiased Asymptotically Optimum Estimator (AOE)

From (2.6) and (2.11), we can calculate the parameters α and β where our proposed estimator becomes at least up to first approximation- an unbiased *AOE*. We obtain a line with $\beta=\frac{1}{2}$(recall that on this line our estimator always reduces to the sample mean estimator $\bar{y}$)

$$\beta=\frac{1}{2}\ , \qquad C=0 \tag{4.1}$$

or a "curve" $\left(\alpha^*(C),\beta^*(C),C\right)\in R^3$ in the parameter space with

$$\alpha^*(C)=\frac{1}{2}\left(1\pm\sqrt{\frac{C}{2C-1}}\right),\ \beta^*(C)=\frac{1}{2}\pm\sqrt{C(2C-1)}. \tag{4.2}$$

We mention that the parametric "curve" in (4.2) is only defined for $C\leq 0$ or $C>\frac{1}{2}$ in fact, this parametric "curve" is three hyperbolas.

Inserting the values of $\alpha^*(C)$ and $\beta^*(C)$ from (4.2) in (2.1), the proposed estimator takes the form:

$$\bar{y}_e^*(C)=T_{e\left(\alpha^*(C),\beta^*(C)\right)}$$

$$=\frac{\bar{y}}{2}\left[\left(1+\sqrt{\frac{C}{2C-1}}\right)exp\left\{\frac{2\sqrt{C(2C-1)}(\bar{X}-\bar{x})}{(\bar{X}+\bar{x})}\right\}+\left(1-\sqrt{\frac{C}{2C-1}}\right)exp\left\{\frac{2\sqrt{C(2C-1)}(\bar{x}-\bar{X})}{(\bar{X}+\bar{x})}\right\}\right] \tag{4.3}$$

It can be easily shown to the first degree of approximation that

$$\left.\begin{array}{l} B\left(\bar{y}_e^*(C)\right)=0, \\ MSE\left(\bar{y}_e^*(C)\right)=\dfrac{(1-f)}{n}S_y^2\left(1-\rho^2\right) \end{array}\right\} \tag{4.4}$$

Thus, the estimator $\bar{y}_e^*(C)$ of (4.3) is an unbiased AOE**.**

Using (2.11) in (2.5) we get the first degree of approximation of the bias of an *AOE*

$$B\left(T_{e(\alpha,\beta)}\right)=\frac{(1-f)}{8n}\bar{Y}C_x^2\left[4C(1-2C)+(1-2\beta)^2\right] \tag{4.5}$$

It follows from (4.5) and (2.11) that the bias can only be made zero if $C \le 0$ or $C \ge \frac{1}{2}$. Otherwise, there is always a positive contribution coming from the term $4C(1-2C)$ that does not vanish no matter what we select for $\beta \in R$.Further from (4.5) we note that the choice $\beta=\frac{1}{2}$ always yields the least possible bias. Given (2.11) and unless *C=0,* we can only assume β be close to $\frac{1}{2}$ and select α accordingly.

Remark 5.1: It is important to mention that the optimum choice of $\theta=(1-2\alpha)(1-2\beta)$ depends upon the value of $C=\rho_{yx}C_y/C_x$ .Reddy (1978) has shown that the value of '*C'* is more stable than other population parameters such as, the linear regression coefficient, over a period of time and least affected by the sampling fluctuations. So its value can be quite accurately guessed from the past data or a pilot survey or experienced gathered in due course of time. However, if the value of '*C'* is not known, one may estimate it on the basis of sample observations without much loss of efficiency, for instance, see Singh et al (1994, p. 216)**.**

6. Empirical Study

To illustrate the performance of the proposed class of estimators $T_{e(\alpha,\beta)}$ over sample mean estimator $\bar{y}$, Bahl and Tuteja's (1991) ratio-type $(\bar{y}_{Re})$ and product-type $(\bar{y}_{Pe})$ exponential estimators, classical ratio estimator $\bar{y}_R$ and product estimator $\bar{y}_P$ we have two natural population data sets.

Population-I [Source: Kadilar and Cingi (2003)]

y: Apple production amount.

x: Number of apples trees.

$N=106$, $n=20$, $\bar{Y}=2212.59$, $\bar{X}=27421.70$,

$C_y=5.22$, $C_x=2.10$, $\rho=0.86$

Population-II [Source: Gupta and Kothwala (1990)]

y: Population of irrigated area

x: Area under crop gram and mixture during 1983-1984 in a village of Rajasthan

$N=400$, $n=100$, $\bar{Y}=36.7183$, $\bar{X}=6.5638$,

$C_y=0.9928$, $C_x=0.9617$, $\rho=-0.4020$

We have computed the percent relative efficiency *(PRE)* of the proposed class of estimators $T_{e(\alpha,\beta)}$ with respect to $\bar{y}$, $\bar{y}_{Re}$, $\bar{y}_{Pe}$, $\bar{y}_R$ and $\bar{y}_P$, using the following formulae:

$$PRE\left(T_{e(\alpha,\beta)},\bar{y}\right)=\frac{C_y^2}{\left[C_y^2+\left(C_x^2/4\right)\left\{(1-2\alpha)^2(1-2\beta)^2-4C(1-2\alpha)(1-2\beta)\right\}\right]}\times 100 \qquad (6.1)$$

$$PRE\left(T_{e(\alpha,\beta)},\bar{y}_R\right)=\frac{\left\{C_y^2+C_x^2(1-2C)\right\}}{\left[C_y^2+\left(C_x^2/4\right)\left\{(1-2\alpha)^2(1-2\beta)^2-4C(1-2\alpha)(1-2\beta)\right\}\right]}\times 100 \qquad (6.2)$$

$$PRE\left(T_{e(\alpha,\beta)},\bar{y}_{Re}\right)=\frac{\left\{C_y^2+\left(C_x^2/4\right)(1-4C)\right\}}{\left[C_y^2+\left(C_x^2/4\right)\left\{(1-2\alpha)^2(1-2\beta)^2-4C(1-2\alpha)(1-2\beta)\right\}\right]}\times 100 \qquad (6.3)$$

$$PRE\left(T_{e(\alpha,\beta)},\bar{y}_P\right)=\frac{\left[\left\{C_y^2+C_x^2(1-2C)\right\}\right]}{\left[C_y^2+\left(C_x^2/4\right)\left\{(1-2\alpha)^2(1-2\beta)^2-4C(1-2\alpha)(1-2\beta)\right\}\right]}\times 100 \qquad (6.4)$$

$$PRE\left(T_{e(\alpha,\beta)},\bar{y}_{Pe}\right)=\frac{\left\{C_y^2+\left(C_x^2/4\right)(1+4C)\right\}}{\left[C_y^2+\left(C_x^2/4\right)\left\{(1-2\alpha)^2(1-2\beta)^2-4C(1-2\alpha)(1-2\beta)\right\}\right]}\times 100 \qquad (6.5)$$

Note 1: We have computed the values of $PRE\left(T_{e(\alpha,\beta)},\bar{y}\right)$, $PRE\left(T_{e(\alpha,\beta)},\bar{y}_R\right)$, and $PRE\left(T_{e(\alpha,\beta)},\bar{y}_{Re}\right)$ for population I as the correlation coefficient between study variate *y* and auxiliary variate *x* is positive.

Note 2: We have computed the values of $PRE\left(T_{e(\alpha,\beta)},\bar{y}\right)$, $PRE\left(T_{e(\alpha,\beta)},\bar{y}_P\right)$ and $PRE\left(T_{e(\alpha,\beta)},\bar{y}_{Pe}\right)$ for population II as the correlation between the study variable *y* and variable *x* is negative.

Findings are shown in tables 6.1 and 6.2.

Table 6.1: *PREs* of the proposed class of estimators $T_{e(\alpha,\beta)}$ with respect to the usual unbiased estimator $\bar{y}$, ratio estimator $\bar{y}_R$ and ratio-type exponential estimator $\bar{y}_{Re}$ for population I

(α,β)	$PRE\left(T_{e(\alpha,\beta)},\bar{y}\right)$	$PRE\left(T_{e(\alpha,\beta)},\bar{y}_R\right)$	$PRE\left(T_{e(\alpha,\beta)},\bar{y}_{Re}\right)$
(-2.00,0.00)	355.06	246.58	166.84
(-2.00,0.25)	257.77	179.02	121.12
(-1.75,0.00)	381.04	264.63	179.05
(-1.75,0.25)	234.53	162.88	110.20
(-1.50,-0.25)	262.65	182.41	123.42
(-1.50,-0.00)	379.55	263.59	178.35
(-1.50,0.25)	212.82	147.80	100.00

Table 6.1 continued…

(α, β)	$PRE\left(T_{e(\alpha,\beta)}, \bar{y}\right)$	$PRE\left(T_{e(\alpha,\beta)}, \bar{y}_R\right)$	$PRE\left(T_{e(\alpha,\beta)}, \bar{y}_{Re}\right)$
(-1.25,-0.25)	334.64	232.40	157.24
(-1.25,0.00)	351.21	243.91	165.03
(-1.00,0.50)	262.65	182.41	123.42
(-1.00,0.25)	381.04	264.63	179.05
(-1.00,0.00)	306.54	212.89	144.04
(-0.75,-0.75)	239.15	166.08	112.37
(-0.75,-0.50)	355.06	246.58	166.84
(-0.75,-0.25)	368.23	255.73	173.03
(-0.75,0.00)	257.77	179.02	121.12
(-0.50,-1.00)	262.65	182.41	123.42
(-0.50,-0.75)	355.06	246.58	166.84
(-0.50,-0.50)	379.55	263.59	178.35
(-0.50,-0.25)	306.54	212.89	144.04
(-0.50,-0.00)	212.82	147.80	100.00
(-0.25,-1.50)	262.65	182.41	123.42
(-0.25,-1.25)	334.64	232.40	157.24
(-0.25,-1.00)	381.04	264.63	179.03
(-0.25,-0.75)	368.23	255.73	173.03
(-0.25,-0.50)	306.54	212.89	144.04
(-0.25,-0.25)	234.53	162.88	110.20
(0.00,-2.00)	355.06	246.58	166.84
(0.00,-1.75)	381.04	264.63	179.05
(0.00,-1.50)	379.55	263.59	178.35
(0.00,-1.25)	351.21	243.91	165.03
(0.00,-1.00)	306.54	212.89	144.04
(0.00,-0.75)	257.77	179.02	121.12
(0.00,-0.50)	212.82	147.80	100.00
(0.25,-2.00)	257.77	179.02	121.12
(0.25,-1.75)	234.53	162.88	110.20
(0.25,-1.50)	212.82	147.80	100.00
(1.25,1.25)	234.53	162.88	110.20
(1.25,1.50)	306.54	212.89	144.04
(1.25,1.75)	368.23	255.73	173.03
(1.25,2.00)	381.04	264.63	179.05
(1.25,2.25)	334.64	232.40	157.24
(1.50,1.00)	212.82	147.80	100.00
(1.50,1.25)	306.54	212.89	144.04
(1.50,1.50)	379.55	263.59	178.35
(1.50,1.75)	355.06	246.58	166.84
(1.50,2.00)	262.65	182.41	123.42
(1.75,1.00)	257.77	179.02	121.12
(1.75,1.25)	368.23	255.73	173.03

Table 6.1 continued…

(α,β)	$PRE(T_{e(\alpha,\beta)},\bar{y})$	$PRE(T_{e(\alpha,\beta)},\bar{y}_R)$	$PRE(T_{e(\alpha,\beta)},\bar{y}_{Re})$
(1.75,1.50)	355.06	246.58	166.84
(1.75,1.75)	239.15	166.08	112.37
(2.00,1.00)	306.54	212.89	144.04
(2.00,1.25)	381.04	264.63	179.05
(2.00,1.50)	262.65	182.41	123.42
(2.25,1.00)	351.21	243.91	165.03
(2.25,1.25)	334.64	232.40	157.24
(2.50,0.75)	212.82	147.80	100.00
(2.50,1.00)	379.55	263.59	178.35
(2.50,1.25)	262.65	182.41	123.42
opt (α,β)	384.02	266.70	180.45

Table 6.2: Percent relative efficiencies (*PREs*) of the proposed class of estimators $T_{e(\alpha,\beta)}$ with respect to usual unbiased estimator $\bar{y}$, product estimator $\bar{y}_P$ and Product- type exponential estimator $\bar{y}_{Pe}$ for population II

(α,β)	$PRE(T_{e(\alpha,\beta)},\bar{y})$	$PRE(T_{e(\alpha,\beta)},\bar{y}_p)$	$PRE(T_{e(\alpha,\beta)},\bar{y}_{pe})$
(-2.00,0.75)	163.25	132.43	108.97
(-2.00,1.00)	197.16	159.93	131.60
(-1.75,0.75)	156.60	127.02	104.52
(-1.75,1.00)	197.74	160.40	131.99
(-1.75,1.25)	166.29	134.89	110.99
(-1.50,0.75)	149.82	121.53	100.00
(-1.50,1.00)	193.99	157.36	129.48
(-1.50,1.25)	183.80	149.09	122.68
(-1.25,1.00)	186.38	151.18	124.40
(-1.25,1.25)	195.24	158.37	130.31
(-1.25,1.50)	159.72	129.56	106.61
(-1.00,1.00)	175.78	142.58	117.32
(-1.00,1.25)	197.74	160.40	131.99
(-1.00,1.50)	183.80	149.09	122.68
(-0.75,1.00)	163.25	132.43	108.97
(-0.75,1.25)	190.62	154.62	127.23
(-0.75,1.50)	197.16	159.93	131.60
(-0.75,1.75)	178.46	144.76	119.12
(-0.50,1.00)	149.82	121.53	100.00
(-0.50,1.25)	175.78	142.58	117.32
(-0.50,1.50)	193.99	157.36	129.48
(-0.50,1.75)	197.16	159.93	131.60
(-0.50,2.00)	183.80	149.09	122.68
(-0.50,2.25)	159.72	129.56	106.61

Table 6.2 continued…

(α,β)	$PRE\left(T_{e(\alpha,\beta)},\bar{y}\right)$	$PRE\left(T_{e(\alpha,\beta)},\bar{y}_p\right)$	$PRE\left(T_{e(\alpha,\beta)},\bar{y}_{pe}\right)$
(-0.25,1.25)	156.60	127.02	104.52
(-0.25,1.50)	175.78	142.58	117.32
(-0.25,1.75)	190.62	154.62	127.23
(-0.25,2.00)	197.74	160.40	131.99
(-0.25,2.25)	195.24	158.37	130.31
(0.75,-2.00)	163.25	132.43	108.97
(0.75,-1.75)	156.60	127.02	104.52
(0.75,-1.50)	149.82	121.53	100.00
(1.00,-2.00)	197.16	159.93	131.60
(1.00,-1.75)	197.74	160.40	131.99
(1.00,-1.50)	193.99	157.36	129.48
(1.00,-1.25)	186.38	151.18	124.40
(1.00,-1.00)	175.78	142.58	117.32
(1.00,-0.75)	163.25	132.43	108.97
(1.00,-0.50)	149.82	121.53	100.00
(1.25,-1.75)	166.29	134.89	110.99
(-1.75,-1.50)	183.80	149.09	122.68
(-1.25,-1.25)	195.24	158.37	130.31
(-1.25,-1.00)	197.47	160.40	131.99
(-1.25,-0.75)	190.62	154.62	127.23
(-0.75,-0.50)	175.78	142.58	117.32
(-0.50,-0.25)	156.60	127.02	104.52
(1.50,-1.25)	159.72	129.56	106.61
(1.50,-1.00)	183.80	149.09	122.68
(1.50,-0.75)	197.16	159.93	131.60
(1.50,-0.50)	193.99	157.36	129.48
(1.50,-0.25)	175.78	142.58	117.32
(1.50,0.00)	149.82	121.53	100.00
(1.75,-0.75)	178.46	144.76	119.12
(1.75,-0.50)	197.16	159.93	131.60
(1.75,-0.25)	190.62	154.62	127.23
(1.75,0.00)	163.25	132.43	108.97
(2.00,-0.50)	183.80	149.09	122.68
(2.00,-0.25)	197.74	160.40	131.99
(2.00,0.00)	175.78	142.58	117.32
(2.25,-0.50)	159.72	129.56	106.61
(2.25,-0.25)	195.24	158.37	130.31
(2.25,0.00)	186.38	151.18	124.40
(2.50,-0.25)	183.80	149.09	122.68
(2.50,0.00)	193.99	157.36	129.48
(2.50,0.25)	149.82	121.53	100.00
$opt(\alpha,\beta)$	198.04	160.64	132.18

Tables 6.1 and 6.2 exhibit that

(i) there is considerable gain in efficiency by using the proposed class of estimators $T_{e(\alpha,\beta)}$ over $(\bar{y},\bar{y}_R,\bar{y}_{Re})$ for population I and $(\bar{y},\bar{y}_P,\bar{y}_{Pe})$ for population II;

(ii) larger gain in efficiency is seen by using $T_{e(\alpha,\beta)}$ over $\bar{y}$ as compared to $\bar{y}_R$ $(\bar{y}_P)$ and $\bar{y}_{Re}$ $(\bar{y}_{Pe})$.

(iii) largest gain in efficiency is observed at optimum (α,β);

(iv) there is considerable gain in efficiency by using the proposed class of estimators $T_{e(\alpha,\beta)}$ over $(\bar{y},\bar{y}_R,\bar{y}_{Re})$ for population I and $(\bar{y},\bar{y}_P,\bar{y}_{Pe})$ for population II even when the values of (α,β) deviates from their optimum values of (α,β).Thus there is enough scope of choosing the values of scalars (α,β) for obtaining estimators better than $(\bar{y},\bar{y}_R,\bar{y}_{Re},\bar{y}_P,\bar{y}_{Pe})$ from the proposed class of estimators $T_{e(\alpha,\beta)}$.

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